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Transient natural convection of non-Newtonian fluids about a vertical surface embedded in an anisotropic porous medium

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Abstract

An analytical method is carried out to investigate transient free convection boundary layer flow along a vertical surface embedded in an anisotropic porous medium saturated by a non-Newtonian fluid. The porous medium is anisotropic in permeability with its principal axes oriented in a direction that is non-coincident with the gravity force. A step increase in wall temperature or in surface heat flux is considered. On the basis of the modified Darcy power-law model proposed by Pascal [H. Pascal, Rheological behaviour effect of non-Newtonian fluids on steady and unsteady flow through porous media, Int. J. Numer. Anal. Methods in Geomech. 7 (1983) 207–224] and the generalized Darcy's law described by Bear [J. Bear, Dynamics of fluids in porous media. Dover Publications, Elsevier, New York (1972)], boundary-layer equations are solved exactly by the method of characteristics. Scale analysis is applied to predict the order-of-magnitudes involved in the boundary layer regime. Analytical expressions are obtained for the limiting time required to reach steady-state, the boundary-layer thickness and the local Nusselt number in terms of the modified-Darcy Rayleigh number, the power-law index, the anisotropic permeability ratio, and the orientation angle of the principal axes. It is demonstrated that both the power-law index and the anisotropic properties have a strong influence on the heat transfer rate.

1. Introduction

The flow through a porous medium under the influence of temperature differences, is one of most considerable and contemporary subjects, because it finds great applications in geothermy, geophysics and technology. The practical interest in convective heat transfer in porous medium is expanded rapidly, due to the wide range of applications in engineering fields. These important applications include such areas as geothermal energy utilization, thermal energy storage and recoverable systems, petroleum reservoirs, insulation of high temperature gas–solid reaction vessels, chemical catalytic convectors, storage of grain, fruits and vegetables, pollutant dispersion in aquifers, industrial and agricultural water distribution, buried electrical cables, combustion in situ in underground reservoirs for the enhancement of oil recovery, ceramic radiant porous burners used in industrial firms as efficient heat transfer devices and the reduction of hazardous combustion products using catalytic porous beds. In a review article, Cheng [1] has discussed various works done in this field as applied to geothermal.

A cursory inspection of the existing references on convective external flow in porous media reveals that, in general, steady-state phenomena have been extensively studied, whereas unsteady phenomena have received relatively much less attention. The mechanism causing an unsteady flow may either act at the boundaries, and this may be through a change in one or more of the dependent variables, or it may be present within the fluid volume. Examples of the former include unsteadiness resulting from the movement of the system boundaries relative to the fluid and changing the upstream or the inlet conditions. Unsteadiness of the latter type results from changing the

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Nomenclature

a, b, c	constants, Eq. (10)
A	boundary-layer thickness, δ/H
B	constant, Eq. (77)
C	constant, Eq. (86)
c_p	specific heat of fluid at constant pressure
$\vec{g}$	gravitational acceleration
k	thermal conductivity
$\bar{\bar{K}}$	flow permeability tensor, Eq. (8)
K_1, K_2	flow permeability along the principal axes
K^*	anisotropic permeability ratio, K_1/K_2
H	height of the surface
$I_{1/n}$	constant, Eq. (73)
n	power-law index of non-Newtonian fluid
Nu_x	local Nusselt number, Eqs. (71) and (88)
p	pressure
q	local heat flux, Eq. (68)
Q	dimensionless heat flux, Eqs. (69) and (87)
Ra_x	modified Darcy–Rayleigh number for isothermal wall
Ra_H	modified Darcy–Rayleigh number, Eq. (29)
R_H	modified Darcy–Rayleigh number, Eq. (42)
R_x	modified Darcy–Rayleigh number for wall with constant heat flux
t	time
T	temperature
$\vec{V}$	seepage velocity
u, v	velocity components in x, y directions
U, V	dimensionless velocity components in X, Y directions
x, y	cartesian coordinates
X, Y	dimensionless cartesian coordinates

Greek symbols

α	thermal diffusivity
β	thermal expansion coefficient of the fluid
δ	thickness of the vertical boundary-layer
ΔT	characteristic scale of temperature
ϵ	constant, Eq. (3)
η	similarity variable, Eqs. (54) and (75)
γ	porosity of the porous medium
$\hbar$	consistency index of the fluid
μ_a	apparent dynamic viscosity of the fluid
ψ	stream function
Ψ	dimensionless stream function
τ	dimensionless time
θ	inclination angle of principal axes
Θ	temperature profile
ϱ	density of the fluid
(ϱc_p)	heat capacity of the fluid
ζ	constant, Eq. (57)

Subscripts

∞	refers to the distance far from the vertical boundary
a	apparent
s	refers to steady-state
w	refers to wall

body forces, wall and internal energy generation rates, or the pressure gradients. More complex unsteadiness may include several of these effects simultaneously. Johnson and Cheng [2] have done a systematic analysis on the basis of the boundary-layer and Darcy approximations, regarding the possibility of similarity solutions for various wall temperature functions. These authors were the first to show that only very specific solutions exist for unsteady free convection about an inclined flat plate in a porous medium, and to summarise all of the physical realizable similarity solutions. Raptis [3] has studied analytically unsteady two-dimensional free-convective flow through a porous medium bounded by an infinite vertical plate, when the temperature of the plate is oscillating with the time about a constant non-zero mean value. The effects the parameter of frequency on the velocity field were considered. Another investigation on the subject was conducted by Raptis and Perdikis [4] when the free stream velocity and the temperature in the free stream are considered constant. These authors have observed that when the permeability parameter or the Grashof number increases, the velocity increases and when the frequency parameter increases, the velocity decreases. Singh et al. [5] have extended these analyses in solving the problem by asymptotic expansions development in powers of the frequency parameter, and discussed the effects of physical parameters on the velocity and the temperature fields. Cheng and Pop [6] have used the method of integral relations to study the transient free convection about a vertical flat plate embedded in a porous medium and demonstrated the growth of the boundary-layer thickness for the case of a step increase in wall temperature. Ingham and Brown [7] were the first to present detailed solutions for the problem of the free convection from a vertical flat plate embedded in a porous medium when the temperature is suddenly raised. This analysis was extended by Merkin and Zhang [8] who have presented numerical solutions for the boundary-layer flow generated on a vertical surface in a porous medium in the case when the wall is heated by a heat flux rate which is varying as a power function of the distance from the origin.

In all the published studies discussed earlier, the fluid saturating the porous medium was assumed to be Newtonian. However, in several of the engineering applications listed at the beginning of this section (such as oil recovery, food processing, the spreading of contaminants in the environment and in various processes in the chemical and materials industry) the fluid saturating the porous matrix is not necessarily Newtonian. For example, in the literature, the number of existing works in the limit of thermal convection in a porous medium saturated with a non-Newtonian fluid driven by temperature gradients alone is very limited. To this end, Chen and Chen [9] studied numerically the problem of boundary layer free convection about an isothermal vertical plate in a porous medium saturated by a power-law index fluid. Poulikakos and Spatz [10] investigated the effect of non-Newtonian natural convection at a melting front in a permeable matrix. Their results documented the dependence of the local heat transfer rate at the melting front on the type of power-law fluid saturating the porous matrix. Recently, Rastogi and Poulikakos [11] studied the double diffusion from a vertical surface embedded in a porous medium saturated by a non-Newtonian fluid. These authors have found that the variation of the wall temperature and concentration necessary to yield a constant heat and species flux at the wall, depended strongly on the power-law index.

Moreover, in all the above studies, the porous medium is assumed to be isotropic whereas, in the several applications, the porous materials are anisotropic. Despite this fact, natural convection in such anisotropic porous media has received relatively little attention. The effects of an anisotropic permeability on thermal convection in a porous medium began with the investigation of Castinel and Combarnous [12], concerning the onset of motion in a horizontal layer heated from below, and continued with the works of Epherre [13], of Kvernvold and Tyvand [14] and of Kibbin [15]. Natural convection within enclosures heated from the side was investigated by Kimura et al. [16] and Ni and Beckermann [17], for the case when one of the principal axes of anisotropy of permeability is aligned with gravity and by Zhang [18], Degan et al. [19] and Degan and Vasseur [20] when the principal axes are inclined with respect to gravity. It was demonstrated by these authors that the effects of the anisotropy considerably modify the convective heat transfer. Recently, the effects of anisotropy on the boundary-layer free convection over an impermeable vertical plate, were investigated by Ene [21], using the method of integral relations, for the case when the principal axes of anisotropy of permeability are coincident with coordinate axes, and by Vasseur and Degan [22] when the principal axes are arbitrary oriented. It was concluded that, if the permeability in the direction normal to the plate is greater than the permeability along the plate, then there is an increase in the temperature field.

The present paper describes an analytical procedure for obtaining an exact solution for transient natural convection from a vertical plate embedded in an anisotropic porous medium saturated by a non-Newtonian fluid. A step increase in wall temperature or in surface heat flux is considered . The porous medium is anisotropic in permeability with its principal axes oriented in a direction that is oblique to the gravity vector. Combining the modified Darcy power-law model proposed by Pascal [23,24] and the generalized Darcy's law proposed by Bear [25], a characterization of the saturating flow through the porous matrix is used to describe the non-Newtonian fluid behavior. In the large Rayleigh number limit, the boundary layer equations are solved analytically upon introducing an scale analysis to predict the order-of-magnitudes involved in the boundary layer regime.

2. Mathematical formulation

We consider here the problem of unsteady heating of a vertical impermeable plate embedded in a saturated porous medium characterized by an anisotropic permeability. The x and y axes are aligned with the vertical and the horizontal coordinates, respectively. The saturating fluid is a non-Newtonian fluid of power-law behaviour and the porous medium is at a uniform temperature T^{∞}. At $t=0$, the temperature of the plate or the heat flux at the plate, is suddenly increased to the constant value T_w or q_w, respectively. The anisotropy of the porous medium is characterized by the anisotropy ratio $K^* = K_1/K_2$ and the orientation angle θ, defined as the angle between the horizontal direction and the principal axis with the permeability K_2. It is assumed that the fluid and the porous medium are everywhere in local thermodynamic equilibrium. The pressure and the temperature are such that the fluid remains in the liquid phase. The thermophysical properties of the fluid are assumed constant, except for the density in the buoyancy term in the momentum equation (i.e., the Boussinesq approximation).

In accordance with previous reports given by Pascal [23,24] and following Bear [25], the model of laminar flow of a non-Newtonian power-law fluid through the porous medium, describing the generalized Darcy's law, can be written as follows

$$\vec{V} = -\frac{\overline{\overline{K}}}{\mu_a}\nabla p, \tag{1}$$

$$\mu_a = \epsilon(u^2 + v^2)^{(n-1)/2}, \tag{2}$$

$$\epsilon = \frac{2\hbar}{8^{(n+1)/2}(\sqrt{K_1K_2\gamma})^{(n-1)/2}(1+3n)^n} \tag{3}$$

Many of the inelastic non-Newtonian fluids encountered in engineering processes are known to follow a power-law model in which the pressure drop is proportional to the mass flow rate.

In the above equations, $\vec{V}$ is the superficial velocity, γ the porosity of the porous medium, μ_a the apparent viscosity, $\hbar$ the consistency index and n the power-law index. In the above model, the rheological parameters $\hbar$ and n are assumed to be temperature independent.

The equations describing conservation of mass, momentum and energy for the present problem are, respectively

$$\nabla \cdot \vec{V} = 0, \tag{4}$$

$$\vec{V} = -\frac{\overline{\overline{K}}}{\mu_a}(\nabla p + \varrho \vec{g}), \tag{5}$$

$$\sigma \frac{\partial T}{\partial t} + \nabla \cdot \left(\vec{V}T - \alpha \nabla T\right) = 0, \tag{6}$$

$$\varrho = \varrho_\infty[1 - \beta(T - T_\infty)]. \tag{7}$$

In the above equations T is the local equilibrium temperature of the fluid and the porous matrix, $\vec{g}$ the gravitational acceleration, p the pressure, t the time, $\alpha = k/(\varrho c_p)_f$ the thermal diffusivity (k the thermal conductivity of fluid/porous matrix combination, $(\varrho c_p)_f$ the heat capacity of the fluid), $\sigma = (\varrho c_p)_p/(\varrho c_p)_f$ the heat capacity ratio, β the coefficient of thermal expansion of the fluid and ϱ the density. The symmetrical second-order permeability tensor $\overline{\overline{K}}$ is defined as

$$\overline{\overline{K}} = \begin{bmatrix} K_1\cos^2\theta + K_2\sin^2\theta & (K_1 - K_2)\sin\theta\cos\theta \\ (K_1 - K_2)\sin\theta\cos\theta & K_2\cos^2\theta + K_1\sin^2\theta \end{bmatrix} \tag{8}$$

Eliminating the pressure term by taking the curl of Eq. (5) and making use of Eq. (4), we obtain a single momentum, which reads

$$a\frac{\partial u}{\partial y} + c\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) - b\frac{\partial v}{\partial x} = \frac{1}{\mu_a}\left\{\frac{\partial \mu_a}{\partial x}(-cu + bv) + \frac{\partial \mu_a}{\partial y}(-au + cv) + K_1\varrho_\infty \beta g\frac{\partial T}{\partial y}\right\}, \tag{9}$$

where

$$\begin{cases} a = \cos^2\theta + K^*\sin^2\theta \\ b = \sin^2\theta + K^*\cos^2\theta \\ c = \frac{1}{2}(1 - K^*)\sin 2\theta \end{cases} \tag{10}$$

3. Scale analysis

In this section, as t increases, the convection effect increases and we consider the boundary layer regime for which most of the fluid motion is restricted to a thin layer δ along the vertical plate. From the momentum equation (9), it is clear that we may use the boundary-layer hypothesis only when the following conditions:

$$a\frac{\partial u}{\partial y} \gg c\frac{\partial v}{\partial y}, \tag{11}$$

$$a\frac{\partial u}{\partial y} \gg c\frac{\partial u}{\partial x}, \tag{12}$$

$$a\frac{\partial u}{\partial y} \gg b\frac{\partial v}{\partial x}, \tag{13}$$

$$K_1\varrho_\infty\beta g\frac{\partial T}{\partial y} \gg \frac{\partial \mu_a}{\partial x}(-cu + bv), \tag{14}$$

$$K_1\varrho_\infty\beta g\frac{\partial T}{\partial y} \gg \frac{\partial \mu_a}{\partial y}(-au + cv), \tag{15}$$

are satisfied. So, under the boundary-layer approximations, at large modified Darcy–Rayleigh number, the governing equations become:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{16}$$

$$\frac{\partial}{\partial y}(u^n) = \frac{n}{a}\frac{K_1\varrho_\infty g\beta}{\epsilon}\frac{\partial T}{\partial y}, \tag{17}$$

$$\sigma\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{18}$$

which are to be solved subject to the following initial condition:

$$t = 0: \quad T(x, y, 0) = 0. \tag{19}$$

The boundary conditions associated with previous governing equations are:

$$y = 0: \; v = 0, \; T(x, 0, t) = T_w \quad \text{(a)} \qquad \frac{\partial T}{\partial y}(x, 0, t) = -\frac{q_w}{k} \quad \text{(b)} \tag{20}$$

$$y \to \infty: \; u = 0, \; T(x, \infty, t) = T_\infty. \tag{21}$$

Following Bejan [26] and recognizing H and δ as the x and y scales, respectively, in the boundary layer of interest ($\delta \ll H$), the conservation Eqs. (16)–(18) and (20) require the following balances:

$$\frac{u}{H} \sim \frac{v}{\delta}, \tag{22}$$

$$a\frac{u}{\delta} \sim \frac{1}{\epsilon u^{n-1}}K_1\varrho_\infty g\beta\frac{\Delta T}{\delta}, \tag{23}$$

$$\sigma\frac{\Delta T}{t}, \quad u\frac{\Delta T}{H}, \quad v\frac{\Delta T}{\delta} \sim \alpha\frac{\Delta T}{\delta^2}, \tag{24}$$

where $\Delta T = (T_w - T_\infty)$ for constant wall temperature, $\Delta T \sim q_w\delta/k$ for constant wall heat flux, is the characteristic scale of temperature. It is noticed that the temperature drop across the boundary layer is of the order of one. In the next subsections, it will distinguish the result of the scale analysis for two cases, according to the heating process of the wall.

3.1. Isothermal wall

Solving the balances between Eqs. (22)–(24) for δ, u, v and t, we obtain the following results:

$$\delta \sim HRa_H^{-1/(2n)}a^{1/(2n)}, \tag{25}$$

$$u \sim \frac{\alpha}{H}Ra_H^{1/n}a^{-1/n}, \tag{26}$$

$$v \sim \frac{\alpha}{H}Ra_H^{1/(2n)}a^{-1/(2n)}, \tag{27}$$

$$t \sim \frac{\sigma H^2}{\alpha}Ra_H^{-1/n}a^{1/n}, \tag{28}$$

where the modified Darcy–Rayleigh number Ra_H, based on the height of the plate, is defined as

$$Ra_H = \frac{K_1\varrho_\infty g\beta\Delta T H^n}{\epsilon\alpha^n}. \tag{29}$$

Defining the stream function ψ related to the velocity components by

$$u = \frac{\partial\psi}{\partial y} \quad v = -\frac{\partial\psi}{\partial x} \tag{30}$$

such that the continuity equation, Eq. (4), is automatically satisfied, the scale for the stream function can be obtained as follows:

$$\psi \sim \alpha Ra_H^{1/(2n)} a^{-1/(2n)} \tag{31}$$

The local Nusselt number, Nu_H defined as the heat transfer over the pure heat conduction through the vertical plate, has the following scale:

$$Nu_H = \frac{hH}{k} \sim Ra_H^{1/(2n)} a^{-1/(2n)} \tag{32}$$

where $h = q/(T_w - T_\infty)$ is the local heat transfer coefficient, $q = -k(\partial T/\partial y)|_{y=0}$ the local heat flux at the heated surface.

For the special case of an isotropic porous medium ($K^* = 1$, i.e., $a = 1$), the scales above reduce to those predicted by Rastoghi and Poulikakos [11] while studying the double diffusion boundary layer regime over a vertical surface embedded in a porous region saturated by a non-Newtonian fluid.

The conditions of validity of the present boundary layer analysis now will be discussed. These results are expected to be valid only when the vertical boundary-layer is slender ($\delta \ll H$), i.e., for $Ra_H \gg a$. Furthermore, from Eqs. (11)–(15), (25)–(27), and making use of the results of the above order-of-magnitude analysis developed in this section, the boundary layer hypothesis is valid only when the conditions

$$b \ll Ra_H^{1/n} a^{(n-1)/n} \tag{33}$$

and

$$c \ll Ra_H^{1/(2n)} a^{(2n-1)/(2n)} \tag{34}$$

are satisfied.

3.2. Wall with uniform heat flux

Solving the balances of quantities of interest from previous equations, using the same calculus procedure when the plate is heated by a constant heat flux, one can have:

$$\delta \sim H R_H^{-1/(2n+1)} a^{1/(2n+1)}, \tag{35}$$

$$u \sim \frac{\alpha}{H} R_H^{2/(2n+1)} a^{-1/(2n+1)}, \tag{36}$$

$$v \sim \frac{\alpha}{H} R_H^{1/(2n+1)} a^{-1/(2n+1)}, \tag{37}$$

$$\Delta T \sim \frac{q_w H}{k} R_H^{-1/(2n+1)} a^{1/(2n+1)}, \tag{38}$$

$$t \sim \frac{\sigma H^2}{\alpha} R_H^{-2/(2n+1)} a^{2/(2n+1)}, \tag{39}$$

$$\psi \sim \alpha R_H^{1/(2n+1)} a^{-1/(2n+1)}, \tag{40}$$

$$Nu_H \sim R_H^{1/(2n+1)} a^{-1/(2n+1)}, \tag{41}$$

where the modified Darcy–Rayleigh number R_H, based on heat flux, is defined as

$$R_H = \frac{K_1 \varrho_\infty g \beta H^{n+1} q_w}{\epsilon \alpha^n k}. \tag{42}$$

Taking into account the previous scales obtained in this case, the condition of existence of the vertical boundary-layer hypothesis formulated by ($\delta \ll H$) becomes $R_H \gg a$. Moreover, making use of Eqs. (11)–(14), (15), (36)–(38), the boundary-layer hypothesis is valid only when the conditions:

$$b \ll R_H^{2/(2n+1)} a^{(2n-1)/(2n+1)} \tag{43}$$

and

$$c \ll R_H^{1/(2n+1)} a^{(2n-1)/(2n+1)} \tag{44}$$

are satisfied.

4. Resolution

On the one hand, taking H, $H/Ra_H^{1/(2n)}$, $\alpha Ra_H^{1/n}/H$, $\alpha Ra_H^{1/(2n)}/H$, ΔT and $\sigma H^2/(\alpha Ra_H^{1/n})$ as respective dimensional scales for length, velocities in x and y directions, temperature and time, concerning the case of isothermal wall and on the other hand, setting H, $H/R_H^{1/(2n+1)}$, $\alpha R_H^{2/(2n+1)}/H$, $\alpha R_H^{1/(2n+1)}/H$, ΔT and $\sigma H^2/(\alpha R_H^{2/(2n+1)})$ as respective dimensional scales for length, velocities in x and y directions, temperature and time, for the case of wall with uniform heat flux, it is found that the dimensionless boundary-layer equations are

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0, \tag{45}$$

$$\frac{\partial}{\partial Y}(U^n) = \frac{n}{a}\frac{\partial \Theta}{\partial Y}, \tag{46}$$

$$\frac{\partial \Theta}{\partial \tau} + U\frac{\partial \Theta}{\partial X} + V\frac{\partial \Theta}{\partial Y} = \frac{\partial^2 \Theta}{\partial Y^2}. \tag{47}$$

Integrating Eq. (46) from $Y = 0$ to $Y = \infty$ (region situated in the free stream), one can have

$$U = \left(\frac{n\Theta}{a}\right)^{1/n} \tag{48}$$

Making use of the continuity Eq. (45), the integration of equation of energy (47) yields:

$$\frac{\partial}{\partial \tau}\int_0^\infty \Theta \, dY + \frac{\partial}{\partial X}\int_0^\infty U\Theta \, dY = -\left(\frac{\partial \Theta}{\partial Y}\right)_{Y=0} \tag{49}$$

Substituting Eq. (48) into Eq. (49) and rearranging the resulting expression, we obtain finally:

$$\frac{\partial}{\partial \tau}\int_0^\infty \Theta \mathrm{d}Y + \left(\frac{n}{a}\right)^{1/n}\frac{\partial}{\partial X}\int_0^\infty \Theta^{(n+1)/n}\mathrm{d}Y = -\left(\frac{\partial \Theta}{\partial Y}\right)_{Y=0} \tag{50}$$

The problem of transient natural convection in a porous medium about a vertical, semi-infinite flat plate with a step increase in wall temperature or surface heat flux, considered here, gives rise, as the classical problem of a viscous boundary layer in a free fluid, to the singularity problem in passing from the initial stage when the leading edge is not felt to the steady state defined for large time. For small values of time, the solutions for velocity and temperature are independent of X; for large values of time the solutions are independent of time. The singularity value of time depends on the vertical distance X. As pointed out by Ene and Polisevski [27], the heat transfer characteristics change suddenly from transient, one-dimensional heat conduction to steady two-dimensional natural convection. So, Eq. (50) is to be solved subject to the initial condition (19) which becomes

$$\tau = 0: \quad \Theta(X, Y, 0) = 0 \tag{51}$$

and the dimensionless boundary conditions prevailing at the vertical surface are:

$$\begin{cases} Y = 0: \Theta(X, 0, \tau) = 1 \quad \text{(a)}, \\ \frac{\partial \Theta}{\partial Y}(X, 0, \tau) = -1 \quad \text{(b)}, \end{cases} \tag{52}$$

In the following subsections, Eq. (50) will be differentially solved by considering the two kinds of boundary conditions (52) imposed in the present analysis.

4.1. Isothermal wall

Following Cheng and Pop [6], with the boundary condition (52a), we assume a temperature profile of the form

$$\Theta = \mathrm{erfc}(\eta), \tag{53}$$

where erfc is the complementary error function and η is expressed by

$$\eta = \frac{Y}{A}Ra_H^{-1/(2n)}, \tag{54}$$

where $A = \delta/H$ is the dimensionless boundary-layer thickness. Substituting Eqs. (53) and (54) into Eq. (50) and after integrating yields

$$\frac{\partial A}{\partial \tau} + \zeta\left(\frac{n}{a}\right)^{1/n}\frac{\partial A}{\partial X} = 2\frac{Ra_H^{-1/n}}{A} \tag{55}$$

subject to the initial and boundary conditions

$$\begin{cases} \tau = 0: \quad A(X, 0) = 0 \quad \text{(a)}, \\ \tau \geqslant 0: \quad A(X, \tau) = 0 \quad \text{at } X = 0 \quad \text{(b)}, \end{cases} \tag{56}$$

where the constant ζ depending on the power-law index of the non-Newtonian fluids is expressed as follows

$$\zeta = \sqrt{\pi}\int_0^\infty [\mathrm{erfc}(x)]^{(n+1)/n}\mathrm{d}x. \tag{57}$$

Eq. (55) is a first-order, linear, partial differential equation of the hyperbolic type and it has been solved exactly by the method of characteristics and approximately by the Karman–Polhausen integral-method. Solving by the method of characteristics, the differential system equivalent to Eq. (55) is expressed as

$$\frac{\mathrm{d}X}{\zeta\left(\frac{n}{a}\right)^{1/n}} = \mathrm{d}\tau = Ra_H^{1/n}\frac{A\mathrm{d}A}{2}, \tag{58}$$

which has the characteristics

$$\mathrm{d}X = \zeta\left(\frac{n}{a}\right)^{1/n}\mathrm{d}\tau. \tag{59}$$

On each characteristic, A is related by:

$$Ra_H^{1/n}A\mathrm{d}A = 2\mathrm{d}\tau \tag{60}$$

or

$$Ra_H^{1/n}\zeta\left(\frac{n}{a}\right)^{1/n}A\mathrm{d}A = 2\mathrm{d}X, \tag{61}$$

depending on whether the characteristic intercepts the τ- or X-axis. Integrating Eq. (60) with the initial condition (56(a)) gives

$$A = 2\sqrt{\frac{\tau}{Ra_H^{1/n}}} \tag{62}$$

and solving Eq. (61) subject to the boundary condition (56(b)) yields

$$A = \frac{2}{\sqrt{\zeta}}\left(\frac{a}{nRa_H}\right)^{1/(2n)}\sqrt{X}. \tag{63}$$

As shown by Cheng and Pop [6] and revisited by Ene and Polisevki [27] in their analyses when studying the same problem in the case of Newtonian fluid saturating an isotropic porous medium, the expression for A changes from Eqs. (62),(63) along the limiting line characteristic

$$\tau = \tau_s = \left(\frac{a}{n}\right)^{1/n}\frac{X}{\zeta}, \tag{64}$$

so that, Eq. (64) is a straight line which divides the $X - \tau$ plane into two regions: a lower region for which A is given by Eq. (62) and an upper region for which A is given by Eq. (63). That limiting line characteristic provides the limit time reached in steady-state regime τ_s expressed by Eq. (64).

Under these considerations, we have two expressions of temperature profile corresponding to the two regions. For $\tau < \tau_s$ (in the lower region), one can find

$$\Theta = \mathrm{erfc}\left(\frac{Y}{2\sqrt{\tau}}\right) \tag{65}$$

and for $\tau > \tau_s$ (in the upper region), the temperature distribution is expressed as

$$\Theta = \mathrm{erfc}\left\{\frac{Y}{2}\sqrt{\frac{\zeta}{X}\left(\frac{n}{a}\right)^{1/n}}\right\} \tag{66}$$

As expected, Eq. (65) is independent of X, and the solution represents the transient heat conduction in a semi-infinite porous medium while Eq. (66), however, is independent of τ, and its solution represents steady-state natural convection.

The local Nusselt number for the region where $\tau > \tau_s$, Nu_x through the vertical surface is defined, in physical variables, by

$$Nu_x = \frac{qx}{k(T_w - T_\infty)} \tag{67}$$

Making use of Eqs. (53), (54), (62) and (65), one can rewrite q as

$$q = \frac{2k(T_w - T_\infty)}{AH\sqrt{\pi}}, \tag{68}$$

which becomes, after substitution of Eq. (62) valid in the lower region where transient heat conduction is predominant, what follows

$$Q = \frac{1}{\sqrt{\pi\tau}}, \tag{69}$$

where $Q = q/[k\Delta T Ra_H^{1/(2n)}/H]$ is the dimensionless local surface heat flux.

Taking into account Eqs. (53), (54), (63) and (66) and Eq. (68) becomes

$$q = \frac{k(T_w - T_\infty)}{x}\sqrt{\frac{\zeta}{\pi}\left(\frac{nRa_x}{a}\right)^{1/n}}, \tag{70}$$

valid in the upper zone where convection heat transfer is predominant (i.e., for $\tau > \tau_s$). So, substituting Eq. (70) into Eq. (67), we can express the local Nusselt number Nu_x as

$$Nu_x = \sqrt{\frac{\zeta n^{1/n}}{\pi}}Ra_x^{1/(2n)}a^{-1/(2n)}, \tag{71}$$

where $Ra_x = K_1 \varrho_\infty g\beta(T_w - T_\infty)x^n/(\epsilon\alpha^n)$ is the modified Darcy–Rayleigh number, based on the distance x from the edge of the surface.

On the other hand, the convective flow is described by the stream function which can be expressed from Eqs. (30), (48), (53), (54) and (63) as

$$\Psi = \frac{2}{\sqrt{\zeta}}n^{1/(2n)}I_{1/n}Ra_x^{1/(2n)}a^{-1/(2n)}, \tag{72}$$

where $\Psi = \psi/\alpha$ and

$$I_{1/n} = \int_0^\infty [\text{erfc}\,(x)]^{1/n}\text{d}x. \tag{73}$$

It is obvious that by setting $n = 1$ in the limit of a Newtonian fluid saturating the porous matrix assumed isotropic in permeability for which the value of the anisotropic parameter a equals to unity, the results presented in the present analysis are found to be in a good agreement with those obtained by Cheng and Pop [6] and by Ene and Polisevski [27].

4.2. Wall with constant heat flux

The problem given by Eq. (50) with the conditions (51) and (52(b)) may be treated for the case of constant plate heat flux in similar fashion. Seeking the analogous form as previously of the solution which satisfies conditions evoked in this case, one can find after algebraic manipulation what follows

$$\Theta = \left\{\frac{A\sqrt{\pi}}{2}R_H^{1/(2n+1)}\right\}\text{erfc}(\eta), \tag{74}$$

where

$$\eta = \frac{Y}{A}R_H^{-1/(2n+1)} \tag{75}$$

Substituting Eqs. (74) and (75) into Eq. (50), the equation for the boundary-layer thickness is obtained by the relation:

$$\frac{\partial A}{\partial \tau} + B\left\{\frac{AR_H^{1/(2n+1)}}{a}\right\}^{1/n}\frac{\partial A}{\partial X} = \left[A(R_H)^{2/(2n+1)}\right]^{-1}, \tag{76}$$

where B is expressed by

$$B = \frac{\zeta(2n+1)}{2n}\left(\frac{n\sqrt{\pi}}{2}\right)^{1/n}. \tag{77}$$

The resolution of Eq. (76) subject to conditions (56(a)) and (56(b)) by the method of characteristics yields the following line expressed by

$$\frac{\text{d}X}{B\left[AR_H^{1/(2n+1)}/a\right]^{1/n}} = \text{d}\tau = R_H^{2/(2n+1)}A\text{d}A. \tag{78}$$

In this case, on each characteristic, A is related by:

$$(R_H)^{2/(2n+1)}A\text{d}A = \text{d}\tau \tag{79}$$

and

$$(R_H)^{1/n}Ba^{-1/n}A^{(n+1)/n}\text{d}A = \text{d}X. \tag{80}$$

Integrating Eqs. (79) and (80) subject to conditions (56(a)) and (56(b)), respectively, A is evaluated by

$$A = \sqrt{2\tau}R_H^{-1/(2n+1)}, \tag{81}$$

and

$$A = \left\{\frac{(2n+1)a^{1/n}X}{nB(R_H)^{1/n}}\right\}^{n/(2n+1)}, \tag{82}$$

such that, the expression of A changes from Eqs. (81) to (82) along the limiting line characteristic

$$\tau = \tau_s = \frac{1}{2}\left\{\frac{(2n+1)a^{1/n}X}{nB}\right\}^{2n/(2n+1)} \tag{83}$$

As expected, the $X-\tau$ plane is divided into two regions, a lower one for which A is calculated by Eq. (81) and an upper one for which A is evaluated by Eq. (82). Then, the limit time τ_s corresponding to the steady-state regime is predicted here by Eq. (83).

In the other terms, one can derive the temperature profile valid for each region considered as follows

in a lower region (for which $\tau < \tau_s$):

$$\Theta = \sqrt{\frac{\pi\tau}{2}}\mathrm{erfc}\left(\frac{Y}{\sqrt{2\tau}}\right) \tag{84}$$

and in an upper region (for which $\tau > \tau_s$):

$$\Theta = \frac{\sqrt{\pi}}{2}(CX)^{n/(2n+1)}\mathrm{erfc}\left\{Y(CX)^{-n/(2n+1)}\right\}, \tag{85}$$

where

$$C = \frac{(2n+1)a^{1/n}}{nB}. \tag{86}$$

As in the previous case, we have a solution for transient heat conduction in a semi-infinite plane with a step increase in surface heat which is in good agreement with the temperature obtained from Eqs. (84) and (74). The solution, Eq. (85), represents steady heat convection on the plate, and it is also in good agreement with the exact solution (see Cheng and Minkowycz [28]).

The heat flux over the vertical surface in the lower region for which the transient heat conduction prevails is then, after substitution of Eq. (81) into Eq. (68):

$$Q = \sqrt{\frac{2}{\pi\tau}}, \tag{87}$$

where $Q = q/[k\Delta T Ra_H^{1/(2n+1)}/H]$ is the dimensionless local surface heat flux.

Considering the situation dominated by convection pattern, when $\tau > \tau_s$, and taking into account Eqs. (82) and (68), the local Nusselt number, Nu_x, defined by Eq. (67) is calculated here by the expression

$$Nu_x = \frac{2}{\sqrt{\pi}}\left[\zeta\left(\frac{n\sqrt{\pi}}{2^{n+1}}\right)^{1/n}\right]^{n/(2n+1)} R_x^{1/(2n+1)}a^{-1/(2n+1)} \tag{88}$$

In the same way, from Eqs. (30), (48), (74), (75) and (82) the stream function is calculated by the expression:

$$\Psi = \left[\left(\frac{2}{\zeta}\right)^{n+1}\left(\frac{n\sqrt{\pi}}{2}\right)\right]^{1/(2n+1)} I_{1/n}Ra_x^{1/(2n+1)}a^{-1/(2n+1)}, \tag{89}$$

where $R_x = K_1\varrho_\infty g\beta q x^{n+1}/(\epsilon\alpha^n k)$ is the modified Darcy–Rayleigh number, based on the distance x from the edge of the surface, for the case of a wall heated by a constant heat flux.

5. Results and discussion

In order to carry out subsequent analysis of the effects of anisotropic parameters and to investigate the influence of power-law indexes of non-Newtonian fluids saturating the porous matrix on thermal convective flow in the neighborhood of the vertical surface with a sudden increase in the temperature, it is convenient to evaluate the constants ζ and $I_{1/n}$ expressed by Eqs. (57) and (73), respectively. These equations are solved numerically by Romberg procedure coupled with extrapolation Richardson method to have the best approximation of the exact solution. The results found here are presented in Table 1.

Moreover, Table 2 is presented here to prove the validity of the present analysis, when compared the result obtained for the parameter $Nu_x/(Ra_x)^{1/2}$ according Eq. (71), for the case of a Newtonian fluid (i.e., $n = 1$) and for isothermal wall, with that which has been found by Vasseur and Degan [22], solving the problem by a consistent numerical procedure coupled with similarity method in an anisotropic porous medium case, and by Cheng and Minkowycz [28], using the similarity transformation for isotropic porous medium situation. It is seen that the result found here by the method of characteristics is in excellent agreement with that obtained by Cheng and Pop [6] when the porous medium is hydrodynamically isotropic and saturated by a Newtonian fluid.

Fig. 1 shows the effect of the power-law index of non-Newtonian fluid on the limiting line characteristic given par Eq. (64) and expressed by the time taken to reach steady-state flow τ_s, when the transient free convection in the porous medium occurs as a result of a step increase in wall temperature. In Fig. 1, it is observed that, when the anisotropic parameters are held constant, for example, for $\theta = 45°$ and $K^* = 0.1$, each limiting line characteristic corresponding to a fixed power-law index n is a straight line which divides, as expected, the $X-\tau$ plane into two regions, a lower one dominated by a pure conduction regime for $\tau < \tau_s$ and the upper one dominated by convective heat transfer for $\tau \geqslant \tau_s$. Moreover, it is seen that, as the power-law index increases from $n = 0.6$ (corresponding to a shear-thinning fluid) to $n = 1.4$ attributed to a shear-thickening fluid, the upper region becomes progressively larger. This behavior can be explained by the fact that in

Table 1
Evaluation of constants

n	ζ	$I_{1/n}$	n	ζ	$I_{1/n}$
0.5	0.41974	0.33049	1.	0.58578	0.56414
0.6	0.46311	0.38206	1.2	0.62799	0.64439
0.8	0.53254	0.47727	1.4	0.66232	0.71913

Table 2
Values of $Nu_x/Ra_x^{1/2}$ obtained by the different methods for $n = 1$ and isothermal wall situation

$a = K^*$ ($\theta = 90°$)	$Nu_x/Ra_x^{1/2}$	
	Similarity method [22]	Present study, Eq. (71)
0.1(<1.0)	1.404	1.3653
1.0(isotropic)	0.444[28]	0.4317[6]
10.(>1.0)	0.140	0.1365

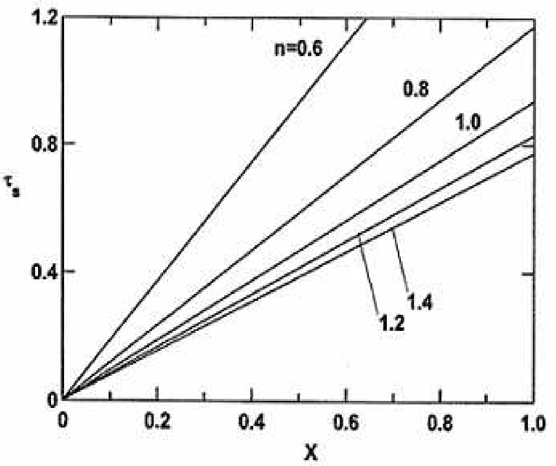

Fig. 1. Effect of the power-law index n on the limiting line characteristic for $\theta = 45°$, $K^* = 0.1$ and for a step increase in wall temperature situation.

the upper region where convection effect is considerable over the limiting time, the local heat transfer rate calculated by Eq. (71) becomes $Nu_x/Ra_x^{0.83} = 0.4127$ when $n = 0.6$ and $Nu_x/Ra_x^{0.35} = 0.6408$ when $n = 1.4$. So, convection motion is enhanced, taking place in the upper region which becomes more and more important than the lower one, as the power-law n index is made higher.

The limiting steady-state time characteristic τ_s is presented in Fig. 2 as a function of the position X from the edge of the vertical surface to investigate the effect of the anisotropic permeability ratio K^* for $n = 1.2$ and $\theta = 30°$, when the wall is heated isothermally. It is seen that, for a given value of the distance X considered from the edge of the wall, the time taken by the heating process transfer to reach steady-state pattern for which convection occurs increases with an increase in permeability ratio K^*. So, convection becomes more and more considerable and occupies upper regions which become more and more large than the lower regions, as K^* is made smaller. This trend comes from the fact that, according to Eq. (64), when the parameters n, θ and X are held constant, τ_s depends solely on $(K^{*0.83})$ and is proportional to this latter. On the other hand, the same behavior is observed in Fig. 3 illustrating the variation of τ_s versus X, for different values of θ, $n = 0.8$ and $K^* = 10$, and when a step of increase in wall temperature is considered. Therefore, the limiting time to reach steady-state regime increases with an increase in orientation angle of the principal axes of the porous matrix, when other parameters are held constant.

Fig. 4 illustrates the effects of varying the modified Darcy–Rayleigh number Ra_H and the time τ (lower than the limiting time required to reach steady-state τ_s) on the boundary-layer thickness A for $n = 0.5$, when the surface is heated isothermally. As expected, the boundary-layer thickness A decreases drastically as Ra_H is made higher, giving rise to a channeling of convective heat flow near the vertical surface. Moreover, it is seen that, for a given

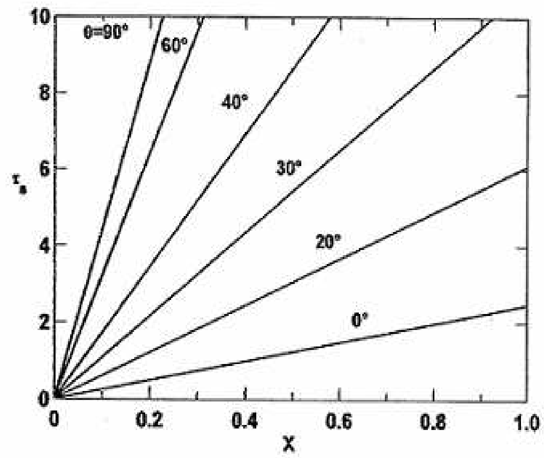

Fig. 3. Effect of the orientation angle θ of the principal axes on the limiting line characteristic for $n = 0.8$, $K^* = 10$ and for a step increase in wall temperature case.

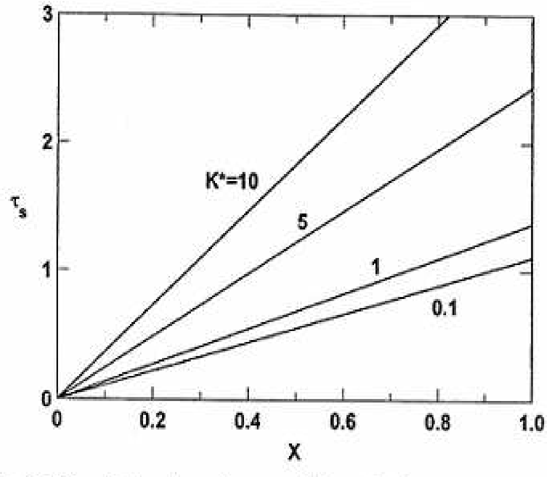

Fig. 2. Effect of the anisotropic permeability ratio K^* on the limiting line characteristic for $n = 1.2$, $\theta = 30°$ and for isothermal wall case.

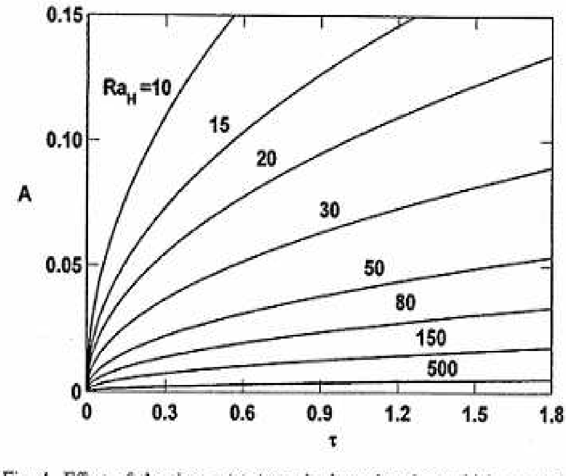

Fig. 4. Effect of the time τ ($<\tau_s$) on the boundary-layer thickness A for $n = 0.5$ and various values of Ra_H, when the surface is isothermally heated.

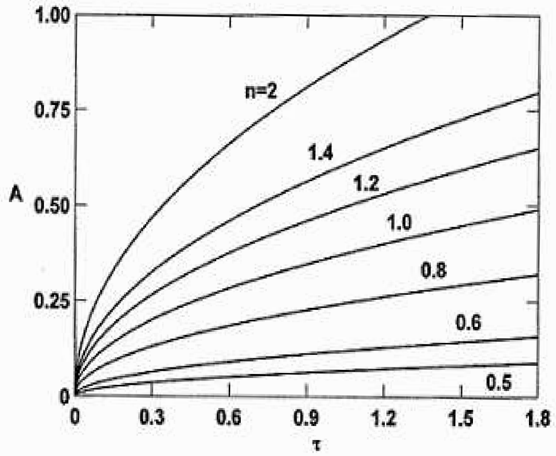

Fig. 5. Effect of the time τ ($<\tau_s$) on the boundary-layer thickness A for $Ra_H = 30$ and various values of n, when the surface is isothermally heated.

value of the time τ (lower than τ_s), the boundary-layer thickness A decreases with an in increase in the modified Darcy–Rayleigh Ra_H. This trend follows from the fact that, according to Eq. (62), the boundary-layer thickness A is proportional to $\tau^{1/2}$ and inversely proportional to $Ra_H^{1/(2n)}$ such that, upon increasing Ra_H, A drops progressively and becomes less and less affected by τ.

Another view of the effects of the varying of the power-law index n and the time τ ($<\tau_s$) for $Ra_H = 30$ and for a step increase in wall temperature, is depicted in Fig. 5. It is observed that the boundary-layer thickness A increases with an increase in power-law index n of non-Newtonian fluids. This can be explained that for a fixed value of τ, according to Eq. (62), when $n \to 0$, $Ra_H^{-1/(2n)} \to 0$, and therefore $A \to 0$. So, in Fig. 5, the boundary-layer thickness A drops progressively as n is made weaker, independently of τ.

6. Conclusion

The problem of transient natural convection in a porous medium adjacent to a vertical, semi-infinite surface with a step increase in wall temperature or surface heat flux. The porous medium is anisotropic in permeability whose principal axes are non-coincident with the gravity vector. The porous matrix is saturated by non-Newtonian fluids. With the formulation of the problem on the basis of the modified Darcy power-law model of Pascal [23,24] and the generalized Darcy's law proposed by Bear [25], boundary-layer equations are solved analytically by the method of characteristics, as time is taken into account in equation of energy. From the results, the following remarks are

(1) The problem considered gives rise to the singularity problem in passing from the initial stage when the leading edge effect is not felt to the steady state defined for large times. The limiting line characteristic represents the time required to reach the steady state τ_s and the singularity value of time for which the heat transfer characteristics change suddenly from transient one-dimensional heat conduction to steady two-dimensional natural convection.

(2) For small values of time ($\tau < \tau_s$), the solutions for flow and temperature fields are dependent solely of time and the heat transfer due by pure conduction in transient regime are independent on the anisotropic parameters of the porous matrix and the power-law indexes of non-Newtonian fluids.

(3) For large values of time ($\tau > \tau_s$), the solutions for velocity, temperature and heat transfer rate valid in steady regime are independent of time, and depend strongly on anisotropic parameters, on the power-law indexes, and on the vertical distance from the edge of the surface.

(4) At steady regime from the limiting line characteristic, the convection heat transfer is enhanced when the power-law index n of non-Newtonian fluids is increased.

(5) The limiting time to reach steady-state increases with an increase in anisotropic permeability ratio and with an increase in orientation angle of the principal axes of the porous medium, and that, independently of the heating process of the vertical surface.

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