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LIGHTLIKE EINSTEIN HYPERSURFACES IN LORENTZIAN MANIFOLDS WITH CONSTANT CURVATURE

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Abstract

We consider a class of lightlike hypersurfaces M, of semi-Riemannian manifolds, whose shape operator is conformal to the shape operator of its screen distributions. Integrability and characterization results are then established.

1. Introduction and results

The classification of Einstein hypersurfaces (M, g) in Euclidean spaces $\mathbf{R}^{n+1}$ was first studied by A. Fialkow [1], T. Y. Thomas [2] in the middle of 1930'*s*. It has been proved that if $f : M^n \to \mathbf{R}^{n+1}$ $n \geq 3$ is a connected Einstein hypersurface, that is $Ric = \rho g$, for some constant ρ, then ρ is non negative.

- if $\rho = 0$ then $f(M^n)$ is locally isometric to $\mathbf{R}^n$
- if $\rho > 0$ then $f(M^n)$ is contained in an n-sphere.

In this paper, a class of lightlike hypersurfaces in a semi-Riemannian manifold $(\overline{M}(c), \bar{g})$ with constant sectional curvature c is considered: the class of lightlike hypersurfaces with conformal screen distribution. Some properties of those hypersurfaces have been studied in ([3]) where a physical model of screen globally conformal lightlike hypersurfaces is presented. Our text is organized as follows. In section 2 we summarize notations and basic facts concerning geometric objects on lightlike hypersurfaces, using notations in [4]. Section 3 deals with the Ricci tensor of a lightlike hypersurface and we establish its equation. It is then proved that screen locally (or globally) conformal totally umbilic lightlike hypersurfaces of a Lorentzian manifold of constant curvature are Einstein manifolds. Finally, section 4 is devoted to the proof of the following characterization theorem which extends Fialkow and Thomas results.

Theorem 1. *Let $(\overline{M}^{n+2}(c), \bar{g})$ be a Lorentzian manifold of constant curvature $c \geq 0$, and $(M^{n+1}, g, S(TM))$ a screen locally (or globally) conformal lightlike hypersurface. If $(M^{n+1}, g, S(TM))$ is Einstein, that is $Ric = \rho g$ for some constante ρ, then ρ is non negative*

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- *If $\rho = nc$, then M is locally a product $\mathbf{L} \times M^*$ where M^* is a Riemannian n-space form with the same sectional curvature c as $\overline{M}$ and $\mathbf{L}$ is an open subset of a lightlike geodesic ray in $\overline{M}$.*
- *If $\rho > nc$, then M is locally a product $\mathbf{L} \times M^*$ where M^* is a Riemannian n-space form of positive constant curvature which is isometric to an n-sphere.*

2. Preliminaries and basic facts

Let $(\overline{M}, \bar{g})$ be a real $(n+2)$-dimensional semi-Riemannian manifold, where $n \geq 1$ and $\bar{g}$ is a non degenerate tensor field of type $(0, 2)$ and of constant index $q \geq 1$, on $\overline{M}$. Consider a submanifold M of $\overline{M}$ of codimension p with $p \geq 1$. The metric $\bar{g}$ might be non degenerate or degenerate on the tangent bundle TM of M. The case $\bar{g}$ is non degenerate has many similarities with Riemannian submanifolds [6].

In the degenerate case, basic differences occur mainly because of the fact that the normal vector bundle $TM^{\perp}$ intersect with the tangent bundle along a nonzero differentiable distribution called radical distribution of M and denoted by $Rad(TM)$

$$Rad(TM) : x \to Rad(T_xM) = T_xM \cap T_xM^{\perp}. \tag{1}$$

The dimension r $(r > 0)$ of fibres of $Rad(TM)$ is the degree of nullity of M. For $r = 0$ the submanifold M is non degenerate.

Given an integer $r > 0$, the submanifold M is said to be r-lightlike if $rank(Rad(T_xM)) = r$ at every point $x \in M$. Lightlike hypersurfaces M of $\overline{M}$ are one-codimensionnal submanifolds for which $r = 1$. Given an $(n+1)$-dimensional lightlike hypersurface M, a *screen distribution*, denoted $S(TM)$, on M is a subbundle of TM which is complementary to the radical distribution. It is non degenerate and has constant rank n. It plays an important role in the study of induced geometric objects on M. For the lightlike hypersurfaces, equation (1) becomes $Rad(TM) = TM^{\perp}$ and we have the orthogonal direct sum

$$TM = S(TM) \perp TM^{\perp}. \tag{2}$$

Throughout this paper we denote by $\mathscr{F}(M)$ the algebra of differentiable functions on M and $\Gamma(E)$ the $\mathscr{F}(M)$-module of differentiable sections of vector bundle E over M. The manifolds we consider are supposed to be paracompact, smooth and connected. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/Rad\,TM$. The screen distribution $S(TM)$ exits because of the paracompactness of M. The following normalization result is now well known:

Theorem (Duggal-Bejancu, [4, page 79]). *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighbourhood $\mathscr{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathscr{U}$ satisfying*

$$\bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, W) = 0, \quad \forall W \in \Gamma(S(TM|_{\mathcal{U}})). \tag{3}$$

Now, consider $\bar{\nabla}$ the Levi-Civita connection of $\overline{M}$ and ∇ the induced connection on the lightlike hypersurface (M, g) where g is the induced metric on M by $\bar{g}$. With decomposition (2) and relations (3), we have

$$T\overline{M}|_M = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM). \tag{4}$$

Gauss and Weingarten formulae provide (see details in [4, page 82])

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM) \tag{5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X \in \Gamma(TM), \forall V \in \Gamma(tr(TM)), \tag{6}$$

where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while h is a $\Gamma(tr(TM))$-valued symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$, A_V is an $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$ and ∇^t is a linear connection on $tr(TM)$. In general, induced connection ∇ is not unique and depends on the triplet $(M, g, S(TM))$. Define a symmetric $\mathcal{F}(M)$-bilinear form B and a 1-form τ on the coordinate neighbourhood $\mathcal{U}$ by

$$B(X, Y) = \bar{g}(h(X, Y), \xi), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{7}$$

$$\tau(X) = \bar{g}(\nabla^t_X N, \xi), \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{8}$$

It follows that,

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{9}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{10}$$

Let P denote the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to decomposition (2). We obtain

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY), \quad \forall X, Y \in \Gamma(TM), \tag{11}$$

$$\nabla_X U = -A^*_U X + \nabla^{*t}_X U, \quad \forall X \in \Gamma(TM), \forall U \in \Gamma(TM^{\perp}). \tag{12}$$

where $\nabla^*_X Y$ and $A^*_U X$ belong to $\Gamma(S(TM))$, ∇ and ∇^{*t} are linear connections on $\Gamma(S(TM))$ and $TM^{\perp}$ respectively, h^* is a $\Gamma(TM^{\perp})$-valued $\mathcal{F}(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$ and A^*_U is $\Gamma(S(TM))$-valued $\mathcal{F}(M)$-linear operator on $\Gamma(TM)$. They are the second fundamental form and shape operator of the screen distribution $S(TM)$ respectively. Define on $\mathcal{U}$

$$C(X, PY) = \bar{g}(h^*(X, PY), N), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{13}$$

$$\varepsilon(X) = \bar{g}(\nabla^{*t}_X \xi, N), \quad \forall X \in \Gamma(TM|_{\mathcal{U}}). \tag{14}$$

One can show that $\varepsilon(X) = -\tau(X)$. Thus, locally we obtain

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \quad \forall X, Y \in \Gamma(TM), \tag{15}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \quad \forall X \in \Gamma(TM). \tag{16}$$

The linear connection ∇^* is a metric connection on $S(TM)$. Hence if $S(TM)$ is integrable and S denotes a leaf of $S(TM)$ then ∇^* represents the Levi-Civita connection of S, intrinsically linked with its geometry. In general, the induced connection ∇ on M is not compatible with the induced metric g. Indeed, we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \quad \forall X, Y \in \Gamma(TM|_{\mathscr{U}}) \tag{17}$$

where

$$\eta(X) = \bar{g}(X,N), \quad \forall X \in \Gamma(TM|_{\mathscr{U}}). \tag{18}$$

Finally, it is easy to check that

$$B(X,\xi) = 0, \quad \forall X \in \Gamma(TM|_{\mathscr{U}}). \tag{19}$$

3. The Ricci tensor of a lightlike hypersurface

In this section we study the Ricci tensor of a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\bar{g})$ with Lorentzian signature. We have the following:

PROPOSITION 1. *Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\bar{g})$ with Lorentzian signature. Then the Ricci tensor of M denoted by Ric is given for X, Y tangent to M by*

$$\begin{aligned} Ric(X,Y) = \overline{R}ic(X,Y) + B(X,Y)\, tr\, A_N - \eta(\overline{R}(\xi,Y)X) \\ - g(A_N X, A^*_\xi Y) \end{aligned} \tag{20}$$

where $\overline{R}ic$ is the Ricci tensor of $\overline{M}$ and $tr\, A_N$ the trace of the operator A_N.

Proof. For X and Y in $\Gamma(TM)$, we have

$$Ric(X,Y) = \sum_{i=1}^{n} g(R(E_i,X)Y,E_i) + \bar{g}(\overline{R}(\xi,X)Y),N)$$

where $\{\xi,E_1,\dots,E_n\}$ is an orthonormal frame field on M adapted to decomposition (2) and ε_i is the causal character of the vector field E_i, of the orthonormal frame field $\{E_1,\dots,E_n\}$ of $S(TM)$. The ambient space being Lorentzian, the screen distribution is Riemannian so that $\varepsilon_i = 1$, $1 \le i \le n$. Then from Gauss-Codazzi equations

$$\begin{aligned} g(R(E_i,X)Y,E_i) &= \bar{g}(\overline{R}(E_i,X)Y,E_i) + \bar{g}(h(X,Y),h^*(E_i,E_i)) \\ &\quad - \bar{g}(h(E_i,Y),h^*(X,E_i)) \\ &= \bar{g}(\overline{R}(E_i,X)Y,E_i) + B(X,Y)C(E_i,E_i) - B(E_i,Y)C(X,E_i) \end{aligned}$$

hence

$$\begin{aligned}
\sum_{i=1}^{n} g(R(E_i, X)Y, E_i) &= \sum_{i=1}^{n} \bar{g}(\overline{R}(E_i, X)Y, E_i) + B(X, Y)\sum_{i=1}^{n} C(E_i, E_i) \\
&\quad - \sum_{i=1}^{n} B(E_i, Y)C(X, E_i) \\
&= \sum_{i=1}^{n} \bar{g}(\overline{R}(E_i, X)Y, E_i) + B(X, Y)\sum_{i=1}^{n} g(A_N E_i, E_i) \\
&\quad - \sum_{i=1}^{n} g(A_\xi^* Y, E_i) g(A_N X, E_i) \\
&= \sum_{i=1}^{n} \bar{g}(\overline{R}(E_i, X)Y, E_i) + B(X, Y)\left(\sum_{i=1}^{n} g(A_N E_i, E_i)\right. \\
&\quad \left. + \bar{g}(A_N \xi, N)\right) - g(A_N X, A_\xi^* Y)
\end{aligned}$$

As $\bar{g}(R(\xi, X)Y, N) = \bar{g}(\overline{R}(\xi, X)Y, N)$ we obtain

$$\begin{aligned}
\sum_{i=1}^{n} g(R(E_i, X)Y, E_i) + \bar{g}(R(\xi, X)Y, N) &= \sum_{i=1}^{n} \bar{g}(\overline{R}(E_i, X)Y, E_i) + \bar{g}(\overline{R}(\xi, X)Y, N) \\
&\quad + B(X, Y)\ tr\ A_N - g(A_N X, A_\xi^* Y)
\end{aligned}$$

Hence

$$Ric(X, Y) = \overline{R}ic(X, Y) + B(X, Y)\ tr\ A_N - \eta(\overline{R}(\xi, Y)X) - g(A_N X, A_\xi^* Y) \qquad \blacksquare$$

It is important to stress that the Ricci tensor does not depend on the choice of ξ. Indeed, if $(\xi', E_1, \ldots, E_n, N')$ is an other frame field adapted to (2) and Ric' denotes the Ricci tensor with respect to (ξ', N') we have for X, Y tangent to M

$$Ric(X, Y) = \sum_{i=1}^{n} \varepsilon_i g(R(E_i, X)Y, E_i) + \bar{g}(R(\xi', X)Y, N')$$

But $\xi' = \alpha\xi$ and $N' = \frac{1}{\alpha}N$ for some smooth function $\alpha > 0$. Then

$$\bar{g}(R(\xi', X)Y, N') = \bar{g}(R(\xi, X)Y, N)$$

so that $Ric(X, Y) = Ric'(X, Y)$.

Corollary 1. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an $(n+2)$-dimensional $(n \geq 2)$ Lorentzian manifold $(\overline{M}, \bar{g})$ of constant curvature c. Then*

$$Ric(X, Y) = ncg(X, Y) + B(X, Y)\ tr\ A_N - g(A_N X, A_\xi^* Y) \tag{21}$$

Proof. In the case $\overline{M}$ is of constant sectional curvature c, we have

$$\overline{Ric} = (n+1)c\bar{g}, \quad \text{and} \quad \eta(\overline{R}(\xi, Y)X) = cg(X, Y),$$

and the result follows. ■

3.1. Conformal screen on lightlike hypersurfaces

Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold, A_N and A_ξ^* the shape operators of M and its screen distribution $S(TM)$ respectively. In the srquel, we will use the following

Definition 1 ([3]). *The hypersurface M is said to be screen locally* (*resp. globally*) *conformal if on any coordinate neighborhood $\mathscr{U}$* (*rep. $\mathscr{U} = M$*), *there is a smooth function φ such that*

$$A_N = \varphi A_\xi^* \tag{22}$$

holds; *the function φ is referred to as the conformal weight.*

A large class of lightlike submanifolds satisfy relation (22):

- The lightlike cone Λ_0^{n+1} of $\mathbf{R}_1^{n+2}$ is such that

$$A_N X = \frac{1}{2(x^0)^2} A_\xi^* X, \quad X \in \Gamma(T\Lambda_0^{n+1}).$$

  So Λ_0^{n+1} is a screen globally conformal lightlike hypersurface of $\mathbf{R}_1^{n+2}$, whose conformal weight is globally defined on Λ_0^{n+1}.
- Lightlike Monge hypersurfaces, endowed with the canonical screen distribution, are screen globally conformal in Lorentz spaces, with constant conformal weight $\varphi(x) = \frac{1}{2}$.

Conformal screen lightlike hypersurfaces have the outstanding feature that their screen distributions are always integrable and their geometry essentially reduces to the one of leaves of these screen distributions. More precisely, a screen locally (or globally) conformal lightlike hypersurface is locally (or globally) a product $\mathscr{C} \times M'$ where $\mathscr{C}$ is a null curve and M' is a leaf of $S(TM)$ ([3]).

Proposition 2. *Any screen conformal totally umbilical lightlike hypersurface $(M^{n+1}, g, S(TM))$ of a Lorentzian space form $(\overline{M}(c), \bar{g})$ with $c > 0$ is an Einstein manifold.*

Proof. The assumption yield $B(X, Y) = \rho g(X, Y)$ and

$$g(A_N X, A_\xi^* Y) = \rho^2 \varphi g(X, Y)$$

for some smooth function ρ on $\mathscr{U} \subset M$ so that ([redacted]) reads

$$Ric(X, Y) = [nc + \rho s - \varphi\rho^2] g(X, Y)$$

where $s = tr\, A_N$. The result follows. ■

Since A_N and A^*_ξ commute when M is screen conformal, Corollary 1 shows that the Ricci tensor of any screen conformal lightlike hypersurface in a Lorentzian manifold of constant curvature is symmetric.

4. Proof of Theorem 1

For the proof of our main theorem, we will make use of the following facts and lemmas. Let x denote an arbitrary point on M.

FACT 1. The operator $A^*_{\xi_x}$ is diagonalizable and ξ is an eigenvector with eigenvalue 0. Since $S(TM)$ is Riemannian for a Lorentzian $\overline{M}$, consider a frame field of eigenvectors of A^*_ξ, say $\{E_0 = \xi, E_1, \ldots, E_n\}$ such that $\{E_i\}_{i=1,\ldots,n}$ represents an orthonormal basis of the screen distribution $S(TM)$ on M. Then $A^*_\xi E_0 = A^*_\xi \xi = 0$ and

$$A^*_\xi E_i = \lambda_i E_i \quad 1 \leq i \leq n.$$

Now we use the Ricci tensor equality ([redacted]) of corollary 1 to have

$$g(A^*_\xi X, A^*_\xi Y) - sg(A^*_\xi X, Y) + \varphi(\rho - nc)g(X, Y) = 0 \tag{23}$$

where $s = tr\, A^*_\xi$. In (23) put $X = Y = E_i$ to obtain

$$\lambda_i^2 - s\lambda_i + -\varphi(\rho - nc) = 0 \quad 1 \leq i \leq n \tag{24}$$

For each integer i, this equation admits no more than two distinct real solutions. We may assume there exists $p \in \{1, \ldots, n\}$ such that $\lambda_1 = \cdots = \lambda_p$ and $\lambda_{p+1} = \cdots = \lambda_n$, by renumbering if necessary. We now follow the general scheme in [5].

Assume $\rho < nc$ and argue by contradiction. Set

$$\lambda_1 = \cdots = \lambda_p = v, \quad \lambda_{p+1} = \cdots = \lambda_n = \mu,$$

from (24) we have

$$s = \mu + v = pv + (n - p)\mu$$

that is

$$(p - 1)v + (n - p - 1)\mu = 0 \quad \text{with } \mu v = -\varphi(\rho - nc) < 0.$$

Note that condition $n \geq 3$ implies $p > 1$ and $p < n - 1$. Then,

$$v^2 = \varphi \frac{n - p - 1}{p - 1}(\rho - nc). \tag{25}$$

Since $(x \mapsto s = tr\, A^*_{\xi_x})$ is a differentiable function, from (24) the roots μ and v are differentiable. From (25) we deduce that v is a finite valued smooth function, so it is a constant function near x. The same for μ and p in a neighborhood of x.

Now, consider the following distributions D_μ and D_v

$$D_\nu(x) = \{X \in T_xM, A_\xi^* X = \nu PX\}$$
$$D_\mu(x) = \{X \in T_xM, A_\xi^* X = \mu PX\}$$

where P is the projection operator on $\Gamma(S(TM))$ in the orthogonal direct sum of the bundle resulting from

$$T_xM = S(T_xM) \perp T_xM^\perp$$

FACT 2. In a neighborhood of $x \in M$, D_μ and D_ν are smooth distributions. Indeed from (23) we deduce

$$A_\xi^{*2} - (\mu + \nu)A_\xi^* + \mu\nu P = 0. \tag{26}$$

Consider $F_1, \dots, F_n$ differentiable vector fields in a neighborhood of x such that $\{PF_i(x)\}_{1 \le i \le p}$ and $\{PF_j(x)\}_{p+1 \le j \le n}$ constitute a basis of $D_\nu(x)$ and $D_\mu(x)$, respectively.

Put

$$Y_i = (A_\xi^* - \mu P)F_i, \quad 1 \le i \le p$$
$$Y_j = (A_\xi^* - \nu P)F_j \quad p + 1 \le j \le n$$

From (26) we have

$$(A_\xi^* - \nu P)Y_i = (A_\xi^* - \nu P)(A_\xi^* - \mu P)F_i = 0$$

and

$$(A_\xi^* - \mu P)Y_j = (A_\xi^* - \mu P)(A_\xi^* - \nu P)F_j = 0$$

that is $Y_i \in D_\nu$ and $Y_j \in D_\mu$. Therefore, since

$$Y_i(x) = (A_\xi^* - \mu P)F_i(x) = (\nu - \mu)PF_i(x)$$

and

$$Y_j(x) = (\nu - \mu)PF_j(x)$$

we deduce that $(Y_i)_{1 \le i \le p}$ and $(Y_j)_{p+1 \le j \le n}$ are basis of D_ν and D_μ, respectively in a coordinate neighborhood of x; thus they are smooth distributions on M.

FACT 3.

$$\begin{cases} Im(A_\xi^* - \mu P) \subset D_\nu \\ Im(A_\xi^* - \nu P) \subset D_\mu. \end{cases} \tag{27}$$

Indeed, let $Y \in Im(A_\xi^* - \mu P)$, there exists $X \in T_xM$ such that $Y = (A_\xi^* - \mu P)X$. Then (27) is immediate from (26).

LEMMA 1. *For all $X \in D_\nu$ and $Y \in D_\mu$, we have*

$$\nabla_X Y \in D_\mu, \quad \nabla_Y X \in D_\nu \tag{28}$$

Proof. Let $X \in D_\nu$, $Y \in D_\mu$. Remind first that

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y) \quad X, Y \in \Gamma(TM) \tag{29}$$

Then

$$\begin{aligned}\nabla_X(g(A_\xi^* Y, Z)) &= B(X, A_\xi^* Y)\eta(Z) + B(X,Z)\eta(A_\xi^* Y)\\ &\quad + g(\nabla_X A_\xi^* Y, Z) + g(A_\xi^* Y, \nabla_X Z) \quad Z \in \Gamma(TM)\end{aligned} \tag{30}$$

where η is the local 1-form dual of the local section ξ; that is

$$\eta(X) = \bar{g}(X, N)$$

Note that $\eta(X) = 0$ if and only if $X \in \Gamma(S(TM))$. Using the stability of D_μ and D_ν with respect to P and also their orthogonality with respect to B, we deduce that $B(X, A_\xi^* Y) = 0$. As $\eta(A_\xi^* Y) = 0$, (30) becomes

$$\begin{aligned}X \cdot g(A_\xi^* Y, Z) &= g(\nabla_X A_\xi^* Y, Z) + g(A_\xi^* Y, \nabla_X Z)\\ &\qquad X \in D_\nu \; Y \in D_\mu \; Z \in \Gamma(TM)\end{aligned} \tag{31}$$

By Codazzi equation,

$$\begin{aligned}X \cdot g(A_\xi^* Y, Z) &- g(\nabla_X A_\xi^* Y, Z) - g(A_\xi^* Y, \nabla_X Z)\\ &= Y \cdot g(A_\xi^* X, Z)\\ &= g(\nabla_Y A_\xi^* X, Z) + g(A_\xi^* X, \nabla_Y Z),\end{aligned}$$

which yields

$$g((A_\xi^* - \mu P)\nabla_X Y, Z) = g((A_\xi^* - \nu P)\nabla_Y X, Z)$$

that is

$$g((A_\xi^* - \mu P)\nabla_X Y - (A_\xi^* - \nu P)\nabla_Y X, Z) = 0, \quad \forall Z \in TM|_{\mathscr{U}}.$$

Then for all $X \in D_\nu$, $Y \in D_\mu$,

$$(A_\xi^* - \mu P)\nabla_X Y - (A_\xi^* - \nu P)\nabla_Y X \in T_x M^\perp$$

But $(A_\xi^* - \mu P)$ and $(A_\xi^* - \nu P)$ are $S(TM)$-valued, so

$$(A_\xi^* - \mu P)\nabla_X Y - (A_\xi^* - \nu P)\nabla_Y X \in S(TM) \cap TM^\perp = \{0\},$$

then

$$(A_\xi^* - \mu P)\nabla_X Y = (A_\xi^* - \nu P)\nabla_Y X, \quad \forall X \in D_\nu, \forall Y \in D_\mu. \tag{32}$$

From (27) and (32) and taking into account $D_\nu \cap D_\mu = \{0\}$ we have

$$(A_\xi^* - \mu P)\nabla_X Y = 0 = (A_\xi^* - \nu P)\nabla_Y X$$

that is for all $X \in D_\nu$ and for all $Y \in D_\mu$,

$$\nabla_X Y \in D_\mu, \quad \text{and} \quad \nabla_Y X \in D_\nu. \qquad \blacksquare \tag{33}$$

LEMMA. *For any* $X, Z \in D_\nu$ *and any* $U, Y \in D_\mu$, *the following holds*

$$g(\nabla_Z X, Y) = g(X, \nabla_U Y) = 0, \tag{34}$$

Proof. Let $X, Z \in D_\nu$, $Y \in D_\mu$. From

$$\begin{aligned} g(X, PY) &= \frac{1}{\nu} g(A^*_\xi X, PY) \\ &= \frac{1}{\nu} B(X, PY) = 0 \end{aligned}$$

we have

$$\begin{aligned} \nabla_Z(g(X, PY)) &= 0 \\ &= B(Z, X)\eta(PY) + B(Z, PY)\eta(X) + g(\nabla_Z X, PY) \\ &\quad + g(X, \nabla_Z PY) \\ &= g(\nabla_Z X, Y). \end{aligned} \tag{35}$$

where we used the orthogonality of D_μ and D_ν with (respect to g) and (33). Then from (19) we get

$$g(\nabla_Z X, Y) = 0, \quad \forall X, Z \in D_\nu, \forall Y \in D_\mu. \tag{36}$$

Similarly we get

$$g(X, \nabla_U Y) = 0, \quad \forall X \in D_\nu\, U, Y \in D_\mu \qquad \blacksquare \tag{37}$$

FACT 4. For $X \in D_\nu$ and $Y \in D_\mu$, we have

$$g(R(PX, PY)PY, PX) = -\varphi\mu\nu g(X, X)g(Y, Y)$$

Indeed for X in D_ν and Y in D_μ we have

$$g(R(X, Y)Y, X) = g(\nabla_X \nabla_Y Y, X)$$

since

$$g(\nabla_Y \nabla_X Y, X) = g(\nabla_{\nabla_Y X} Y, X) = g(\nabla_{\nabla_X Y} Y, X) = 0$$

by using (34) and (33) and the g-orthogonality of D_μ and D_ν. Let us consider P_μ the projection on D_μ. From (36) we know that $\nabla_Y Y$ has no component in D_ν; so

$$\nabla_Y Y = P_\mu \nabla_Y Y + \eta(\nabla_Y Y)\xi.$$

It follows that

$$g(\nabla_X \nabla_Y Y, X) = g(\nabla_X P_\mu(\nabla_Y Y), X) + \nabla_X(\eta(\nabla_Y Y))g(\xi, X) + \eta(\nabla_Y Y)g(\nabla_X \xi, X)$$

and using again (34) and (33) and the g-orthogonality of D_μ and D_ν we get

$$g(\nabla_X \nabla_Y Y, X) = -\nu\eta(\nabla_Y Y)g(X, X), \quad X \in D_\nu,\ Y \in D_\mu \tag{38}$$

One can check that $\eta(\nabla_Y Y) = Y \cdot (\eta(Y)) + \varphi\mu g(Y, Y)$ so that

$$g(R(X, Y)Y, X) = -\nu Y \cdot (\eta(Y))g(X, X) - \varphi\mu\nu g(X, X)g(Y, Y) \quad X \in D_\nu,\ Y \in D_\mu.$$

In particular

$$g(R(PX, PY)PY, PX) = -\varphi\mu\nu g(X, X)g(Y, Y) \quad X \in D_\nu,\ Y \in D_\mu \tag{39}$$

Proof of Theorem 1. Now, using Gauss equation with (39) leads to

$$\begin{aligned} g(R(PX, PY)PY, PX) = c\{g(Py, PY)g(PX, PX) - (g(PX, PY))^2\} \\ + \varphi\{-(B(PX, PY))^2 + B(PY, PY)B(PX, PX)\} \end{aligned}$$

hence

$$g(R(PX, PY)PY, PX) = (c + \varphi)\mu\nu g(X, X)g(Y, Y) \tag{40}$$

where we use the orthogonality of D_μ and D_ν with respect to both B and g and relation

$$C(X, PY) = \varphi g(X, PY).$$

Choosing $X \in D_\nu$ and $Y \in D_\mu$ such that $g(X, X) \neq 0$ and $g(Y, Y) \neq 0$ and taking into account (39) and (40) leads to $\varphi\mu\nu = \frac{1}{2}c$. But we have also that $\varphi\mu\nu = (\rho - nc)$. This is a contradiction with $\rho - nc < 0$ since $c \geq 0$. So $\rho \geq nc$ is proved. In the sequel we distinguish the two cases $\rho = nc$ and $\rho > nc$

Case $\rho = nc$

From (24) we have

$$\lambda_i(\lambda_i - s) = 0 \quad 1 \leq i \leq n$$

- Assume $p = n$, then $\lambda_1 = \cdots = \lambda_n = \lambda$, and

  $$\lambda(\lambda - n\lambda) = -(n-1)\lambda^2 = 0$$

  which implies

  $$\lambda = 0 \quad \text{and} \quad A_\xi^* = 0$$

  or equivalently

  $$h = 0$$

  that is M is totally geodesic in $\overline{M}$. Recall that for X, Y in $\Gamma(S(TM))$ we have

  $$\overline{\nabla}_X Y = \nabla_X^* Y + C(X, Y)\xi + B(X, Y)N \tag{41}$$

  Hence for a totally geodesic M, by (22) we have $\overline{\nabla}_X Y = \nabla_X^* Y$ for X and Y tangent to a leaf M^* of the screen distribution. Therefore M^* is a Riemannian space form of same constant sectional curvature c as $\overline{M}$ and

M is locally a product $\mathbf{L} \times M^*$ where $\mathbf{L}$ is a leaf of the rank one distribution $Rad(TM)$.

- Assume $1 \leq p \leq n-1$. We have

$$\lambda_1 = \cdots = \lambda_p = 0 \neq \lambda_{p+1} = \cdots = \lambda_n = \lambda$$

Then

$$s = (n-p)\lambda$$

and

$$(n-p-1)\lambda^2 = 0$$

So

$$p = n-1$$

that is

$$A_\xi^* = \begin{pmatrix} 0 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \lambda \end{pmatrix}. \tag{42}$$

Let K denote the sectional curvature of M^*. From Gauss equation and considering $\{E_0 = \xi, E_1, \ldots, E_n\}$ such that $\{E_1, \ldots, E_n\}$ represents an orthonormal frame field of $S(TM)$, we have for $1 \leq i < j \leq n$

$$K(E_i, E_j) = \frac{g(R^*(E_i, E_j)E_j, E_i)}{\langle E_i, E_i\rangle\langle E_j, E_j\rangle - \langle E_i, E_j\rangle^2}.$$

Because of (42), $B(E_i, E_j) = 0$ for $1 \leq i < j \leq n$. Hence from (41) we have

$$g(R^*(E_i, E_j)E_j, E_i) = \bar{g}(\overline{R}(E_i, E_j)E_j, E_i) = c.$$

M^* is a Riemannian manifold of same constant sectional curvature c as $\overline{M}$, and M is locally a product $\mathbf{L} \times M^*$.

Case $\rho > nc$. Assume first that $1 \leq p \leq n$ that is

$$\lambda_1 = \cdots = \lambda_p = \alpha \neq \beta = \lambda_{p+1} = \cdots = \lambda_n$$

From (24) we have

$$\begin{cases} \alpha + \beta = s \\ \varphi\alpha\beta = (\rho - nc) > 0. \end{cases} \tag{43}$$

$$\alpha + \beta = s = p\alpha + (n-p)\beta$$

$$(p-1)\alpha = -(n-p-1)\beta$$

But, from the second equality of (43), α and β have the same sign. Then, since

$$p - 1 \leq 0 \quad \text{and} \quad (-(n - p - 1) \leq 0)$$

the only possibility is $p = 1$ and $n \geq 2$ which is a contradiction with $n \geq 3$.

So $p = n$ and then $\lambda = \lambda_1 = \cdots = \lambda_n$

$$A^*_\xi X = \lambda PX, \quad \forall X \in \Gamma(TM|_{\mathscr{U}})$$

with $\lambda \neq 0$ (for we have $\lambda^2 = \rho - nc > 0$)

Using the fact that M is screen locally conformal we infer

$$A_N X = \varphi\lambda PX, \quad \forall X \in \Gamma(TM)|_{\mathscr{U}} \tag{44}$$

It follows that M is locally a product $\mathbf{L} \times M^*$ where M^* is a totally umbilical Riemannian manifold which is not totally geodesic ($\lambda \neq 0$) of codimension 2 in $\overline{M}(c)$.

In fact, we show that the leaves M^* are space forms of constant curvature $c + 2\varphi\lambda^2$. Indeed, for X, Y in $\Gamma(S(TM))$ one checks that

$$\overline{\nabla}_X Y = \nabla^*_X Y + \lambda g(X, Y)(\varphi\xi + N)$$

so that

$$\overline{\nabla}_{[X, Y]} Z = \nabla^*_{[X, Y]} Z + \lambda g([X, Y], Z)(\varphi\xi + N)$$

and

$$\begin{aligned}\overline{\nabla}_X \overline{\nabla}_Y Z = \nabla^*_X \nabla^*_Y Z + \lambda g(\nabla^*_X Y, Z)(\varphi\xi + N) + \lambda[g(X, \nabla^*_Y Z) \\ + g(Y, \nabla^*_X Z)](\varphi\xi + N) + \lambda g(Y, Z)\overline{\nabla}_X(\varphi\xi + N)\end{aligned}$$

then

$$\overline{R}(X, Y)Z = R^*(X, Y)Z + \lambda g(Y, Z)\overline{\nabla}_X(\varphi\xi + N) - \lambda g(X, Z)\overline{\nabla}_Y(\varphi\xi + N)$$

It is easy to see that

$$\overline{\nabla}_X(\varphi\xi + N) = -2\lambda\varphi X - \tau(X)(\varphi\xi - N).$$

Hence, using the fact that $\overline{M}$ is of constant sectional curvature c and $R^*(X, Y)Z$ is $\Gamma(S(TM))$-valued, we obtain

$$R^*(X, Y)Z = (c + 2\lambda^2\varphi)\{g(Y, Z)X - g(X, Z)Y\} \tag{45}$$

and

$$\lambda[g(Y, Z)\tau(X) - g(X, Z)\tau(Y)] = 0 \tag{46}$$

The assertion that the leafs are space forms of positive constant curvature follows from (45). Now, in (46), let Y be such that $Y \neq 0$ and $\tau(Y) = 0$. Then (46) implies $g(Y, Y)\tau(X) = 0$ for all X tangent to the leaf M^*. Thus τ vanishes identically on the leafs of the screen distribution on M.

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