

C. Atindogbé, M. Gutiérrez* and R. Hounnonkpe

Compact null hypersurfaces in Lorentzian manifolds

DOI 10.1515/advgeom-2021-0001. Received 26 March, 2019; revised 9 September, 2019

Abstract: We show how the topological and geometric properties of the family of null hypersurfaces in a Lorentzian manifold are related with the properties of the ambient manifold itself. In particular, we focus in how the presence of global symmetries and curvature conditions restrict the existence of compact null hypersurfaces. We use these results to show the influence on the existence of compact totally umbilic null hypersurfaces which are not totally geodesic. Finally we describe the restrictions that they impose in causality theory.

Keywords: Compact null hypersurfaces, rigging technique, causality, null convergence condition, non-imprisoning Lorentzian manifold.

2000 Mathematics Subject Classification: 53C50, 53C80, 83C75

Communicated by: T. Leistner

1 Introduction

Given a Lorentzian manifold $(\overline{M}, \overline{g})$, that is, a differentiable manifold furnished with a metric with signature $(-, +, \dots, +)$, and a point $p \in \overline{M}$, a hyperplane in $T_p\overline{M}$ is spacelike, null or timelike if the restriction of \overline{g}_p to it is positive definite, degenerate or indefinite, respectively. A hypersurface is said to be spacelike, null or timelike if its tangent space at each point is of the corresponding type. On non-degenerate hypersurfaces one can consider all the fundamental intrinsic and extrinsic geometric notions. In particular, we have a canonical splitting of the ambient tangent space into two factors, a tangent and an orthogonal one. Therefore, by respective projections, one has fundamental equations such as Gauss, Codazzi, and Weingarten equations, along with the second fundamental form, shape operator, induced connection, etc. The null hypersurface case is precisely when the normal vector field is null (also called lightlike) and contrary to the non-degenerate counterpart, the normal vector bundle is tangent to the null hypersurface. One cannot find a natural projector, and hence there is no preferred induced connection to define induced geometric objects as usual. This forces to develop specific techniques and tools. Despite the degeneracy of the induced metric, null hypersurfaces have geometric interest and play an important role in general relativity, as they represent horizons of various sorts. Also, it is well-known that restrictions are obtained on the topology of a null hypersurface in the presence of certain symmetries or geometric conditions on the Lorentzian manifold. For instance in [30] the idea of collapsing Riemannian manifolds introduced in [11], [12] on smooth compact Cauchy horizons is used to establish restrictions to their existence, and in [20] Riemannian techniques are used to show an obstruction to the existence of compact strictly totally umbilic null hypersurfaces. Both examples show that the topological or geometric properties of the family of null hypersurfaces contains valuable information of the ambient space. This is specially relevant because the family of null hypersurfaces is a genuine Lorentzian object, so it is interesting to exploit it systematically.

Our objective in this paper is to study compact null hypersurfaces in the presence of some kind of symmetries, some of which are present in locally generalized Robertson–Walker spaces. Theorem 4 uses the inverse null convergence condition to exhibit a new obstruction to the existence of compact strictly totally umbilic

C. Atindogbé, R. Hounnonkpe, Université d'Abomey-Calavi, Bénin, email: atincyr@gmail.com, rhounnonkpe@ymail.com

*Corresponding author: M. Gutiérrez, Universidad de Málaga, Spain, email: m_gutierrez@uma.es

null hypersurfaces. Using the null convergence condition we obtain a stronger result. In fact, Theorem 7 uses the null convergence condition to show that any compact null hypersurface is totally geodesic, see also [26, Theorem 18]. Observe that both conditions on the curvature are satisfied by Ricci flat spacetimes. In the last section we combine the techniques developed in the first part of the paper with techniques of dynamical systems to obtain some implications in causality theory, see Theorems 30, 33 and 35. For this kind of results see also [6].

2 Compact totally umbilic null hypersurfaces

We use the rigging technique introduced in [20] which complements the usual tools given by the choice of a null section and a screen distribution introduced in [13] to handle null hypersurface geometry. Let (\bar{M}, \bar{g}) be a Lorentzian manifold, $\bar{\nabla}$ the Levi–Civita connection and M a null hypersurface. The choice of a null section ξ and a screen distribution \mathcal{S} on M (a complementary distribution to ξ in TM) allows us to determine an auxiliary null transversal vector field N defined on M by the conditions $\bar{g}(N, \xi) = 1$, $\bar{g}(N, X) = 0$, for any $X \in \mathcal{S}$. The transversality of N leads to the decomposition of the covariant derivative

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad (1)$$

for any $X, Y \in \mathfrak{X}(M)$, where $\nabla_X Y \in \mathfrak{X}(M)$ is the projection of $\bar{\nabla}_X Y$ on TM using the transverse vector field N . The above identity defines the symmetric tensor B , called the null second fundamental form associated to the pair (ξ, \mathcal{S}) , and a projected connection ∇ on M .

The null hypersurface M is said to be totally umbilic if $B = \sigma g$ for some function $\sigma \in C^\infty(M)$, with g being the restriction of \bar{g} on M , strictly totally umbilic if σ never vanishes, and totally geodesic if $\sigma = 0$. Those notions do not depend on the choice of the pair (ξ, \mathcal{S}) .

The covariant derivation of the transverse vector field N along $U \in \mathfrak{X}(M)$ decomposes as

$$\bar{\nabla}_U N = -A(U) + \tau(U)N,$$

where τ is a one-form and $A(U) \in TM$. The covariant derivation, using the projected connection ∇ , of the null vector field ξ is

$$\nabla_U \xi = -A_\xi^*(U) - \tau(U)\xi, \quad (2)$$

for any $U \in \mathfrak{X}(M)$. The shape operator of \mathcal{S} with respect to ξ is A_ξ^* . The following identities are trivial:

$$B(U, V) = -\bar{g}(\bar{\nabla}_U \xi, V) = \bar{g}(A_\xi^*(U), V), \quad (3)$$

for any $U, V \in \mathfrak{X}(M)$. The trace of A_ξ^* is called the null mean curvature H of M . It holds that

$$H_p = \sum_{i=1}^n B(e_i, e_i)$$

where $\{e_1, \dots, e_n\}$ is an orthonormal basis of \mathcal{S}_p . Combining Equations (1) and (2) we obtain $B(\xi, U) = 0$, $A_\xi^*(\xi) = 0$ and $\bar{\nabla}_\xi \xi = -\tau(\xi)\xi$, that is, the null section ξ is a pregeodesic vector field.

Whereas B encodes useful information on M , the projected connection ∇ is an auxiliary connection with poor properties. A rigging vector field ζ for M is a vector field defined in an open set containing M and transverse to it. The pair (M, ζ) is called a normalization of M . The choice of a rigging ζ determines a pair (ξ, \mathcal{S}) as above, ξ being the unique null vector field on M with the normalization $\bar{g}(\zeta, \xi) = 1$, and $\mathcal{S} = TM \cap \zeta^\perp$, allowing us to use the standard tools in the literature. Additionally it provides a Riemann metric \bar{g} on M called the rigged metric induced by ζ ,

$$\bar{g} = \omega \otimes \omega + i^* \bar{g}$$

where $i : M \rightarrow \bar{M}$ is the canonical inclusion and $\omega = i^* \alpha$ where α is the g -equivalent 1-form to ζ . In particular, we can use the Levi–Civita connection of \bar{g} instead of the above projected connection ∇ , leading us to use

all the powerful techniques of Riemannian geometry, see also [4], [5], [6]. All those notions can be extended easily to null submanifolds.

Symmetries in a Lorentzian manifold and energy conditions have many implications for null hypersurfaces. For instance in [20] it is shown that if the reverse null convergence condition (that is $\overline{\text{Ric}}(U) \leq 0$ for all lightlike vector field U) holds on a Lorentzian manifold admitting a timelike conformal vector field, then any compact totally umbilic null hypersurface is totally geodesic. In this section we will see a variety of situations which lead to the same conclusion.

We introduce the following definition which is present in several examples of spacetimes, like in [15] and in locally generalized Robertson–Walker spaces [18].

Definition 1. Let ζ be a reference frame, that is, a unit timelike vector field on \overline{M} . It is spatially conformal stationary if $(L_\zeta \overline{g})(X, Y) = \rho \overline{g}(X, Y)$ for all $X, Y \in \zeta^\perp$ and for some function $\rho \in C^\infty(\overline{M})$. If the function ρ is constant, ζ is called spatially homothetic stationary, if $\rho = 0$ it is called spatially stationary. We call ρ the conformal function of ζ .

In General Relativity, spatially stationary reference frames model a family of observers which see a constant space metric, and spatially conformal stationary reference frames model observers which see an expansion or compression along their worldline.

Note that for a spatially conformal stationary reference frame ζ , if $\dim \overline{M} = n + 2$ then $\overline{\text{div}} \zeta = \frac{1}{2}(n + 1)\rho$.

Lemma 2. Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface M such that ζ is a geodesic spatially conformal stationary reference frame with the conformal function ρ . Then $\tau(\xi) = \frac{1}{2}\rho|_M$.

Proof. First, observe that $\xi + \zeta$ is orthogonal to ζ . Since ζ is spatially conformal stationary, we have

$$(L_\zeta \overline{g})(\xi + \zeta, \xi + \zeta) = \rho|_M \overline{g}(\xi + \zeta, \xi + \zeta) = \rho|_M.$$

As ζ is unitary and geodesic, we have

$$(L_\zeta \overline{g})(U, \zeta) = \overline{g}(\overline{\nabla}_U \zeta, \zeta) + \overline{g}(\overline{\nabla}_\zeta \zeta, U) = 0$$

for all $U \in T\overline{M}$. Hence we get $\rho|_M = (L_\zeta \overline{g})(\xi + \zeta, \xi + \zeta) = (L_\zeta \overline{g})(\xi, \xi)$. From [20, Corollary 3.6], we have $(L_\zeta \overline{g})(\xi, \xi) = 2\overline{g}(\overline{\nabla}_\zeta \zeta, \xi) = 2\tau(\xi)$. It follows that $2\tau(\xi) = \rho|_M$. \square

Lemma 3. Let $(\overline{M}^{n+2}, \overline{g})$ be a Lorentzian manifold and (M, ζ) a normalized null hypersurface M such that ζ is a geodesic spatially conformal stationary reference frame. Suppose that $\overline{\nabla}(\overline{\text{div}} \zeta) = \lambda \zeta$ for some $\lambda \in C^\infty(\overline{M})$. Then $\xi(\tau(\xi)) = \frac{1}{n+1}\lambda|_M$.

Proof. From Lemma 2, $\tau(\xi) = \frac{1}{2}\rho|_M = \frac{1}{n+1}(\overline{\text{div}} \zeta)|_M$. Then $\lambda|_M = \overline{g}(\lambda \zeta, \xi) = \overline{g}(\overline{\nabla}(\overline{\text{div}} \zeta), \xi) = (n + 1)\xi(\tau(\xi))$. \square

Let (M, ζ) be a normalization of the null hypersurface M . Using the decomposition $TM = \text{span}\{\zeta\} \oplus \mathcal{S}$ we have the identity $\overline{\nabla}_U \xi = -\tau(U)\xi - A_\zeta^*(U)$ for any $U \in \mathfrak{X}(M)$. This defines the shape operator A_ζ^* of \mathcal{S} . Observe that the following theorem applies to Ricci flat spacetimes. Moreover, the hypothesis concerning the reference frame, although it seems some artificial, is weaker than the hypothesis leading to a locally generalized Robertson–Walker space, see [18, Corollary 2.3]. In physics, a spacetime with a geodesic spatially conformal stationary reference frame is called Hubble-isotropic, see [22].

Theorem 4. Let $(\overline{M}, \overline{g})$ be an orientable Lorentzian manifold of dimension $\dim \overline{M} \geq 4$ with $\overline{\text{Ric}}(U) \leq 0$ for all lightlike vector fields U . Suppose that it admits a reference frame ζ which is geodesic spatially conformal stationary and $\overline{\nabla}(\overline{\text{div}} \zeta) = \lambda \zeta$ where λ is a non-negative function. Then any compact totally umbilic null hypersurface is totally geodesic.

Proof. Let M be a compact null hypersurface. By the equality $H = -\overline{\text{div}}(\xi)$, where H is the null mean curvature associated to B , see [20], and the divergence theorem, we have

$$\int_M \tau(\xi) H d\overline{g} = \int_M \xi(\tau(\xi)) d\overline{g} \geq 0$$

where the inequality is due to Lemma 3 and $\lambda_{|M}$ is non-negative by hypothesis. Recall the Raychaudhuri equation [3]

$$\overline{\text{Ric}}(\xi) = \xi(H) + \tau(\xi)H - |A_\xi^*|^2. \quad (4)$$

Integrating we obtain

$$\int_M \overline{\text{Ric}}(\xi) d\bar{g} \geq \int_M (\xi(H) - |A_\xi^*|^2) d\bar{g}.$$

By the divergence theorem again,

$$\int_M \xi(H) d\bar{g} = - \int_M H \overline{\text{div}}(\xi) d\bar{g} = \int_M H^2 d\bar{g}, \quad \text{hence} \quad \int_M \overline{\text{Ric}}(\xi) d\bar{g} \geq \int_M (H^2 - |A_\xi^*|^2) d\bar{g}.$$

Now we use that M is totally umbilic, so $B = \sigma g$ for some function σ on M . Then A_ξ^* has one zero eigenvalue $\lambda_{n+1} = 0$, corresponding to the eigenvector ξ , and all the other eigenvalues $\lambda_i = \sigma$ for $i = 1, \dots, n$. Because $\dim \bar{M} = n + 2$ we can write $H^2 - |A_\xi^*|^2 = \left(\sum_{i=1}^n \sigma\right)^2 - \sum_{i=1}^n \sigma^2 = (n-1)n\sigma^2 \geq 0$. This implies that $\int_M \overline{\text{Ric}}(\xi) d\bar{g} \geq (n-1)n \int_M \sigma^2 d\bar{g} \geq 0$. Using $\overline{\text{Ric}}(\xi) \leq 0$, we get $\sigma = 0$, so $B = 0$. \square

Remark 5. For geodesic spatially homothetic stationary reference frames ζ , Lemmas 2 and 3 yield $\lambda_{|M} = 0$.

Corollary 6. *Let (\bar{M}, \bar{g}) be an orientable Lorentzian manifold of dimension $\dim \bar{M} \geq 4$ with $\overline{\text{Ric}}(U) \leq 0$ for all lightlike vector field U . Suppose it admits a geodesic spatially homothetic stationary reference frame. Then any compact totally umbilic null hypersurface is totally geodesic.*

The null convergence condition is a classical hypothesis in general relativity, meaning that gravity attracts on average. In this case we show that any compact null hypersurface is totally geodesic, not only totally umbilic ones.

Theorem 7. *Let (\bar{M}, \bar{g}) be an orientable Lorentzian manifold on which the null convergence condition holds, and admits a geodesic spatially conformal stationary reference frame with conformal function that never vanishes. Then any compact null hypersurface is totally geodesic.*

Proof. Let M be a compact null hypersurface in \bar{M} and take the reference frame ζ in the hypothesis as a rigging for it. From Lemma 2 we know that $\tau(\xi) = \frac{1}{2}\rho_{|M}$ where ρ is the conformal function of ζ . We know that it has some sign by hypothesis.

The null mean curvature function H has a maximum and a minimum respectively at some x_1 and x_2 in M and then $\bar{\nabla}H$ vanishes at x_1 and x_2 . Recall from [3] that

$$\overline{\text{Ric}}(\xi) = \xi(H) + \tau(\xi)H - |A_\xi^*|^2. \quad (5)$$

But $\xi(H) = \bar{g}(\bar{\nabla}H, \xi)$ and then $\xi(H)(x_1) = \xi(H)(x_2) = 0$. It follows that for $i = 1, 2$

$$\overline{\text{Ric}}(\xi)(x_i) = \tau(\xi)(x_i)H(x_i) - |A_\xi^*|^2(x_i).$$

Since $\overline{\text{Ric}}(\xi) \geq 0$, we have $\tau(\xi)(x_i)H(x_i) \geq |A_\xi^*|^2(x_i)$, so the sign of $\tau(\xi)$ determines that the sign of both $H(x_1)$ and $H(x_2)$ are the same, so H has sign. Without loss of generality we suppose that $H(x) \leq 0$ for all $x \in M$. Recall that $H = -\overline{\text{div}}(\xi)$, so it follows that $\overline{\text{div}}(\xi) \geq 0$. Since \bar{M} is orientable and ζ is transverse to M , the latter is also orientable which leads to $H = -\overline{\text{div}}(\xi) = 0$ by the Stokes theorem.

Then Equation (5) implies that $\overline{\text{Ric}}(\xi) = -|A_\xi^*|^2$ which leads to $|A_\xi^*|^2 = 0$ as $\overline{\text{Ric}}(\xi) \geq 0$. It follows that $A_\xi^* = 0$ and Equation (3) implies $B = 0$, so M is totally geodesic. \square

If ζ is a geodesic spatially homothetic stationary reference frame, it is obvious that $\overline{\text{div}}\zeta$ has some sign, so we have:

Corollary 8. *Let (\bar{M}, \bar{g}) be an orientable Lorentzian manifold on which the null convergence condition holds. If (\bar{M}, \bar{g}) admits a geodesic spatially homothetic stationary reference frame, then any compact null hypersurface is totally geodesic.*

There are some geometric properties of a null hypersurface that are independent of the conformal class of the metric, for example total umbilicity, see [19, Lemma 5.1]. It seems natural to ask for properties of null hypersurfaces under the hypothesis of staying in the conformal class of the metric. For a Lorentzian manifold $(\overline{M}, \overline{g})$ we denote by $[\overline{g}]$ the conformal class of \overline{g} .

Proposition 9. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold with $\dim \overline{M} \geq 3$ furnished with a timelike conformal vector field. If the null convergence condition holds on (\overline{M}, g^*) for some $g^* \in [\overline{g}]$, then any compact null hypersurface in $(\overline{M}, \overline{g})$ is totally umbilic.*

Proof. Consider a compact null hypersurface M in $(\overline{M}, \overline{g})$ where $\dim \overline{M} = n + 2$. It is also a compact null hypersurface in (\overline{M}, g^*) . Since $g^* \in [\overline{g}]$, (\overline{M}, g^*) admits a timelike conformal vector field too. So, using (\overline{M}, g^*) as the ambient space, we can write

$$\overline{\text{Ric}}(\xi) = \xi(H) - |A_\xi^*|^2 \leq \xi(H) - \frac{1}{n}H^2, \quad (6)$$

see [20, Proposition 3.11]. By hypothesis the null convergence condition holds on (\overline{M}, g^*) , so $0 \leq \xi(H) - \frac{1}{n}H^2$. As M is compact, the unique solution is $H = 0$, so Equation (6) implies that M is totally geodesic in (\overline{M}, g^*) . Using [19, Lemma 5.1] we conclude that M is totally umbilic in $(\overline{M}, \overline{g})$. \square

Corollary 10. *Let \overline{M} be an orientable manifold with $\dim M \geq 3$. If \overline{g}, g^* are two Lorentzian metrics conformally related, admitting a timelike conformal vector field such that for g^* holds the null convergence condition and for \overline{g} holds the reverse one, then any compact null hypersurface in $(\overline{M}, \overline{g})$ is totally geodesic.*

Proof. Consider a compact null hypersurface M in $(\overline{M}, \overline{g})$. By Proposition 9, M is totally umbilic. Using [20, Theorem 5.1], it follows that M is totally geodesic in $(\overline{M}, \overline{g})$. \square

Proposition 11. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold with $\dim \overline{M} = n$ furnished with a timelike conformal vector field. Suppose there exists g_1, g_2 in $[\overline{g}]$ such that $\text{Ric}_1(U) < \text{Ric}_2(U)$ for all null vector fields U . Then $(\overline{M}, \overline{g})$ does not contain compact null hypersurfaces.*

Proof. Suppose that M is a compact null hypersurface in $(\overline{M}, \overline{g})$ and that $g_1 = e^{2f}g_2$. Then M is also a null hypersurface in (\overline{M}, g_2) . Take a timelike conformal vector field ζ on (\overline{M}, g_1) as a rigging for M and call ξ its associated rigged vector field. Then ξ is a geodesic vector field for (\overline{M}, g_1) . It is well known (see [9]) that the Ricci tensors are related by

$$\text{Ric}_1 - \text{Ric}_2 = -n(\text{Hess}_2 f - df \otimes df) + (\Delta_2 f - ng_2(\nabla^2 f, \nabla^2 f))g_2 \quad (7)$$

where Δ_2 and ∇^2 are the Laplacian and the gradient operators with respect to the metric g_2 respectively. Writing $\phi = e^{-f}$ it is easy to see that

$$\text{Hess}_2 f - df \otimes df = -\frac{1}{\phi} \text{Hess}_1 \phi - g_2(\nabla^2 f, \nabla^2 f)g_2.$$

Using that ξ is lightlike in (7), we get

$$\text{Ric}_1(\xi) - \text{Ric}_2(\xi) = \frac{n}{\phi} \text{Hess}_1 \phi(\xi, \xi).$$

As $\text{Ric}_1(U) < \text{Ric}_2(U)$ for all lightlike vector fields U , it follows that $\text{Hess}_1 \phi(\xi, \xi) < 0$. So $\xi(\xi(\phi)) - \nabla_\xi^1 \xi(\phi) < 0$. Since $\nabla_\xi^1 \xi = 0$, we get that $\xi(\xi(\phi)) < 0$, which contradicts the compactness of M . \square

In [20] it is shown that the presence of a timelike gradient vector field in a Lorentzian manifold prevents the existence of compact null hypersurface. We use a similar argument to show that the same conclusion holds in the spacelike case.

Proposition 12. *Let $(\overline{M}, \overline{g})$ be a time orientable Lorentzian manifold admitting a never vanishing spacelike gradient vector field. Then $(\overline{M}, \overline{g})$ contains no compact null hypersurfaces.*

Proof. Suppose that there exists a compact null hypersurface M in $(\overline{M}, \overline{g})$. We can take any timelike vector field ζ as a rigging for M . Let $f \in C^\infty(M)$ with gradient $\overline{\nabla}f$ never vanishing and spacelike on \overline{M} . Let \tilde{f} be the restriction of f to M . Then, along M one has $\overline{\nabla}f = a\xi + bN + X$ where N is the transverse null vector field, a and b are smooth functions on M and $X \in \mathcal{S}(\zeta)$ is the screen distribution associated to ζ . Using the Riemannian metric \overline{g} , we get $\overline{\nabla}\tilde{f} = b\xi + X$. By compactness of M there is a point in M where $\overline{\nabla}\tilde{f} = 0$. But at this point $\overline{\nabla}f = a\xi$, in contradiction with the fact that $\overline{\nabla}f$ is a never vanishing spacelike vector field. \square

As we can see in the following corollary, Proposition 12 has strong implications and shows that many Lorentzian manifolds cannot admit compact null hypersurfaces.

Corollary 13. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold. Suppose that \overline{M} splits diffeomorphically as $\overline{M} = M_1 \times M_2$ with M_2 non-compact. If there exists $U \in TM_1$ which is timelike, then \overline{M} admits no compact null hypersurface.*

Proof. Since M_2 is non-compact, it carries a submersion function f , see [24], so $\overline{\nabla}(f \circ \pi_2)$ is never zero, where π_2 is the projection on the factor M_2 . Since $U \in TM_1$, we have $\overline{g}(\overline{\nabla}(f \circ \pi_2), U) = U(f \circ \pi_2) = 0$ which implies that $\overline{\nabla}(f \circ \pi_2)$ is spacelike since U is timelike. We apply now Proposition 12. \square

Example 14. Let $\overline{M} = M_1 \times_{(f_1, f_2)} M_2$ be a time orientable Lorentzian doubly twisted product with (M_1, g_1) Lorentzian and (M_2, g_2) Riemannian and non-compact. Then \overline{M} admits no compact null hypersurface. This family includes examples of generalized Robertson–Walker spaces and standard static Lorentzian manifolds of type $S^1 \times_{(f,1)} F$ with F non-compact. Another family is the Lorentzian product $\overline{M} = \mathbb{R}^3 \times \mathbb{R}$ with metric $\overline{g} = -(dt + fdx)^2 + hdx^2 + dy^2 + dz^2$, where f and h are smooth positive functions on \mathbb{R}^3 . In particular, taking $f = e^x$ and $h = \frac{1}{2}e^{2x}$, it follows that the Gödel spacetime admits no compact null hypersurface.

3 Convex functions and compact null hypersurfaces

Convex functions play an important role in semi-Riemannian geometry and put restrictions on the properties of the semi-Riemannian manifold, see [10] and [16]. For example, in [16] the authors recall that the approach of Gibbs [17] to thermodynamic is based on the idea that the free energy should be a convex function. In this section, we explore how convex functions and spacetime convex functions (see Definition 19) restrict the topological and geometrical properties of null hypersurfaces in a Lorentzian manifold. Using these results we show in the next section how convex functions influence the causal structure, see Theorems 31 and 35.

Definition 15. By a convex (respectively strictly convex) function on a Lorentzian manifold $(\overline{M}, \overline{g})$, we mean a smooth real-valued function whose restriction to every geodesic has nonnegative (respectively positive) second derivative. Equivalently, f is convex (respectively strictly convex) if and only if the Hessian $\overline{\text{Hess}}f$ is positive semidefinite (respectively positive definite) where the Hessian is given by

$$\overline{\text{Hess}}f(X, Y) = X(Y(f)) - (\overline{\nabla}_X Y)(f). \quad (8)$$

Proposition 16. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold admitting a strictly convex function f and a timelike conformal vector field ζ . Then \overline{M} does not contain compact null hypersurfaces.*

Proof. Suppose that \overline{M} admits a compact null hypersurface M . Take the normalization (M, ζ) . We have $\overline{\text{Hess}}f(\xi, \xi) = \xi(\xi(f)) - (\overline{\nabla}_\xi \xi)(f) > 0$. But $\overline{\nabla}_\xi \xi = 0$ since ζ is conformal, see [20], and then $\xi(\xi(f)) > 0$, which is a contradiction to the compactness of M . \square

In the case where \overline{M} admits no timelike conformal vector field but a geodesic spatially conformal stationary reference frame, we get the following.

Proposition 17. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold admitting a strictly convex function and a geodesic spatially conformal stationary reference frame ζ such that $\overline{\nabla}(\overline{\text{div}}\zeta) = \lambda\zeta$ where λ is a non-negative function on \overline{M} . Then \overline{M} does not contain compact null hypersurfaces.*

Proof. Suppose that \overline{M} admits a compact null hypersurface M . Take the normalization (M, ζ) . Consider the restriction of the strictly convex function f on M that we call \tilde{f} . Adding a constant to f if necessary, we can suppose that $\tilde{f} \geq 0$. Observe that $\xi(\xi(\tilde{f}) + \tau(\tilde{f})) > \xi(\tau(\tilde{f}))\tilde{f}$. In fact, f is strictly convex, so $\overline{\text{Hess}}\tilde{f}(\xi, \xi) = \xi(\xi(\tilde{f})) - (\nabla_{\xi}\tilde{f}) > 0$, but $\nabla_{\xi}\tilde{f} = -\tau(\tilde{f})\xi$ and then $\xi(\xi(\tilde{f})) + \tau(\tilde{f})\xi > 0$.

By Lemma 3, $\xi(\tau(\tilde{f})) = \frac{1}{n+1}\lambda|_M$ is non-negative. It follows that $\xi(\xi(\tilde{f}) + \tau(\tilde{f})) > 0$, which is a contradiction since M is compact. \square

Using Remark 5, the following holds directly.

Corollary 18. *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold admitting a strictly convex function and a geodesic spatially homothetic reference frame. Then \overline{M} does not contain compact null hypersurfaces.*

Gibbons and Ishibashi [16] introduce "space-time convex" functions on Lorentzian manifolds.

Definition 19. A smooth function $f : \overline{M} \rightarrow \mathbb{R}$ is called a spacetime convex function if the Hessian $\overline{\text{Hess}}f$ has Lorentzian signature and satisfies the condition

$$\overline{\text{Hess}}f(X, X) \geq c\overline{g}(X, X) \quad \text{for all } X \in T\overline{M}, \quad \text{with a positive constant } c. \quad (9)$$

The geometrical interpretation of this definition is that the light cone C_f defined by the Hessian of f lies inside the light cone $C_{\overline{g}}$ defined by the spacetime metric \overline{g} . This means, in an intuitive sense, that the Condition (9) prevents the collapse of the light cone defined by \overline{g} . In [16], the authors find examples of the existence or non-existence of such functions on regions in cosmological space-times and black-hole space-times.

In a spacetime, isolated horizons (see [1]) are totally geodesics, so they are examples of minimal null hypersurfaces, that is the null mean curvature H is identically zero. For a Lorentzian manifold admitting a spacetime convex function, we get a relation with this kind of null hypersurfaces.

Theorem 20. *Let $(\overline{M}, \overline{g})$ be an orientable Lorentzian manifold admitting a spacetime convex function and a closed and spatially homothetic reference frame. Then \overline{M} does not contain compact minimal null hypersurfaces.*

Proof. Suppose that \overline{M} has dimension $n + 2$ and admits a compact minimal null hypersurface M . Take the normalization (M, ζ) where ζ is the reference frame in the hypothesis. Let $(\xi, E_2, \dots, E_{n+1})$ be a local \overline{g} -orthonormal basis in TM and let f be the spacetime convex function. Then $\overline{\text{Hess}}f(E_i, E_i) = E_i(E_i(f)) - (\nabla_{E_i}E_i)f$. From the definition of the null second fundamental form, we get

$$\overline{\text{Hess}}f(E_i, E_i) = E_i(E_i(f)) - (\nabla_{E_i}E_i)f - B(E_i, E_i)N(f).$$

From [20, Proposition 3.7, Remark 3.8 and Proposition 3.15], we have $\nabla_{E_i}E_i - \tilde{\nabla}_{E_i}E_i = (C(E_i, E_i) - B(E_i, E_i))\xi$. It follows that

$$\overline{\text{Hess}}f(E_i, E_i) = E_i(E_i(f)) - (\tilde{\nabla}_{E_i}E_i)f - (C(E_i, E_i) - B(E_i, E_i))\xi(f) - B(E_i, E_i)N(f).$$

Since ζ is a closed, spatially homothetic reference frame (with conformal factor a constant ρ), from [20, Corollary 3.6 (4)] we obtain $C(E_i, E_i) = -\frac{1}{2}\rho + \frac{1}{2}B(E_i, E_i)$, hence we have

$$\overline{\text{Hess}}f(E_i, E_i) = \widetilde{\text{Hess}}f(E_i, E_i) - \left(-\frac{1}{2}\rho + \frac{1}{2}B(E_i, E_i) - B(E_i, E_i)\right)\xi(f) - B(E_i, E_i)N(f).$$

Using the fact that M is minimal we get from the above expression the following

$$\sum_{i=2}^{n+1} \overline{\text{Hess}}f(E_i, E_i) = \sum_{i=2}^{n+1} \widetilde{\text{Hess}}f(E_i, E_i) + \frac{1}{2}n\rho\xi(f) > 0,$$

since f is spacetime convex, so we have

$$\sum_{i=2}^{n+1} \widetilde{\text{Hess}}f(E_i, E_i) > -\frac{1}{2}n\rho\xi(f). \quad (10)$$

Moreover, using again that f a spacetime convex function, we have $\overline{\text{Hess}}f(\xi, \xi) = \xi(\xi(f)) - (\nabla_{\xi}\xi)(f) \geq 0$. As ζ is closed and unitary, it is geodesic. By Lemma 2, $\nabla_{\xi}\xi = -\tau(\xi)\xi = -\frac{1}{2}\rho\xi$, hence $\xi(\xi(f)) + \frac{\rho}{2}\xi(f) \geq 0$. Since ζ is closed, $\tilde{\nabla}_{\xi}\xi = 0$ and then $\widetilde{\text{Hess}}f(\xi, \xi) = \xi(\xi(f))$. It follows that $\widetilde{\text{Hess}}f(\xi, \xi) + \frac{\rho}{2}\xi(f) \geq 0$ and then

$$\widetilde{\text{Hess}}f(\xi, \xi) \geq -\frac{1}{2}\rho\xi(f). \quad (11)$$

From the relations (10) and (11) we get

$$\bar{\Delta}f > -\frac{1}{2}(n+1)\rho\xi(f). \quad (12)$$

The computations have been done in an open set where the local orthonormal basis is defined, but it is clear that the inequality holds in all of M . Recall that ρ is constant, so integrating (12) and using the divergence theorem, we get

$$\int_M \bar{\Delta}f \, d\bar{g} > \frac{1}{2}(n+1)\rho \int_M f \bar{\operatorname{div}}(\xi) \, d\bar{g}. \quad (13)$$

Recall also that $-\bar{\operatorname{div}}(\xi) = H$ and, since M is minimal, $H = 0$. It follows that $\int_M \bar{\Delta}f \, d\bar{g} > 0$, which is a contradiction because M is compact. \square

Similar to Definition 1, we call a vector field ζ orthogonally killing if $L_\zeta g(X, Y) = 0$ for any $X, Y \in \zeta^\perp$.

Theorem 21. *Let (\bar{M}, \bar{g}) be an orientable Lorentzian manifold. Suppose it admits*

- (1) *a timelike irrotational and orthogonally killing vector field ζ with closed (as subsets) orthogonal leaves*
- (2) *and a strictly convex function or a spacetime convex function f .*

Then (\bar{M}, \bar{g}) does not contain compact minimal null hypersurfaces.

Proof. Suppose that \bar{M} admits a compact minimal null hypersurface M . Take the normalization (M, ζ) . By hypothesis, the screen distribution $\mathcal{S}(\zeta)$ is integrable. Take a leaf L of $\mathcal{S}(\zeta)$. It is the intersection of M with an orthogonal leaf of ζ , so L is compact since orthogonal leaves of ζ are closed and M is compact. Let $(\xi, E_2, \dots, E_{n+1})$ be a local \bar{g} -orthonormal basis in TM . The Hessian of f satisfies $\bar{\operatorname{Hess}} f(E_i, E_i) = E_i(E_i(f)) - (\bar{\nabla}_{E_i} E_i)(f)$. From the definition of the null second fundamental form, we get

$$\bar{\operatorname{Hess}} f(E_i, E_i) = E_i(E_i(f)) - (\nabla_{E_i} E_i)(f) - B(E_i, E_i)N(f).$$

From [20, Proposition 3.7 and Remark 3.8] we have

$$\nabla_{E_i} E_i - \bar{\nabla}_{E_i} E_i = (C(E_i, E_i) + \bar{g}(\bar{\nabla}_{E_i} \xi, E_i))\xi \quad \text{and} \quad \bar{\nabla}_{E_i} E_i = \nabla_{E_i}^* E_i - \bar{g}(\bar{\nabla}_{E_i} \xi, E_i)\xi$$

where ∇^* is the Levi-Civita connection induced on L . It follows that

$$\bar{\operatorname{Hess}} f(E_i, E_i) = E_i(E_i(f)) - (\nabla_{E_i}^* E_i)(f) - C(E_i, E_i)\xi(f) - B(E_i, E_i)N(f).$$

Since ζ is orthogonally killing, from [20, Corollary 3.6] we have $C(E_i, E_i) = -\frac{1}{2}\bar{g}(\zeta, \zeta)B(E_i, E_i)$, then

$$\bar{\operatorname{Hess}} f(E_i, E_i) = E_i(E_i(f)) - (\nabla_{E_i}^* E_i)(f) + \frac{1}{2}\bar{g}(\zeta, \zeta)B(E_i, E_i)\xi(f) - B(E_i, E_i)N(f) > 0$$

since f is strictly convex or spacetime convex. This means that $\bar{\operatorname{Hess}}^* f(E_i, E_i) + \frac{1}{2}\bar{g}(\zeta, \zeta)B(E_i, E_i)\xi(f) - B(E_i, E_i)N(f) > 0$.

Since M is minimal we get $\Delta^* f > 0$, and this inequality holds in all of M . This is a contradiction as L is compact. \square

4 Causality and compact null hypersurfaces

Causality is one of the main features of Lorentzian manifolds. We recall some definitions used here. For general reference on causality see for instance [28], [29], [8]. Causality has obviously influence on the existence of compact null hypersurface. For instance, in [20] it is shown that if \bar{M} is a stably causal Lorentzian manifold, then it does not contain compact null hypersurfaces. We improve this result showing that compact null submanifolds (not only hypersurfaces) cannot exist in non-total imprisoning Lorentzian manifold, see Proposition 23. We determine conditions that can be added to a causal or chronological Lorentzian manifold to be a space with a higher causality step in the causal hierarchy, see Theorems 30, 33 and 35.

In this section, all Lorentzian manifolds are time oriented, in particular there always exists a rigging vector field for any null submanifold.

Definition 22. A point $p \in \bar{M}$ is a future endpoint of a future-directed causal curve $\gamma : I \rightarrow \bar{M}$ if, for every neighborhood O of p , there exists a point $t_0 \in I$ such that $\gamma(t) \in O$ for all $t > t_0$. A causal curve is future inextendible (respectively, past inextendible) if it has no future (respectively, past) endpoint.

A future inextendible causal curve $\gamma : I \rightarrow \bar{M}$ is totally future imprisoned in the compact set C if there is $t_0 \in I$ such that $\gamma(t) \in C$ for every $t > t_0$, i.e. if it enters and remains in C . It is partially future imprisoned if for every $t_0 \in I$, there is $t > t_0$ such that $\gamma(t) \in C$, i.e. if it continually returns to it. The curve escapes to infinity in the future if it is not partially future imprisoned in any compact set.

A spacetime is non-total future imprisoning if no future inextendible causal curve is totally future imprisoned in a compact set. A spacetime is non-partial future imprisoning if no future inextendible causal curve is partially future imprisoned in a compact set. Beem [7] proved that a spacetime is non-total future imprisoning if and only if it is non-total past imprisoning, thus in the non-total case one can simply speak of the non-total imprisoning property (condition N in Beem's terminology).

A space-time M is future-distinguishing at $p \in \bar{M}$ if $I^+(p) \neq I^+(q)$ for each $q \in \bar{M}$ with $q \neq p$. \bar{M} is future-distinguishing if it is future-distinguishing at every point. This property of being future-distinguishing is called future-distinction. The concept of past-distinction is defined similarly.

The following result is an obstruction to the existence of compact null submanifolds, not only compact null hypersurfaces.

Proposition 23. Let (\bar{M}, \bar{g}) , be a non-total imprisoning Lorentzian manifold. Then \bar{M} does not contain compact null submanifolds.

Proof. Suppose that \bar{M} admit a compact null submanifold M . The possibility that M is a null curve is excluded by the non-imprisonment assumption. Then \bar{M} has dimension at least two. Take a normalization (M, ζ) . The integral curves of the rigged vector field ξ are null curves and are defined on the whole of \mathbb{R} (as M is compact). Let $\gamma : [0, \infty) \rightarrow M \subset \bar{M}$ be an integral curve of ξ . We are going to show that γ is future inextendible. Suppose that γ admit a future endpoint $p \in \bar{M}$. As $\gamma(t) \in M$ for all $t \in [0, \infty)$ and M is compact, it follows that $p \in M$. But in this case $\xi(p) = 0$, see [28, Page 33], which is not possible as ξ never vanishes. Hence γ is a future inextendible null curve imprisoned in the compact set M , which contradicts the non-total future imprisoning assertion. Thus \bar{M} contains no compact null submanifold. \square

By the standard causal hierarchy for a Lorentzian manifold, distinguishing, strongly causal, stable causal and globally hyperbolic Lorentzian manifolds cannot contain compact null submanifolds.

Remark 24. We show that there exists a causal spacetime which contains compact null hypersurfaces. Consider the following space called Carter space. The manifold is $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$ described by coordinates (t, y, z) where (t, y, z) and $(t, y, z + 1)$ are identified, and (t, y, z) and $(t, y + 1, z + a)$ are identified, where a is an irrational number. The Lorentz metric is given by $\bar{g} = -(\cosh t - 1)^2(dt^2 - dy^2) - dt dy + dz^2$. This space is totally imprisoning but causal, see [23, Page 195]. Nevertheless, $\{0\} \times \mathbb{S}^1 \times \mathbb{S}^1$ is a compact null hypersurface.

Definition 25. (See [8]) A Lorentzian manifold (\bar{M}, \bar{g}) is called disprisoning if for each inextendible geodesic $\gamma : (a, b) \rightarrow \bar{M}$ and any fixed $t_0 \in (a, b)$ the images of each of the two maps $\gamma|_{(a, t_0]}$ and $\gamma|_{[t_0, b)}$ fail to have compact closure. It is null disprisoning if this property is satisfied for each inextendible null geodesic.

Proposition 26. Let (\bar{M}, \bar{g}) be a null-disprisoning Lorentzian manifold admitting a timelike conformal vector field ζ . Then \bar{M} does not contain compact null hypersurfaces.

Proof. Suppose that \bar{M} admits a compact null hypersurface M . Take the normalization (M, ζ) . The integral curves of the rigged vector field ξ are null geodesic curves since ζ is conformal, see [20], and are defined on the whole of \mathbb{R} (as M is compact). The conclusion follows as in the proof of Proposition 23. \square

Definition 27. (See [8]) Let (\bar{M}, \bar{g}) be a Lorentzian manifold. A function $f : \bar{M} \rightarrow \mathbb{R}$ is a generalized time function if $f(p) < f(q)$ for all $p, q \in \bar{M}$ with $p < q$.

Note that in the above definition f is not supposed to be continuous. It is known that when f is continuous then (\bar{M}, \bar{g}) is stably causal, see [27].

Proposition 28. *Let (\bar{M}, \bar{g}) be a Lorentzian manifold admitting a lower semi-continuous (or upper semi-continuous) generalized time function. Then (\bar{M}, \bar{g}) does not contain compact null hypersurfaces.*

Proof. We give the proof in the lower semi-continuous case, the upper semi-continuous case is similar. Suppose that there is a compact null hypersurface M in (\bar{M}, \bar{g}) . Take any rigging ζ of it. Let f be a lower semi-continuous generalized time function on (\bar{M}, \bar{g}) . We still denote by f its restriction on M . Since M is compact and f is lower semi-continuous, f reaches its minimum at some point x_0 in M . Let γ_{x_0} be the integral curve of ζ such that $\gamma_{x_0}(0) = x_0$. Since M is compact, γ_{x_0} is defined on the whole of \mathbb{R} . As ζ never vanishes, γ_{x_0} is not a constant curve. We can take ζ future directed, so $\gamma_{x_0}(t) < x_0$ for all $t \in (-\infty, 0)$, hence $f(\gamma_{x_0}(t)) < f(x_0)$. But x_0 is a minimum point of f on M ; this is a contradiction. \square

Recall the following well known definitions from dynamical systems.

Definition 29. Let M be a manifold, X a complete vector field on M and Φ its flow. Let $\gamma : \mathbb{R} \rightarrow M$ be an integral curve of X . The sets

$$\omega(\gamma) = \{p \in M : \gamma(t_n) \rightarrow p; t_n \rightarrow \infty\} \quad \text{and} \quad \alpha(\gamma) = \{p \in M : \gamma(t_n) \rightarrow p; t_n \rightarrow -\infty\}$$

are called respectively the ω -limit set and the α -limit set of the orbit γ . A point p is called positively recurrent if $p \in \omega(\gamma_p)$ and negatively recurrent if $p \in \alpha(\gamma_p)$, where γ_p is the unique integral curve of X through p . A subset $A \subset M$ is invariant if $\Phi_t(A) \subset A$ for all $t \in \mathbb{R}$. It is known that for any integral curve γ , $\omega(\gamma)$ and $\alpha(\gamma)$ are closed (possibly empty) invariant subsets. A closed, non-empty, invariant subset $A \subset M$ is a minimal set if it contains no proper, closed, non-empty, invariant subset.

We will exploit previous results on the non-existence of compact null hypersurfaces to determine conditions that can be added to a causal or chronological Lorentzian manifold to be a space with a higher causality step in the causal hierarchy.

Theorem 30. *Let (\bar{M}, \bar{g}) be a time orientable, null complete 3-dimensional Lorentzian manifold satisfying the null convergence condition. Suppose that there exists a never vanishing gradient spacelike vector field. If (\bar{M}, \bar{g}) is causal, then it is non-total imprisoning.*

Proof. Suppose that (\bar{M}, \bar{g}) is totally imprisoning. Since (\bar{M}, \bar{g}) is chronological, from [25, Theorem 3.9] it contains a null line η contained in a compact minimal invariant set Ω (in the sense of [25, Definition 3.6]) such that $\bar{\eta} = \Omega$ and through each point of Ω , there passes one and only one null line contained in Ω . Using the null completeness and the null convergence condition, η is contained in a smooth (topologically) closed achronal totally geodesic null surface M , see [14, Theorem IV.1]. Take any timelike vector field as a rigging for M and rescale its associated rigged vector field ξ such that ξ is complete. Then, as any null line through a point $p \in \Omega$ shares the same trace with the integral curve of ξ through p , it follows that Ω is a compact minimal invariant set of the flow of ξ . It is known that a compact minimal set of a C^2 differentiable dynamical system on a 2-surface S is either a fixed point, a periodic orbit or all of S , in which case S is a torus, see [31] and observe that a priori S need not be compact, compare [21, Chapter VII, 12.1]. Since ξ is a lightlike vector field and (\bar{M}, \bar{g}) is causal, in our case the compact minimal set Ω is neither a fixed point nor a periodic orbit, so Ω is the whole surface M and then M is a compact null surface. This is in contradiction with the existence of a never vanishing gradient spacelike vector field, see Proposition 12. \square

The key point in the above proof is the existence of a compact null hypersurface which leads to a contradiction with the existence of a gradient spacelike vector field. So we can replace the existence of this vector field by other alternative conditions thanks to Proposition 16 and Corollary 18.

Corollary 31. *Let (\bar{M}, \bar{g}) be a null complete 3-dimensional Lorentzian manifold satisfying the null convergence condition. Suppose that one of the following hypotheses is satisfied:*

- (1) *There exists a conformal timelike vector field and a strictly convex function on \bar{M} .*
- (2) *There exists a geodesic spatially homothetic reference frame and a strictly convex function on \bar{M} .*

If (\bar{M}, \bar{g}) is causal, then it is non-total imprisoning.

Moreover, observe that in the above proof, the surface M is compact and totally geodesic, in particular it is minimal, so we can use Theorems 20 and 21 to get a small variation of the above corollary.

Corollary 32. *Let (\bar{M}, \bar{g}) be a orientable null complete 3-dimensional Lorentzian manifold satisfying the null convergence condition. Suppose one of the following hypotheses is satisfied:*

- (1) *There exists a spacetime convex function and a closed, spatially homothetic reference frame.*
- (2) *There exists a timelike irrotational orthogonally killing vector field with closed (as subsets) orthogonal leaves and a strictly convex function (or a spacetime convex function).*

If (\bar{M}, \bar{g}) is causal, then it is non-total imprisoning.

The arguments from dynamical systems used here can be modified to avoid the use of the existence of compact null hypersurfaces, allowing to improve the hypothesis from causal to **chronological**. In fact:

Theorem 33. *Let (\bar{M}, \bar{g}) be a null complete Lorentzian manifold of dimension greater than 2 satisfying the null convergence condition. Suppose that (\bar{M}, \bar{g}) admits*

- (1) *a timelike conformal vector field ζ_1 and*
- (2) *a geodesic spatially conformal stationary reference frame ζ_2 such that $\bar{\text{div}}\zeta_2$ never vanishes.*

If (\bar{M}, \bar{g}) is chronological, then it is non-total imprisoning.

Proof. Suppose that (\bar{M}, \bar{g}) is totally imprisoning. Since (\bar{M}, \bar{g}) is chronological, by [25, Theorem 3.9] it contains a null line η contained in a compact minimal invariant set Ω (in the sense of [25, Definition 3.6]) such that $\bar{\eta} = \Omega$. By the null completeness and the null convergence condition, η is contained in a smooth (topologically) closed achronal totally geodesic null hypersurface M , see [14, Theorem IV.1]. Now consider the flow of ζ_2 on M . Take a point $p \in \eta$ and let γ_p be the integral curve of ζ_2 such that $\gamma_p(0) = p$. Then from [25, Theorem 3.9], $\bar{\gamma}_p = \omega(\gamma_p)$ and it follows that p is a positively recurrent point, that is, there exist $t_n \rightarrow \infty$ such that $\gamma_p(t_n) \rightarrow p$. We prove that $f \circ \gamma_p$ is strictly increasing to get a contradiction.

Consider the two normalizations (M, ζ_1) and (M, ζ_2) . From [2, Lemma 2.1 (g)] there exists a function $f \in C^\infty(M)$ such that $\tau_2(\xi_2) = \tau_1(\xi_2) + \xi_2(f)$. But $\tau_1(\xi_2) = 0$ because ζ_1 is conformal (see [20]) and $\tau_2(\xi_2) = \frac{1}{n+1}\bar{\text{div}}\zeta_2$ by Lemma 2. It follows that $\xi_2(f) = \frac{1}{n+1}\bar{\text{div}}\zeta_2$. Without loss of generality, suppose that $\bar{\text{div}}\zeta_2 > 0$. Then $\xi_2(f) > 0$ and the claim follows. \square

Corollary 34. *Let (\bar{M}, \bar{g}) be a null complete Lorentzian manifold of dimension greater than 2 satisfying the null convergence condition. Suppose that (\bar{M}, \bar{g}) admits a pregeodesic timelike conformal vector field ζ_1 with nowhere vanishing divergence. If (\bar{M}, \bar{g}) is chronological, then it is non-total imprisoning.*

Proof. Take ζ_2 to be the unitary vector field of ζ_1 and note that ζ_2 geodesic. Let $a := (-g(\zeta_1, \zeta_1))^{-1/2}$. Since ζ_1 is conformal, we get $\zeta_1(a) = -\frac{1}{2}a\rho$ and $\bar{\text{div}}\zeta_1 = \frac{1}{2}\rho\dim\bar{M} \neq 0$ where ρ is the conformal function of ζ_1 . Thus $\bar{\text{div}}\zeta_2 = -\frac{1}{2}a\rho + \frac{1}{2}a\rho\dim\bar{M} \neq 0$, so ζ_2 satisfies point (2) of Theorem 33. \square

Finally, we show how to take advantage of the existence of a convex function with the same objective as in the previous results in this section.

Theorem 35. *Let (\bar{M}, \bar{g}) be a null complete Lorentzian manifold of dimension greater than 2 satisfying the null convergence condition. Suppose that one of the following holds:*

- (1) *There exists a timelike conformal vector field (or a geodesic spatially homothetic reference frame) and a strictly convex function.*
- (2) *There exists a timelike conformal vector field and some $g^* \in [\bar{g}]$ such that $\text{Ric}_{g^*}(U) < 0$ for all lightlike vector fields U .*

If (\bar{M}, \bar{g}) is chronological, then it is non-total imprisoning.

Proof. Suppose that (\bar{M}, \bar{g}) is totally imprisoning. Following the proof of Theorem 33, we know that there exists a null line η contained in a totally geodesic null hypersurface M .

Suppose that (1) holds. Consider the normalized null hypersurface (M, ζ) where ζ is the vector field of the hypothesis (1). Let ξ be the associated rigged vector field. Its flow admits a positively recurrent point p (see the proof of Theorem 33). We consider first the case where ζ is conformal. Call f the function in (1). Since f is strictly convex, we have $\text{Hess } f(\xi, \xi) = \xi(\xi(f)) - (\bar{\nabla}_\xi \xi)(f) > 0$. But $\bar{\nabla}_\xi \xi = 0$ since ζ is conformal, see [20]). Hence $\xi(\xi(f)) > 0$, which contradicts the presence of a recurrent point.

Now suppose that ζ is a geodesic spatially homothetic reference frame. In this case $L_\zeta \bar{g} = 2\rho \bar{g}$ in the direction ζ^\perp , so $\bar{\nabla}_\xi \xi = -\tau(\xi)\xi$ with $\tau(\xi)$ constant, see Lemma 2. We have $\text{Hess } f(\xi, \xi) = \xi(\xi(f)) - (\bar{\nabla}_\xi \xi)(f) = \xi(\xi(f) - \tau(\xi)f) > 0$, which again contradicts the presence of a recurrent point.

Now suppose that (2) holds. Our hypothesis implies that $\text{Ric}_{g^*}(U) < \text{Ric}_{\bar{g}}(U)$ for any lightlike vector field U . From the proof of Proposition 11 there exists a function, say h , such that $\xi(h) > 0$, and the contradiction follows again. \square

Observe that due to Proposition 23 all the results in this section do not admit compact null hypersurfaces.

Funding: The second and third authors have been partially supported by a Ministerio de Economía y Competitividad MTM2016-78647-P grant. The third author has been partially supported by an African Centre of Excellence in Mathematical Sciences and Applications ACE-SMA grant.

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