

Just Renormalizable TGFT's on $U(1)^d$ with Gauge Invariance

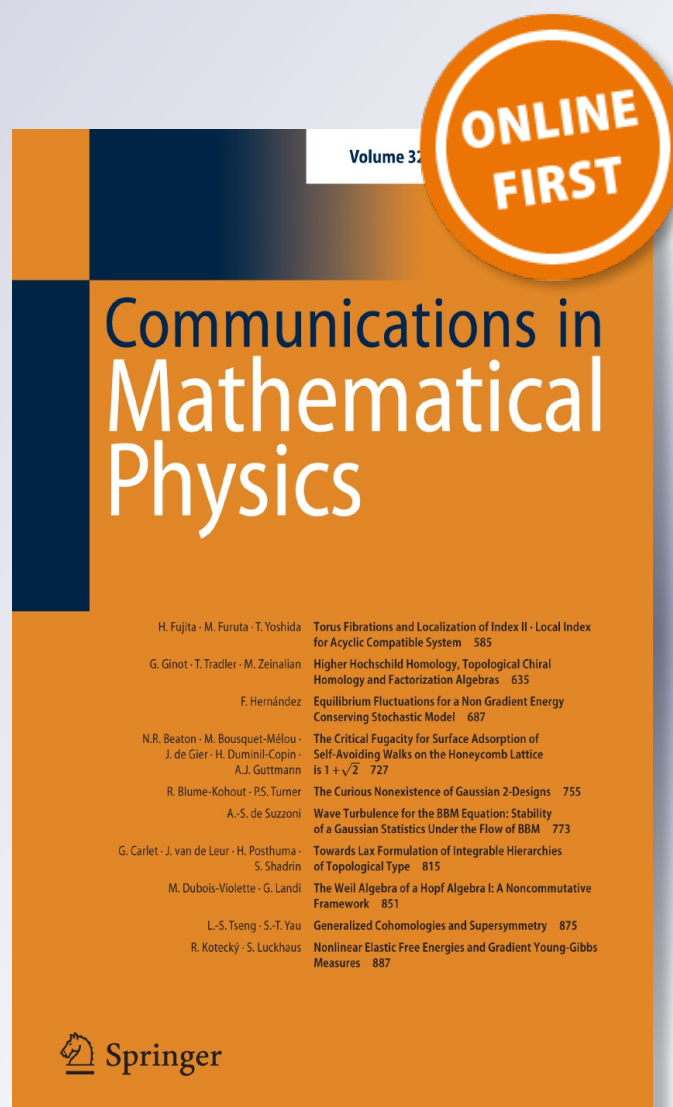
Dine Ousmane Samary & Fabien Vignes-Tourneret

**Communications in Mathematical
Physics**

ISSN 0010-3616

Commun. Math. Phys.

DOI 10.1007/s00220-014-1930-3



Your article is protected by copyright and all rights are held exclusively by Springer-Verlag Berlin Heidelberg. This e-offprint is for personal use only and shall not be self-archived in electronic repositories. If you wish to self-archive your article, please use the accepted manuscript version for posting on your own website. You may further deposit the accepted manuscript version in any repository, provided it is only made publicly available 12 months after official publication or later and provided acknowledgement is given to the original source of publication and a link is inserted to the published article on Springer's website. The link must be accompanied by the following text: "The final publication is available at link.springer.com".

Just Renormalizable TGFT's on $U(1)^d$ with Gauge Invariance

Dine Ousmane Samary^{1,2}, Fabien Vignes-Tourneret¹

¹ Institut Camille Jordan, Université de Lyon, CNRS UMR 5208, Bât. Braconnier, 43 bd du
11 novembre 1918, F-69622 Villeurbanne cedex, France.

E-mail: vignes@math.univ-lyon1.fr

² International Chair in Mathematical Physics and Applications (ICMPA-UNESCO Chair),
072 BP 50 Cotonou, Republic of Benin. E-mail: dine.ousmanesamary@cipma.uac.bj

Received: 30 January 2013 / Accepted: 26 October 2013

© Springer-Verlag Berlin Heidelberg 2014

Abstract: We study the polynomial Abelian or $U(1)^d$ Tensorial Group Field Theories equipped with a gauge invariance condition in any dimension d . We prove the just renormalizability at all orders of perturbation of the φ_6^4 and φ_5^6 random tensor models. We also deduce that the φ_5^4 tensor model is super-renormalizable.

Contents

1.	Introduction	
1.1	Motivations	
1.2	Colored graphs	
1.3	Uncolored graphs	
2.	The Models	
3.	Multi-Scale Analysis and Power Counting Theorem	
3.1	Multiscale analysis	
3.2	Analysis of the divergence degree	
3.3	Classification of divergent graphs	
4.	Renormalization	
4.1	Resolution of the delta functions	
4.2	Taylor expansions	
4.3	Traciality of the counterterms	
5.	The Super-Renormalizable φ_5^4 -Model	
6.	Conclusion and Discussion	
A.	Paths in a Graph	
B.	Combinatorial Analysis of $\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})$	
	References	

1. Introduction

1.1. Motivations. The complete definition of a quantum theory of gravity is probably one of the most fundamental problems of theoretical physics. According to several theoreticians, such a theory should obviously be background independent. As a consequence, spacetime has to be reconstructed from more fundamental degrees of freedom which may very well be of a discrete nature.

Tensor Group Field Theory (TGFT) is quite a recent framework which aims at describing such a pre-geometric phase [28,29]. Such an approach stands at the intersection of random tensor models and Group Field Theory (GFT, see [15,26,30] for general introductions). GFT is a second quantized version of Loop Quantum Gravity [31,32]. Random Tensors, especially colored ones, allow one to define probability measures on simplicial pseudo-manifolds (see [22] and references therein). Let us recall quickly that a random tensor of rank d represents a $(d-1)$ -simplex. Each of its d indices corresponds to a $(d-2)$ -simplex defining its boundary. The typical interaction part of a tensor model is given by the gluing of $d+1$ $(d-1)$ -simplices to get a d -simplex. GFT equips those tensors with some crucial group theoretical data regarded as the seeds of a post geometric phase [13]. TGFT could potentially relate a discrete quantum pre-geometric phase to a classical continuum limit consistent with Einstein General Relativity through a phase transition dubbed geometrogenesis (see [1,23–25,33,34] for different approaches). Such a scenario is conceivable using the fact that several TGFTs have been proven asymptotically free [2–6].

$2D$ quantum gravity via matrix models is a successful example of such a program. Matrix models indeed are theories of discrete surfaces yielding (after a phase transition) in the continuum a theory of gravity dominated by sphere geometries [14]. It can be stressed that the crucial analytical ingredient for achieving this result is the t'Hooft $1/N$ expansion. Until recently there was no analogue of such an expansion in higher dimensions or for tensors of higher rank. Then, Gurau discovered a genuine way to generalize the matrix $1/N$ expansion to any dimension and any rank but for particular tensors [18,19,21]. Indeed, the new $1/N$ expansion relies on the so-called *colored* random tensor models [17,22]. The net result of this analysis is that the partition and correlation functions of the colored models admit perturbative expansions which are dominated by peculiar triangulations of spheres called *melons* [10]. This result has been extended to any dimension.

Moreover, it has been realized [11] that colored models can be used to construct effective actions (and, then later [7], renormalizable actions!) for *uncolored* tensor fields. In dimension d , there are $d+1$ colored fields. By integrating over d of them, one obtains an effective action for the last one, whose interactions are dominated by terms corresponding precisely to those spheres which dominate the tensor $1/N$ expansion.

The first TGFT proved to be (just) renormalizable is a complex φ^6 tensor field theory on four copies of $U(1)$ [7]. Since then, other examples have been discovered [5,6,13]. In particular, the contribution [13] deals with a propagator which implements the so-called closure or gauge invariance condition on tensors. Such an additional symmetry is necessary, for instance, in order to interpret the Feynman amplitudes of the tensor model as the amplitudes of a discretized simplicial manifold issued from topological BF theories. We mention also that the model considered in [13] is super-renormalizable. Let us for short call these models as φ_d^n , where $\varphi : U(1)^d \rightarrow \mathbb{C}$ is the rank d tensor and n is the maximal coordination (or valence) of the vertices of the theory. Our aim in this paper is to exhibit the first examples of *just* renormalizable Abelian TGFTs on $U(1)^d$ with gauge invariance.

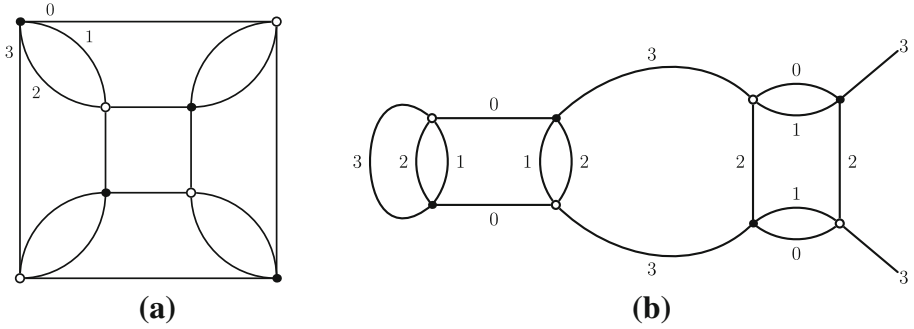


Fig. 1. Colored graphs, **a** closed, **b** open

The paper is organized as follows. We first recall the basic definitions of colored graph theory in Sect. 1.2 and their effective (faded) counterpart in Sect. 1.3. In Sect. 2, we present two main models analyzed in this paper, namely the φ_6^4 and φ_5^6 models. Section 3 is the core of our contribution. It deals with the multi-scale analysis and the power counting theorem of some general polynomial TGFTs. Using this study, we provide the classification of divergent graphs appearing in φ_6^4 and φ_5^6 which yields a control on the divergent amplitudes of these models. Section 4 is devoted to the renormalization of these divergent graphs providing, finally, the proof of the renormalizability of the φ_6^4 and φ_5^6 models. Section 5 discusses, in a streamlined analysis, the super-renormalizability of the φ_6^4 model followed by a conclusion and two technical appendices.

1.2. Colored graphs. The Feynman graphs of the colored tensor model are $(d + 1)$ -colored graphs [17, 22]. For the sake of completeness, we recall here few facts about these graphs, their representation as stranded graphs and their uncolored version.

The graphs that we consider possibly bear external edges, that is to say, half-edges hooked to a unique vertex. We denote $\mathcal{G}_c$ a colored graph, $\mathcal{L}(\mathcal{G}_c)$ the set of its internal edges ($L(\mathcal{G}_c) = |\mathcal{L}(\mathcal{G}_c)|$) and $\mathcal{L}_e(\mathcal{G}_c)$ the set of its external edges ($L_e(\mathcal{G}_c) = |\mathcal{L}_e(\mathcal{G}_c)|$). For all $n \in \mathbb{N}$, let $[n]$ be the set $\{0, 1, \dots, n\}$ and $[n]^*$ be $\{1, 2, \dots, n\}$.

Definition 1.1 (Colored graphs). *Let $d \in \mathbb{N}^*$. A $(d + 1)$ -colored graph $\mathcal{G}_c$ is a $(d + 1)$ -regular bipartite graph equipped with a proper edge-coloring. In other words, there exists a map $\eta : \mathcal{L}(\mathcal{G}_c) \cup \mathcal{L}_e(\mathcal{G}_c) \rightarrow [d]$ such that if e and e' are adjacent edges, $\eta(e) \neq \eta(e')$.*

A colored graph is said to be closed if it has no external edges and open otherwise.

Examples of four-colored graphs are given in Fig. 1. White and black dots represent the bipartition of the vertices of the colored graphs.

Definition 1.2 (Faces). *Let $\mathcal{G}_c$ be a $(d + 1)$ -colored graph and S a subset of $\{0, \dots, d\}$. We note $\mathcal{G}_c^S$ the spanning subgraph of $\mathcal{G}_c$ induced by the edges of colors in S . Then for all $0 \leq i, j \leq d$, $i \neq j$, a face of colors i, j is a connected component of $\mathcal{G}_c^{\{i, j\}}$.*

A face is open (or external) if it contains an external edge and closed (or internal) otherwise. The set of closed faces of a graph $\mathcal{G}_c$ is written $\mathcal{F}(\mathcal{G}_c)$ ($F(\mathcal{G}_c) = |\mathcal{F}(\mathcal{G}_c)|$).

Any $(d + 1)$ -colored graph has an alternative stranded representation. Any edge is made of d parallel strands. If the edge is of color i , its strands are bicolored ij with

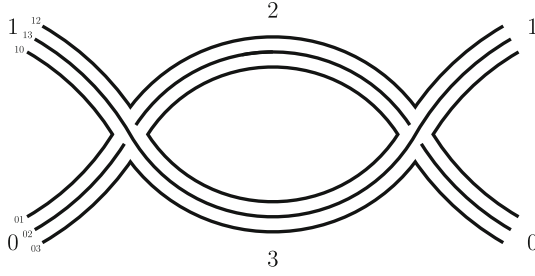


Fig. 2. The stranded representation of Fig. 4a

$j \in \hat{i} := [d] \setminus \{i\}$. The connecting pattern of any $(d + 1)$ -valent vertex is the complete graph K_{d+1} . A (closed) face is then represented as a (closed) curve made of one strand. An example of such a representation is given in Fig. 2.

Definition 1.3 (Jackets). Let σ be a cyclic permutation on $[d]$, up to orientation. The jacket J_σ of a $(d + 1)$ -colored graph $\mathcal{G}_c$ is the ribbon subgraph of $\mathcal{G}_c$ whose faces are colored $\sigma^q(0)$, $\sigma^{q+1}(0)$ for $0 \leq q \leq d$.

To any jacket $J \subset \mathcal{G}_c$, we associate its closed version $\tilde{J}$ obtained from J by pinching its external legs. See Fig. 3.

The numerous applications of random matrices originate in the possibility to control (at least partially but non perturbatively) the perturbative series of the partition function of these models. This interesting feature is due to the existence of the $1/N$ -expansion of matrix models (N denoting the size of the matrix) which provides in return a topological

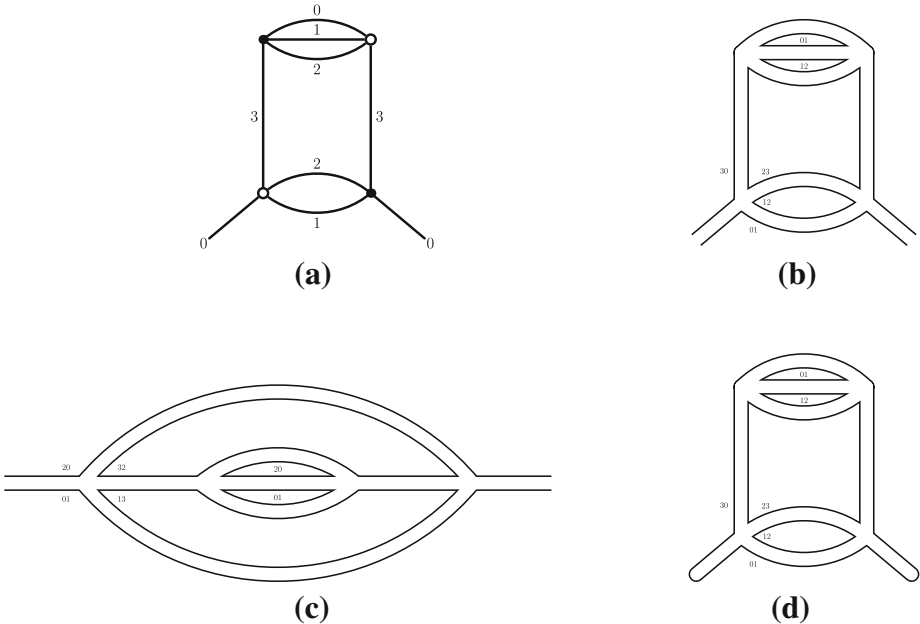


Fig. 3. Jackets, **a** A two-point colored graph, **b** its jacket (0123), **c** its jacket (0132), **d** the closed jacket (0123)

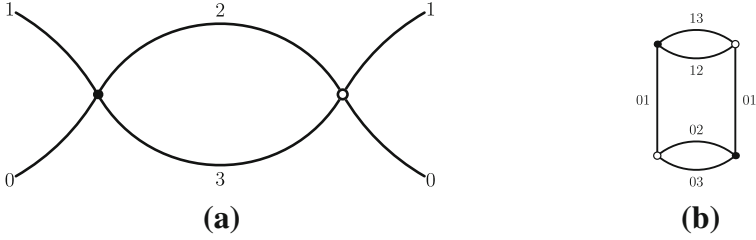


Fig. 4. The boundary operation, **a** an open colored graph $\mathcal{G}_c$, **b** the boundary graph $\partial\mathcal{G}_c$

expansion of the partition function in terms of the genus. In higher dimensions, the generalization of such an $1/N$ -expansion (where N will denote the typical size of the tensor) does not yield a topological expansion but rather a combinatorial expansion in terms of the degree of the graph [18, 19, 21]. For a colored closed graph $\mathcal{G}_c$, the degree is defined as

$$\omega(\mathcal{G}_c) := \sum_{J \text{ jacket of } \mathcal{G}_c} g_J, \quad (1.1)$$

where g_J is the genus of the jacket J .

A $(d+1)$ -colored graph $\mathcal{G}_c$ is dual to a $(d+1)$ -triangulated space (or *Delta-complex*) corresponding to a pseudo-manifold [16]. The boundary of this manifold is triangulated by a complex dual to the boundary graph of $\mathcal{G}_c$.

Definition 1.4 (Boundary graph). *Let $\mathcal{G}_c$ be a $(d+1)$ -colored graph. Its boundary graph $\partial\mathcal{G}_c$ is the d -colored graph whose vertex-set is the set $\mathcal{L}_e(\mathcal{G}_c)$ of external edges of $\mathcal{G}_c$ and edge-set the bi-colored paths linking two external edges of $\mathcal{G}_c$. In other words, an (internal) edge of $\partial\mathcal{G}_c$ corresponds to an external face of $\mathcal{G}_c$.*

Note that the boundary graph of a closed colored graph is the empty graph.

1.3. Uncolored graphs. As explained in Sect. 1.1, we are interested in effective actions obtained from the iid model [17] by integrating over the fields of colors from 1 to d . The effective vertices correspond to open melonic graphs [10] whose external edges are of color 0. The Feynman graphs of such models are so-called uncolored graphs. In fact, a close inspection of these uncolored graphs show that they still possess a colored structure. Indeed, they are colored graphs but whose edges of colors $1, \dots, d$ are made of only one strand whereas edges of color 0 still contain d strands. Such graphs actually represent the connecting pattern of the indices of the tensor field of color 0 [11]. Generally uncolored graphs map onto tensor trace invariant objects [20].

An uncolored graph $\mathcal{G}$ has a unique colored extension $\mathcal{G}_c$ which contains all the faces ij , $0 \leq i, j \leq d$ of a $(d+1)$ -colored graph. The faces of the uncolored graph are the $0i$ -faces of its colored extension. Figure 5a depicts an uncolored open graph. The mono-stranded lines of color $i \geq 1$ are faded. Its colored extension $\mathcal{G}_c$ is shown in Fig. 3a and its (partially) stranded representation is drawn in Fig. 5b.

Connectedness. In graph theory, there is a well-known notion of connectedness. A graph G is connected if there exists at least one path in G between any two of its vertices. Another way of defining connectedness is the following. Let us choose an orientation of the edges of G and consider the incidence matrix I between edges and vertices whose

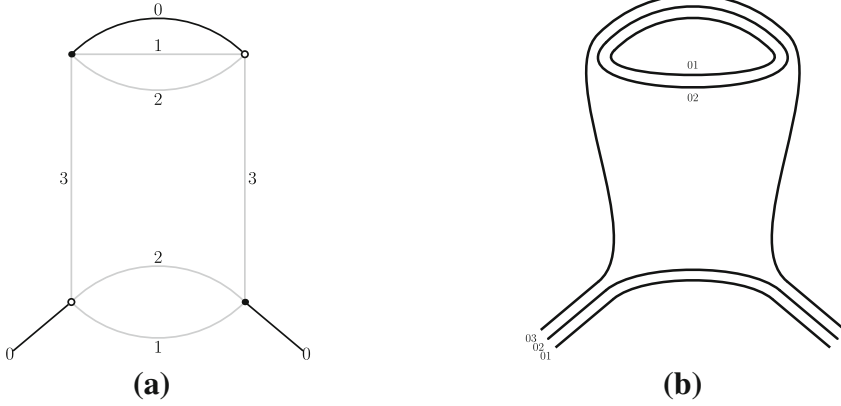


Fig. 5. An open uncolored graph $\mathcal{G}$, **a** an open uncolored graph $\mathcal{G}$, **b** the stranded representation of $\mathcal{G}$

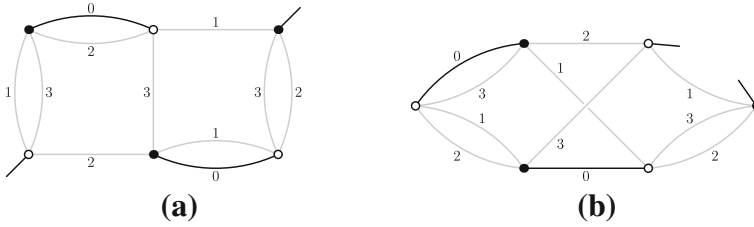


Fig. 6. Face-connectedness, **a** a face-disconnected graph, **b** a face-connected graph

element I_{lv} is 1 if l enters v , -1 if l exits from v and 0 otherwise (in the case of a loop, we also choose 0). Then a graph is connected if it is not possible to put its incidence matrix I into a block diagonal form, after possible reordering of its rows and columns.

There is another notion of connectedness that is relevant for tensor graphs, colored or not. It uses the incidence matrix between internal edges and faces:

Definition 1.5 (Matrix $(\epsilon)_{lf}$ [13]). *Let $\mathcal{G}$ be a (un)colored graph. Let us pick an arbitrary orientation for all of its edges and for all of its faces. We define the $L(\mathcal{G}) \times F(\mathcal{G})$ matrix $\epsilon(\mathcal{G})$ as follows:*

$$\epsilon_{lf}(\mathcal{G}) := \begin{cases} 1 & \text{if } l \in f \text{ and their orientations match,} \\ -1 & \text{if } l \in f \text{ and their orientations do not match,} \\ 0 & \text{otherwise.} \end{cases} \quad (1.2)$$

The matrix $\epsilon(\mathcal{G})$ depends on the chosen orientations but one easily checks that its rank does not. A tensor graph will be said to be *face-connected* if its incidence matrix ϵ cannot be put into a block diagonal form by a permutation of its rows and columns. In other words, a (un)colored graph is face-connected if it is not possible to split the set of internal lines $\mathcal{L}$ into two subsets $\mathcal{L}_1$ and $\mathcal{L}_2$ with no face containing a line of $\mathcal{L}_1$ and a line of $\mathcal{L}_2$. Examples of face-connected and disconnected graphs are given in Fig. 6.

To distinguish between the two notions of connectedness, let us call a graph *vertex-connected* if it is connected in the usual sense. Face-connectedness is a relevant notion in our context because the power counting factorizes into the face-connected components

of the tensor graphs. But note that the amplitude themselves do not enjoy such a factorization, and the usual notion of vertex-connectedness remains relevant for renormalization (i.e., for locality or better here traciality, see Sect. 4.3).

Remark. The notion of face-connectedness is useful only for open uncolored graphs. For colored and closed uncolored graphs, vertex-connectedness is indeed equivalent to face-connectedness.

2. The Models

Let us start now the study of quantum tensorial field theories on $U(1)^d$. The field in the present context is a tensor $\varphi : U(1)^d \rightarrow \mathbb{C}$. We will mainly assume that the field φ satisfies the following translation invariance under a diagonal group action also called gauge condition:

$$\varphi(hg_1, \dots, hg_d) = \varphi(g_1, \dots, g_d), \quad \forall h \in U(1). \quad (2.1)$$

For $1 \leq j \leq d$, let $g_j = e^{i\theta_j} \in U(1)$. By Fourier transform and writing $P := (p_1, \dots, p_d)$, one has

$$\varphi(g_1, \dots, g_d) = \sum_{p \in \mathbb{Z}^d} \varphi_{p_1, \dots, p_d} \prod_{j=1}^d e^{i\theta_j p_j}. \quad (2.2)$$

We sometimes further simplify the notations as $\varphi_{p_1, \dots, p_d} =: \varphi_{1\dots d}$ when we want to emphasize the tensorial nature of the model.

We will concentrate on two models namely for φ^4 on $U(1)^6$ and for φ^6 on $U(1)^5$, or simply the φ_6^4 and φ_5^6 models, respectively. We want to prove that they both are renormalizable. Rather than separating the renormalizability proofs, we perform the analysis in a row for both of these models because of their similar features.

There is a unique type of (vertex-)connected melonic quartic vertex in any dimension. In $d = 6, 5$, these are given by:

$$V_{4,1}^{d=6} := \sum_{\mathbb{Z}^{12}} \bar{\varphi}_{654321} \varphi_{12'3'4'5'6'} \bar{\varphi}_{6'5'4'3'2'1'} \varphi_{1'23456} + \text{permutations}, \quad (2.3)$$

$$V_{4,1}^{d=5} := \sum_{\mathbb{Z}^{12}} \bar{\varphi}_{54321} \varphi_{12'3'4'5'} \bar{\varphi}_{5'4'3'2'1'} \varphi_{1'2345} + \text{permutations}, \quad (2.4)$$

which are depicted in Fig. 7. The permutations are taken on the color numbers (from 1 to 6 or 5, respectively).

For the φ_5^6 model, there are other interactions: two connected melonic uncolored graphs with six external edges of color 0 (see Fig. 8):

$$V_{6,1} := \sum_{\mathbb{Z}^{15}} \bar{\varphi}_{54321} \varphi_{1'2345} \bar{\varphi}_{5'4'3'2'1'} \varphi_{1''2'3'4'5'} \bar{\varphi}_{5''4''3''2''1''} \varphi_{12''3''4''5''} + 9 \text{ permutations}, \quad (2.5)$$

$$V_{6,2} := \sum_{\mathbb{Z}^{15}} \bar{\varphi}_{54321} \varphi_{1'2345} \bar{\varphi}_{5'4'3'2'1'} \varphi_{1''2''3''4''5'} \bar{\varphi}_{5''4''3''2''1''} \varphi_{12'3'4'5''} + \text{permutations}. \quad (2.6)$$

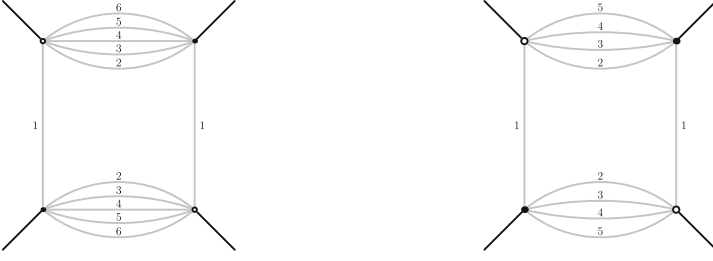


Fig. 7. Melonic quartic vertex for $d = 6$ left and $d = 5$ right

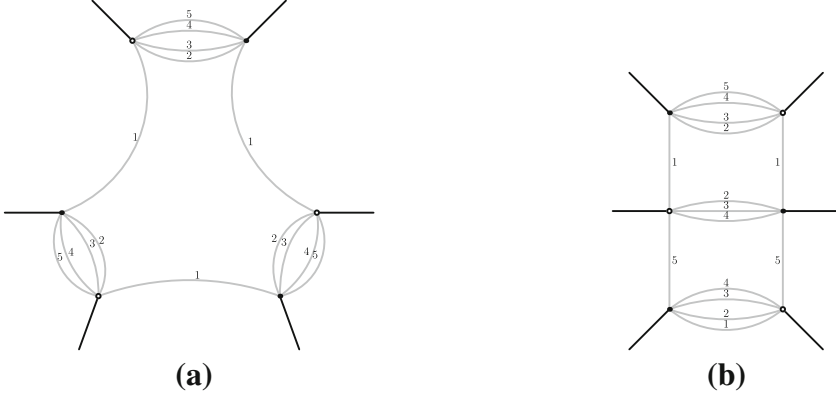


Fig. 8. Melonic Vertices of degree 6, **a** Type I ($V_{6,1}$), **b** Type II ($V_{6,2}$)

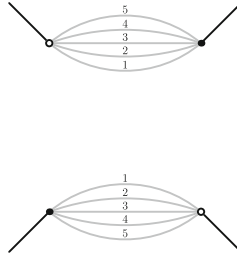


Fig. 9. Graph corresponding to the vertex $V_{4,2}$

As realized in [7], the renormalization of the four-point function of the φ_5^6 model will generate a disconnected *anomalous* vertex of degree 4 (see Fig. 9) that we need to incorporate in the action:

$$V_{4,2} := \left(\sum_{\mathbb{Z}^5} \bar{\varphi}_{54321} \varphi_{12345} \right)^2. \quad (2.7)$$

Feynman graphs. The Feynman graphs of the models we consider are properly represented as uncolored graphs (see Sect. 1.3). Each vertex of a Feynman graph consists in an open uncolored graph (all of its external edges being of color 0) taken among the ones depicted in Figs. 7, 8, 9. As a consequence, all the edges of a Feynman graph are

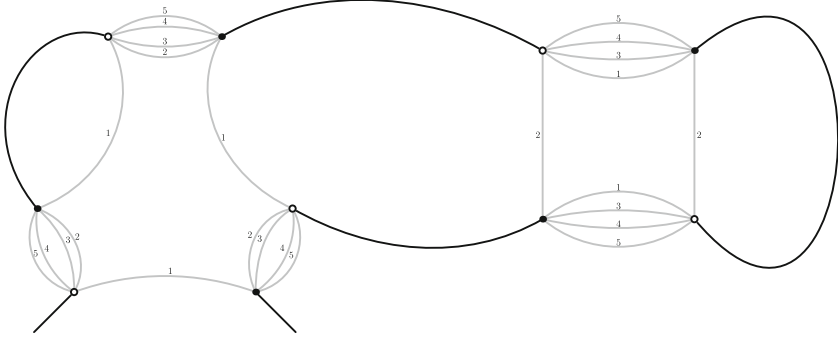


Fig. 10. A Feynman graph with 2 vertices, 2 external edges and 4 internal edges.

of color 0. In particular, if G is a Feynman graph, $\mathcal{L}(G)$ denotes the set of its internal 0-colored lines. An example of a Feynman graph is given in Fig. 10.

Note that white (resp. black) dots represent φ (resp. $\bar{\varphi}$) fields. Also, each edge of G (thus 0-colored) bears a full d -momentum P whereas the lines of color i , $1 \leq i \leq d$, only carry the i th component p_i of a d -vector.

Let v be a vertex of degree $2n$ of the theory, we will generically denote by K_v the corresponding kernel which is of the form

$$V_{2n,i} = \sum_{p_1, \dots, p_{2nd}} K_v(p_1, \dots, p_{2nd}) \prod_{k=1}^n \bar{\varphi}_{p_{k,1}, \dots, p_{k,d}} \varphi_{p_{k+n,1}, \dots, p_{k+n,d}}. \quad (2.8)$$

For both models, the propagator $C(p, p')$ is the usual one $(aP^2 + m^2)^{-1} \delta(P - P')$ (where $P^2 = \sum_{i=1}^d p_i^2$, a is the wave-function “coupling constant”) supplemented by the gauge condition $\delta(\sum_i p_i)$.¹ Feynman amplitudes take indeed the form of lattice gauge theories on the cellular complex dual to the TGFT Feynman diagram.

The two actions that we consider are:

$$S_4[\bar{\varphi}, \varphi] = \sum_{\mathbb{Z}^6} \bar{\varphi}_{654321} \delta(\sum_i p_i) (aP^2 + m^2) \varphi_{123456} + \lambda_{4,1}^{(4)} V_{4,1}^6, \quad (2.9)$$

$$S_6[\bar{\varphi}, \varphi] = \sum_{\mathbb{Z}^5} \bar{\varphi}_{54321} \delta(\sum_i p_i) (aP^2 + m^2) \varphi_{12345} \\ + \lambda_{4,1}^{(6)} V_{4,1}^5 + \lambda_{4,2} V_{4,2} + \lambda_{6,1} V_{6,1} + \lambda_{6,2} V_{6,2}. \quad (2.10)$$

Let μ_C be the Gaussian measure associated to the covariance C :

$$\int d\mu_C \bar{\varphi}(P) \varphi(P') = C(P, P'), \quad \int d\mu_C \bar{\varphi}(P) \bar{\varphi}(P') = \int d\mu_C \varphi(P) \varphi(P') = 0. \quad (2.11)$$

The correlation functions are formally given by

$$S_{2N}(g_{1,1}, g_{1,2}, \dots, g_{2N,d}) \\ = \int d\mu_C \left(\prod_{i=1}^N \bar{\varphi}(g_{i,1}, \dots, g_{i,d}) \varphi(g_{2i,1}, \dots, g_{2i,d}) \right) e^{-S_{\text{int}}[\bar{\varphi}, \varphi]}. \quad (2.12)$$

¹ Note that δ here is understood as the Kronecker symbol.

where $S_{\text{int}}[\bar{\varphi}, \varphi]$ is either $\lambda_{4,1}^{(4)} V_{4,1}^6$ or $\lambda_{4,1}^{(6)} V_{4,1}^5 + \lambda_{4,2} V_{4,2} + \lambda_{6,1} V_{6,1} + \lambda_{6,2} V_{6,2}$, depending on the model under consideration. Our aim here is to define these correlation functions as formal power series. In other words, we prove that the models (2.9) and (2.10) are renormalizable to all orders of perturbation:

Theorem 2.1. *There exist formal power series F_1, F_2, F_3 in a parameter $\lambda_{4,1}^{(4),r}$ and multi-power series $\{G_i\}_{1 \leq i \leq 6}$ in parameters $\lambda^r := \{\lambda_{4,1}^{(6),r}, \lambda_{4,2}^r, \lambda_{6,1}^r, \lambda_{6,2}^r\}$ such that if we fix*

$$\lambda_{4,1}^{(4)} = F_1(\lambda_{4,1}^{(4),r}), \quad m^2 = F_2(\lambda_{4,1}^{(4),r}), \quad a = F_3(\lambda_{4,1}^{(4),r}), \quad (2.13)$$

$$\lambda_{4,1} = G_1(\lambda^r), \quad \lambda_{4,2} = G_2(\lambda^r), \quad \lambda_{6,1} = G_3(\lambda^r), \quad (2.14)$$

$$\lambda_{6,2} = G_4(\lambda^r), \quad m^2 = G_5(\lambda^r), \quad a = G_6(\lambda^r), \quad (2.15)$$

all correlation functions are well-defined formal power series in $\lambda_{4,1}^{(4),r}$, and λ^r , respectively.

The rest of the paper is devoted to the proof of this theorem. For simplicity and when no confusion may occur, both $\lambda_{4,1}^{(4)}$ and $\lambda_{4,1}^{(6)}$ will be simply denoted by $\lambda_{4,1}$.

3. Multi-Scale Analysis and Power Counting Theorem

The goal of this section is the classification of all the primitively divergent graphs generated by both models (2.9) and (2.10). Our main tool is the multiscale analysis. This will help to the proof of an upper bound on the amplitude of a general graph implying the existence of a power counting theorem.

All the framework of Sect. 2 directly extends to models with arbitrary rank tensors with polynomial interactions $P(\bar{\varphi}, \varphi)$. We perform our multi-scale analysis in this general setting and only at the end we will specialize the rank and maximal degree of the vertices. This leads us to some models of interest (φ_6^4 , φ_5^5 and φ_5^4 models).

3.1. Multiscale analysis. In the following, as $C(P, P')$ is proportional to $\delta(P, P')$, we will abusively denote $C(P)$ the coefficient of proportionality.

Multiscale analysis allows us to study precisely the amplitudes of Feynman graphs through the glass of a discrete version of Hepp sectors [27]. To this aim, the first step consists in slicing the propagator into different scales.

Let $M \in \mathbb{R}$, $M > 1$, we have:

$$C(P) = \delta\left(\sum_{i=1}^d p_i\right) \int_{\mathbb{R}_+} e^{-\alpha(P^2+m^2)} d\alpha =: \sum_{j=0}^{\infty} C^j(P) \quad (3.1)$$

with

$$C^0(P) := \delta\left(\sum_{i=1}^d p_i\right) \int_1^{\infty} e^{-\alpha(P^2+m^2)} d\alpha, \quad (3.2)$$

$$\forall j \geq 1, \quad C^j(P) := \delta\left(\sum_{i=1}^d p_i\right) \int_{M^{-2j}}^{M^{-2(j-1)}} e^{-\alpha(P^2+m^2)} d\alpha. \quad (3.3)$$

We regularize the models with an ultraviolet cutoff by restricting the sum over j to the range 0 to $\rho < \infty$. The momenta are thus (smoothly) bounded by $M^{2\rho}$. The sliced propagator admits a simple upper bound:

$$C^i(P) \leq K M^{-2i} e^{-M^{-2i}(P^2+m^2)} \delta\left(\sum_j p_j\right), \quad (3.4)$$

where $K = M^2 - 1$.

The next stage is to bound any graph amplitude. Consider then $\mathcal{G}$ a Feynman graph. Its amplitude writes

$$A_{\mathcal{G}} = \sum_{p_1, \dots, p_{dL(\mathcal{G})}} \prod_{l \in \mathcal{L}(\mathcal{G})} C_l(P_l, P'_l) \prod_{v \in \mathcal{V}(\mathcal{G})} K_v, \quad (3.5)$$

where $\mathcal{V}(\mathcal{G})$ ($V(\mathcal{G}) = |\mathcal{V}(\mathcal{G})|$) denotes the set of vertices of $\mathcal{G}$ and $p_1, \dots, p_{dL(\mathcal{G})}$ the momenta associated to the strands of lines in $\mathcal{G}$. As each propagator is sliced according to (3.1), the amplitude can be decomposed as a sum over the so-called momentum attributions:

$$A_{\mathcal{G}} = \sum_{i_1, \dots, i_L} \sum_{p_1, \dots, p_{dL(\mathcal{G})}} \prod_{l \in \mathcal{L}(\mathcal{G})} C_l^{i_l}(P_l, P'_l) \prod_{v \in \mathcal{V}(\mathcal{G})} K_v =: \sum_{\mu \in \mathbb{N}^L} A_{\mathcal{G}}^{\mu}. \quad (3.6)$$

We focus on $A_{\mathcal{G}}^{\mu}$. The significant upper bound of the following will be expressed in terms of certain special subgraphs of $\mathcal{G}$ called *dangerous* subgraphs defined as follows. Let $\mathcal{G}^{\mu}$ be a Feynman graph with a scale attribution μ . For all $i \in \mathbb{N}$, let $\mathcal{G}^i$ be the (uncolored) subgraph of $\mathcal{G}$ induced by $\mathcal{L}^i(\mathcal{G}) := \{l \in \mathcal{L}(\mathcal{G}) : i_l \geq i\}$. $\mathcal{G}^i$ may have several (say of number $C_i(\mathcal{G})$) vertex-connected components in which case we note them $\mathcal{G}_k^i$, $1 \leq k \leq C_i(\mathcal{G})$. These connected subgraphs are the dangerous subgraphs in the sense that the power counting will be written only in terms of those subgraphs and no other. There is a simple way to determine if a given subgraph $\mathcal{H}$ is dangerous or not. Let $i_{\mathcal{H}}(\mu) := \inf_{l \in \mathcal{L}(\mathcal{H})} i_l$ and $e_{\mathcal{H}}(\mu) := \sup_{l \in \mathcal{L}_c(\mathcal{H})} i_l$. $\mathcal{H}$ is dangerous if and only if $i_{\mathcal{H}}(\mu) > e_{\mathcal{H}}(\mu)$.

The $\mathcal{G}_k^i$'s are partially ordered by inclusion and form in fact a forest, i.e. a set of connected graphs such that any two of them are either disjoint or included one in the other [27]. If $\mathcal{G}$ is itself connected, the forest is in fact a tree whose root is the full graph $\mathcal{G} = \mathcal{G}^0$. This abstract tree is named the Gallavotti–Nicolò (GN) tree.

Our goal is to find an optimal (with respect to a scale attribution) upper bound on the amplitude of a general graph $\mathcal{G}^{\mu}$.

Theorem 3.1 (Power counting). *Let $\mathcal{G}$ be a Feynman graph of a polynomial $P(\overline{\varphi}, \varphi)$ model with propagator (3.1) on $U(1)^d$. There exist constants $K_1, K_2, K_3 \in \mathbb{R}_+^*$ such that*

$$A_{\mathcal{G}}^{\mu} \leq K_1^{V(\mathcal{G})} K_2^{N(\mathcal{G})} K_3^{F(\mathcal{G})} \prod_{i=0}^{\rho} \prod_{k=1}^{C_i(\mathcal{G})} M^{\omega_d(\mathcal{G}_k^i)}, \text{ where } \omega_d(\mathcal{G}_k^i) = -2L(\mathcal{G}_k^i) + F(\mathcal{G}_k^i) - R_k^i \quad (3.7)$$

and R_k^i is the rank of $\epsilon(\mathcal{G}_k^i)$.

The proof of this theorem is already available in the literature. In [9], the *superficial* degree of divergence of a TGFT graph amplitude for an Abelian theory and without $(p^2 + m^2)$ term is computed and proven to be $F - R$, with $R = rk\epsilon(\mathcal{G})$. In [7] an *optimized* bound on the amplitudes of a φ^6 -type model with $(p^2 + m^2)^{-1}$ as propagator (but without gauge invariant condition) is proven. The degree of divergence is there $-2L + F$. More recently, Abelian theories with both $(p^2 + m^2)^{-1}$ and gauge invariant condition has been finally proven in [13]. It is precisely the bound (3.7). However, we think that it may be instructive to collect here all the arguments and rewrite the complete proof, in momentum space.

Proof of theorem 3.1. We want to bound the amplitude of graph $\mathcal{G}$ with scale attribution μ :

$$A_{\mathcal{G}}^{\mu} = \sum_{p_1, \dots, p_{dL(\mathcal{G})}} \prod_{l \in \mathcal{L}(\mathcal{G})} C_l^i(p_l, p'_l) \prod_{v \in \mathcal{V}(\mathcal{G})} K_v. \quad (3.8)$$

The specific forms of the vertex kernels K_v considered in such models imply that there is actually one independent sum per closed face of $\mathcal{G}$ (as it is the case in matrix models). Let us pick an arbitrary orientation of the faces and define the unique momentum of the face f to be p_f in the direction of the chosen orientation. The orientations (signs) of the line momenta are similarly fixed by choosing an orientation of the edges of $\mathcal{G}$. For each line $l \in \mathcal{L}$, the delta function $\delta_l(\sum_{i=1}^d p_{l,i})$ can be rewritten as $\delta_l(\sum_{f \in \mathcal{F}} \epsilon_{lf} p_f + p_{l,e})$ where $p_{l,e}$ is the sum of momenta of the line l which belong to external faces.

Using the bound (3.4) on the sliced propagator, we get

$$\begin{aligned} A_{\mathcal{G}}^{\mu} &\leq K^L \sum_{p_{f_1}, \dots, p_{f_F}} \prod_{l \in \mathcal{L}(\mathcal{G})} [M^{-2i_l} e^{-M^{-2i_l} p_l^2} \delta_l(\sum_{f \in \mathcal{F}(\mathcal{G})} \epsilon_{lf} p_f + p_{l,e})] \\ &\leq K_1^V K_2^N \sum_{p_{f_1}, \dots, p_{f_F}} [(\prod_{f \in \mathcal{F}(\mathcal{G})} e^{-M^{-2i_f} p_f^2}) \prod_{l \in \mathcal{L}} M^{-2i_l} \delta_l(\sum_{f \in \mathcal{F}(\mathcal{G})} \epsilon_{lf} p_f + p_{l,e})], \end{aligned} \quad (3.9)$$

$$(3.10)$$

where $i_f := \inf_{l \in f} i_l$, $L = L(\mathcal{G})$, $V = V(\mathcal{G})$ and $N = N(\mathcal{G})$. We choose subsets $\mathcal{F}_{\mu} \subseteq \mathcal{F}$ and $\mathcal{L}_{\mu} \subseteq \mathcal{L}$ such that $|\mathcal{F} \setminus \mathcal{F}_{\mu}| + |\mathcal{L}_{\mu}| = F(\mathcal{G})$. If $f \in \mathcal{F} \setminus \mathcal{F}_{\mu}$, the sum over p_f is performed using the corresponding exponential function. If not, the sum over p_f is performed using a δ_l function corresponding to a line $l \in \mathcal{L}_{\mu}$:

$$A_{\mathcal{G}}^{\mu} \leq K_1^V K_2^N \prod_{l \in \mathcal{L}} M^{-2i_l} \sum_{p_{f_1}, \dots, p_{f_F}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-M^{-2i_f} p_f^2} \prod_{l \in \mathcal{L}_{\mu}} \delta_l(\sum_{f \in \mathcal{F}(\mathcal{G})} \epsilon_{lf} p_f + p_{l,e}). \quad (3.11)$$

The maximal number of sums we can perform with the δ_l functions is precisely $rk\epsilon(\mathcal{G})$. A sum performed with an exponential function brings a factor M^{i_f} whereas a sum performed with a delta function gives 1. It is thus necessary to optimize the choice of the sets $\mathcal{F}_{\mu}$ and $\mathcal{L}_{\mu}$ with respect to the scale attribution μ . In [13] it is proven that such an optimal choice is possible and given by:

1. There exists a subset $\mathcal{L}_{\mu} \subset \mathcal{L}$ with $|\mathcal{L}_{\mu}| = rk\epsilon(\mathcal{G})$ and the arguments of the corresponding $\delta_{l \in \mathcal{L}_{\mu}}$ functions are independent.
2. For all i, k , $|\mathcal{L}_{\mu} \cap \mathcal{L}(\mathcal{G}_k^i)| = rk\epsilon(\mathcal{G}_k^i)$.

Let us rephrase the proof of carrozza et al. in the following way. For all $\mathcal{F}' \subset \mathcal{F}(\mathcal{G})$, let us denote by $\epsilon_{|\mathcal{F}'}$ the matrix $\epsilon(\mathcal{G})$ with columns restricted to faces in $\mathcal{F}'$. We first choose $\mathcal{F}_\mu$, inductively from the leaves of the GN tree towards its root. Consider a leaf of the GN tree. It corresponds to a certain $\mathcal{G}_k^i$. We choose $rk\epsilon(\mathcal{G}_k^i)$ independent columns of $\epsilon(\mathcal{G}_k^i)$. The corresponding faces of $\mathcal{G}$ are put in $\mathcal{F}_\mu$. Note that these columns are also independent in $\epsilon_{|\mathcal{F}(\mathcal{G}_k^i)}$. This later matrix contains indeed only zeros on the lines $l \notin \mathcal{L}(\mathcal{G}_k^i)$. We proceed similarly for all the leaves of the GN tree. Then, when several $\mathcal{G}_k^i$'s merge into a $\mathcal{G}_{k'}^j$, $j < i$, we add to $\mathcal{F}_\mu$ as many faces as necessary to have $|\mathcal{F}_\mu \cap \mathcal{F}(\mathcal{G}_{k'}^j)| = rk\epsilon(\mathcal{G}_{k'}^j)$. At the last step of this process, when one reaches the root of the GN tree, the cardinal of $\mathcal{F}_\mu$ is clearly equal to the rank of $\epsilon(\mathcal{G})$. Moreover for all i, k , $|\mathcal{F}_\mu \cap \mathcal{F}(\mathcal{G}_k^i)| = rk\epsilon(\mathcal{G}_k^i)$.

It remains to choose the set $\mathcal{L}_\mu$. The matrix $\epsilon(\mathcal{G})_{|\mathcal{F}_\mu}$ has the same rank as $\epsilon(\mathcal{G})$. There exist $|\mathcal{F}_\mu|$ lines of $\epsilon(\mathcal{G})_{|\mathcal{F}_\mu}$ such that the restricted square matrix has still the rank of $\epsilon(\mathcal{G})$. These lines form the set $\mathcal{L}_\mu$. The point is that it is possible to choose these $|\mathcal{F}_\mu|$ lines such that $|\mathcal{L}_\mu \cap \mathcal{L}(\mathcal{G}_k^i)| = rk\epsilon(\mathcal{G}_k^i)$ for all i and k . Indeed, if there exists a $\mathcal{G}_k^i$ such that $|\mathcal{L}_\mu \cap \mathcal{L}(\mathcal{G}_k^i)| < rk\epsilon(\mathcal{G}_k^i)$ then there is a line $l \in \mathcal{L}(\mathcal{G}_k^i) \setminus \mathcal{L}_\mu$ such that the corresponding line-vector is independent of the $|\mathcal{L}_\mu|$ other ones (remember that $\epsilon(\mathcal{G})_{lf} = 0$ if $l \notin \mathcal{L}(\mathcal{G}_k^i)$ and $f \in \mathcal{F}(\mathcal{G}_k^i)$). And the set $\mathcal{L}_\mu$ of line-vectors is not maximally independent.

The proof of theorem (3.1) is achieved by the following. Start from Eq. (3.11) and write

$$A_{\mathcal{G}}^\mu \leq K_1^V K_2^N K_3^F \prod_{l \in \mathcal{L}} M^{-2il} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_\mu} M^{if} \quad (3.12)$$

$$\leq K_1^V K_2^N K_3^F \prod_{i,k} \prod_{l \in \mathcal{L}(\mathcal{G}_k^i)} M^{-2} \prod_{i,k} \prod_{f \in \mathcal{F}(\mathcal{G}_k^i) \setminus \mathcal{F}_\mu} M \quad (3.13)$$

$$\leq K_1^V K_2^N K_3^F \prod_{i,k} M^{-2L(\mathcal{G}_k^i) + F(\mathcal{G}_k^i) - R_k^i}. \quad (3.14)$$

□

3.2. Analysis of the divergence degree. The divergence degree is $\omega_d = -2L + F - R$. In this section, we scrutinize this quantity and re-express it in terms of more useful quantities. We develop as well new tools for this task. This allows us to go beyond the analysis in dimension $d = 4$ as performed in [13] and find renormalizable theories.

The following lemma involves the notion of a spanning tree of a graph $\mathcal{G}$. In our context of uncolored graphs, such a tree is a subgraph of $\mathcal{G}$ thus made of 0-colored edges without cycles. It is spanning if it contains all the vertices of $\mathcal{G}$. Let us emphasize once more that vertices of $\mathcal{G}$ are open uncolored graphs depicted in Figs.7, 8, 9.

Lemma 3.2 (Contraction of a tree). *Let $\mathcal{G}$ be a connected uncolored graph and $\mathcal{T}$ be any of its spanning trees. Under contraction of $\mathcal{T}$, neither F nor R changes:*

$$F(\mathcal{G}) = F(\mathcal{G}/\mathcal{T}), \quad R(\mathcal{G}) = rk\epsilon(\mathcal{G}) = rk\epsilon(\mathcal{G}/\mathcal{T}) = R(\mathcal{G}/\mathcal{T}). \quad (3.15)$$

Proof. The fact that $F(\mathcal{G})$ does not change under contraction is quite obvious: under contraction of an internal line, faces can only get shorter. This is true both for open and closed faces. Moreover, if the contracted line is a tree line, the face cannot disappear.

Let $\ell \in \mathcal{L}(\mathcal{G})$ be any line of $\mathcal{G}$ (not necessarily a tree line). The matrix $\epsilon(\mathcal{G}/\ell)$ is obtained from $\epsilon(\mathcal{G})$ by erasing the row ℓ and the columns full of zeros corresponding to the faces which disappeared under the contraction. In the case of a tree line, this second step does not happen, as explained just above. As a consequence to prove that R is invariant under the contraction of a tree line l , we need to prove that erasing this row does not change the rank of ϵ that is to say that the row l is a linear combination of the other rows of the matrix:

$$\forall f \in \mathcal{F}(\mathcal{G}), \quad \epsilon_{lf} = \sum_{\substack{\ell \in \mathcal{L}, \\ \ell \neq l}} a_{\ell} \epsilon_{\ell f}, \quad (3.16)$$

where the a_{ℓ} 's are independent of f .

Any oriented line ℓ links a vertex v_{ℓ} to another (different) one v'_{ℓ} . There is a unique oriented path $\mathcal{P}_{\mathcal{T}}(\ell)$ in $\mathcal{T}$ from v_{ℓ} to v'_{ℓ} (see Appendix A). Thus $\mathcal{P}_{\mathcal{T}}$ is, in particular, a map from $\mathcal{L}(\mathcal{G})$ to $2^{\mathcal{L}(\mathcal{T})}$. For any internal face f , the set of lines of $\mathcal{G}$ contributing to this face forms a cycle. This cycle can be *projected* onto a path in $\mathcal{T}$ thanks to the map $\mathcal{P}_{\mathcal{T}}$. The face f being a cycle, the corresponding path in $\mathcal{T}$ begins and ends at the same vertex. But as $\mathcal{T}$ is acyclic, each edge has to be covered an even number of times and in *opposite* directions. Thus if we go all over an internal face, and count with signs the number of times a given tree line appears in the projected path, we find zero.

Let us pick up a face f and go all over it according to its orientation². For all $\ell \in f$ and all $l \in \mathcal{T}$, let $\varepsilon_{\ell l}(f)$ be +1 if $l \in \mathcal{P}_{\mathcal{T}}(\ell)$ and its orientation in $\mathcal{P}_{\mathcal{T}}(\ell)$ matches its chosen orientation in $\mathcal{G}$, -1 if $l \in \mathcal{P}_{\mathcal{T}}(\ell)$ and the two orientations do not match, and 0 otherwise. We have

$$\sum_{\ell \in f} \varepsilon_{\ell l}(f) = 0. \quad (3.17)$$

For all $\ell \in \mathcal{L}$ and $l \in \mathcal{T}$, let us define $\eta_{\ell l}$ as +1 if $l \in \mathcal{P}_{\mathcal{T}}(\ell)$ and the orientation of l in $\mathcal{P}_{\mathcal{T}}(\ell)$ (fixed by the chosen orientation of ℓ in $\mathcal{G}$) matches its orientation in $\mathcal{G}$, -1 if $l \in \mathcal{P}_{\mathcal{T}}(\ell)$ and the orientations do not match, 0 otherwise. It is not difficult to check that $\varepsilon_{\ell l}(f) = \eta_{\ell l} \epsilon_{\ell f}$. As $\eta_{ll} = 1$, we get

$$\sum_{\ell \in f} \varepsilon_{\ell l}(f) = \sum_{\ell \in \mathcal{L}} \eta_{\ell l} \epsilon_{\ell f} = 0 \iff \epsilon_{lf} = - \sum_{\ell \in \mathcal{L}, \ell \neq l} \eta_{\ell l} \epsilon_{\ell f} \quad (3.18)$$

which is of the form of Eq. 3.16 and achieves the proof. (For an example treated in detail, see Appendix A.) $\square$

Definition 3.3 (*k-dipole*). Let $\mathcal{G}$ be an uncolored graph. A *k-dipole* is a line ℓ of $\mathcal{G}$ such that it belongs to exactly k faces of length 1. In other words, if ℓ joins two vertices v and v' of the colored extension $\mathcal{G}_c$, there are exactly k edges in $\mathcal{G}_c$ of colors $i > 0$ linking v and v' , see Fig. 11.

Definition 3.4 (Rosettes [8]). Let $\mathcal{G}$ be a connected graph and $\mathcal{T}$ any of its spanning trees. The contracted graph $\mathcal{G}/\mathcal{T}$ is called a *rosette*. A rosette with external lines³ is fully melonic if there exists an order on its $L - V + 1$ lines such that l_1 is a $(d - 1)$ -dipole in $\mathcal{G}/\mathcal{T}$ and for all $2 \leq i \leq L - V + 1$, l_i is a $(d - 1)$ -dipole in $\mathcal{G}/(\mathcal{T} \cup \{l_1, \dots, l_{i-1}\})$.

² Remember that an orientation has been chosen for each face and each line of $\mathcal{G}$ in order to define the matrix ϵ . We will refer to this choice as an orientation in $\mathcal{G}$.

³ For the vacuum rosette the definition is the same except that the last line $i = L - V + 1$ corresponds to a d -dipole not a $d - 1$.

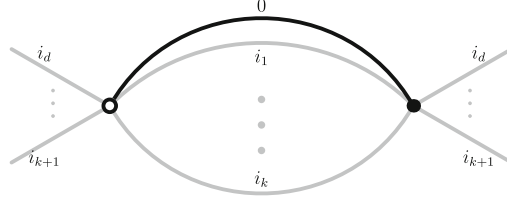


Fig. 11. A k -dipole

Let us consider a polynomial $P(\overline{\varphi}, \varphi)_{U(1)^d}$ model and $\mathcal{G}$ one of its graphs. For all $i \geq 2$, we denote by V_i its number of vertices of degree i and $n \cdot V := \sum_{i \geq 2} i V_i$. The following statement holds.

Lemma 3.5. *Let $\mathcal{G}$ be a connected Feynman graph and $\mathcal{T}$ one of its spanning trees. If the rosette $\mathcal{G}/\mathcal{T}$ is fully melonic,*

$$F(\mathcal{G}) = (d-1)(L-V+1), \quad (3.19a)$$

$$R(\mathcal{G}) = R_{\max}(\mathcal{G}) := L-V+1, \quad (3.19b)$$

$$2\omega_d(\mathcal{G}) = -(d-4)N + (d-4)n \cdot V - 2(d-2)V + 2(d-2). \quad (3.19c)$$

Proof. We contract successively all the lines of $\mathcal{G}/\mathcal{T}$. The rosette being fully melonic, we contract only $(d-1)$ -dipoles. Then for all $i \in [L-V+1]^*$,

$$F(\mathcal{G}/(\mathcal{T} \cup \{l_1, \dots, l_i\})) = F(\mathcal{G}/(\mathcal{T} \cup \{l_1, \dots, l_{i-1}\})) - (d-1), \quad (3.20)$$

$$R(\mathcal{G}/(\mathcal{T} \cup \{l_1, \dots, l_i\})) = R(\mathcal{G}/(\mathcal{T} \cup \{l_1, \dots, l_{i-1}\})) - 1, \quad (3.21)$$

$$F(\mathcal{G}) = F(\mathcal{G}/\mathcal{T}) = (d-1)(L-V+1), \quad (3.22)$$

$$R(\mathcal{G}) = R(\mathcal{G}/\mathcal{T}) = L-V+1. \quad (3.23)$$

Using $\omega_d = -2L + F - R$ and $2L + N = n \cdot V$, one gets the desired result. $\square$

We are in position to understand why the φ_6^4 and φ_5^6 are just renormalizable. Indeed, applying formula (3.19c) to the models (2.9) and (2.10), we get

$$\omega_{d,4} = -(N-4), \quad \omega_{d,6} = -\frac{N-6}{2} - V_4, \quad (3.24)$$

which are typical divergence degrees of just renormalizable models. In the following, we will prove that the divergence degree of a graph is bounded from above by the divergence degree of the graphs with fully melonic rosettes. Moreover we will see that the model (2.10) contains subdivergent contributions, i.e., divergent graphs with non fully melonic rosettes.

Let $\rho(\mathcal{G})$ be defined as $F(\mathcal{G}) - R(\mathcal{G}) - (d-2)\tilde{L}(\mathcal{G})$ with $\tilde{L}(\mathcal{G}) := L(\mathcal{G}) - V(\mathcal{G}) + 1$. Note that for any spanning tree $\mathcal{T}$ in $\mathcal{G}$, $\tilde{L}(\mathcal{G}) = L(\mathcal{G}/\mathcal{T}) = \tilde{L}(\mathcal{G}/\mathcal{T})$ so that thanks to Lemma 3.2, $\rho(\mathcal{G}) = \rho(\mathcal{G}/\mathcal{T})$, $\forall \mathcal{T}$. If $\mathcal{G}$ is face-disconnected, $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}^i$, then $\rho(\mathcal{G}) = \sum_{i \in I} \rho(\mathcal{G}^i)$. Moreover Carrozza, Oriti, and Rivasseau have proven the following [12]

Lemma 3.6. *Let $\mathcal{G}$ be a face-connected rosette.*

1. *If $N(\mathcal{G}) = 0$, then $\rho(\mathcal{G}) \leq 1$ and $\rho(\mathcal{G}) = 1$ iff $\mathcal{G}$ is fully melonic.*

Table 1. Potentially divergent graphs

	φ_6^4				φ_5^6					
N	2	2	2	4	2	2	2	4	4	6
ρ	0	-1	-2	0	0	-1	-2	0	-1	0
ω_d	2	1	0	0	2	1	0	1	0	0

2. If $N(\mathcal{G}) > 0$, then $\rho(\mathcal{G}) \leq 0$ and $\rho(\mathcal{G}) = 0$ iff $\mathcal{G}$ is fully melonic.

The divergence degree of a graph rewrites as

$$\omega_d(\mathcal{G}) = -2L(\mathcal{G}) + (d-2)\tilde{L}(\mathcal{G}) + \rho(\mathcal{G}) \quad (3.25)$$

which leads to

$$\omega_{d,4}(\mathcal{G}) = 4 - N + \rho(\mathcal{G}), \quad \omega_{d,6}(\mathcal{G}) = 3 - \frac{N(\mathcal{G})}{2} - V_4 + \rho(\mathcal{G}). \quad (3.26)$$

The list of potentially divergent graphs is thus given by Table 1:

In the next section, we will characterize fully melonic graphs ($\rho(\mathcal{G}) = 0$) and explain how to deal with the non fully melonic ones ($\rho(\mathcal{G}) < 0$).

3.3. Classification of divergent graphs. We now describe the graphs of Table 1 such that $\rho = 0, -1, -2$. To this aim, we first re-express the divergence degree as follows.

Let $\mathcal{G}$ be an uncolored graph and $\mathcal{G}_c$ be its colored extension. We define $\tilde{\omega}(\mathcal{G}) := \sum_{J \subset \mathcal{G}_c} g_{\tilde{J}}$, where $\tilde{J}$ is the pinched jacket associated with a jacket J of $\mathcal{G}_c$.

Proposition 3.7 (Divergence degree). *The degree of divergence ω_d of a $P(\bar{\varphi}, \varphi)_{U(1)^d}$ model with propagator (3.1) is given by*

$$\omega_d(\mathcal{G}) = (-2L + F - R)(\mathcal{G}) \quad (3.27)$$

$$= -\frac{2}{(d-1)!}(\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})) - (C_{\partial\mathcal{G}} - 1) - \frac{d-3}{2}N + (d-1) + \frac{d-3}{2}n \cdot V - (d-1) \cdot V - R \quad (3.28)$$

where $C_{\partial\mathcal{G}}$ is the number of vertex-connected components of $\partial\mathcal{G}$.

Proof. The number of vertices $V(\mathcal{G}_c)$ of the colored extension $\mathcal{G}_c$ of $\mathcal{G}$ can be given in terms of $L(\mathcal{G})$ and $N(\mathcal{G})$ by the relation $V(\mathcal{G}_c) = n \cdot V = 2L + N$. The number of its lines is $L(\mathcal{G}_c) = L + L_{i,\mathcal{G}_c} := \frac{1}{2}[(d+1)n \cdot V - N]$, where $L_{i,\mathcal{G}_c}$ is the number of internal lines of $\mathcal{G}_c$ which do not appear in $\mathcal{G}$. In the same way $F(\mathcal{G}_c) = F + F_{i,\mathcal{G}_c}$. There exist $d!/2$ jackets of $\mathcal{G}_c$. Each face is shared by $(d-1)!$ jackets. Then $\sum_J F_J = (d-1)!F(\mathcal{G}_c)$. The numbers of vertices (resp. lines, resp. external edges) of $\mathcal{G}_c$, J and $\tilde{J}$ are equal. The graph $\tilde{J}$ is a vacuum ribbon graph and its parameters $F_{\tilde{J}}$, $V_{\tilde{J}}$ and $L_{\tilde{J}}$ satisfy the following relation

$$F_{\tilde{J}} = F_{i,\tilde{J}} + F_{e,\tilde{J}} = 2 - 2g_{\tilde{J}} - V(\mathcal{G}_c) + L(\mathcal{G}_c), \quad L(\mathcal{G}_c) = L_{\tilde{J}}, \quad V(\mathcal{G}_c) = V_{\tilde{J}}, \quad (3.29)$$

where $F_{i,\tilde{J}}$ is the number of internal faces of $\tilde{J}$, and $F_{e,\tilde{J}}$ is the number of faces of $\tilde{J}$ which are made of external faces of J . Denote by $F_{i,\tilde{J},\mathcal{G}}$ the number of internal faces of $\tilde{J}$ colored $0i$, $1 \leq i \leq d$ and $F_{i,\tilde{J},\mathcal{G}_c}$ the number of internal faces colored ij , $1 \leq i, j \leq d$. We get $F_{i,\tilde{J}} = F_{i,\tilde{J},\mathcal{G}} + F_{i,\tilde{J},\mathcal{G}_c}$. Then

$$\sum_J F_{i,\tilde{J}} = (d-1)!(F + F_{i,\mathcal{G}_c}). \quad (3.30)$$

The number $F_{i,\mathcal{G}_c}$ can be easily computed [19]

$$F_{i,\mathcal{G}_c} = \left[\frac{(d-1)(d-2)}{2} \frac{n}{2} + d - 1 \right] \cdot V. \quad (3.31)$$

The quantity $\sum_J (-V_{\tilde{J}} + L_{\tilde{J}})$ can be written as using (3.29)

$$\sum_J -V_{\tilde{J}} + L_{\tilde{J}} = \frac{n \cdot V}{4} d!(d-1) - \frac{d!}{4} N(\mathcal{G}). \quad (3.32)$$

Then

$$F = -\frac{1}{(d-1)!} \sum_J F_{e,\tilde{J}} - \frac{2}{(d-1)!} \sum_J g_{\tilde{J}} - \frac{(d-1)}{4} (4-2n) \cdot V - \frac{d}{4} N + d. \quad (3.33)$$

The next stage consists in re-expressing $\sum_J F_{e,\tilde{J}}$ in terms of the parameters of the boundary graph $\partial\mathcal{G}$ of $\mathcal{G}$. For any jacket J_{∂} of $\partial\mathcal{G}$, note that $V_{\partial\mathcal{G}} = V_{J_{\partial}} = N$, $L_{\partial\mathcal{G}} = L_{J_{\partial}} = F_e$, $dV_{\partial\mathcal{G}} = 2L_{\partial\mathcal{G}} \Rightarrow F_e = \frac{d}{2}N$. There exist $(d-1)!/2$ boundary jackets of $\mathcal{G}_c$. Each face of the graph $\partial\mathcal{G}$ is shared by exactly $(d-2)!$ boundary jackets. Using the fact that the Euler characteristic $\chi(J_{\partial}) = 2C_{J_{\partial}} - 2g_{J_{\partial}} = V_{J_{\partial}} - L_{J_{\partial}} + F_{J_{\partial}}$, we arrive at

$$F_{\partial\mathcal{G}} = \frac{2}{(d-2)!} \sum_{J_{\partial}} C_{J_{\partial}} - \frac{2}{(d-2)!} \sum_{J_{\partial}} g_{J_{\partial}} + \frac{(d-1)}{2} \frac{(d-2)}{2} N. \quad (3.34)$$

Noting that $C_{J_{\partial}} = C_{\partial\mathcal{G}}$. Finally

$$\begin{aligned} \sum_J F_{e,\tilde{J}} &= (d-2)! F_{\partial\mathcal{G}} \\ &= (d-1)!(C_{\partial\mathcal{G}} - 1) - 2 \sum_{J_{\partial}} g_{J_{\partial}} + \frac{(d-1)!(d-2)}{4} N + (d-1)! \end{aligned} \quad (3.35)$$

and

$$\begin{aligned} F &= -\frac{2}{(d-1)!} \left(\sum_J g_{\tilde{J}} - \sum_{J_{\partial}} g_{J_{\partial}} \right) - (C_{\partial\mathcal{G}} - 1) \\ &\quad - \frac{d-1}{2} N + d - 1 - \frac{d-1}{4} (4-2n) \cdot V. \end{aligned} \quad (3.36)$$

Using $L = \frac{1}{2}(n \cdot V - N)$ and equation (3.36), we get (3.28). $\square$

According to Eq. 3.36, the number of internal faces of a graph is given by

$$F(\mathcal{G}) = -\frac{2}{(d-1)!}(\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})) - (C_{\partial\mathcal{G}} - 1) + \frac{d-1}{2}(2 - N + (n-2) \cdot V). \quad (3.37)$$

We define

$$F_{\max}(\mathcal{G}) := \frac{d-1}{2}(2 - N + (n-2) \cdot V) = (d-1)(L - V + 1) \quad (3.38)$$

such that

$$F_{\max}(\mathcal{G}) - F(\mathcal{G}) = \frac{2}{(d-1)!}(\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})) + (C_{\partial\mathcal{G}} - 1). \quad (3.39)$$

According to Lemma 5 of [7] (or to Corollary B.4 in Appendix B),

$$F(\mathcal{G}) = F_{\max}(\mathcal{G}) \iff \tilde{\omega}(\mathcal{G}) = \omega(\partial\mathcal{G}) = C_{\partial\mathcal{G}} - 1 = 0. \quad (3.40)$$

Before giving the topological properties of the graphs with $\rho = 0, -1, -2$, we need the following definitions and technical lemma. Let us denote the number of vacuum face-connected components of a graph $\mathcal{G}$ by $C_0^f(\mathcal{G})$. Let $\mathcal{G}$ be a graph and E a subset of its edges equipped with a total order. We can thus write $E = \{l_1, \dots, l_{|E|}\}$. For all $i \in [|E|]^* \setminus \{1\}$, we define $\mathcal{G}_i := \mathcal{G}/\{l_1, \dots, l_{i-1}\}$ and $\mathcal{G}_1 := \mathcal{G}$.

Lemma 3.8 (Non-foaming 0-dipoles). *Let $\mathcal{G}$ be a vertex-connected non-vacuum ($N(\mathcal{G}) > 0$) uncolored d -tensor graph and $\mathcal{T}$ any of its spanning trees. If there exists an order on the $\tilde{L}$ lines of $\mathcal{R} := \mathcal{G}/\mathcal{T}$ such that:*

1. *there exists $i_0 \in [\tilde{L}]^*$ such that l_{i_0} is a 0-dipole in $\mathcal{R}_{i_0}$, and*
2. *$C_0^f(\mathcal{R}_{i_0+1}) = C_0^f(\mathcal{R}_{i_0})$,*

then l_{i_0} is called a non-foaming 0-dipole, and $\rho(\mathcal{G}) \leq -(d-2)$.

The proof requires another lemma proven in [12]:

Lemma 3.9 (Foaming 0-dipoles). *Let $\mathcal{R}$ be a rosette (i.e. a one-vertex uncolored tensor graph) and l a 0-dipole in $\mathcal{R}$. If $C_0^f(\mathcal{R}/l) > C_0^f(\mathcal{R})$, then $\rho(\mathcal{R}) = \rho(\mathcal{R}/l) - (d-1)$.*

Proof of Lemma 3.8. Let us first suppose that the lemma is proven for face-connected graphs. Consider then a vertex-connected but face-disconnected graph $\mathcal{G}$: $\mathcal{G} = \bigcup_{i \in I} \mathcal{G}^i$ and $\rho(\mathcal{G}) = \sum_{i \in I} \rho(\mathcal{G}^i)$. At least one of the $\mathcal{G}^i$'s contains a non-foaming 0-dipole. The lemma is thus proven if all the other face-connected components satisfy $\rho \leq 0$. Fortunately, a vertex-connected but face-disconnected graph cannot have vacuum face-connected components. The color structure of the tensor graphs ensures it. And we conclude using Lemma 3.6.

So let us assume that $\mathcal{G}$ is face-connected and let us prove the lemma by induction on the number $\tilde{L}$ of lines of $\mathcal{R}$. If $\tilde{L} = 1$, l_1 is a 0-dipole in $\mathcal{R}$. In this case, $F(\mathcal{R}) = F(\mathcal{G}) = 0 = R(\mathcal{G})$ so that $\rho = -(d-2)$.

Let us now assume that the lemma holds for all graphs with at most $\tilde{L} = n$ lines and let us consider a graph with $\tilde{L} = n+1$ edges. If l_1 is a 0-dipole which does not create additional vacuum connected components, $\rho(\mathcal{R}) = \rho(\mathcal{R}/l_1) - (d-2)$ (if $R(\mathcal{R}/l_1) = R(\mathcal{R})$) or $\rho(\mathcal{R}) = \rho(\mathcal{R}/l_1) - (d-1)$ (if $R(\mathcal{R}/l_1) = R(\mathcal{R}) - 1$). Moreover

$C_0^f(\mathcal{R}/l_1) = C_0^f(\mathcal{R}) = C_0^f(\mathcal{G}) = 1$. Thus, according to Lemma 3.6, $\rho(\mathcal{R}/l_1) \leq 0$ and $\rho(\mathcal{R}) = \rho(\mathcal{G}) \leq -(d-2)$.

If l_1 is a k -dipole, $0 \leq k \leq d-1$, which does not satisfy the conditions of the lemma, then $\rho(\mathcal{G}) = \rho(\mathcal{R}) = \rho(\mathcal{R}/l_1) - (d-k-1)$. The contraction of l_1 may have created q connected components (i.e. the number $C^v(\mathcal{R}/l_1)$ of vertex-connected components of $\mathcal{R}/l_1$ is q) with $1 \leq q \leq d-k$. But by assumption, at least one of these q components obey the induction hypothesis. Then,

$$\rho(\mathcal{G}) \leq q - 1 - (d-2) - (d-k-1) \leq -(d-2) \quad (3.41)$$

which proves the lemma. $\square$

We are now in position to give the topological properties of the divergent graphs of the models (2.9) and (2.10).

Proposition 3.10. *The divergent graphs of the models (2.9) and (2.10) are classified in the following table*

Proof. Let $\mathcal{G}$ be a graph of one of the types listed in Table 1. If $\rho(\mathcal{G}) = 0$, according to Lemma 3.6, $\mathcal{G}$ is fully melonic and by Lemma 3.5 and Eq. 3.40, $\tilde{\omega}(\mathcal{G}) = \omega(\partial\mathcal{G}) = C_{\partial\mathcal{G}} - 1$. Let us now assume that $\rho(\mathcal{G}) < 0$. If $R(\mathcal{G}) < R_{\max}(\mathcal{G})$, according to Lemma 3.9, for any tree $\mathcal{T}$ in $\mathcal{G}$ and any order on the lines of $\mathcal{G}/\mathcal{T}$, there must be a non-foaming 0-dipole in $\mathcal{G}$ and by Lemma 3.8, $\rho(\mathcal{G}) \leq -(d-2) \leq -3$ for both models 1 and 1.

We can thus assume that $R(\mathcal{G}) = R_{\max}(\mathcal{G})$. In this case [see Eq. 3.39],

$$\rho(\mathcal{G}) = F(\mathcal{G}) - (d-1)\tilde{L}(\mathcal{G}) = -\frac{2}{(d-1)!}(\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})) - (C_{\partial\mathcal{G}} - 1). \quad (3.42)$$

But Ben Geloun and Rivasseau have proven that for any d -tensor graph $\mathcal{G}$, the quantity $\frac{2}{(d-1)!}(\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G}))$ is either equal to zero or bigger or equal to $d-2$ [8]. Thus for $d \geq 5$, graphs $\mathcal{G}$ such that $\rho \geq -2$ and $R = R_{\max}$ must satisfy $\tilde{\omega}(\mathcal{G}) = \omega(\partial\mathcal{G}) = 0$. Consequently, graphs with $\rho = -1$ (resp. -2) have a boundary graph with two (resp. three) (vertex-)connected components. We simply conclude the proof by noting that the boundary graph of a two-point graph is necessarily connected. $\square$

4. Renormalization

Let us consider an arbitrary divergent graph $\mathcal{G}$ with N external legs. This graph has N external propagators. We denote by p_{f_e} , the external momentum of $\mathcal{G}$ associated to the external face f_e , and $P_j = (p_{j,f_{e_1}}, p_{j,f_{e_2}}, \dots, p_{j,f_{e_d}})$, $1 \leq j \leq N$ the d -vectors associated to the external edges of $\mathcal{G}$. In the same manner, the d -dimensional momentum of an internal line l of $\mathcal{G}$ will be denoted by a capital letter: $P_l = (p_{f_1(l)}, \dots, p_{f_d(l)})$.

In this section, we will complete the proof of the finiteness, order by order, of the usual *effective series* which express any connected function of the theory in terms of an infinite set of effective couplings, related one to each other by a discretized flow [27]. Reexpressing these effective series in terms of the renormalized couplings would reintroduce in the usual way the Zimmermanns forests of “useless” counterterms and build the standard renormalized series. The most explicit way to check finiteness of these renormalized series in order to complete the “BPHZ theorem” is to use the standard “classification of forests” which distributes Zimmermanns forests into packets such that the sum over assignments in each packet is finite [27]. This part is completely

standard and will not be repeated here. As a consequence, we can focus our attention on (primitively divergent) dangerous graphs (see Sect. 3.1).

The truncated amplitude of a graph $\mathcal{G}$ with a scale attribution μ is given by

$$\overline{A}_{\mathcal{G}}^{\mu} = \sum_{\{P_j\}} \overline{\varphi}_{P_1} \varphi_{P_2} \cdots \overline{\varphi}_{P_{N-1}} \varphi_{P_N} A_{\mathcal{G}}^{\mu}(\{P_j\}), \quad (4.1)$$

where

$$A_{\mathcal{G}}^{\mu}(\{P_j\}) = \sum_{P_l, l \in \mathcal{L}} \int \prod_{l \in \mathcal{L}} (e^{-\alpha_l(aP_l^2+m^2)} \delta_l(\sum_{j=1}^d p_{l,j})) \prod_{v \in \mathcal{V}} K_v(\{P_l\}) \prod_{l \in \mathcal{L}} d\alpha_l \quad (4.2)$$

and the φ 's and $\overline{\varphi}$'s are fields of scales strictly lower than the lowest internal scale of $\mathcal{G}^{\mu}$.

Each delta function $\delta_l(\sum_{j=1}^d p_{l,j})$ can be re-expressed in the form

$$\delta_l(\sum_{j=1}^d p_{l,j}) = \delta_l(\sum_{f \in \mathcal{F}} \epsilon_{lf} p_f + \sum_{f_e \in \mathcal{F}_e} \tilde{\epsilon}_{lf_e} p_{f_e}), \quad (4.3)$$

where the tensor $\tilde{\epsilon}_{lf_e}$ is the tensor analogous to ϵ_{lf} but associated with the external faces of $\mathcal{G}$. Remark also that

$$\prod_{l \in \mathcal{L}} e^{-\alpha_l a P_l^2} = \prod_{f \in \mathcal{F}} e^{-a(\sum_{l \in f} \alpha_l) p_f^2} \prod_{f_e \in \mathcal{F}_e} e^{-a(\sum_{l \in f_e} \alpha_l) p_{f_e}^2}. \quad (4.4)$$

In the rest of this work, we set $\alpha_f := \sum_{l \in f} \alpha_l$.

4.1. Resolution of the delta functions. Let l be an arbitrary internal line of $\mathcal{G}$ such that $l \in \mathcal{L}_{\mu}$, see Sect. 3.1. Recall that the subset $\mathcal{L}_{\mu}$ of $\mathcal{L}$ is defined such that $|\mathcal{L}_{\mu}| = r \in (\mathcal{G}) = R$. The number of delta functions, such that the one in Eq. 4.3, that will interest us is exactly the rank $R \leq L$ of the matrix $(\epsilon)_{lf}$. The remaining of the delta functions, i.e. the $L - R$ delta functions, will be put to 1 i.e. $\delta_l = \delta(0) = 1$ after summation.

The kernels K_v are such that the momenta are conserved along the strands:

$$A_{\mathcal{G}}^{\mu}(\{P_j\}) = K_{\partial \mathcal{G}}(\{P_j\}) \sum_{p_f, f \in \mathcal{F}} \int \prod_{l \in \mathcal{L}} (e^{-\alpha_l(aP_l^2+m^2)} \delta_l(\sum_{j=1}^d p_{l,j})) \prod_{l \in \mathcal{L}} d\alpha_l \quad (4.5)$$

$$=: K_{\partial \mathcal{G}}(\{P_j\}) A_{\mathcal{G}}^{\mu}(\{P_j\}). \quad (4.6)$$

The kernel $K_{\partial \mathcal{G}}$ identifies the momenta at the two ends of each of the $Nd/2$ external faces. Thus it precisely reproduces the structure of the boundary graph $\partial \mathcal{G}$ of $\mathcal{G}$.

According to Eqs. 4.3, 4.4 (see also Sect. 3.1),

$$\begin{aligned} \sum_{p_f, f \in \mathcal{F}} \prod_{l \in \mathcal{L}} e^{-\alpha_l a P_l^2} \delta_l(\sum_{j=1}^d p_{l,j}) &= \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-a\alpha_f p_f^2} \prod_{f_e \in \mathcal{F}_e} e^{-a\alpha_{f_e} p_{f_e}^2} \\ &\times \prod_{f \in \mathcal{F}_{\mu}} e^{-a\alpha_f (\sum_{f' \in \mathcal{F}, f' \neq f} \epsilon_{l(f)f'} p_{f'} + \sum_{f_e \in \mathcal{F}_e} \tilde{\epsilon}_{l(f)f_e} p_{f_e})^2}. \end{aligned} \quad (4.7)$$

Finally

$$\begin{aligned} \bar{A}_{\mathcal{G}}^{\mu} &= \sum_{P_j, j \in [N]^*} K_{\partial \mathcal{G}}(\{P_j\}) \bar{\varphi}_{P_1} \varphi_{P_2} \cdots \bar{\varphi}_{P_{N-1}} \varphi_{P_N} \prod_{f_e \in \mathcal{F}_e} e^{-a\alpha_{f_e} p_{f_e}^2} \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-a\alpha_l m^2} \\ &\times \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-a\alpha_f p_f^2} \prod_{f \in \mathcal{F}_{\mu}} e^{-a\alpha_f (\sum_{f' \in \mathcal{F}, f' \neq f} \epsilon_{l(f)} f' p_{f'} + \sum_{f_e \in \mathcal{F}_e} \tilde{\epsilon}_{l(f)} f_e p_{f_e})^2}. \end{aligned} \quad (4.8)$$

4.2. Taylor expansions. The aim of this section is to expose general features of the Taylor expansion of the Feynman amplitudes.

Let $\mathcal{G}$ be any Feynman graph of the models (2.9) and (2.10). $\mathcal{G}$ may not have a divergent amplitude. We define the parametrized amplitude $\mathcal{A}_{\mathcal{G}}^{\mu}(\{P_j\}, t)$ which depends on a parameter $t \in [0, 1]$ such that $\mathcal{A}_{\mathcal{G}}^{\mu}(\{P_j\}, t) := \mathcal{A}_{\mathcal{G}}^{\mu}(\{t P_j\})$. Obviously, $\mathcal{A}_{\mathcal{G}}^{\mu}(\{P_j\}) = \mathcal{A}_{\mathcal{G}}^{\mu}(\{P_j\}, 1)$. We will perform a Taylor expansion (in t) of $\mathcal{A}_{\mathcal{G}}^{\mu}(\{P_j\}, 1)$ around $t = 0$.

4.2.1. Zeroth order.

$$\mathcal{A}_{\mathcal{G}}^{\mu}(\{P_j\}) = \mathcal{A}_{\mathcal{G}}^{\mu}(\{P_j\}, t)|_{t=1} \quad (4.9)$$

$$\begin{aligned} &= \prod_{f_e \in \mathcal{F}_e} e^{-a\alpha_{f_e} t^2 p_{f_e}^2} \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-a\alpha_l m^2} \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-a\alpha_f p_f^2} \\ &\times \prod_{f \in \mathcal{F}_{\mu}} e^{-a\alpha_f (\sum_{f' \in \mathcal{F}, f' \neq f} \epsilon_{l(f)} f' p_{f'} + \sum_{f_e \in \mathcal{F}_e} \tilde{\epsilon}_{l(f)} f_e t p_{f_e})^2} \Big|_{t=1}. \end{aligned} \quad (4.10)$$

The zeroth order term of the Taylor expansion of $\mathcal{A}_{\mathcal{G}}^{\mu}(\{P_j\})$ is

$$\begin{aligned} \mathcal{A}_{\mathcal{G},0}^{\mu}(\{P_j\}) &:= \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-a\alpha_l m^2} \\ &\times \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-a\alpha_f p_f^2} \prod_{f \in \mathcal{F}_{\mu}} e^{-a\alpha_f (\sum_{f' \in \mathcal{F}, f' \neq f} \epsilon_{l(f)} f' p_{f'})^2}. \end{aligned} \quad (4.11)$$

Note that it is independent of the P_j 's. The Taylor expansion of $\mathcal{A}_{\mathcal{G}}^{\mu}$ induces an expansion of $\bar{A}_{\mathcal{G}}^{\mu}$ whose zeroth order takes the following form:

$$\bar{A}_{\mathcal{G},0}^{\mu} := \mathcal{A}_{\mathcal{G},0}^{\mu} \sum_{\{P_j\}} K_{\partial \mathcal{G}}(\{P_j\}) \bar{\varphi}_{P_1} \varphi_{P_2} \cdots \bar{\varphi}_{P_{N-1}} \varphi_{P_N}. \quad (4.12)$$

In conclusion, the zeroth order term of $\bar{A}_{\mathcal{G}}^{\mu}$ has the form of a vertex whose connecting pattern is given by the boundary graph of $\mathcal{G}$.

Table 2. Classification of divergent graphs

	N	$\tilde{\omega}(\mathcal{G})$	$\omega(\partial\mathcal{G})$	$C_{\partial\mathcal{G}} - 1$	$\omega_d(\mathcal{G})$
φ_6^4	2	0	0	0	2
	4	0	0	0	0
φ_5^6	2	0	0	0	2
	4	0	0	0	1
	4	0	0	1	0
	6	0	0	0	0

4.2.2. First order. The first order of the Taylor expansion of $\mathcal{A}_{\mathcal{G}}^\mu$ is

$$\mathcal{A}_{\mathcal{G},1}^\mu(\{P_j\}) := \left. \frac{d\mathcal{A}_{\mathcal{G}}^\mu(\{P_j, \cdot\})}{dt} \right|_{t=0}. \quad (4.13)$$

To simplify notations, let us introduce, for all $f \in \mathcal{F}_\mu$

$$\mathfrak{p}_f := \sum_{f' \in \mathcal{F}, f' \neq f} \epsilon_{l(f)f'} p_{f'} \text{ and } \mathfrak{p}_{e(f)} := \sum_{f_e \in \mathcal{F}_e} \tilde{\epsilon}_{l(f)f_e} p_{f_e}. \quad (4.14)$$

Thus we get

$$\begin{aligned} \mathcal{A}_{\mathcal{G},1}^\mu(\{P_j\}) &= -2a \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-a\alpha_l m^2} \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_\mu} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_\mu} e^{-a\alpha_f p_f^2} \\ &\times \prod_{f \in \mathcal{F}_\mu} e^{-a\alpha_f \mathfrak{p}_f^2} \left(\sum_{f \in \mathcal{F}_\mu} \alpha_f \mathfrak{p}_{e(f)} \mathfrak{p}_f \right). \end{aligned} \quad (4.15)$$

The sums on the p_f 's are performed over $\mathbb{Z}$ and the summands are odd so that $\mathcal{A}_{\mathcal{G},1}^\mu$ vanishes identically.

4.3. Traciality of the counterterms. In [11], it has been realized that the effective action for a single tensor field, obtained by the integration of d tensor fields out of the $d+1$ fields of an iid model, is dominated by invariant traces indexed by melonic d -colored graphs. The vertices of the model (2.9) [resp. (2.10)] correspond to all the vacuum *connected* melonic six-colored (resp. five-colored) graphs up to order four (resp. six) plus a so-called anomaly namely a product of two quadratic traces.

We consider the divergent graphs of the φ_6^4 and φ_5^6 models, listed in Table 2. For simplicity, let us start with the graphs $\mathcal{G}$ such that $\omega_d(\mathcal{G}) = 0$ or 1. Those graphs have four or six external legs. According to the discussion of Sect. 4.2, $\bar{\mathcal{A}}_{\mathcal{G},0}^\mu$ corresponds to a vertex whose structure is given by the boundary graph of $\mathcal{G}$. All the divergent graphs in our models have melonic boundary graphs. If $N(\mathcal{G}) = 4$ and $C_{\partial\mathcal{G}} = 1$, $\partial\mathcal{G}$ is one of the graphs depicted in Fig. 7. If $N(\mathcal{G}) = 6$ and $C_{\partial\mathcal{G}} = 1$, $\partial\mathcal{G}$ is one of the graphs of Fig. 8. Finally, there are 4-point divergent graphs with a disconnected melonic boundary. They correspond to the disconnected invariant trace of Fig. 9. Such an “anomaly” has also been observed in [7].

Just Renormalizable TGFT's

As $\mathcal{A}_{\mathcal{G},1}^\mu = 0$, we have

$$\mathcal{A}_{\mathcal{G}}^\mu(\{P_j\}) = \mathcal{A}_{\mathcal{G},0}^\mu + \int_0^1 (1-s) \frac{d^2 \mathcal{A}_{\mathcal{G}}^\mu(\{P_j, \cdot\})}{dt^2} \Big|_{t=s} ds =: \mathcal{A}_{\mathcal{G},0}^\mu + \mathcal{R}_2, \quad (4.16a)$$

$$\begin{aligned} \mathcal{R}_2 = & \int_0^1 (1-s) \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-\alpha_l m^2} \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_\mu} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_\mu} e^{-\alpha_f p_f^2} \prod_{f_e \in \mathcal{F}_e} e^{-\alpha_{f_e} s^2 p_{f_e}^2} \\ & \times \prod_{f \in \mathcal{F}_\mu} e^{-\alpha_f (\mathfrak{p}_f + s \mathfrak{p}_{e(f)})^2} \left[\left(\sum_{f_e \in \mathcal{F}_e} -2\alpha_{f_e} s p_{f_e}^2 + \sum_{f \in \mathcal{F}_\mu} -2\alpha_f \mathfrak{p}_{e(f)} (\mathfrak{p}_f + s \mathfrak{p}_{e(f)}) \right)^2 \right. \\ & \left. + \sum_{f_e \in \mathcal{F}_e} -2\alpha_{f_e} p_{f_e}^2 + \sum_{f \in \mathcal{F}_\mu} -2\alpha_f \mathfrak{p}_{e(f)}^2 \right]. \quad (4.16b) \end{aligned}$$

$\mathcal{R}_2$ is the renormalized amplitude of $\mathcal{G}^\mu$. Let us prove that it is finite (in fact summable with respect to its scale index). Using the simple upper bound

$$|p_f| e^{-\alpha_f p_f^2} \leq \frac{e^{-\alpha_f p_f^2/2}}{\sqrt{\alpha_f}}, \quad (4.17)$$

one easily gets that the terms between square bracket in Eq. 4.16b are bounded by $c M^{-2(i_{\mathcal{G}}(\mu) - e_{\mathcal{G}}(\mu))}$ where c is a positive constant. The rest of the summand/integrand reproduces the power counting of $\mathcal{G}$ (see Sect. 3.1). Thus for logarithmically or linearly divergent graphs, $\mathcal{R}_2$ is finite.

Let us now consider the divergent two-point graphs of the models (2.9) and (2.10). Their degree of divergence ω_d equals 2. In consequence, their amplitude has to be expanded upto order 2:

$$\mathcal{A}_{\mathcal{G}}^\mu(\{P_j\}) = \mathcal{A}_{\mathcal{G},0}^\mu + \mathcal{A}_{\mathcal{G},2}^\mu(\{P_j\}) + \mathcal{R}_3, \quad (4.18a)$$

$$\mathcal{R}_3 = \frac{1}{2} \int_0^1 (1-s)^2 \frac{d^3 \mathcal{A}_{\mathcal{G}}^\mu(\{P_j, \cdot\})}{dt^3} \Big|_{t=s} ds. \quad (4.18b)$$

Let us recall that (see Eq. 4.16b)

$$\begin{aligned} \frac{d^2 \mathcal{A}_{\mathcal{G}}^\mu(\{P_j, t\})}{dt^2} = & \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-\alpha_l m^2} \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_\mu} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_\mu} e^{-\alpha_f p_f^2} \\ & \times \prod_{f_e \in \mathcal{F}_e} e^{-\alpha_{f_e} t^2 p_{f_e}^2} \prod_{f \in \mathcal{F}_\mu} e^{-\alpha_f (\mathfrak{p}_f + t \mathfrak{p}_{e(f)})^2} [E(t)^2 + E'], \quad (4.19a) \end{aligned}$$

$$E(t) := \sum_{f_e \in \mathcal{F}_e} -2\alpha_{f_e} t p_{f_e}^2 + \sum_{f \in \mathcal{F}_\mu} -2\alpha_f \mathfrak{p}_{e(f)} (\mathfrak{p}_f + t \mathfrak{p}_{e(f)}), \quad (4.19b)$$

$$E' := \sum_{f_e \in \mathcal{F}_e} -2\alpha_{f_e} p_{f_e}^2 + \sum_{f \in \mathcal{F}_\mu} -2\alpha_f \mathfrak{p}_{e(f)}^2. \quad (4.19c)$$

Note that E' does not depend on t . As a consequence,

$$\begin{aligned} \frac{d^3 \mathcal{A}_{\mathcal{G}}^{\mu}(\{P_j, t\})}{dt^3} &= \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-a\alpha_l m^2} \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-a\alpha_f p_f^2} \\ &\quad \times \prod_{f_e \in \mathcal{F}_e} e^{-a\alpha_{f_e} t^2 p_{f_e}^2} \prod_{f \in \mathcal{F}_{\mu}} e^{-a\alpha_f (\mathfrak{p}_f + t \mathfrak{p}_e(f))^2} (E[E^2 + E'] + 2EE'). \end{aligned} \quad (4.20)$$

We have already seen that $|E(t)| \sim M^{-(i_{\mathcal{G}}(\mu) - e_{\mathcal{G}}(\mu))}$ and $|E'| \sim M^{-2(i_{\mathcal{G}}(\mu) - e_{\mathcal{G}}(\mu))}$. Thus $|\mathcal{R}_3|$ is bounded by $M^{-3(i_{\mathcal{G}}(\mu) - e_{\mathcal{G}}(\mu))}$ times the power counting of $\mathcal{G}^{\mu}$ and is therefore summable for $i_{\mathcal{G}}(\mu) > e_{\mathcal{G}}(\mu)$.

$\bar{A}_{\mathcal{G},0}^{\mu}$ has the structure of the boundary graph of $\mathcal{G}$. As $N(\mathcal{G}) = 2$, its boundary is the unique melon with two vertices and $\bar{A}_{\mathcal{G},0}^{\mu}$ thus contributes to the renormalization of the mass.

There only remains to prove that $\bar{A}_{\mathcal{G},2}^{\mu}$ renormalizes the wave function. The argument is a bit subtle and twofold.

$$\mathcal{A}_{\mathcal{G},2}^{\mu} = \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-a\alpha_l m^2} \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-a\alpha_f p_f^2} \prod_{f \in \mathcal{F}_{\mu}} e^{-a\alpha_f \mathfrak{p}_f^2} [E(0)^2 + E'] \quad (4.21a)$$

$$=: \sum_{f_1, f_2 \in \mathcal{F}_{\mu}} \mathfrak{p}_e(f_1) \mathfrak{p}_e(f_2) F_1(f_1, f_2) + \sum_{f_e \in \mathcal{F}_e} p_{f_e}^2 F_2(f_e) + \sum_{f \in \mathcal{F}_{\mu}} \mathfrak{p}_e(f)^2 F_3(f), \quad (4.21b)$$

$$\begin{aligned} F_1(f_1, f_2) &= 4a^2 \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-a\alpha_l m^2} \alpha_{f_1} \alpha_{f_2} \\ &\quad \times \sum_{\substack{p_f \\ f \in \mathcal{F} \setminus \mathcal{F}_{\mu}}} \mathfrak{p}_{f_1} \mathfrak{p}_{f_2} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-a\alpha_f p_f^2} \prod_{f \in \mathcal{F}_{\mu}} e^{-a\alpha_f \mathfrak{p}_f^2}, \end{aligned} \quad (4.21c)$$

$$F_2(f_e) = -2a \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-a\alpha_l m^2} \alpha_{f_e} \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-a\alpha_f p_f^2} \prod_{f \in \mathcal{F}_{\mu}} e^{-a\alpha_f \mathfrak{p}_f^2}, \quad (4.21d)$$

$$F_3(f) = -2a \int \prod_{l \in \mathcal{L}} d\alpha_l e^{-a\alpha_l m^2} \alpha_f \sum_{p_f, f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_{\mu}} e^{-a\alpha_f p_f^2} \prod_{f \in \mathcal{F}_{\mu}} e^{-a\alpha_f \mathfrak{p}_f^2}. \quad (4.21e)$$

$\mathcal{A}_{\mathcal{G},2}^{\mu}$ contributes to the renormalization of the wave function if it is of the form $F \sum_{f_e \in \mathcal{F}_e} p_{f_e}^2$ where F is a constant independent of the f_e 's. We will see in the sequel that is not but that the models are still renormalizable. We will need to exploit the fully melonic character of the two-point divergent graphs and a non-perturbative argument.

First of all, let us remark that none of the F_i 's are constant. Moreover the first and third terms in Eq. 4.21b do not seem to be sums of squares of p_{f_e} 's. Let us first study

the third term. According to its definition, Eq. 4.14, $\mathfrak{p}_{e(f)}$ is *in general* a sum of external momenta. Let us prove that in the case of fully melonic graphs, this sum contains at most one term. Indeed, according to the definition of the sets $\mathcal{F}_\mu$ and $\mathcal{L}_\mu$ (see Sect. 3.1), to any internal face $f \in \mathcal{F}_\mu$, we associate a unique internal line $l(f) \in \mathcal{L}_\mu$ such that $l(f) \in f$. According to the definition of the matrix $\tilde{\epsilon}$, $\mathfrak{p}_{e(f)}$ is the (possibly alternating) sum of momenta of the external faces to which the line $l(f)$ contributes. So we have to prove that a line in $\mathcal{L}_\mu$ contributes to at most one external face.

As proven in Lemma 3.2, for any spanning tree $\mathcal{T}$ in $\mathcal{G}$, the rows of ϵ corresponding to tree lines are linear combinations of the loop lines. In other words, $\mathcal{L}_\mu \subset \mathcal{L}(\mathcal{G}) \setminus \mathcal{L}(\mathcal{T})$ (remember that $\mathcal{L}_\mu$ is a set of maximally independent edges). Let us then contract a spanning tree and consider the rosette $\mathcal{G}/\mathcal{T}$. This contraction does not change the nature (internal or external) of the faces to which the lines of $\mathcal{L}_\mu$ contribute. As $\mathcal{G}$ is fully melonic, there exists an order on the edges of $\mathcal{G}/\mathcal{T}$ such that for all $i \in [L(\mathcal{G}/\mathcal{T})]$, l_i is a $(d-1)$ -dipole in $\mathcal{G}_i$, see page 19. Thus each l_i contributes to $d-1$ internal faces (of length 1) and to possibly one external face.

In consequence, for any internal face $f \in \mathcal{F}_\mu$, there exists at most one external face $f_e(f)$ such that $\mathfrak{p}_{e(f)} = \tilde{\epsilon}_{l(f)f_e(f)} p_{f_e(f)}$. The third and first term of Eq. 4.21b rewrites

$$\sum_{f \in \mathcal{F}_\mu} \mathfrak{p}_{e(f)}^2 F_3(f) = \sum_{f \in \mathcal{F}_\mu} p_{f_e(f)}^2 F_3(f) = \sum_{f_e \in \mathcal{F}_e} p_{f_e}^2 \sum_{\substack{f \in \mathcal{F}_\mu, \\ f_e(f) = f_e}} F_3(f), \quad (4.22a)$$

$$\sum_{f_1, f_2 \in \mathcal{F}_\mu} \mathfrak{p}_{e(f_1)} \mathfrak{p}_{e(f_2)} F_1(f_1, f_2) = \sum_{f_{e,1}, f_{e,2} \in \mathcal{F}_e} p_{f_{e,1}} p_{f_{e,2}} \sum_{\substack{f_1, f_2 \in \mathcal{F}_\mu, \\ f_e(f_1) = f_{e,1}, \\ f_e(f_2) = f_{e,2}}} F_1(f_1, f_2). \quad (4.22b)$$

Unfortunately, the term with F'_1 still does not seem to be a sum of squares of external momenta. In fact it is and it is once more due to the fact that $\mathcal{G}$ is fully melonic. Let us prove the following simple result:

Lemma 4.1. *Let $\mathcal{G}$ be a fully melonic d -tensor graph. Let $f_{e,1}$ and $f_{e,2}$ be two (not necessarily different) external faces of $\mathcal{G}$. Let l (resp. l') be a loop line contributing to $f_{e,1}$ (resp. $f_{e,2}$). Then,*

$$\{f \in \mathcal{F} : l \in f\} \cap \{f \in \mathcal{F} : l' \in f\} = \emptyset. \quad (4.23)$$

In words, if, in a fully melonic graph, there are two loop lines contributing to two external faces, then they contribute to no common internal face.

Proof. It goes by induction on the lines of $\mathcal{G}/\mathcal{T}$. There exists an order on $\mathcal{L}(\mathcal{G}/\mathcal{T})$ such that for all $i \in [L(\mathcal{G}/\mathcal{T})]$, l_i is a $(d-1)$ -dipole in $\mathcal{G}_i$. Without loss of generality, let us assume that $l = l_i$ and $l' = l_j$ with $i < j$. In $\mathcal{G}_i$, l_i is a $(d-1)$ -dipole. Then all the internal faces to which l_i contributes are of length 1 in $\mathcal{G}_i$. In particular l_j does contribute to no internal face of l_i . $\square$

Let us now consider Eq. 4.22b. Let $f_{e,1}, f_{e,2}$ be two *different* external faces of $\mathcal{G}$. Let f_1, f_2 be two internal faces of $\mathcal{G}$ such that $f_e(f_i) = f_{e,i}$ for $i = 1, 2$. Then $l(f_1) \neq l(f_2)$ and these lines do not share any internal face. As a consequence, the sums in $\mathfrak{p}_{f_1}$ and in $\mathfrak{p}_{f_2}$ have no term in common. The summand in $F_1(f_1, f_2)$ is thus odd under the

simultaneous change of sign of all the momenta in $\mathfrak{p}_{f_1}$ (say) and $F_1(f_1, f_2) = 0$ in this case.

Equation (4.22) rewrites

$$\sum_{f_1, f_2 \in \mathcal{F}_\mu} \mathfrak{p}_{e(f_1)} \mathfrak{p}_{e(f_2)} F_1(f_1, f_2) = \sum_{f_e \in \mathcal{F}_e} p_{f_e}^2 \sum_{\substack{f_1, f_2 \in \mathcal{F}_\mu, \\ f_e(f_1) = f_e(f_2) = f_e}} F_1(f_1, f_2). \quad (4.24)$$

All three terms in Eq. 4.21b have now been proven to be sums of squares of external momenta. But the coefficients of these quadratic polynomials still depend on the external faces. And this is not an artefact. These sums contain only external faces which are made of internal lines. In other words, external faces of length 0 do not appear. And there are, of course, graphs with external faces of length 0 (see Fig. 5a for an example). $\mathcal{A}_{\mathcal{G},2}^\mu$ cannot in general reproduce a p^2 term.

Fortunately, the interactions we have considered are symmetric under any permutation of the colors 1 to d (the *positive* colors). The external faces of a two-point graph are indexed by the colors from 1 to d : $\{f_e \in \mathcal{F}_e\} = \{f_{e,01}, f_{e,02}, \dots, f_{e,0d}\}$. Moreover the set of permutations on $[d]$ (or the set of a given type of interaction) can be partitioned into the equivalence classes under the action of the cyclic permutations. We say that two graphs are equivalent if the colored extension of one of them can be obtained from the colored extension of the other by a cyclic permutation of the positive colors. Let $[\mathcal{G}]$ be the set of representatives of such an equivalence class (thus $\mathcal{G}, \mathcal{G}' \in [\mathcal{G}]$ are such that $\mathcal{G}_c$ can be obtained from $\mathcal{G}'_c$ by a cyclic permutation of the positive colors).

According to the discussion above, $\mathcal{A}_{\mathcal{G},2}^\mu$ is of the form

$$\mathcal{A}_{\mathcal{G},2}^\mu = \sum_{f_e \in \mathcal{F}_e} p_{f_e}^2 F_{\mathcal{G}}(f_e). \quad (4.25)$$

Thus,

$$\sum_{\mathcal{G}' \in [\mathcal{G}]} \mathcal{A}_{\mathcal{G}',2}^\mu = p^2 \sum_{f_e \in \mathcal{F}_e} F_{\mathcal{G}}(f_e). \quad (4.26)$$

The second order of the taylor expansion of the sum of the amplitudes of all the graphs in $[\mathcal{G}]$ contribute to the wave function renormalization which finally concludes the proof of the perturbative renormalizability of the models 1 and 1.

5. The Super-Renormalizable φ_5^4 -Model

The analysis of the divergence degree in Sect. 3.2 provides us with another model of potential interest that we now describe. Let us consider the φ_5^4 tensor model with the same dynamics described so far and quartic interaction as given by (2.4). This model can be viewed as well as a truncation of the φ_5^6 to a smaller set of interactions.

Using Eq. (3.19c), we can deduce that the divergence degree of a fully melonic graph is

$$2\omega_d(\mathcal{G}) = -(N - 6) - 2V. \quad (5.1)$$

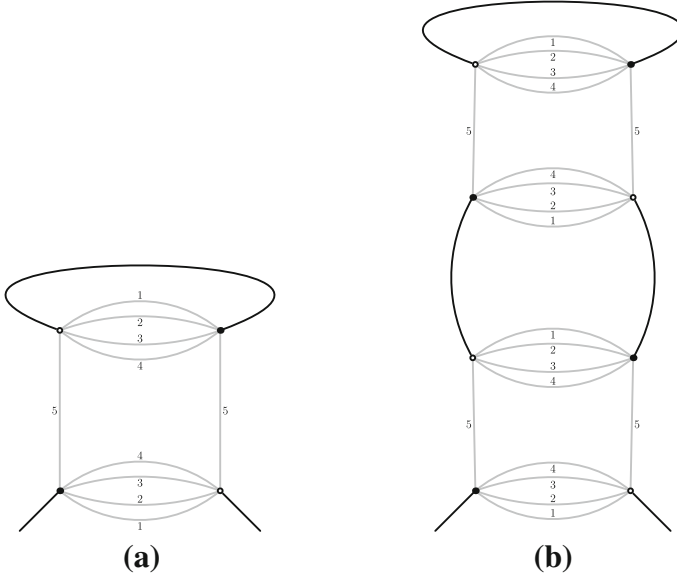


Fig. 12. Divergent graphs of φ_5^4 , **a** $\omega_d = 1$, **b** $\omega_d = 0$

Proposition 5.1. *The rank-5 φ_5^4 tensor model is super-renormalizable.*

Proof. If the quantity $\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G}) > 0$ i.e. not all jackets of $\mathcal{G}_c$ are planar, then

$$\omega_d(\mathcal{G}) \leq 2 - (C_{\partial\mathcal{G}} - 1) - N - 2V_2 - R. \quad (5.2)$$

Using the fact that $V_2 \geq 0$, $C_{\partial\mathcal{G}} \geq 1$ and $R \geq 1$, we get $\omega_d(\mathcal{G}) \leq 1 - N$. This shows that non melonic graphs are convergent graphs. In contrast, if the quantity $\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G}) = 0$ then $C_{\partial\mathcal{G}} = 1$ and we get $\omega_d(\mathcal{G}) \leq 4 - N - R$. The divergent graphs have exactly two external legs. Therefore $\omega_d(\mathcal{G}) = 2 - V$. The divergent graphs of this model are given in Fig. 12. So we infer that the φ_5^4 tensor model is super-renormalizable like the φ_4^4 model studied in [13]. $\square$

6. Conclusion and Discussion

Just renormalizability is a property shared by all physical interactions except (until now) gravity. In the renormalization group sense, it is natural. Indeed just renormalizable interactions survive long-lived renormalization group flow. They can be considered the result of a kind of Darwinian selection associated to such flows. Therefore if quantum gravity can be renormalized, it will rely on the same powerful technique that applies successfully to all other interactions of the standard model [29].

In this work, we have shown that the φ_6^4 and φ_5^6 tensor models are renormalizable at all orders of perturbation. The central point of this proof is given by the multiscale analysis. Our result sheds more light on the power counting in TGFTs with the gauge invariance condition. This gauge condition had already been introduced in the previous work of Carrozza et al. [13], who showed that the generic rank-four models are super-renormalizable. The hurdle which can appear in the power counting due to the emergence

of connected components in the k -dipole contraction is fully resolved now. This work and previous results [7, 12, 13] show that there is indeed a neat family of renormalizable TGFT.

Having defined the first just renormalizable tensor models satisfying the gauge invariance, it remains to address the interesting question about how one can recover General Relativity in the continuum limit from such renormalizable models. A phase transition discrete to continuum geometries, from discrete degrees of freedom in the form of basic simplex (dual to tensors) presented here to more elaborate ones, should be understood. This phase transition would be a conceivable scenario if, for instance, the models described here can be proved asymptotically free in the UV such that the renormalized coupling constants become larger and larger in the opposite direction. Some tensor models without gauge invariance have been proved to be asymptotically free [3, 5, 6]. The study of the β -functions of the φ_6^4 and φ_5^6 characterizing the UV limit of these models will be addressed in forthcoming works.

Acknowledgements. The authors are indebted to Vincent Rivasseau for having proposed the problem treated here and for his guidance through the stranded meanders of TGFT. They also sincerely thank Joseph Ben Geloun for his numerous, complete and rapid explanations of his work. D. Ousmane Samary thanks the Centre International de Mathématiques Pures et Appliquées (CIMPA), the Association pour la Promotion Scientifique de l'Afrique (APSA) and the Laboratoire de Physique Théorique d'Orsay (LPT) for financial supports. This work was supported by the LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

A. Paths in a Graph

This section aims at illustrating the different definitions introduced for the proof of Lemma 3.2. We choose a graph and depicts its vertices as black dots, see Fig. 13.

Let us consider the oriented face $f = (\ell_1, \ell_2, \ell_3)$. We have:

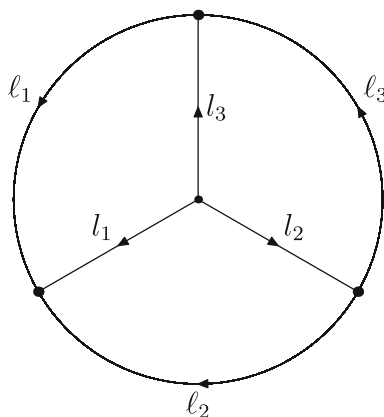


Fig. 13. An oriented graph

$$\varepsilon(f) = \begin{matrix} & l_1 & l_2 & l_3 \\ \begin{matrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{matrix} & \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{matrix}, \quad \eta = \begin{matrix} & l_1 & l_2 & l_3 \\ \begin{matrix} l_1 \\ l_2 \\ l_3 \\ \ell_1 \\ \ell_2 \\ \ell_3 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \end{matrix}, \quad \epsilon = \begin{matrix} & f \\ \begin{matrix} l_1 \\ l_2 \\ l_3 \\ \ell_1 \\ \ell_2 \\ \ell_3 \end{matrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \end{matrix}. \quad (\text{A.1})$$

Note that we have three paths denoted by $\mathcal{P}_T(\ell_1) = \{l_{3-}, l_{1+}\}$, $\mathcal{P}_T(\ell_2) = \{l_{2-}, l_{1+}\}$, $\mathcal{P}_T(\ell_3) = \{l_{2-}, l_{3+}\}$. The signs $+$ and $-$ are used to identify the direction on the path $\mathcal{P}_T(\ell_i)$, $i = 1, 2, 3$ of the path-lines l_i with respect to the direction of ℓ_i . This is well illustrated in the first formula of Eq. (A.1). If $\varepsilon(f)_{l\ell} = 0$ then $l \notin \mathcal{P}_T(\ell)$. One sees easily that

$$\epsilon_{\ell_1 f} \eta_{\ell_1 l_1} + \epsilon_{\ell_2 f} \eta_{\ell_2 l_1} + \epsilon_{\ell_3 f} \eta_{\ell_3 l_1} = 0 \quad (\text{A.2})$$

$$\epsilon_{\ell_1 f} \eta_{\ell_1 l_2} + \epsilon_{\ell_2 f} \eta_{\ell_2 l_2} + \epsilon_{\ell_3 f} \eta_{\ell_3 l_2} = 0 \quad (\text{A.3})$$

$$\epsilon_{\ell_1 f} \eta_{\ell_1 l_3} + \epsilon_{\ell_2 f} \eta_{\ell_2 l_3} + \epsilon_{\ell_3 f} \eta_{\ell_3 l_3} = 0. \quad (\text{A.4})$$

Therefore $\sum_{\ell} \epsilon_{\ell f} \eta_{\ell l} = 0$ and then relation $\epsilon_{lf} = -\sum_{\ell \in \mathcal{L}, \ell \neq l} \eta_{\ell l} \epsilon_{\ell f}$ is well satisfied.

B. Combinatorial Analysis of $\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})$

We propose here an alternative purely combinatorial proof of the fact that $\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G}) \geq 0$. This proof is simpler than the analysis of [8]. However it only proves a weaker bound when $\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G}) > 0$ and $d > 4$. In the case where $d = 4$ the bounds of [8] and this appendix $((d-1)!)$ happen to coincide. The sign of $\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})$ can be analyzed using the so-called dipole contraction. We immediately remind the reader with the definition of a $0k$ -dipole [7].

Definition B.1 ($0k$ -dipole). A $0k$ -dipole (where $k = 0, 1, \dots, d-1$) of a colored graph $\mathcal{G}_c$ is a set of $k+1$ lines, one of which of color 0, joining the same two vertices and such that no other lines connect the same two vertices.

The contraction of a $0k$ -dipole erases the $k+1$ lines of the dipole and connects the remaining $d-k$ lines on both sides of the dipole by respecting the colors. See Fig. 14. Let us denote by $\mathcal{G}'_c$ the graph obtained after contraction of a $0k$ -dipole of $\mathcal{G}_c$. We have

$$V(\mathcal{G}'_c) = V(\mathcal{G}_c) - 2, \quad L(\mathcal{G}'_c) = L(\mathcal{G}_c) - (d+1). \quad (\text{B.1})$$

Let us consider a $0k$ -dipole inside the colored graph $\mathcal{G}_c$. A “pair” is a couple of colors (i, j) , $i, j = 0, 1, 2, \dots, d$. If none of the $k+1$ lines of the dipole bears color i or j , the pair is said to be “outer”. If exactly one of the lines of the dipole bears color i or j , the pair is “mixed”. If one line of the dipole has color i and another one color j , the pair is “inner”. An outer pair (i, j) is said to be of type A or disconnected by the dipole contraction if the half-edges of lines i and j at each corner on the left- and on the right-hand side of the dipole belong to two different connected components of the graph after the dipole contraction. Outer pairs belonging necessarily to closed faces, they are

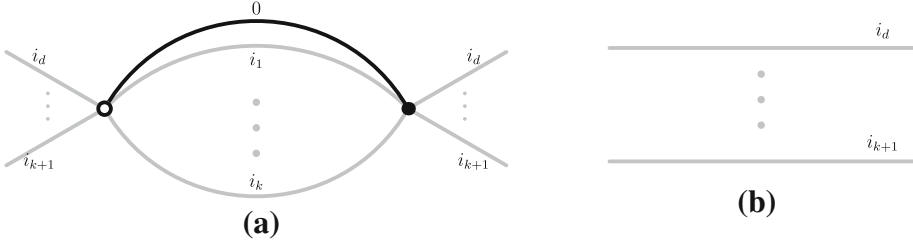


Fig. 14. Contraction of a $0k$ -dipole, **a** $0k$ -dipole, **b** after contraction

single-faced in $\mathcal{G}_c$. The pair (i, j) is said to be special if the half-edges of lines i and j belong to one single connected component of $\mathcal{G}'_c$. There are two types of special pairs. Type B outer pairs are single-faced in $\mathcal{G}_c$ (hence double-faced in $\mathcal{G}'_c$). Type C outer pairs are double-faced in $\mathcal{G}_c$ (single-faced in $\mathcal{G}'_c$).

After contraction, $F(\mathcal{G}_c)$ has increased by 1 for each pair of type A or B and decreased by 1 for pairs of type C. Remark that the mixed pairs preserve the number of faces. In the same manner the number of faces decreases by 1 for each internal pair. We then arrive at

$$F(\mathcal{G}'_c) - F(\mathcal{G}_c) = A + B - C - I \quad (\text{B.2})$$

where $X \in \{A, B, C\}$ is the number of faces of type X and I is the number of inner faces.

The strategy is the same as the one in [7]. We will bound the difference between $\tilde{\omega}(\mathcal{G})$ and $\tilde{\omega}(\mathcal{G}')$. Then we apply the same bound all along a sequence of dipole contractions from $\mathcal{G}$ to $\partial\mathcal{G}$ (remember that the graph obtained after contraction of all the dipoles of $\mathcal{G}$ is essentially $\partial\mathcal{G}$ [7]).

Proposition B.2 (Bound on genera). *Let J be a jacket of $\mathcal{G}_c$. We note J' the jacket of $\mathcal{G}'_c$ corresponding to the same permutation as J . With c' the number of connected components of $\mathcal{G}'_c$,*

$$\sum_J (g_{\tilde{J}} - g_{\tilde{J}'}) \geq \frac{(d-1)!}{2} (d-k-c')(c'+k-1) \geq 0. \quad (\text{B.3})$$

Proof Using relations (B.1), the Euler characteristics of $\tilde{J}$ and $\tilde{J}'$ are given by

$$2 - 2g_{\tilde{J}} = V - L + F_{\tilde{J}}, \quad 2c' - 2g_{\tilde{J}'} = V - 2 - (L - d - 1) + F_{\tilde{J}'}. \quad (\text{B.4})$$

We get

$$\sum_J (g_{\tilde{J}} - g_{\tilde{J}'}) = \frac{1}{2} \left[\sum_J (F_{\tilde{J}'} - F_{\tilde{J}}) + \frac{d!(d-1)}{2} - d!(c'-1) \right]. \quad (\text{B.5})$$

Recall that $\sum_J F_{\tilde{J}} = (d-1)!F(\mathcal{G}_c)$ and $\sum_{J'} F_{\tilde{J}'} = (d-1)!F(\mathcal{G}'_c)$. Then

$$\begin{aligned} \sum_j (g_{\tilde{j}} - g_{\tilde{j}'}) &= \frac{(d-1)!}{2} \left[F(\mathcal{G}'_c) - F(\mathcal{G}_c) + \frac{d(d-1)}{2} - d(c'-1) \right] \\ &= \frac{(d-1)!}{2} (A + B - C - I) + \frac{d!(d-1)}{4} - \frac{d!}{2} (c'-1). \end{aligned} \quad (\text{B.6})$$

The rest of the proof will be devoted to find a lower bound on the quantity $A + B - C - I$. This can be done using the formalism of integer partitions. The number of connected components c' of $\mathcal{G}'_c$ being fixed, the $d - k$ external lines of the dipole are distributed among c' connected colored graphs [7]. Each such configuration corresponds to a partition of $d - k$ into c' parts. Let $\mathcal{P}_p(n)$ be the set of partitions of n in p parts:

$$\mathcal{P}_p(n) := \left\{ (n_i)_{1 \leq i \leq p}, n_1 \geq n_2 \geq \dots \geq n_p : \sum_{i=1}^p n_i = n \right\}. \quad (\text{B.7})$$

For all $n \in \mathbb{N}^*$ and $1 \leq p \leq n$, we denote by λ_1 the following partition of $\mathcal{P}_p(n)$:

$$\lambda_1 := (n - p + 1, \underbrace{1, \dots, 1}_{p-1 \text{ terms}}). \quad (\text{B.8})$$

Given a configuration of the external lines of a $0k$ -dipole, that is to say a partition $\lambda = (n_i)$ of $d - k$ into c' parts, we have

$$(B + C)(\lambda) = \sum_{i=1}^{c'} \frac{n_i(n_i - 1)}{2} = \frac{1}{2} \sum_{i=1}^{c'} n_i^2 - \frac{1}{2} (d - k), \quad (\text{B.9a})$$

$$A(\lambda) = \frac{1}{2} (d - k)(d - k - 1) - (B + C), \quad (\text{B.9b})$$

$$I = \frac{1}{2} (k + 1)k. \quad (\text{B.9c})$$

For a $0k$ -dipole and a fixed c' , $\tilde{\omega}(\mathcal{G}) - \omega(\partial\mathcal{G})$ is minimal when $B + C$ is maximal and $B = 0$. Therefore,

$$\begin{aligned} \tilde{\omega}(\mathcal{G}) - \tilde{\omega}(\mathcal{G}') &\geq \frac{(d-1)!}{2} \left(\frac{1}{2} (d - k)(d - k - 1) - 2C_M - \frac{1}{2} (k + 1)k \right) \\ &\quad + \frac{d!(d-1)}{4} - \frac{d!}{2} (c' - 1), \end{aligned} \quad (\text{B.10})$$

$$C_M := \max_{\lambda \in \mathcal{P}_{c'}(d-k)} (B + C)(\lambda). \quad (\text{B.11})$$

It remains to determine C_M . To this aim, we note that

Proposition B.3. *Any partition of $\mathcal{P}_p(n)$ can be obtained from λ_1 by a (possibly empty) sequence of the following basic operation D_{ij} : let $\lambda = (n_i) \in \mathcal{P}_p(n)$. If there exists a couple $(i, j) \in ([p]^*)^2$, $i < j$ such that $n_i - n_j \geq 2$, we define $D_{ij}\lambda = \lambda^{(1)} = (n_i^{(1)}) \in \mathcal{P}_p(n)$ by $n_i^{(1)} = n_i - 1$, $n_j^{(1)} = n_j + 1$, and for all $k \neq i, j$, $n_k^{(1)} = n_k$. We potentially need to reorder the $n_k^{(1)}$'s to get a proper partition.*

Proof. Let us consider a partition $\lambda = (n_i) \in \mathcal{P}_p(n)$. If $\lambda = \lambda_1$, we are done. If not, it is enough to prove that there exists $\lambda^{(-1)} \in \mathcal{P}_p(n)$ and $(i, j) \in ([p]^*)^2$ such that $D_{ij}\lambda^{(-1)} = \lambda$. We get the proposition simply by iterating that result.

The construction of $\lambda^{(-1)}$ goes as follows. As $\lambda \neq \lambda_1$, there is $(i, j) \in ([p]^*)^2$, $i < j$ such that $n_i, n_j \geq 2$. $\lambda^{(-1)} = (n'_k)$ is then defined as: $n'_i = n_i + 1$, $n'_j = n_j - 1$, for all $k \neq i, j$, $n'_k = n_k$. As $n'_i - n'_j \geq 2$, $D_{ij}\lambda^{(-1)} = \lambda$. $\square$

If $\mathcal{P}_p(n)$ is equipped with the lexicographical (total) order, λ_1 is the highest partition. Moreover for all λ , $D_{ij}\lambda < \lambda$. But $\sum_{k=1}^p (n_k^2 - (n_k^{(1)})^2) = 2(n_i - n_j - 2) \geq 0$. Thus the maximum over $\mathcal{P}_p(n)$ of $\sum_{i=1}^p n_i^2(\lambda)$ is reached for the highest partition in the lexicographical order, namely λ_1 . As a consequence,

$$C_M = \frac{1}{2}((d - k - c' + 1)^2 + c' - 1) - \frac{1}{2}(d - k) \quad (\text{B.12})$$

and

$$\begin{aligned} \tilde{\omega}(\mathcal{G}) - \tilde{\omega}(\mathcal{G}') &\geq \frac{(d-1)!}{2} \left(\frac{1}{2}(d-k)(d-k-1) - (d-k-c'+1)(d-k-c') \right. \\ &\quad \left. - \frac{1}{2}k(k+1) \right) + \frac{d!(d-1)}{4} - \frac{d!}{2}(c'-1) \end{aligned} \quad (\text{B.13})$$

$$= \frac{(d-1)!}{2}(d-k-c')(k+c'-1). \quad (\text{B.14})$$

As $1 \leq c' \leq d - k$, $\tilde{\omega}(\mathcal{G}) - \tilde{\omega}(\mathcal{G}') \geq 0$. $\square$

Corollary B.4. *For any graph $\mathcal{G}$, $\tilde{\omega}(\mathcal{G}) - d \omega(\partial\mathcal{G}) \geq 0$.*

Proof. Let us denote $\mathcal{G}/\mathcal{L}(\mathcal{G})$ the graph obtained after a complete sequence of contractions of the dipoles of $\mathcal{G}$. Iterating the bound (B.3), we get

$$\tilde{\omega}(\mathcal{G}) - \tilde{\omega}(\mathcal{G}/\mathcal{L}) \geq 0. \quad (\text{B.15})$$

The colored extension $(\mathcal{G}/\mathcal{L}(\mathcal{G}))_c$ of $\mathcal{G}/\mathcal{L}$ is $\partial\mathcal{G}_c$ equipped with external legs of color 0. Let $\sigma = (\sigma(1) \cdots \sigma(d))$ be a cyclic permutation on $[d]^*$ and $J_\partial(\sigma)$ the corresponding jacket of $\partial\mathcal{G}_c$. Any permutation τ on $[d]$ of the following set (of cardinality d):

$$P_\sigma := \{(0\sigma(1) \cdots \sigma(d)), (\sigma(1)0\sigma(2) \cdots \sigma(d)), \dots, (\sigma(1) \cdots \sigma(d-1)0\sigma(d))\} \quad (\text{B.16})$$

gives rise to a jacket $J(\tau)$ of $\mathcal{G}_c$ such that $g_{\tilde{J}(\tau)} = g_{J_\partial(\sigma)}$. Moreover the set of cyclic permutations on $[d]$ can be partitioned as $\cup_{\sigma \text{ on } [d]^*} P_\sigma$. Thus,

$$\tilde{\omega}(\mathcal{G}/\mathcal{L}) = \sum_{J \subset (\mathcal{G}/\mathcal{L})_c} g_{\tilde{J}} = d \sum_{J_\partial \subset \partial\mathcal{G}_c} g_{J_\partial}, \quad (\text{B.17})$$

which ends the proof. $\square$

References

1. Ambjorn, J., Durhuus, B., Jonsson, T.: *Quantum geometry. a statistical field theory approach*. In: Cambridge monographs on mathematical physics. Cambridge: Cambridge University Press, 2005
2. Ben Geloun, J.: Asymptotic Freedom of Rank 4 Tensor Group Field Theory. Contribution to the XXIXth International Colloquium on Group-Theoretical Methods in Physics, Nankai (2012). arXiv:1210.5490
3. Ben Geloun, J.: Two and four-loop β -functions of rank 4 renormalizable tensor field theories. *Class. Quant. Grav.* **235011**, (2012). arXiv:1205.5513
4. Ben Geloun, J.: Renormalizable Models in Rank $d \geq 2$ Tensorial Group Field Theory (2013). arXiv:1306.1201
5. Ben Geloun, J., Livine, E.R.: Some classes of renormalizable tensor models. *J. Math. Phys.* **54**, 082303 (2013). arXiv:1207.0416
6. Ben Geloun, J., Ousmane Samary, D.: 3D tensor field theory: renormalization and one-loop β -functions. *Ann. H. Poincaré* **14**(6), 1599–1642 (2013). arXiv:1201.0176
7. Ben Geloun, J., Rivasseau, V.: A renormalizable 4-dimensional tensor field theory. *Commun. Math. Phys.* **318**(1), 69–109 (2013). arXiv:1111.4997
8. Ben Geloun, J., Rivasseau, V.: Addendum to a renormalizable 4-dimensional tensor field theory. *Commun. Math. Phys.* **322**(3), 957–965 (2013). arXiv:1209.4606
9. Ben Geloun, J., Krajewski, T., Magnen, J., Rivasseau, V.: Linearized group field theory and power-counting theorems. *Class. Quant. Grav.*, **27**(15), 155012 (2010). arXiv:1002.3592
10. Bonzom, V., Gurau, R., Riello, A., Rivasseau, V.: Critical behavior of colored tensor models in the large N limit. *Nucl. Phys. B.*, **853**, 174–195 (2011). arXiv:1105.3122
11. Bonzom, V., Gurau, R., Rivasseau, V.: Random tensor models in the large N limit: uncoloring the colored tensor models. *Phys. Rev. D*, **85**(8), 084037 (2012). arXiv:1202.3637
12. Carrozza, S., Oriti, D., Rivasseau, V.: Renormalization of an $SU(2)$ tensorial group field theory in three dimensions. To appear in *Commun. Math. Phys.* arXiv:1303.6772
13. Carrozza, S., Oriti, D., Rivasseau, V.: Renormalization of tensorial group field theories: abelian $U(1)$ models in four dimensions. To appear in *Commun. Math. Phys.* arXiv:1207.6734
14. Di Francesco, P., Ginsparg, P., Zinn-Justin, J.: 2D gravity and random matrices. *Phys. Rept.* **254**, 1–133 (1995)
15. Freidel, L.: Group field theory: an overview. *Int. J. Theor. Phys.*, **44**, 1769 (2005). arXiv:hep-th/0505016
16. Gurau, R.: Lost in translation: topological singularities in group field theory. *Class. Quant. Grav.* **27**(23) (2010). arXiv:1006.0714, doi:[10.1088/0264-9381/27/23/235023](https://doi.org/10.1088/0264-9381/27/23/235023)
17. Gurau, R.: Colored group field theory. *Commun. Math. Phys.* **304**, 69–93 (2011). arXiv:0907.2582, doi:[10.1007/s00220-011-1226-9](https://doi.org/10.1007/s00220-011-1226-9)
18. Gurau, R.: The $1/N$ expansion of colored tensor models. *Ann. H. Poincaré*, **12**(5), 829–847 (2011). arXiv:1011.2726, doi:[10.1007/s00023-011-0101-8](https://doi.org/10.1007/s00023-011-0101-8)
19. Gurau, R.: The Complete $1/N$ expansion of colored tensor models in arbitrary dimension. *Ann. H. Poincaré*, **13**, 399–423 (2012). arXiv:1102.5759, doi:[10.1007/s00023-011-0118-z](https://doi.org/10.1007/s00023-011-0118-z)
20. Gurau, R.: The Schwinger Dyson equations and the algebra of constraints of random tensor models at all orders. *Nucl. Phys. B* **865**, 133 (2012). arXiv:1203.4965
21. Gurau, R., Rivasseau, V.: The $1/N$ expansion of colored tensor models in arbitrary dimension. *Eur. Phys. Lett.*, **95**(5), 50004 (2011). arXiv:1101.4182
22. Gurau, R., Ryan, J.P.: Colored tensor models—a review. *SIGMA*, **8**(020), 78 (2012). arXiv:1109.4812, doi:[10.3842/SIGMA.2012.020](https://doi.org/10.3842/SIGMA.2012.020)
23. Konopka, T., Markopoulou, F., Severini, S.: Quantum graphity: a model of emergent locality. *Phys. Rev. D*, **77**, 104029 (2008). arXiv:0801.0861
24. Loll, R., Ambjorn, J., Jurkiewicz, J.: The universe from Scratch. *Contemp. Phys.*, **47**, 103–117, (2006), arXiv:hep-th/0509010
25. Oriti, D.: Group field theory as the microscopic description of the quantum spacetime fluid: a new perspective on the continuum in quantum gravity. In: P. of Science, editor From quantum to emergent gravity: theory and phenomenology, volume 030 (2007). arXiv:0710.3276
26. Oriti, D.: *Approaches to quantum gravity: toward a new understanding of space, time and matter*. In: The group field theory approach to quantum gravity, chapter 17. Cambridge: Cambridge University Press 2009, p 310 arXiv:gr-qc/0607032
27. Rivasseau, V.: From perturbative to constructive renormalization. In: Princeton series in physics. Princeton: Princeton University Press, 1991, p. 336
28. Rivasseau, V.: Quantum gravity and renormalization: the tensor track. *AIP Conf. Proc.* **1444**, 18 (2012). arXiv:1112.5104
29. Rivasseau, V.: The tensor track: an update (2012). arXiv:1209.5284
30. Rovelli, C.: Quantum gravity. Cambridge University Press, Cambridge (2004)

31. Rovelli, C., Reisenberger, M.: Spin foams as Feynman diagrams. 2001: A relativistic Spacetime Odyssey, world Sci. Publishing, 2003, pp 431–448, arXiv:gr-qc/0002083
32. Rovelli, C., Reisenberger, M.: Spacetime as a Feynman diagram: the connection formulation. *Class. Quant. Grav.* **18**, 121–140 (2001). arXiv:gr-qc/0002095, doi:[10.1088/0264-9381/18/1/308](https://doi.org/10.1088/0264-9381/18/1/308)
33. Seiberg, N.: Emergent spacetime. Rapporteur talk at the 23rd Solvay Conference in Physics, December, 2005., January (2006). arXiv:hep-th/0601234
34. Sindoni, L.: Emergent models for gravity: an overview of microscopic models. *SIGMA*, **8**(027) (2012). arXiv:1110.0686, doi:[10.3842/SIGMA.2012.027](https://doi.org/10.3842/SIGMA.2012.027)

Communicated by M. Salmhofer