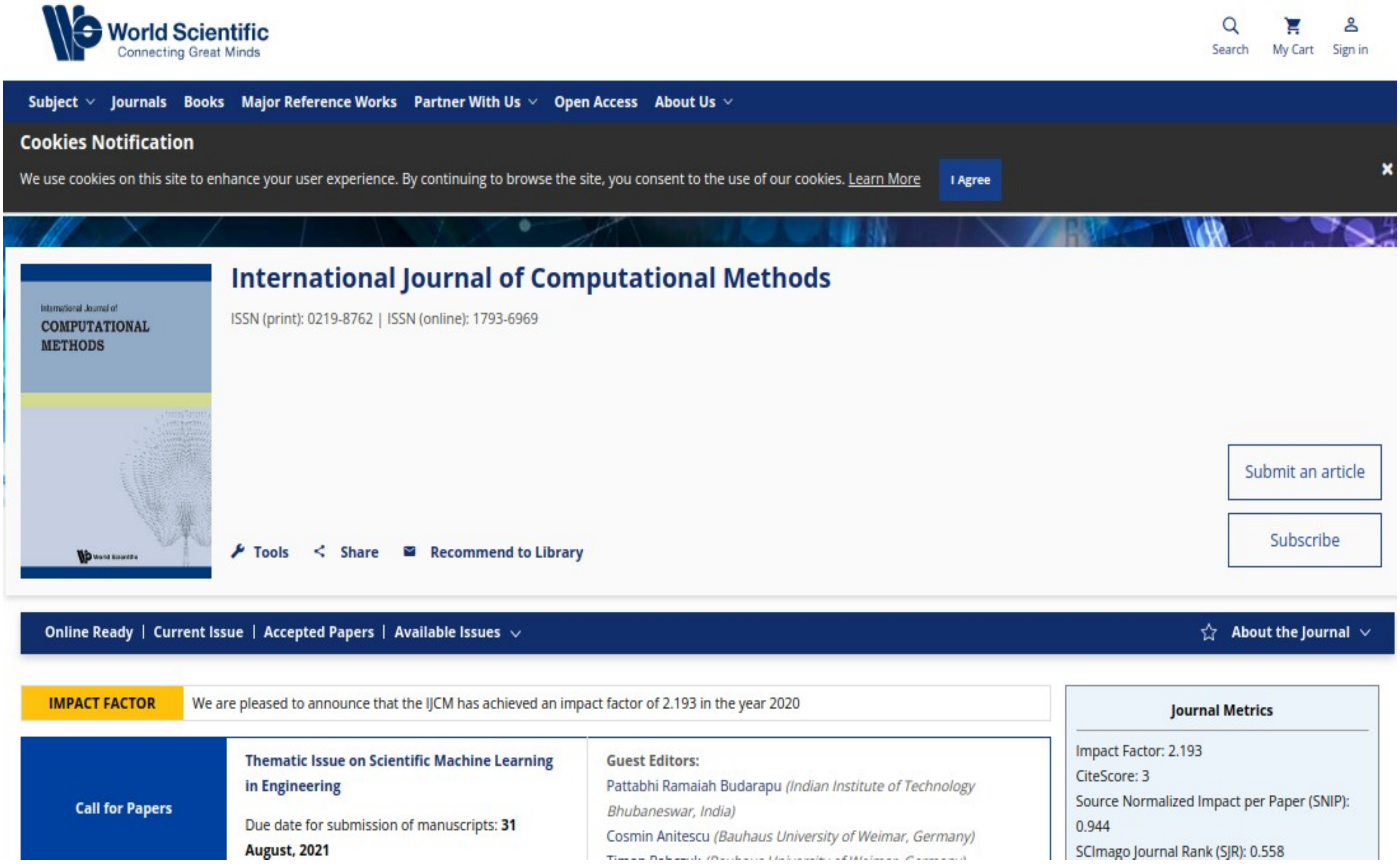

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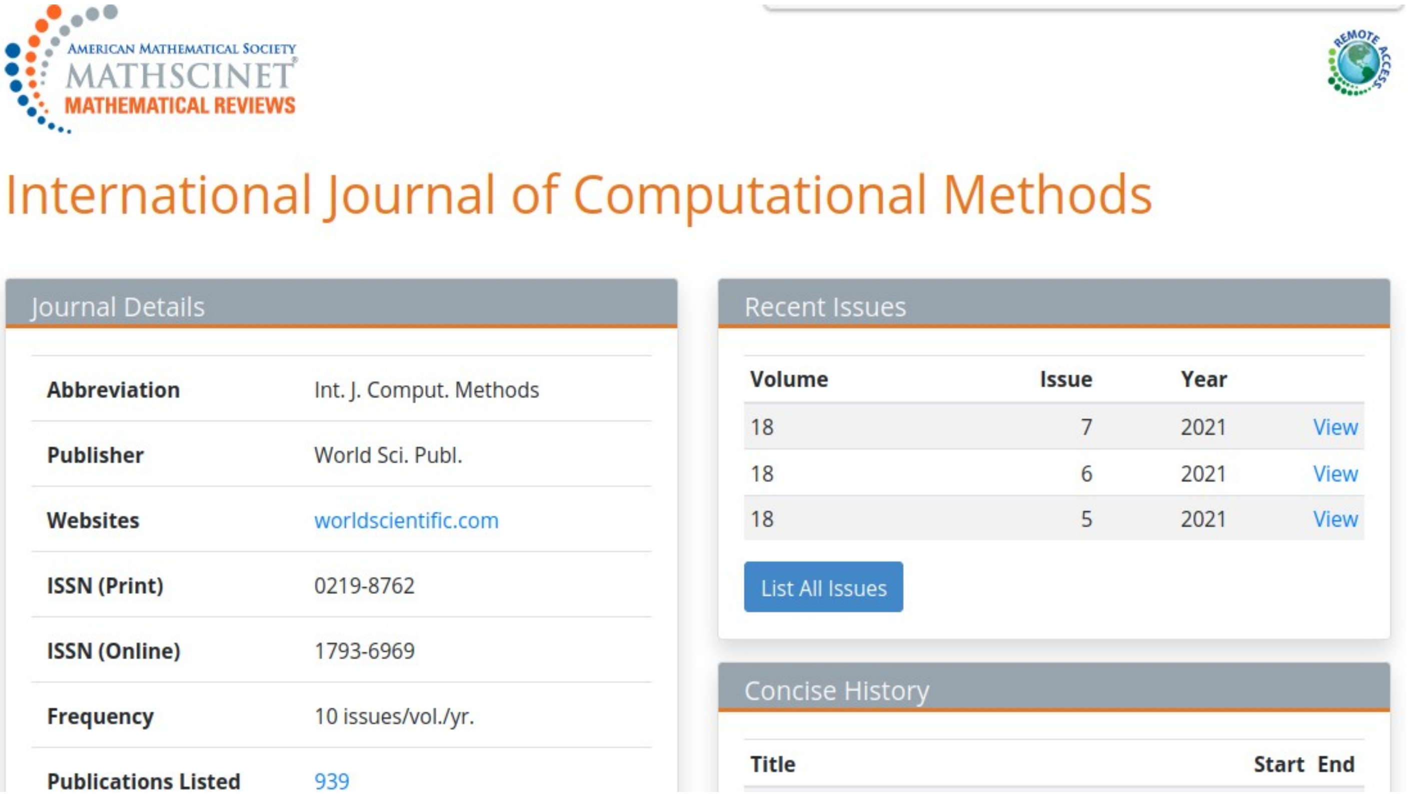

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Robust Finite Difference Method for Singularly Perturbed Two-Parameter Parabolic Convection-Diffusion Problems

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Robust finite difference method is introduced in order to solve singularly perturbed two parametric parabolic convection-diffusion problems. In order to discretize the solution domain, Micken's type discretization on a uniform mesh is applied and then followed by the fitted operator approach. The convergence of the method is established and observed to be first-order convergent, but it is accelerated by Richardson extrapolation. To validate the applicability of the proposed method, some numerical examples are considered and observed that the numerical results confirm the agreement of the method with the theoretical results effectively. Furthermore, the method is convergent regardless of perturbation parameter and produces more accurate solution than the standard methods for solving singularly perturbed parabolic problems.

Keywords: Singularly perturbed parabolic problems; convection-diffusion; robust method; two-parameter; accurate solution.

1. Introduction

Singularly perturbed differential equations arise in diverse areas of applied mathematics and mathematical physics such as fluid dynamics, quantum mechanics, chemical reactor theory, meteorology, oceanography, reaction-diffusion process, Navier–Stokes equation of fluid flow at high Reynolds number, and so on. Differential equation depends on perturbation parameter(s), multiplying the highest-order derivative term(s) is known as singularly perturbed differential equation. When the perturbation parameter tends to zero, the problem has warning solution and the regions of nonuniform convergence lie near the boundary of the domain which are represented

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by boundary layers [Roos *et al.* (2008); Das and Mehrmann (2016)]. For singularly perturbed ordinary differential equations, effective numerical methods like a posteriori error estimation by Linss [2010] and Haar wavelet approach are proposed by Pandit and Kumar [2014] for numerical solution of two parameters singularly perturbed boundary value problems. However, solving singularly perturbed parabolic problems by standard numerical methods like finite difference, finite volume and finite element methods, the obtained solution is inaccurate due to boundary layers. In order to capture the layers, a large number of special methods have been developed by the researchers to provide accurate solution which cover the singularly perturbed parabolic problems with one parameter [Clavero and Gracia (2013); Gracia and O'Riordan (2015); Gowrisankar and Srinivasan (2013, 2014); Chandru *e al.* (2019)].

From many other schemes, few recently developed numerical schemes to solve singularly perturbed two parameter parabolic problems are;arameter-uniform finite element method presented by Kadalbajoo and Yadaw [2012], spline difference scheme by Zahra *et al.* [2014], robust finite difference method by Munyakazi [2015], robust layer adapted difference method by Jha and Kadalbajoo [2015] and a parameter-uniform higher-order finite difference scheme presented by Gupta *et al.* [2019]. All these methods are developed to solve singularly perturbed parabolic problems in which the perturbation parameter is affecting the first- and second-order derivative terms. Altogether the above-mentioned methods provided parameter independent uniform convergence for a class of singularly perturbed parabolic convection-diffusion problems with two-parameters and the obtained result holds even for the limiting case where the perturbation parameters are zero. Further, there are some numerical methods recommended by various authors for solving singularly perturbed parabolic problems with two small parameters. Specifically, numerical treatment of two-parameter singularly perturbed parabolic convection diffusion problems with nonsmooth data by Chandru *et al.* [2018]; numerical method for solving boundary and interior layers dominated parabolic problems with discontinuous convection coefficient and source terms by Chandru *et al.* [2017] and a uniformly convergent hybrid scheme for singularly perturbed system of reaction-diffusion Robin type boundary-value problems are proposed by Das and Natesan [2013a].

More recently, Gupta *et al.* [2019] developed a parameter-uniform numerical method to solve singularly perturbed parabolic problems with two parameters. These authors developed and analyzed the method using asymptotic behavior of the solution and a decomposition of solution into its regular and singular parts. To approximate the solution, they considered the implicit Euler method for time stepping on a uniform mesh and a special hybrid monotone difference operator for spatial discretization on a specially designed piecewise uniform Shishkin mesh. They improved order of convergence by means of Richardson extrapolation technique used in temporal variable only and the resulting scheme was proved to be uniformly convergent of order two in both the spatial and temporal variables. In connection to this, there are more recent numerical methods proposed by various authors for systems of singularly perturbed differential equations like; posteriori-based convergence analysis method by Das [2018]. Comparison of a priori and a posteriori meshes,;igher-order accurate approximations on equidistributed meshes by Das [2015]; Das *et al.* [2020] and optimal error estimate using mesh equidistribution technique developed by Das and Natesan [2014] are some of the applicable methods for solving singularly perturbed parabolic problems with two parameters.

Generally, uniformly convergent numerical methods for solving singular perturbation problems are widely classified into fitted operator and fitted mesh methods [Roos *et al.* (2008); Miller *et al.* (1996)]. Most of the developed uniformly convergent numerical methods for solving the problem under consideration were focused on fitted mesh methods rather than fitted operator finite difference method. Herein, we apply the fitted operator finite difference method obtained through the nonstandard finite difference method. The method is nonstandard owing to the fact that the classical denominator for spatial mesh size h or h^2 of the discrete first- or second-order derivatives in the standard finite difference scheme is replaced by nonnegative function ϕ such that $\phi(z) = h + O(z^2)$ or $\phi(z) = h^2 + O(z^3)$, as $0 < z \to 0$. This denominator function imitates a number of significant properties of the governing differential equations. Interested readers may refer to a comprehensive list of works and detail concept on nonstandard finite difference methods from books and articles in Patidar and Sharma [2006]; Jean *et al.* [2006]; Kadalbajoo *et al.* [2006]; Mickens [1994, 2005].

In the past few years, numerous uniformly convergent numerical schemes have been proposed in the literature for solving singular perturbation parabolic problems, and typically most methods are associated to fitted mesh methods. Thus, the main objective of this paper is to present robust finite difference method which is based on fitted operator for solving singularly perturbed parabolic initial boundary value problems (IBVPs) with two-parameters to produce more accurate numerical solution.

2. Statement of the Problem

We consider the following singularly perturbed two-parameter parabolic IBVP on the solution domain $(x,t) \in Q := \Omega \times (0,T], \Omega = (0,1)$

$$L_{x,t}u \equiv \varepsilon\frac{\partial^2 u}{\partial x^2} + \mu a(x,t)\frac{\partial u}{\partial x} - b(x,t)u(x,t) - \frac{\partial u}{\partial t} = f(x,t) \tag{1}$$

subject to the initial and boundary conditions

$$\begin{aligned} u(x,0) &= s(x), \quad x \in \bar{\Omega} \\ u(0,t) &= q_0(t), \quad u(1,t) = q_1(t), \ \ t \in [0,T] \end{aligned} \tag{2}$$

with two small parameters $0 < \varepsilon, \mu \ll 1$. The coefficient functions $a(x,t), b(x,t)$ and source function $f(x,t)$ are sufficiently regular on $\bar{Q}$ and satisfy $a(x,t) \geq \alpha > 0$, $b(x,t) \geq \beta > 0$; α and β are real numbers. Also, we assume that sufficient regularity and compatibility conditions are imposed on the functions $s(x), q_0(t), q_1(t)$ and $f(x,t)$, so that a unique solution exists.

Problem given by Eqs. (1) and (2) exhibits two boundary layers with different widths depending on the relation between the two parameters ε and μ. For chosen $\gamma \approx \min_{(x,t)\in\bar{Q}} \frac{b(x,t)}{a(x,t)}$ if $\frac{\mu^2}{\varepsilon} \leq \frac{\gamma}{\alpha}$, the reduced problem Eq. (1) is

$$-b(x,t)u_0(x,t) - \frac{\partial u_0}{\partial t} = f(x,t), \quad u_0(x,0) = s(x) \tag{3}$$

Thus, boundary layers of width $O(\sqrt{\varepsilon})$ are expected in both neighborhood of $x = 0$ and $x = 1$ if $u_0(0,t) \neq q_0(t)$ and $u_0(1,t) \neq q_1(t)$.

If $\frac{\mu^2}{\varepsilon} \geq \frac{\gamma}{\alpha}$, then reduced problem

$$\begin{cases} \mu a(x,t)\dfrac{\partial u_\mu}{\partial x} - b(x,t)u_\mu(x,t) - \dfrac{\partial u_\mu}{\partial t} = f(x,t) \\ u_\mu(x,0) = s(x) \quad \text{and} \quad u_\mu(0,t) = q_0(t) \end{cases} \tag{4}$$

is again singularly perturbed problem with perturbation parameter, μ. This is a first-order hyperbolic equation with initial data specified along two sides $t = 0$ and $x = 0$ of the domain $\bar{Q}$. A boundary layer of width $O(\frac{\varepsilon}{\mu})$ is predictable in a right neighborhood of $x = 0$ if $u_\mu(0,t) \neq q_0(t)$, and a boundary layer of width $O(\mu)$ is expected in a left neighborhood of $x = 0$ if $u_\mu(1,t) \neq q_1(t)$ [Kadalbajoo and Yadaw (2012); Miller *et al.* (1996)]. When the parameter $\mu = 1$, the problem is a well-studied parabolic convection-diffusion problem [Chen and Liu (2016); Roos *et al.* (2008); Gowrisankar and Srinivasan (2014)] with boundary layer of width $O(\varepsilon)$ appearing in the neighborhood of $x = 0$ or $x = 1$. While the parameter $\mu = 0$, the problem is parabolic reaction-diffusion problem [Clavero and Gracia (2013); Gracia and O'Riordan (2015)] which has two boundary layers of width $O(\sqrt{\varepsilon})$ appearing near $x = 0$ and $x = 1$. Here, we consider the problem in Eqs. (1) and (2) when the two perturbation parameters satisfy $0 < \varepsilon \ll 1$ and $0 < \mu \ll 1$, for which the problem has different layer widths on the opposite side of the space domain depending on the value of two perturbation parameters, ε and μ.

3. Properties of Continuous Solution

By imposing the compatibility condition $s(0) = q_0(0)$ and $s(1) = q_1(1)$, data match at the two corners $(0,0)$ and $(1,0)$. This condition promises that there exist constants C_1 and C_2 such that $\forall x, t \in \bar{Q}$

$$|u(x,t) - s(x)| \leq C_1 t \tag{5}$$

$$|u(x,t) - q_0(t)| = |u(x,t)| \leq C_2(1-x) \tag{6}$$

For the detailed proof of both Eqs. (5) and (6), readers can refer to Kadalbajoo and Yadaw [2012]. Let $u(x,t) \in C^1(\bar{Q})$, then from Eqs. (5) and (6), it implies that $|u_t(x,0)| \leq C$ and $|u_x(0,t)| \leq C$, where u_t and u_x are derivatives with respect to t and x, respectively.

Lemma 3.1 (Continuous minimum principle). *If $\varphi(x,t) \in C^2(Q) \cap C^0(\bar{Q})$ is such that $\varphi(x,t)|_{\partial Q} \geq 0$ and $L_{x,t}\varphi(x,t)|_Q \leq 0$, then $\varphi(x,t)|_{\bar{Q}} \geq 0$.*

Proof. See Jha and Kadalbajoo [2015]; Gupta *et al.* [2019]. $\square$

Lemma 3.2 (Uniform stability estimate). *Let $u(x,t)$ be the solution of Eqs. (1) and (2). Then, we have*

$$\|u(x,t)\| \leq C(\beta^{-1}\|f(x,t)\| + \max(|q_0(t)| + |q_1(t)|)), \quad \forall (x,t) \in \bar{Q}$$

where $\|\,.\,\|$ denotes the maximum norm on the domain $\bar{Q}$, and β is a positive constant specified under Sec. 2.

Proof. See Gupta *et al.* [2019]. $\square$

Lemma 3.3. *For any nonnegative integers i, j such that $0 \leq i+3j \leq 4$, the solution $u(x,t)$ satisfies*

$$\left\|\frac{\partial^i u}{\partial x^i \partial t^i}\right\|_{\bar{Q}} \leq C \begin{cases} \dfrac{1}{(\sqrt{\varepsilon})^i}, & \text{if } \mu^2 \leq c\varepsilon \\ \left(\dfrac{\mu}{\varepsilon}\right)^i, & \text{if } \mu^2 \geq c\varepsilon \end{cases}$$

where $c = \frac{\gamma}{\alpha}$, the constant C is independent of parameters ε, μ, and is depend on $\|\frac{\partial^p a}{\partial x^p}\|_{\bar{Q}}$, $\|\frac{\partial^p b}{\partial x^p}\|_{\bar{Q}}$ and $\|\frac{\partial^p f}{\partial x^p}\|_{\bar{Q}}$, for $p = 0, 1, 2$

For the proof of this lemma, one can refer Jha and Kadalbajoo [2015]; Gupta *et al.* [2019].

4. Description of the Scheme

To describe the proposed scheme, the argument splits into two steps; the time variable is discretized which leads to a system of boundary value problem and then the discretization of space variable on a uniform mesh followed by the nonstandard methodology of Mickens [1994, 2005].

4.1. *Temporal discretization*

To discretize the time variable with uniform step size k, the interval $[0,T]$ is partitioned into N equal sub-intervals and each nodal point satisfies $0 = t_0 < t_1 < \cdots <$

$t_N = T$, for $t_n = nk$, $k = \frac{T}{N}, n = 0, 1, 2, \ldots, N$. Now, at the point (x, t_{n+1}), Eq. (1) written as

$$L_{x,t_{n+1}} u \equiv \varepsilon \frac{\partial^2 u}{\partial x^2}(x, t_{n+1}) + \mu a(x, t_{n+1}) \frac{\partial u}{\partial x}(x, t_{n+1}) - b(x, t_{n+1}) u(x, t_{n+1}) - \frac{\partial u}{\partial t}(x, t_{n+1}) = f(x, t_{n+1}). \tag{7}$$

From Taylor's series expansion about the point (x, t_{n+1}), we have

$$u(x, t_n) = u(x, t_{n+1}) - k \frac{\partial u}{\partial t}(x, t_{n+1}) + \frac{k^2}{2} \frac{\partial^2 u}{\partial t^2}(x, t_{n+1}) - \frac{k^3}{6} \frac{\partial^3 u}{\partial t^3}(x, t_{n+1}) + \cdots$$

Solving for $\frac{\partial u}{\partial t}(x, t_{n+1})$ from this equation, we get

$$\frac{\partial u}{\partial t}(x, t_{n+1}) = \frac{u(x, t_{n+1}) - u(x, t_n)}{k} + \tau_1, \tag{8}$$

where $\tau_1 = \frac{k}{2} \frac{\partial^2 u}{\partial t^2}(x, t_{n+1}) \equiv O(k)$.

This indicates that the error estimate of time discretization is given by

$$\|E_n\| \le Ck, \tag{9}$$

where C is constant independent of perturbation parameters ε, μ and mesh size k.

Substituting Eq. (8) into Eq. (7) yields boundary value problems in terms of space variable at each time level written as

$$L_x Z \equiv \varepsilon \frac{d^2 Z}{dx^2} + \mu a(x, t_{n+1}) \frac{dZ}{dx} - \left(b(x, t_{n+1}) + \frac{1}{k} \right) Z = z^* \tag{10}$$

subject to the conditions

$$Z(0) = q_0(t_{n+1}), \quad Z(1) = q_1(t_{n+1}), \quad n = 0, 1, \ldots, N, \tag{11}$$

where $Z = u(x, t_{n+1})$ and $z* = f(x, t_{n+1}) - \frac{1}{k} u(x, t_n)$.

Differential operator L_x in boundary value problems in Eq. (10) satisfies the following discrete minimum principle.

Lemma 4.1 (Continuous minimum principle for semi-discretize). *If* $\phi(x) \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *is such that* $\phi(x)|_{\partial\Omega} \ge 0$ *and* $L_x \phi(x)|_{\Omega} \le 0$, *then* $\phi(x)|_{\bar{\Omega}} \ge 0$.

Proof. One can prove this lemma by the same procedure to the proof of Lemma 3.1. □

Lemma 4.2 (Uniform stability estimate for Semidiscretization). *Let* Z *be the solution of Eqs.* (11) *and* (12). *Then, we have*

$$\|Z\| \le C(\beta^{-1} \|z^*\| + \max(|Z(0)| + |Z(1)|)), \quad \forall (x, t_{n+1}) \in \bar{Q}^N.$$

Proof. Consider the comparison function

$$\theta^{*\pm}(x,t_{n+1}) = \beta^{-1}\|z^*\| + \max(|Z(0)| + |Z(1)|) \pm Z, \quad \forall\,(x,t_{n+1}) \in \bar{Q}^N.$$

Then we have

$$\begin{aligned} L_x\theta^{*\pm}(x,t_{n+1}) &= -b(x,t_{n+1})\beta^{-1}\|z^*\| \\ &\qquad - b(x,t_{n+1})\max(|Z(0)| + |Z(1)|) \pm L_x Z, \quad \forall(x,t_{n+1}) \in \bar{Q}^N. \end{aligned}$$

Since $Lu(x,t) \le 0$, from the minimum principle in Lemma 3.1, it follows that $L_x\theta^{*\pm}(x,t_{n+1}) \le 0$, $\forall(x,t_{n+1}) \in \bar{Q}^N$, which implies $\theta^{*\pm}(x,t_{n+1}) \ge 0$, $\forall(x,t_{n+1}) \in \bar{Q}^N$. Thus

$$\theta^{*\pm}(x,t_{n+1}) = \beta^{-1}\|z^*\| + \max(|Z(0)| + |Z(1)|) \pm Z, \quad \forall\,(x,t_{n+1}) \in \bar{Q}^N$$

This completes the required proof. □

The characteristic equation for the homogeneous part of Eq. (10) is

$$\varepsilon\, r^2(x) + \mu a(x,t_{n+1})r(x) - \left(b(x,t_{n+1}) + \frac{1}{k}\right) = 0.$$

Assume this equation has two real solutions $r_1(x) < 0$ and $r_2(x) > 0$, and let

$$\lambda_1 = -\max_{x\in[0,1]} r_1(x) \quad \text{and} \quad \lambda_2 = \min_{x\in[0,1]} r_2(x).$$

The situations of two layers are characterized by the case $\mu^2 \ll \varepsilon$ as $\varepsilon \to 0$, which suggests that $\lambda_1 \approx \lambda_2 \approx \sqrt{\frac{1+kb(x,t_{n+1})}{\varepsilon k}}$ and we have the layer like to the case $\mu = 0$. The other condition, layers comes across in the case where $\varepsilon \ll \mu^2$ as $\mu \to 0$ yields $\lambda_1 \approx \min_{x\in[0,1]} \frac{\mu a(x,t_{n+1})}{\varepsilon}$ and $\lambda_2 \approx 0$. The detailed proof of the following lemma given by the authors in Gupta *et al.* [2019] requires some used bounds on the solution and its derivatives for the convergence analysis of the numerical scheme.

Lemma 4.3. *For any real constant number v, $0 < v < 1$ we have up to a certain order q that depends on the smoothness of data*

$$\left|\frac{\partial^j}{\partial x^j} Z\right| \le C\left\{1 + \lambda_1^j e^{-v\lambda_1 x} + \lambda_2^j e^{-v\lambda_2(1-x)}\right\}, \quad \textit{for } 0 \le j \le q \tag{12}$$

The detailed proof of this lemma is given in Kadalbajoo and Yadaw [2012].

4.2. *The full discrete problem*

Assume that $\bar{\Omega}^M$denotes partition of $[0,1]$ into M subintervals such that $0 = x_0 < x_1 < \cdots < x_M = 1$ and $x_m = mh, h = \frac{1}{M}, 1 = 0,1,2,\ldots M$, which can be decomposed as $[0,\frac{1}{2}] \cup [\frac{1}{2},1]$ and then the tensor-product grids $\bar{Q}^{M,N}$. Using the decomposition of space domain, we consider the following two cases to construct the scheme for solving Eq. (10) sideways with appropriate boundary conditions.

Case 1. On the domain $[0, \frac{1}{2}] \times (0, T]$, (for left boundary layer case)

By undertaking the notation $U_m^{n+1} \approx u(x_m, t_{n+1}) = Z(x_m, t_{n+1})$ and using the nonstandard finite difference methodology of Mickens [1994, 2005], Eq. (10) can be written as

$$\begin{aligned} L_{\varepsilon,\mu}^{M,N} U_m^n \equiv \varepsilon \frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{\phi_m^2} + \mu a_m^{n+1} \frac{U_{m+1}^{n+1} - U_m^{n+1}}{h} \\ - \left(b_m^{n+1} + \frac{1}{k}\right) U_m^{n+1} = f_m^{n+1} - \frac{1}{k} U_m^n \end{aligned} \tag{13}$$

with the discrete initial and boundary conditions

$$\begin{aligned} &U(x_m, 0) = s(x_m), \quad x_m \in \bar{\Omega}^{[0, \frac{1}{2}]} \\ &U(0, t_{n+1}) = q_0(t_{n+1}), \quad u(1, t_{n+1}) = q_1(t_{n+1}), \quad t_{n+1} \in [0, T]^N \end{aligned} \tag{14}$$

As Mickens [1994] provided the detail to prove, the denominator function ϕ_m^2 is given by

$$\phi_m^2(\varepsilon, \mu, h) = \frac{h\varepsilon}{\mu a(x_m)} \left(\exp\left(\frac{h\mu a(x_m)}{\varepsilon} \right) - 1 \right). \tag{15}$$

Case 2. On the domain $[\frac{1}{2}, 1] \times (0, T]$, (right boundary layer)

Similarly, nonstandard finite difference scheme to solve Eq. (10) as

$$\begin{aligned} L_{\varepsilon,\mu}^{M,N} U_m^n \equiv \varepsilon \frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{\phi_m^2} + \mu a_m^{n+1} \frac{U_m^{n+1} - U_{m-1}^{n+1}}{h} \\ - \left(b_m^{n+1} + \frac{1}{k}\right) U_m^{n+1} = f_m^{n+1} - \frac{1}{k} U_m^n \end{aligned} \tag{16}$$

Subject to the discrete initial and boundary conditions

$$\begin{aligned} &U(x_m, 0) = s(x_m), \quad x_m \in \bar{\Omega}^{[\frac{1}{2}, 1]}, \\ &U(0, t_{n+1}) = q_0(t_{n+1}), \quad U(1, t_{n+1}) = q_1(t_{n+1}), \quad t_{n+1} \in [0, T]^N, \ n = 0, 1, \ldots, N \end{aligned} \tag{17}$$

The schemes in both Eqs. (13) and (16) can be re-written as

$$L_{\varepsilon,\mu}^{N,M} = E_m^{n+1} U_{m-1}^{n+1} - F_m^{n+1} U_m^{n+1} + G_m^{n+1} U_{m+1}^{n+1} = H_m^{n+1}, \tag{18}$$

where $E_m^{n+1} = \frac{\varepsilon}{\phi_m^2}$, $F_m^{n+1} = \frac{2\varepsilon}{\phi_m^2} + \frac{\mu a_m^{n+1}}{h} + b_m^{n+1} + \frac{1}{k}$, $G_m^{n+1} = \frac{\varepsilon}{\phi_m^2} + \frac{\mu a_m^{n+1}}{h}$; for $m = 1, 2, \ldots, \frac{M}{2}$ and $n = 0, 1, 2, \ldots, N - 1$.

$$E_m^{n+1} = \frac{\varepsilon}{\phi_m^2} - \frac{\mu a_m^{n+1}}{h}, F_m^{n+1} = \frac{2\varepsilon}{\phi_m^2} - \frac{\mu a_m^{n+1}}{h} + b_m^{n+1} + \frac{1}{k}, \quad G_m^{n+1} = \frac{\varepsilon}{\phi_m^2};$$

for $m = \frac{M}{2} + 1, \frac{M}{2} + 2, \ldots, M - 1$ and $n = 0, 1, 2, \ldots, N - 1$ and $H_m^{n+1} = f_m^{n+1} - \frac{1}{k} U_m^n$, $m = 1, 2, \ldots, M - 1$ and $n = 0, 1, 2, \ldots, N - 1$.

Hence, scheme developed in Eq. (18) is considered as fitted operator finite difference method to solve Eqs. (1) and (2). Here, Eq. (18) is tri-diagonal system of equations with respect to the $x-$direction and the coefficients $E_m^{n+1}, F_m^{n+1}, G_m^{n+1}$ and the right-hand side H_m^{n+1} are given that they satisfy the conditions $|E_m^{n+1}| > 0, |F_m^{n+1}| > 0, |G_m^{n+1}| > 0$ and $|F_m^{n+1}| > |E_m^{n+1}| + |G_m^{n+1}|$ at each $(n+1)^{\text{th}}$ level. These situations guarantee that the system is *diagonally dominant*. Thus, $M \times M$ system of equation given in Eq. (18) can be solved by Thomas algorithm. The discrete operator $L_{\varepsilon,\mu}^{M,N}$ in Eq. (18) satisfies the following minimum principle.

Lemma 4.4 (Discrete minimum principle). *Assume that* $L_{\varepsilon,\mu}^{N,M}$ *is the discrete operator of Eq.* (19) *and* ψ_m^{n+1} *is any mesh satisfying* $\psi_m^0 \geq 0, 0 \leq m \leq M, , \psi_0^{n+1} \geq 0, \psi_M^{n+1} \geq 0, 0 \leq n \leq N$. *If* $L_{\varepsilon,\mu}^{M,N} \psi_m^{n+1} \geq 0$ *in* Q_M^N, *then* $\psi_m^{n+1} \geq 0$ *in* $\bar{Q}_M^N$.

Proof. Let i and j be the indices such that $\psi_i^{j+1} = \min_{(i,j)} \psi_i^{j+1}$ for $\psi_i^{j+1} \in \bar{Q}_M^N$. Assume that $\psi_i^{j+1} < 0$. It is easy to see that $(i,j) \in \{1,2,\ldots,M-1\} \times \{1,2,\ldots,N\}$ because otherwise $\psi_i^{j+1} \geq 0$. It tails that $\psi_{i+1}^{j+1} - \psi_i^{j+1} > 0$. and $\psi_{i-1}^{j+1} - \psi_i^{j+1} > 0$. Thus, $L_{\varepsilon,\mu}^{M,N} \psi_m^{n+1} < 0$ which is a contradiction. Therefore, $\psi_i^{j+1} \geq 0$. The indices i and j being arbitrary, we obtain $\psi_m^{n+1} \geq 0$ in $\bar{Q}_m^n$. □

Lemma 4.5 (Uniform stability estimate for discrete problem). *At any time level* t_{n+1}, *if* U_m^{n+1} *is any function such that* $U_0^{n+1} = U_M^{n+1} = 0$, *for* $n = 0, 1, \ldots, N$, *then*

$$|U_i^{n+1}| \leq \frac{1}{\alpha} \max_{1 \leq m \leq M-1} |L_{\varepsilon,\mu}^{M,N} U_m^{n+1}|, \quad \textit{for } 0 < i < M. \tag{19}$$

This means the operator $L_{\varepsilon,\mu}^{M,N}$ *is uniformly stable.*

Proof. For $m = 1, 2, \ldots, M-1$ and $n = 0, 1, \ldots, N$, Let $(\psi^{\pm})_m^{n+1}$ be the mesh function demarcated by

$$(\psi^{\pm})_m^{n+1} = P + U_m^{n+1} \quad \text{where } P = \frac{1}{\beta} \max_{1 \leq m \leq M-1} \left| L_{\varepsilon,\mu}^{M,N} U_m^{n+1} \right|$$

with $(\psi^{\pm})_0^{n+1} = (\psi^{\pm})_M^{n+1} = P \geq 0$. Moreover, for $1 \leq m \leq M-1$ and $n = 0, 1, \ldots, N$, we have

$$L_{\varepsilon,\mu}^{M,N} (\psi^{\pm})_m^{n+1} = -\frac{b_m^{n+1} + \frac{1}{k}}{\beta} \max_{1 \leq m \leq M-1} |L_{\varepsilon,\mu}^{M,N} U_m^{n+1}| \pm L_{\varepsilon,\mu}^{M,N} U_m^{n+1}.$$

Spending the fact that $0 < \beta \leq b_m^{n+1} \leq b_m^{n+1} + \frac{1}{k}$, we have $L_{\varepsilon,\mu}^{M,N} (\psi^{\pm})_m^{n+1} \leq 0$. By the discrete minimum principle above, we obtain $(\psi^{\pm})_m^{n+1} \geq 0$ for $0 \leq m \leq M$. □

5. Convergence Analysis

The error estimate to approximate the derivative with respect to time in the differential equation was verified in Eq. (9). Thus for now, drop the time level indices for the sake of simplicity and then the following analysis concerns the space variable x. The local truncation error of nonstandard finite difference method which considered as the fitted operator finite difference method in Eqs. (13) and (16) is given by

$$L_x^{M,N}(u(x_m,t_{n+1})-u_m^{n+1})=(L-L_x)u_m^{n+1},$$

which verified as

$$\begin{aligned}L_x^{M,N}\left(u(x_m,t_{n+1})-u_m^{n+1}\right)&=\varepsilon\frac{d^2u_m^{n+1}}{dx^2}+\mu a_m^{n+1}\frac{du_m^{n+1}}{dx}\\&\quad-\varepsilon\frac{u_{m+1}^{n+1}-2u_m^{n+1}-u_{m-1}^{n+1}}{\phi_m^2}-\mu a_m^{n+1}\frac{u_{m+1}^{n+1}-u_m^{n+1}}{h},\end{aligned}\tag{20}$$

$$\begin{aligned}L_x^{M,N}\left(u(x_m,t_{n+1})-u_m^{n+1}\right)&=\varepsilon\frac{d^2u_m^{n+1}}{dx^2}+\mu a_m^{n+1}\frac{du_m^{n+1}}{dx}\\&\quad-\varepsilon\frac{u_{m+1}^{n+1}-2u_m^{n+1}-u_{m-1}^{n+1}}{\phi_m^2}-\mu u_m^{n+1}\frac{u_m^{n+1}-u_{m-1}^{n+1}}{h}.\end{aligned}\tag{21}$$

From Taylor series expansion for u_{m+1}^{n+1} and u_{m-1}^{n+1} at the point x_m on spatial direction, we obtain

$$u_{m+1}^{n+1}-2u_m^{n+1}+u_{m-1}^{n+1}=h^2(u_m^{n+1})''+\frac{h^4}{12}(u_m^{n+1})^{(4)}+\cdots,\tag{22}$$

$$u_{m+1}^{n+1}-u_m^{n+1}=h(u_m^{n+1})'+\frac{h^2}{2}(u_m^{n+1})''+\frac{h^3}{6}(u_m^{n+1})'''+\cdots,\tag{23}$$

$$u_m^{n+1}-u_{m-1}^{n+1}=h(u_m^{n+1})'-\frac{h^2}{2}(u_m^{n+1})''+\frac{h^3}{6}(u_m^{n+1})'''+\cdots.\tag{24}$$

Also, by the truncated Taylor series expansions for $\frac{1}{\phi_m^2}$ given in Munyakazi [2015], we have

$$\frac{1}{\phi_m^2}=\frac{1}{h^2}-\frac{\mu a_m}{2\varepsilon h}+\frac{\mu^2a_m^2}{12\varepsilon^2}\quad\text{or}\quad\frac{\varepsilon h^2}{\phi_m^2}=\varepsilon-\frac{\mu h a_m}{2}+\frac{\mu^2h^2a_m^2}{12\varepsilon}.\tag{25}$$

Substituting Eqs. (22)–(25) into both Eqs. (20) and (21), and after simple algebraic manipulation, we get

$$\begin{aligned}L_x^{M,N}(u(x_m,t_{n+1})-u_m^{n+1})&=h\mu a_m^{n+1}(u_m^{n+1})''\\&\quad-h^2\left(\frac{\mu^2(a_m^{n+1})^2}{12\varepsilon}(u_m^{n+1})''+\frac{\mu a_m^{n+1}}{6}(u_m^{n+1})'''\right)+\cdots.\end{aligned}\tag{26}$$

Using the bounds on derivatives of u (Lemma 4.3) the fact that for small h, $h^4<h^3<h^2<h$ and observing that for both $\lambda_1^je^{-v\lambda_1x_m}$ and $\lambda_2^je^{-v\lambda_2(1-x_m)}$ approach

zero as $\varepsilon \to 0$ for all $j \in \{0, 1, 2, \ldots\}$, we obtain

$$|L_x^{M,N}(u(x_m, t_{n+1}) - u_m^{n+1})| \leq Ch \tag{27}$$

where $C = |\mu a_m^{n+1}(u_m^{n+1})''|$ is constant independent of the parameter ε and mesh size h.

Here, raising the uniform stability estimate (Lemma 4.5) leads to

$$\max_{0 \leq m \leq M} |(u(x_m, t_{n+1}) - u_m^{n+1})| \leq Ch \tag{28}$$

Consequently, form Eqs. (9) and (28), the key and main result of this work are given as follows.

Theorem 1. *Let $u(x,t)$ be the solution of Eqs.* (1) *and* (2) *and U_m^{n+1} its numerical approximation for solution of Eq.* (19). *Then there exists a constant C independent of ε, k and h such that*

$$\max_{0 \leq m \leq M, 0 \leq n \leq N} |u(x_m, t_{n+1}) - U_m^{n+1}| \leq C(h + k). \tag{29}$$

This consequence specifies that the developed method is first-order convergent.

Proof. One can provide the detailed proof for this theorem by following the procedures given by Das and Mehrmann [2016]. □

6. Richardson Extrapolation

For the detail convergence analysis of Richardson extrapolation method, refer Das [2018] and Das and Natesan [2013b]. Since Richardson extrapolation technique is a convergence acceleration technique which involves combination of two computed approximations of a solution, the combination goes out to be an improved approximation. Hence, from Eq. (29), we have

$$|u(x_m, t_{n+1}) - U_m^{n+1}| \leq C(h + k), \tag{30}$$

where $u(x_m, t_{n+1})$ and U_m^{n+1} are exact and approximate solutions, respectively, C is constant free from perturbation parameter ε and mesh sizes h and k.

Let Ω_{2M}^{2N} be the mesh found by dividing each mesh interval in Ω_M^N and symbolize the calculation of the solution on Ω_{2M}^{2N} by $\bar{U}_m^{n+1}$. Consider Eq. (30) works for any $h, k \neq 0$, which implies for all $m = 0, 1, \ldots, M$ and $n = 0, 1, \ldots, N$:

$$u(x_m, t_{n+1}) - U_m^{n+1} \leq C(h + k) + R_M^N \quad (x_m, t_{n+1}) \in \Omega_M^N. \tag{31}$$

So that it works for any $\frac{h}{2}, \frac{k}{2} \neq 0$

$$u(x_m, t_{n+1}) - \bar{U}_m^{n+1} \leq C\left(\frac{h}{2} + \frac{k}{2}\right) + R_{2M}^{2N} \quad (x_m, t_{n+1}) \in \Omega_{2M}^{2N} \tag{32}$$

where the remainders, R_M^N and R_{2M}^{2N} are $O(h^2 + k^2)$. Combination of inequalities in Eqs. (31) and (32) leads to $u(x_m, t_{n+1}) - (2\bar{U}_m^{n+1} - U_m^{n+1}) \leq C(h^2 + k^2)$ which proposes that

$$(U_m^{n+1})^{\text{ext}} = 2\bar{U}_m^{n+1} - U_m^{n+1} \tag{33}$$

is also an approximate solution of $u(x_m, t_{n+1})$. Equation (33) approximates the solution with estimated truncation error

$$|u(x_m, t_{n+1}) - (U_m^{n+1})^{\text{ext}}| \leq C(h^2 + k^2), \tag{34}$$

where C is free of mesh sizes h and k. Thus, using Richardson extrapolation,;irst-order convergent method is accelerated into second-order convergent as provided in Eq. (34). Thus, the proposed method is second-order convergent. To verify the proof for Eq. (34), one can follow the procedures provided by Das [2018] and Das and Natesan [2013a].

7. Numerical Illustrations and Discussion

To demonstrate the efficiency of the proposed scheme, we display numerical results for two test examples. Since the exact solution for such type of problems is not available, the maximum absolute errors are evaluated to before and after extrapolation using the formula

$$E_{\varepsilon,\mu}^{M,N} = \max_{0\leq m\leq M; 0\leq n\leq N} |U_m^{n+1} - U_{2m}^{2n+1}|$$

and

$$E_{\varepsilon,\mu}^{M,N} = \max_{0\leq m\leq M; 0\leq n\leq N} |(U_m^{n+1})^{\text{ext}} - (U_{2m}^{2n+1})^{\text{ext}}|, \quad \text{respectively,}$$

where U_m^{n+1} is the approximate solution obtained using a constant space mash size h and time step k and U_{2m}^{2n+1} is also approximate solution produced using space step $\frac{h}{2}$ with time step $\frac{k}{2}$. Also, its solutions obtained by Richardson extrapolation are $(U_m^{n+1})^{\text{ext}}$ and $t(U_{2m}^{2n+1})^{\text{ext}}$. Likewise, compute the rates of convergence R by

$$R = \frac{\log E_{\varepsilon,\mu}^{M,N} - \log E_{\varepsilon,\mu}^{2M,2N}}{\log 2}.$$

Example 1. Consider the IBVP

$$\varepsilon\frac{\partial^2 u}{\partial x^2} + \mu(1+x)\frac{\partial u}{\partial x} - u(x,t) - \frac{\partial u}{\partial t} = 16x^2(1-x)^2 \quad (x,t) \in \Omega := (0,1)\times(0,T]$$

subject to the conditions

$$u(x,0) = 0, \quad x \in [0,1] \quad \text{and} \quad u(0,t) = 0 = u(1,t), \quad t \in [0,T]$$

For this example, the obtained maximum absolute errors are given in Tables 1 and 2, rate of convergence is given in Table 3 and Log–log plot of maximum absolute errors are given in Fig. 1.

Table 1. Comparison maximum absolute errors for Example 1 at $M = 32$, $k = \frac{0.125}{4}$.

$\mu \downarrow \varepsilon \rightarrow$	10^{-2}	10^{-4}	10^{-6}	10^{-8}
Present method				
10^{-2}	2.4609e-05	1.0585e-05	1.4416e-05	1.4416e-05
10^{-4}	2.0593e-05	1.2422e-05	1.2323e-05	1.2362e-05
10^{-6}	2.0588e-05	1.2390e-05	1.2343e-05	1.2342e-05
10^{-8}	2.0588e-05	1.2389e-05	1.2343e-05	1.2343e-05
Gupta *et al.* [2019]				
10^{-2}	1.7212e-02	1.7507e-02	2.2799e-02	2.2801e-02
10^{-4}	1.7000e-02	1.6928e-02	1.6913e-02	1.6962e-02
10^{-6}	1.6998e-02	1.6923e-02	1.6908e-02	1.6917e-02
10^{-8}	1.6998e-02	1.6923e-02	1.6908e-02	1.6917e-02

Table 2. Maximum absolute errors at $\mu = 10^{-4}, T = 1$, number of intervals M/N for Example 1.

$\varepsilon \downarrow M/N$	8/4	16/8	32/16	64/32	128/64	256/128
After extrapolation						
10^{-2}	2.0324e-03	5.6707e-04	1.5080e-04	3.8842e-05	9.8588e-06	2.4836e-06
10^{-4}	1.5798e-03	4.4201e-04	1.1796e-04	3.0736e-05	7.9840e-06	2.0373e-06
10^{-6}	1.5751e-03	4.3990e-04	1.1681e-04	3.0021e-05	7.6259e-06	1.9373e-06
10^{-8}	1.5754e-03	4.4092e-04	1.1766e-04	3.0349e-05	7.7131e-06	1.9526e-06
10^{-10}	1.5754e-03	4.4092e-04	1.1766e-04	3.0349e-05	7.7131e-06	1.9526e-06
Before extrapolation						
10^{-2}	1.8431e-02	1.0259e-02	5.4059e-03	2.7758e-03	1.4067e-03	7.0813e-04
10^{-4}	1.9808e-02	1.0637e-02	5.5205e-03	2.8136e-03	1.4217e-03	7.1443e-04
10^{-6}	1.9837e-02	1.0649e-02	5.5257e-03	2.8153e-03	1.4227e-03	7.1497e-04
10^{-8}	1.9837e-02	1.0649e-02	5.5265e-03	2.8165e-03	1.4236e-03	7.1552e-04
10^{-10}	1.9837e-02	1.0649e-02	5.5265e-03	2.8165e-03	1.4236e-03	7.1552e-04

Table 3. Rate of convergence at $\mu = 10^{-4}$, $T = 1$ and number of intervals M/N for Example 1.

$\varepsilon \downarrow$	8/4	16/8	32/16	64/32	128/64
After extrapolation					
10^{-2}	1.8416	1.9109	1.9569	1.9781	1.9890
10^{-4}	1.8376	1.9058	1.9403	1.9447	1.9705
10^{-6}	1.8402	1.9130	1.9601	1.9770	1.9769
10^{-8}	1.8371	1.9059	1.9549	1.9763	1.9819
10^{-10}	1.8371	1.9059	1.9549	1.9763	1.9819
Before extrapolation					
10^{-2}	0.8452	0.9243	0.9616	0.9806	0.9902
10^{-4}	0.8970	0.9462	0.9724	0.9848	0.9928
10^{-6}	0.8975	0.9465	0.9729	0.9847	0.9927
10^{-8}	0.8975	0.9463	0.9725	0.9844	0.9925
10^{-10}	0.8975	0.9463	0.9725	0.9844	0.9925

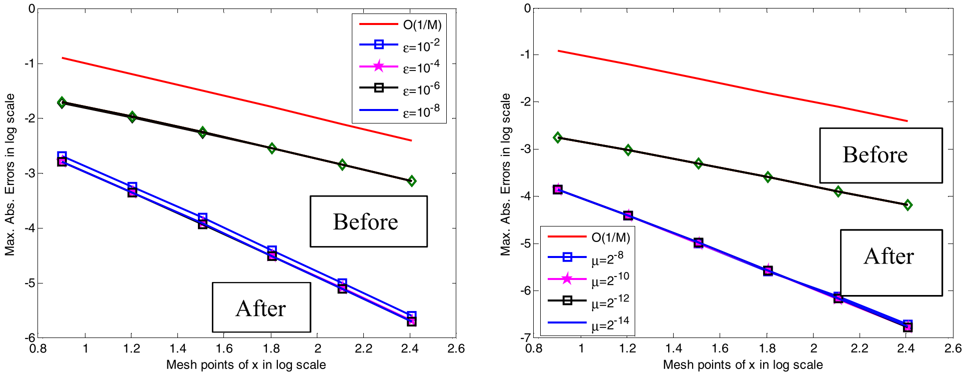

Fig. 1. Log–log plot of maximum absolute errors before and after Richardson extrapolation at $M = \{8, 16, 32, 64, 128, 256\}$ for Example 1 with $\mu = 10^{-4}$ and different values of ε in the left side and for Example 2 with $\varepsilon = 2^{-10}$ and different values of μ in the right adjacent.

Example 2. Consider the singularly perturbed parabolic IBVP

$$\varepsilon\frac{\partial^2 u}{\partial x^2} + \mu[1 + x(1-x) + t^2]\frac{\partial u}{\partial x} - (1 + 5xt)u(x,t) - \frac{\partial u}{\partial t} = x(1-x)(e^t - 1)$$

subject to the conditions: $u(x,0) = 0, x \in [0,1], u(0,t) = 0 = u(1,t), t \in [0,1]$. Computed result given is in Tables 4 and 5 along with Fig. 1.

From the results displayed in Tables 2–5, one can observe the effects of using Richardson extrapolation method to produce more accurate numerical solution corresponding to higher rate of convergence for solving singularly perturbed parabolic convection-diffusion IBVPs with two parameters. In addition, results in Table 1 demonstrate the accelerated fitted operator finite difference method gives more accurate numerical solution than the current method as cited in the literature. Moreover, clearly to verify the contribution of nonstandard or fitted operator finite difference method to get accurate numerical solution without dependent of the perturbation parameters, one can realize from results in Tables 1, 2 and 4 with the

Table 4. Maximum absolute errors at $\varepsilon = 2^{-10}$ and number of intervals $M = N$ for Example 2.

$\mu \downarrow N \rightarrow$	8	16	32	64	128	256
After extrapolation						
2^{-8}	1.3840e-04	3.8046e-05	9.9990e-06	2.5738e-06	7.2761e-07	1.9279e-07
2^{-10}	1.3991e-04	3.8463e-05	1.0119e-05	2.5939e-06	6.5673e-07	1.6520e-07
2^{-12}	1.4044e-04	3.8594e-05	1.0159e-05	2.6058e-06	6.5956e-07	1.6593e-07
2^{-14}	1.4058e-04	3.8629e-05	1.0181e-05	2.6097e-06	6.8344e-07	1.7560e-07
Before extrapolation						
2^{-8}	1.7714e-03	9.5858e-04	4.9804e-04	2.5358e-04	1.2795e-04	6.4275e-05
2^{-10}	1.7705e-03	9.5664e-04	4.9723e-04	2.5314e-04	1.2774e-04	6.4168e-05
2^{-12}	1.7700e-03	9.5602e-04	4.9701e-04	2.5302e-04	1.2769e-04	6.4141e-05
2^{-14}	1.7699e-03	9.5586e-04	4.9695e-04	2.5299e-04	1.2768e-04	6.4134e-05

Table 5. Order of convergence at $\varepsilon = 2^{-10}$ and number of intervals $M = N$ for Example 2.

$\mu \downarrow N \rightarrow$	8	16	32	64	128
After extrapolation					
2^{-8}	1.8630	1.9279	1.9579	1.8227	1.9161
2^{-10}	1.8630	1.9264	1.9639	1.9818	1.9911
2^{-12}	1.8635	1.9256	1.9630	1.9822	1.9909
2^{-14}	1.8636	1.9238	1.9639	1.9330	1.9605
Before extrapolation					
2^{-8}	0.8859	0.9446	0.9738	0.9869	0.9933
2^{-10}	0.8881	0.9441	0.9740	0.9867	0.9933
2^{-12}	0.8886	0.9438	0.9740	0.9866	0.9933
2^{-14}	0.8888	0.9437	0.9740	0.9865	0.9934

corresponding nondisturbance of rate of convergence in Tables 3 and 5 at both before and after extrapolation. Besides, Fig. 1 indicates log–log plotted to show the difference between the obtained maximum absolute errors before and after applying Richardson extrapolation and robust or uniformly convergence of the two schemes.

8. Conclusion

Robust finite difference method is developed for solving singularly perturbed parabolic problems whose two derivative terms are affected by perturbation parameters. Replacing the derivative with respect to time in the differential equation by finite difference approximation yields boundary value problem. The space domain discretization on uniform mesh subsequent the nonstandard procedure of Mickens clues to a full discrete problem whose fundamental operator fulfilled a continuous minimum principle. A convergence analysis grounded on this detail presented that the developed method is robust regardless of the perturbation parameters. To validate the proposed method, we considered test examples and numerical results to care the theoretical results and to determine its effectiveness. Furthermore, the presented method gives more accurate results than the existing methods in the recent literature. In a concise manner, the proposed method is robust, convergent and produces more accurate numerical solution than the existing method for solving singularly perturbed parabolic IBVPs with two parameters.

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