

## Research Article

# A Posteriori Error Analysis for a Lagrange Multiplier Method for a Stokes/Biot Fluid-Poroelastic Structure Interaction Model

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In this work, we develop an a posteriori error analysis of a conforming mixed finite element method for solving the coupled problem arising in the interaction between a free fluid and a fluid in a poroelastic medium on isotropic meshes in  $\mathbb{R}^d$ ,  $d \in \{2, 3\}$ . The approach utilizes a Lagrange multiplier method to impose weakly the interface conditions [Ilona Ambartsumyan et al., *Numerische Mathematik*, 140 (2): 513-553, 2018]. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. It is proven that the a posteriori error estimate provided in this paper is both reliable and efficient. The proof of reliability makes use of suitable auxiliary problems, diverse continuous inf-sup conditions satisfied by the bilinear forms involved, Helmholtz decomposition, and local approximation properties of the Clément interpolant. On the other hand, inverse inequalities and the localization technique based on simplexe-bubble and face-bubble functions are the main tools for proving the efficiency of the estimator. Up to minor modifications, our analysis can be extended to other finite element subspaces yielding a stable Galerkin scheme.

## 1. Introduction

In this paper, we develop an a posteriori error analysis for solving the interaction of a free incompressible viscous Newtonian fluid with a fluid within a poroelastic medium. This is a challenging multiphysics problem with applications to predicting and controlling processes arising in groundwater flow in fractured aquifers, oil and gas extraction, arterial flows, and industrial filters. In these applications, it is important to model properly the interaction between the free fluid with the fluid within the porous medium and to take into account the effect of the deformation of the medium. For example, geomechanical effects play an important role in hydraulic fracturing, as well as in modeling phenomena such as subsidence and compaction.

We adopt the Stokes equations to model the free fluid and the Biot system [1] for the fluid in the poroelastic media. In the latter, the volumetric deformation of the elastic porous matrix is complemented with the Darcy equation that describes the average velocity of the fluid in the pores. The model features two different kinds of coupling across the

interface: Stokes-Darcy coupling [2–10] and fluid-structure interaction (FSI) [11–15].

The well-posedness of the mathematical model based on the Stokes-Biot system for the coupling between a fluid and a poroelastic structure is studied in [16]. A numerical study of the problem, using a Navier-Stokes equations for the fluid, is presented in [11, 17], utilizing a variational multiscale approach to stabilize the finite element spaces. The problem is solved using both a monolithic and a partitioned approach, with the latter requiring subiterations between the two problems.

Finite element analysis of an arbitrary Lagrangian-Eulerian method for Stokes/parabolic moving interface problem with jump coefficients has been studied in [18]. The authors in [19] study a numerical solution of the coupled system of the time-dependent Stokes and fully dynamic Biot equations. They establish stability of the scheme and derive error estimates for the fully discrete coupled scheme. Numerical errors and convergence rates for smooth problems as well as tests on realistic material parameters have been presented. In [20], Wen and He consider a strongly conservative

discretization for the rearranged Stokes-Biot model based on interior penalty discontinuous Galerkin method and mixed finite element method. The existence and uniqueness of solution of the numerical scheme have been presented. Then, the analysis of stability and priori error estimates have been derived. The numerical examples under uniform meshes, which well validate the analysis of convergence and the strong mass conservation, are presented. A staggered finite element procedure for the coupled Stokes-Biot system with fluid entry resistance has been studied by Bergkamp et al. in [21] while Ambartsumyan et al. study in [22] flow and transport in fractured poroelastic media using Stokes flow in the fractures and the Biot model in the porous media. In [23], semidiscrete continuous-in-time approximation has been proposed for the weak coupled mixed formulation. For the discretization of the fluid velocity and pressure, the authors have used the finite elements which include the MINI elements, the Taylor-Hood elements, and the conforming Crouzeix-Raviart elements. For the discretization of the porous medium problem, they choose the spaces that include Raviart-Thomas and Brezzi-Douglas-Marini elements. An a priori error analysis is performed with some numerical tests confirming the convergence rates.

A posteriori error estimators are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. They are essential to design adaptive mesh refinement algorithms which a priori distribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuška and Rheinboldt [24–27], adaptive finite element methods based on a posteriori error estimates have been extensively investigated. To our best knowledge, there is no a posteriori error estimation for the Stokes/Biot fluid-poroelastic structure interaction model for finite element methods. Here, we develop such a posteriori error analysis for the semidiscrete conforming finite element methods. We have got a new family of a local indicator error  $\Theta_K$  (see Definition 5, Equation (48)) and global  $\Theta$  (Equation (52)).

The schedule of the paper is as follows. Section 2 is devoted to notations and basic results that are used throughout the document. Our main results regarding a posteriori error analysis are stated in Section 3. We prove that our indicator errors are efficiency and reliability, and then, are optimal. The global inf-sup condition is the main tool yielding the reliability. In turn, the local efficiency result is derived using the technique of bubble function introduced by Verfürth [28] and used in similar context by Carstensen and Dolzmann [29]. Finally, this paper is summarized with further works in Section 4.

## 2. Preliminaries and Notations

**2.1. Stokes-Biot Model Problem.** We consider a multiphysics model problem for free fluid's interaction with a flow in a deformable porous media, where the simulation domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , is a union of nonoverlapping regions  $\Omega_f$  and  $\Omega_p$ . Here,  $\Omega_f$  is a free fluid region with flow governed by the Stokes equations, and  $\Omega_p$  is a poroelastic material governed by the Biot system. For simplicity of notation, we

assume that each region is connected. The extension to non-connected regions is straightforward. Let  $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$  (see Figure 1).

Let  $(u_*, p_*)$  be the velocity-pressure pair in  $\Omega_*$ ,  $*$  =  $f, p$ , and let  $\eta_p$  be the displacement in  $\Omega_p$ . Let  $\mu > 0$  be the fluid viscosity, let  $f_*$  be the body force terms, and let  $q_*$  be external source or sink terms. Let  $D(u_f)$  and  $\sigma_f(u_f, p_f)$  denote, respectively, the deformation rate tensor and the stress tensor:

$$D(u_f) = \frac{1}{2} (\nabla u_f + \nabla u_f^T), \quad (1)$$

$$\sigma_f(u_f, p_f) = -p_f I + 2\mu D(u_f).$$

In the free fluid region  $\Omega_f$ ,  $(u_f, p_f)$  satisfy the Stokes equations:

$$-\nabla \cdot \sigma_f(u_f, p_f) = f_f \text{ in } \Omega_f \times (0, T], \quad (2)$$

$$\nabla \cdot u_f = q_f \text{ in } \Omega_f \times (0, T], \quad (3)$$

where  $T > 0$  is the final time. Let  $\sigma_e(\eta_p)$  and  $\sigma_p(\eta_p, p_p)$  be the elastic and poroelastic stress tensors, respectively:

$$\sigma_e(\eta_p) = \lambda_p (\nabla \cdot \eta_p) I + 2\mu_p D(\eta_p), \quad (4)$$

$$\sigma_p(\eta_p, p_p) = \sigma_e(\eta_p) - \alpha p_p I,$$

where  $0 < \lambda_{\min} \leq \lambda_p(x) \leq \lambda_{\max}$  and  $0 < \mu_{\min} \leq \mu_p(x) \leq \mu_{\max}$  are the Lamé parameters and  $0 < \alpha \leq 1$  is the Biot-Willis constant. The poroelasticity region  $\Omega_p$  is governed by the quasistatic Biot system [23]:

$$-\nabla \cdot \sigma_p(\eta_p, p_p) = f_p, \mu K^{-1} u_p + \nabla p_p = 0 \text{ in } \Omega_p \times (0, T], \quad (5)$$

$$\frac{\partial}{\partial t} (s_0 p_p + \alpha \nabla \cdot \eta_p) + \nabla \cdot u_p = q_p \text{ in } \Omega_p \times (0, T], \quad (6)$$

where  $s_0 \geq 0$  is a storage coefficient and  $K$  the symmetric and uniformly positive definite rock permeability tensor, satisfying, for some constants  $0 < k_{\min} \leq k_{\max}$ ,

$$\forall \xi \in \mathbb{R}^d, k_{\min} \xi^T \xi \leq \xi^T K(x) \xi \leq k_{\max} \xi^T \xi, \forall x \in \Omega_p. \quad (7)$$

Following [1], the interface conditions on the fluid-poroelasticity interface  $\Gamma_{fp}$  are mass conservation, balance of stresses, and the Beavers-Joseph-Saffman (BJS) condition [30] modeling slip with friction:

$$u_f \cdot n_f + \left( \frac{\partial \eta_p}{\partial t} + u_p \right) \cdot n_p = 0 \text{ on } \Gamma_{fp} \times (0, T], \quad (8)$$

$$-(\sigma_f n_f) \cdot n_f = p_p, \sigma_f n_f + \sigma_p n_p = 0 \text{ on } \Gamma_{fp} \times (0, T], \quad (9)$$

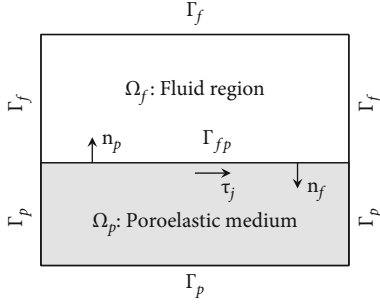


FIGURE 1: Global domain  $\Omega$  consisting of the fluid region  $\Omega_f$  and the poroelastic media region  $\Omega_p$ , separated by the interface  $\Gamma_{fp}$ .

$$-(\sigma_f n_f) \cdot \tau_{f,j} = \mu \alpha_{\text{BJS}} \sqrt{K_j^{-1}} \left( u_f - \frac{\partial \eta_p}{\partial t} \right) \cdot \tau_{f,j} \text{ on } \Gamma_{fp} \times (0, T], \quad (10)$$

where  $n_f$  and  $n_p$  are the outward unit normal vectors to  $\partial\Omega_f$  and  $\partial\Omega_p$ , respectively,  $\tau_{f,j}$ ,  $1 \leq j \leq d-1$ , is an orthogonal system of unit tangent vectors on  $\Gamma_{fp}$ ,  $K_j = (K \tau_{f,j}) \cdot \tau_{f,j}$ , and  $\alpha_{\text{BJS}} \geq 0$  is an experimentally determined friction coefficient. We note that continuity of flux constrains the normal velocity of the solid skeleton, while the BJS condition accounts for its tangential velocity.

The above system of equations needs to be complemented by a set of boundary and initial conditions. Let  $\Gamma_f = \partial\Omega_f \cap \partial\Omega$  and  $\Gamma_p = \partial\Omega_p \cap \partial\Omega$ . Let  $\Gamma_p = \Gamma_p^D \cup \Gamma_p^N$ . We assume for simplicity homogeneous boundary conditions:

$$\begin{aligned} u_f &= 0 \text{ on } \Gamma_f \times (0, T], \\ \eta_p &= 0 \text{ on } \Gamma_p \times (0, T], \\ p_p &= 0 \text{ on } \Gamma_p^D \times (0, T], \\ u_p \cdot n_p &= 0 \text{ on } \Gamma_p^N \times (0, T]. \end{aligned} \quad (11)$$

To avoid the issue with restricting the mean value of the pressure, we assume that  $|\Gamma_p^D| > 0$ . We also assume that  $\Gamma_p^D$  is not adjacent to the interface  $\Gamma_{fp}$ , i.e.,  $\text{dist}(\Gamma_p^D, \Gamma_{fp}) \geq s > 0$ . Nonhomogeneous displacement and velocity conditions can be handled in a standard way by adding suitable extensions of the boundary data. The pressure boundary condition is natural in the mixed Darcy formulation, so nonhomogeneous pressure data would lead to an additional boundary term. We further set the initial conditions:

$$\begin{aligned} p_p(x, 0) &= p_{p,0}(x), \\ \eta_p(x, 0) &= \eta_{p,0}(x) \text{ in } \Omega_p. \end{aligned} \quad (12)$$

Equations (2)–(10) consist of the model of the coupled Stokes and Biot flows problem that we will study below.

**2.2. Weak Formulation.** In this part, we first introduce some Sobolev spaces [31] and norms. If  $W$  is a bounded domain of  $\mathbb{R}^d$  and  $m$  is a nonnegative integer, the Sobolev

space  $H^m(W) = W^{m,2}(W)$  is defined in the usual way with the usual norm  $\|\cdot\|_{m,W}$  and seminorm  $|\cdot|_{m,W}$ . In particular,  $H^0(W) = L^2(W)$ , and we write  $\|\cdot\|_W$  for  $\|\cdot\|_{0,W}$ . Similarly, we denote by  $(\cdot, \cdot)_W$  the  $L^2(W)[L^2(W)]^d$  or  $[L^2(W)]^{d \times d}$  inner product. For shortness if  $W$  is equal to  $\Omega$ , we will drop the index  $\Omega$ , while for any  $m \geq 0$ ,  $\|\cdot\|_{m,\hat{a}} = \|\cdot\|_{m,\Omega_{\hat{a}}}$ ,  $|\cdot|_{m,\hat{a}} = |\cdot|_{m,\Omega_{\hat{a}}}$  and  $(\cdot, \cdot)_{\hat{a}} = (\cdot, \cdot)_{\Omega_{\hat{a}}}$ , for  $\hat{a} = f, p$ . The space  $H_0^m(\Omega)$  denotes the closure of  $C_0^\infty(\Omega)$  in  $H^m(\Omega)$ . Let  $[H^m(\Omega)]^d$  be the space of vector valued functions  $v = (v_1, \dots, v_d)$  with components  $v_i$  in  $H^m(\Omega)$ . The norm and the seminorm on  $[H^m(\Omega)]^d$  are given by

$$\begin{aligned} \|v\|_{m,\Omega} &:= \left( \sum_{i=0}^d \|v_i\|_{m,\Omega}^2 \right)^{1/2}, \\ |v|_{m,\Omega} &:= \left( \sum_{i=0}^d |v_i|_{m,\Omega}^2 \right)^{1/2}. \end{aligned} \quad (13)$$

For a connected open subset of the boundary  $E \subset \partial\Omega_f \cup \partial\Omega_p$ , we write  $\langle \cdot, \cdot \rangle_E$  for the  $L^2(E)$  inner product (or duality pairing), that is, for scalar valued functions  $\lambda, \sigma$  one defines:

$$\langle \lambda, \sigma \rangle_E := \int_E \lambda \sigma ds. \quad (14)$$

In the following, we derive a Lagrange multiplier type weak formulation of the system, which will be the basis for our finite element approximation. Let

$$\begin{aligned} V_f &= \left\{ v_f \in H^1(\Omega_f)^d : v_f = 0 \text{ on } \Gamma_f \right\}, \quad W_f = L^2(\Omega_f), \\ V_p &= \left\{ v_p \in H(\cdot; \Omega_p) : v_p \cdot n_p = 0 \text{ on } \Gamma_p^N \right\}, \quad W_p = L^2(\Omega_p), \\ X_p &= \left\{ \xi_p \in H^1(\Omega_p)^d : \xi_p = 0 \text{ on } \Gamma_p \right\}, \end{aligned} \quad (15)$$

where  $H(\cdot; \Omega_p)$  is the space of  $L^2(\Omega_p)^d$  vectors with divergence in  $L^2(\Omega_p)$  with a norm

$$\|v\|_{H(\cdot; \Omega_p)}^2 := \|v\|_{\Omega_p}^2 + \|\nabla \cdot v\|_{\Omega_p}^2. \quad (16)$$

We define the global velocity and pressure spaces as

$$\begin{aligned} V &= \{v = (v_f, v_p) \in V_f \times V_p\}, \\ W &= \{w = (w_f, w_p) \in W_f \times W_p\}, \end{aligned} \quad (17)$$

with norms

$$\begin{aligned} \|v\|_V^2 &:= \|v_f\|_{\Omega_f}^2 + \|v_p\|_{H(\cdot; \Omega_p)}^2, \\ \|w\|_W^2 &= \|w_f\|_{\Omega_f}^2 + \|w_p\|_{\Omega_p}^2. \end{aligned} \quad (18)$$

The weak formulation is obtained by multiplying the equations in each region by suitable test functions, integrating by parts the second-order terms in space, and utilizing the interface and boundary conditions.

Let define

$$\begin{aligned} a_f(u_f, v_f) &:= (2\mu D(u_f), D(v_f))_{\Omega_f}, \\ a_p^d(u_p, v_p) &:= (\mu K^{-1}u_p, v_p)_{\Omega_p}, \\ a_p^e(\eta_p, \xi_p) &:= \left(2\mu_p D(\eta_p), D(\xi_p)\right)_{\Omega_p} + \left(\lambda_p \nabla \cdot \eta_p, \nabla \cdot \xi_p\right)_{\Omega_p}, \end{aligned} \quad (19)$$

be the bilinear forms related to Stokes, Darcy, and the elastic operator, respectively. Let

$$b_*(v, w) := -(\nabla \cdot v, w)_{\Omega_*}. \quad (20)$$

Integration by parts in (2) and the two equations in (5) lead to the interface term

$$I_{\Gamma_{fp}} = -\langle \sigma_f n_f, v_f \rangle_{\Gamma_{fp}} - \langle \sigma_p n_p, \xi_p \rangle_{\Gamma_{fp}} + \langle p_p, v_p \cdot n_p \rangle_{\Gamma_{fp}}. \quad (21)$$

Using the first condition for balance of normal stress in (9), we set

$$\lambda = -(\sigma_f n_f) \cdot n_f = p_p \text{ on } \Gamma_{fp}, \quad (22)$$

which will be used as a Lagrange multiplier to impose the mass conservation interface condition (8). Utilizing the BJS condition (10) and the second condition for balance of stresses in (9), we obtain

$$I_{\Gamma_{fp}} = a_{\text{BJS}}(u_f, \partial_t \eta_p; v_f, \xi_p) + b_\Gamma(v_f, v_p, \xi_p, \lambda), \quad (23)$$

where

$$\begin{aligned} a_{\text{BJS}}(u_f, \eta_p; v_f, \xi_p) &= \sum_{j=1}^{d-1} \left\langle \mu \alpha_{\text{BJS}} \sqrt{K_j^{-1}} (u_f - \eta_p) \cdot \tau_{f,j}, (v_f - \xi_p) \cdot \tau_{f,j} \right\rangle_{\Gamma_{fp}}, \\ b_\Gamma(v_f, v_p, \xi_p; \mu) &= \langle v_f \cdot n_f + (\xi_p + v_p) \cdot n_p, \mu \rangle_{\Gamma_{fp}}. \end{aligned} \quad (24)$$

For the well-posedness of  $b_\Gamma$ , we require that  $\lambda \in \wedge = (V_p \cdot n_p|_{\Gamma_{fp}})'$ . According to the normal trace theorem, since  $v_p \in V_p \subset H(\cdot; \Omega_p)$ , then  $v_p \cdot n_p \in H^{-1/2}(\partial\Omega_p)$ . Furthermore, since  $v_p \cdot n_p = 0$  on  $\Gamma_p^N$  and  $\text{dis}(\Gamma_p^D, \Gamma_{fp}) \geq s > 0$ , then  $v_p \cdot n_p \in H^{-1/2}(\Gamma_{fp})$  (see, e.g., [32]). Therefore, we take  $\wedge = H^{1/2}(\Gamma_{fp})$ .

The Lagrange multiplier variational formulation is for  $t \in (0, T]$ , find  $u_f(t) \in V_f$ ,  $p_f(t) \in W_f$ ,  $u_p(t) \in V_p$ ,  $p_p(t) \in W_p$ ,  $\eta_p(t) \in X_p$ , and  $\lambda(t) \in \wedge$  such that  $p_p(0) = p_{p,0}$ ,  $\eta_p(0) = \eta_{p,0}$ ,

and for all  $v_f \in V_f$ ,  $w_f \in W_f$ ,  $v_p \in V_p$ ,  $w_p \in W_p$ ,  $\xi_p \in X_p$ , and  $\mu \in \wedge$ ,

$$\begin{aligned} a_f(u_f, v_f) + a_p^d(u_p, v_p) + a_p^e(\eta_p, \xi_p) + a_{\text{BJS}}(u_f, \partial_t \eta_p; v_f, \xi_p), \\ + b_f(v_f, p_f) + b_p(v_p, p_p) + \alpha b_p(\xi_p, p_p) + b_\Gamma(v_f, v_p, \xi_p; \lambda) \\ = (f_f, v_f)_{\Omega_f} + (f_p, \xi_p)_{\Omega_p} (s_0 \partial_t p_p, w_p)_{\Omega_p} \\ - \alpha b_p(\partial_t \eta_p, w_p) - b_p(u_p, w_p) - b_f(u_f, w_f) \\ = (q_f, w_f)_{\Omega_f} + (q_p, w_p)_{\Omega_p}, \end{aligned} \quad (26)$$

$$b_\Gamma(u_f, u_p, \partial_t \eta_p; \mu) = 0. \quad (27)$$

where we used the notation  $\partial_t = \partial/\partial t$ .

The assumptions on the fluid viscosity  $\mu$  and the material coefficients  $K$ ,  $\lambda_p$ , and  $\mu_p$  imply that the bilinear forms  $a_f(\cdot, \cdot)$ ,  $a_p^d(\cdot, \cdot)$ , and  $a_p^e(\cdot, \cdot)$  are coercive and continuous in the appropriate norms. In particular, there exist positive constants  $c^f$ ,  $c^p$ ,  $c^e$ ,  $C^f$ ,  $C^p$ ,  $C^e$  such that:

$$c^f \|v_f\|_{H^1(\Omega_f)}^2 \leq a_f(v_f, v_f), \quad \forall v_f \in V_f, \quad (28)$$

$$a_f(v_f, q_f) \leq C^f \|v_f\|_{H^1(\Omega_f)} \|q_f\|_{H^1(\Omega_f)}, \quad \forall v_f, q_f \in V_f, \quad (29)$$

$$c^p \|v_p\|_{L^2(\Omega_p)}^2 \leq a_p^d(v_p, v_p), \quad \forall v_p \in V_p, \quad (30)$$

$$a_p^d(v_p, q_p) \leq C^p \|v_p\|_{L^2(\Omega_p)} \|q_p\|_{L^2(\Omega_p)}, \quad \forall v_p, q_p \in V_p, \quad (31)$$

$$c^e \|\xi_p\|_{H^1(\Omega_p)}^2 \leq a_p^e(\xi_p, \xi_p) \forall \xi_p \in X_p, \quad (32)$$

$$a_p^e(\xi_p, \zeta_p) \leq C^e \|\xi_p\|_{H^1(\Omega_p)} \|\zeta_p\|_{H^1(\Omega_p)} \forall \xi_p, \zeta_p \in X_p, \quad (33)$$

where (28)–(29) and (32)–(33) hold true thanks to Poincaré inequality and (32)–(33) also relies on Korn's inequality.

In summary, from ([23], Corollary 3.1, Page 7), the following result holds:

**Theorem 1.** *There exists a unique solution  $(u_f, p_f, u_p, p_p, \eta_p, \lambda) \in L^\infty(0, T; V_f) \times L^\infty(0, T; W_f) \times L^\infty(0, T; V_p) \times W^{1,\infty}(0, T; W_p) \times W^{1,\infty}(0, T; X_p) \times L^\infty(0, T; \wedge)$  to the problem (25)–(27).*

**2.3. Finite Element Discretization.** Let  $T_h^f$  and  $T_h^p$  be shape-regular and quasiuniform partition of  $\Omega_f$  and  $\Omega_p$ , respectively, both consisting of affine elements with maximal element diameter  $h$ . The two partitions may be nonmatching at the interface  $\Gamma_{fp}$ . For the discretization of the fluid velocity and pressure, we choose finite element spaces  $V_{f,h} \subset V_f$  and

$W_{f,h} \subset W_f$ , which are assumed to be inf-sup stable. Examples of such spaces include the MINI elements, the Taylor-Hood elements, and the conforming Crouzeix-Raviart elements. For the discretization of the porous medium problem, we choose  $V_{p,h} \subset V_p$  and  $W_{p,h} \subset W_p$  to be any of well-known inf-sup stable mixed finite element spaces, such as the Raviart-Thomas or the Brezzi-Douglas-Marini spaces. The global spaces are:

$$\begin{aligned} V_h &:= \{v_h = (v_{f,h}, v_{p,h}) \in V_{f,h} \times V_{p,h}\}, \\ W_h &:= \{w_h = (w_{f,h}, w_{p,h}) \in W_{f,h} \times W_{p,h}\}. \end{aligned} \quad (34)$$

We employ a conforming Lagrangian finite element space  $X_{p,h} \subset X_p$  to approximate the structure displacement. Note that the finite element spaces  $V_{f,h}$ ,  $V_{p,h}$ , and  $X_{p,h}$  satisfy the prescribed homogeneous boundary conditions on the external boundaries. For the discrete Lagrange multiplier space, we take

$$\Lambda_h = V_{p,h} \cdot n_{p|\Gamma_{fp}}. \quad (35)$$

The semidiscrete continuous-in-time problem reads given  $p_{p,h}(0)$  and  $\eta_{p,h}(0)$ , for  $t \in (0, T]$ , find  $u_{f,h}(t) \in V_{f,h}$ ,  $p_{f,h}(t) \in W_{f,h}$ ,  $u_{p,h}(t) \in V_{p,h}$ ,  $p_{p,h}(t) \in W_{p,h}$ ,  $\eta_{p,h}(t) \in X_{p,h}$ , and  $\lambda_h(t) \in \Lambda_h$  such that for all  $v_{f,h} \in V_{f,h}$ ,  $w_{f,h} \in W_{f,h}$ ,  $v_{p,h} \in V_{p,h}$ ,  $w_{p,h} \in W_{p,h}$ ,  $\xi_{p,h} \in X_{p,h}$ , and  $\mu_h \in \Lambda_h$ ,

$$\begin{aligned} &a_f(u_{f,h}, v_{f,h}) + a_p^d(u_{p,h}, v_{p,h}) + a_p^c(\eta_{p,h}, \xi_{p,h}) \\ &+ a_{BJS}(u_{f,h}, \partial_t \eta_{p,h}; v_{f,h}, \xi_{p,h}) \\ &+ b_f(v_{f,h}, p_{f,h}) + b_p(v_{p,h}, p_{p,h}) + \alpha b_p(\xi_{p,h}, p_{p,h}) \\ &+ b_\Gamma(v_{f,h}, v_{p,h}, \xi_{p,h}; \lambda_h) = (f_{f,h}, v_{f,h})_{\Omega_f} + (f_{p,h}, \xi_{p,h})_{\Omega_p}, \end{aligned} \quad (36)$$

$$\begin{aligned} &(s_0 \partial_t p_{p,h}, w_{p,h})_{\Omega_p} - \alpha b_p(\partial_t \eta_{p,h}, w_{p,h}) - b_p(u_{p,h}, w_{p,h}) \\ &- b_f(u_{f,h}, w_{f,h}) = (q_f, w_{f,h})_{\Omega_f} + (q_p, w_{p,h})_{\Omega_p}, \end{aligned} \quad (37)$$

$$b_\Gamma(u_{f,h}, u_{p,h}, \partial_t \eta_{p,h}; \mu_h) = 0. \quad (38)$$

We will take  $p_{p,h}(0)$  and  $\eta_{p,h}(0)$  to be suitable projections of the initial data  $p_{p,0}$  and  $\eta_{p,0}$ .

We introduce the errors for all variables:

$$\begin{aligned} e_f &:= u_f - u_{f,h}, \\ e_p &:= u_p - u_{p,h}, \\ e_s &:= \eta_p - \eta_{p,h}, \\ e_{fp} &:= p_f - p_{f,h}, \\ e_{pp} &:= p_p - p_{p,h}, \\ e_\lambda &:= \lambda - \lambda_h. \end{aligned} \quad (39)$$

The following results hold cf. [23]:

**Theorem 2** (A priori error estimation). *There exists a unique solution  $(u_{f,h}, p_{f,h}, u_{p,h}, p_{p,h}, \eta_{p,h}, \lambda_h)$  in  $L^\infty(0, T; V_{f,h}) \times L^\infty(0, T; W_{f,h}) \times L^\infty(0, T; V_{p,h}) \times W^{1,\infty}(0, T; W_{p,h}) \times W^{1,\infty}(0, T; X_{p,h}) \times L^\infty(0, T; \Lambda_h)$  of the discrete problem (36)–(38), and if the solution  $(u_f, p_f, u_p, p_p, \eta_p, \lambda) \in V_f \times W_f \times V_p \times W_p \times X_p \times \Lambda$  of the continuous problem (25)–(27) is smooth enough, then we have:*

$$\begin{aligned} &\|e_f\|_{L^2(0,T;H^1(\Omega_f))} + \|e_p\|_{L^2(0,T;L^2(\Omega_p))} + \|e_s\|_{L^\infty(0,T;H^1(\Omega_p))} \\ &+ \|e_{fp}\|_{L^\infty(0,T;L^2(\Omega_f))} + \|e_{pp}\|_{L^2(0,T;L^2(\Omega_p))} \\ &+ \|e_\lambda\|_{L^2(0,T;\Lambda_h)} \leq C \left( u_f, p_f, u_p, p_p, \eta_p, \lambda \right) h^r, \quad r > 0. \end{aligned} \quad (40)$$

For  $W_h = (v_{f,h}, w_{f,h}, v_{p,h}, w_{p,h}, \xi_{p,h}, \lambda_h) \in L^\infty(0, T; V_{f,h}) \times L^\infty(0, T; W_{f,h}) \times L^\infty(0, T; V_{p,h}) \times W^{1,\infty}(0, T; W_{p,h}) \times W^{1,\infty}(0, T; X_{p,h}) \times L^\infty(0, T; \Lambda_h)$ , we can subtract (36)–(38) to (25)–(27) to obtain the Galerkin orthogonality relation for all  $t \in ]0, T]$ :

$$\begin{aligned} &a_f(e_f(t), v_{f,h}(t)) + a_p^d(e_s(t), \xi_{p,h}(t)) \\ &+ a_{BJS}(e_f(t), \partial_t e_s(t); v_{f,h}(t), \xi_{p,h}(t)) \\ &+ b_f(v_{f,h}(t), e_f(t)) + b_p(v_{p,h}(t), e_p(t)) \\ &+ \alpha b_p(\xi_{p,h}(t), e_p(t)) + b_\Gamma(v_f(t), v_p(t), \xi_p(t), e_\lambda(t)) \\ &+ (s_0 \partial_t e_p(t), w_{p,h}(t)) - \alpha b_p(\partial_t e_s(t), w_{p,h}(t)) \\ &- b_p(e_p(t), w_{p,h}(t)) - b_f(e_f(t), w_{f,h}(t)) = 0. \end{aligned} \quad (41)$$

### 3. A Posteriori Error Analysis

In order to solve the Stokes-Biot model problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriated indicators. In this section, we first define the local and global indicators (Section 3.1), and then, the lower and upper error bounds are derived (Sections 3.4 and 3.5).

**3.1. Residual Error Estimators.** The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier

and that involve the data at hand. To this end, define the exact element residuals:

**Definition 3** (Exact Element Residuals). Let  $t \in ]0, T]$  and  $W_h(t) = (v_{f,h}(t), p_{f,h}(t), v_{p,h}(t), w_{p,h}(t), \xi_{p,h}(t), \lambda_h(t)) \in V_{f,h} \times W_{f,h} \times V_{p,h} \times W_{p,h} \times X_{p,h} \times \Lambda_h$  be an arbitrary finite element function. The exact element residuals over a triangle or tetrahedra  $K \in_h$  and over  $E \in_h(\Gamma_{fp})$  are defined for all  $t \in ]0, T]$  by:

$$\begin{aligned} R_{f,K}(W_h(t)) &= f_{f,K} + \nabla \cdot \sigma_f(v_{f,h}(t), w_{f,h}(t)), \\ R_{p,K,1}(W_h(t)) &= f_{p,K} + \nabla \cdot \sigma_p(\xi_{p,h}(t), w_{p,h}(t)), \\ R_{p,K,2}(W_h(t)) &= \mu K^{-1} v_{p,h}(t) + \nabla w_{p,h}(t), \\ R_{f,K}(W_h(t)) &= q_{f,K} - \nabla \cdot v_{f,h}(t), \\ R_{p,K}(W_h(t)) &= q_p - \partial_t(s_0 w_{p,h}(t) + \alpha \nabla \cdot \xi_{p,h}(t)) + \nabla \cdot v_{p,h}(t), \\ R_{E,pf,1}(W_h(t)) &= v_{f,h}(t) \cdot n_{f,E} + (\partial_t \xi_p(t) + v_{p,h}(t)) \cdot n_{p,E}, \\ R_{E,pf,2}(W_h(t)) &= w_{p,h} + (\sigma_f(v_{f,h}(t), w_{f,h}(t)) n_{f,E}) \cdot n_{f,E}, \\ R_{E,pf,3}(W_h(t)) &= \sigma_f(v_{f,h}(t), w_{f,h}(t)) n_{f,E} + \sigma_p(\xi_{p,h}(t), w_{p,h}(t)) n_{p,E}, \\ R_{E,pf,4}(W_h(t))(j) &= (\sigma_f(v_{f,h}, w_{f,h})) \tau_{f,E,j} + \mu \alpha_{BJS} \sqrt{K_j^{-1}} (v_{f,h} - \partial_t \xi_{p,h}) \cdot \tau_{f,E,j}. \end{aligned} \quad (42)$$

As it is common, these exact residuals are replaced by some finite-dimensional approximation called approximate element residual  $r_{*,K}$ ,  $r_{*,K}$ ,  $*$   $\in \{f, p\}$ ,  $R_{E,pf,l}$ ,  $l \in \{1, 2, 3, 4\}$ . This approximation is here achieved by projecting  $f_f$  and  $q_f$  on the space of piecewise constant functions in  $\Omega_f$  and piecewise  $\mathbb{P}^1$  functions in  $\Omega_p$ , more precisely for all  $K \in_h^f$  we take

$$\begin{aligned} f_{f,K} &= P_K(f_f), \\ q_{f,K} &= P_K(q_f), \end{aligned}$$

with  $P_K : L^1(K) \rightarrow \mathbb{R}$  such that  $P_K(\phi) = \frac{1}{|K|} \int_K \phi(x) dx$ .

$$(43)$$

While for all  $K \in_h^p$ , we take  $f_{p,K}$  and  $q_p$  as the unique element of  $[\mathbb{P}^1(K)]^d$ , respectively  $\mathbb{P}^1(K)$  such that:

$$\int_K f_{p,K}(x) \cdot q(x) dx = \int_K f_p(x) q(x) dx, \quad q \in [\mathbb{P}^1(K)]^d, \quad (44)$$

respectively,

$$\int_K q_{p,K}(x) q(x) dx = \int_K q_p(x) q(x) dx, \quad \forall q \in \mathbb{P}^1(K). \quad (45)$$

Thereby, we define the approximate element residuals.

**Definition 4** [Approximate Element Residuals]. Let  $t \in ]0, T]$  and  $W_h(t) = (v_{f,h}(t), p_{f,h}(t), v_{p,h}(t), w_{p,h}(t), \xi_{p,h}(t), \lambda_h(t)) \in V_{f,h} \times W_{f,h} \times V_{p,h} \times W_{p,h} \times X_{p,h} \times \Lambda_h$  be an arbitrary finite element function. Then, the approximate element residuals are defined for all  $t \in ]0, T]$  by:

$$\begin{aligned} r_{f,K}(W_h(t)) &= f_{f,K} + \nabla \cdot \sigma_f(v_{f,h}(t), w_{f,h}(t)), \\ r_{p,K,1}(W_h(t)) &= f_{p,K} + \nabla \cdot \sigma_p(\xi_{p,h}(t), w_{p,h}(t)), \\ r_{p,K,2}(W_h(t)) &= \mu K^{-1} v_{p,h}(t) + \nabla w_{p,h}(t), \\ r_{f,K}(W_h(t)) &= q_{f,K} - \nabla \cdot v_{f,h}(t), \\ r_{p,K}(W_h(t)) &= q_{p,K} - \partial_t(s_0 w_{p,h}(t) + \alpha \nabla \cdot \xi_{p,h}(t)) + \nabla \cdot v_{p,h}(t). \end{aligned} \quad (46)$$

Next, introduce the gradient jump in normal direction by

$$\begin{cases} J_{E,n_{ef}}(U_h) := \left[ (2\mu D(u_{f,h}) - p_{f,h} I) \cdot n_E \right]_E & \text{if } E \in_h(\Omega_f), \\ J_{E,n_{ep}}(U_h) J_{E,n_{ep}}(U_h) := \left[ (2\mu D(\eta_{p,h}) - p_{p,h} I) \cdot n_E \right]_E & \text{if } E \in_h(\Omega_p). \end{cases} \quad (47)$$

where  $I$  is the identity matrix of  $\mathbb{R}^{d \times d}$ .

**Definition 5** [Residual Error Estimators]. Let  $U_h = (u_{f,h}, p_{f,h}, u_{p,h}, p_{p,h}, \eta_{p,h}, \lambda_h)$  be the finite element solution of the problem (36)–(38) in  $L^\infty(0, T; V_{f,h}) \times L^\infty(0, T; W_{f,h}) \times L^\infty(0, T; V_{p,h}) \times W^{1,\infty}(0, T; W_{p,h}) \times W^{1,\infty}(0, T; X_{p,h}) \times L^\infty(0, T; \Lambda_h)$ . Then, the residual error estimator is locally defined by

$$\Theta_K(U_h) := \left[ \Theta_{K,f}^2(U_h) + \Theta_{K,p}^2(U_h) + \Theta_{K,pf}^2(U_h) \right]^{1/2}, \quad (48)$$

where

$$\begin{aligned} \Theta_{K,f}^2(U_h) &:= h_K^2 \|r_{f,K}(U_h)\|_{L^\infty(0,T;L^2(K))}^2 + \|r_{f,K}(U_h)\|_{L^\infty(0,T;L^2(K))}^2 \\ &+ \sum_{E \in_h(\Omega_f)} h_E \|J_{E,n_{ef}}(U_h)\|_{L^\infty(0,T;L^2(E))}^2, \end{aligned} \quad (49)$$

$$\begin{aligned} \Theta_{K,p}^2(U_h) &:= h_K^2 \left( \|r_{p,K,1}(U_h)\|_{L^\infty(0,T;L^2(K))}^2 + \|r_{p,K,2}(U_h)\|_{L^\infty(0,T;L^2(K))}^2 \right) \\ &+ h_K^2 \| [r_{p,K,2}(U_h)] \|_{L^\infty(0,T;L^2(K))}^2 + \|r_{p,K}(U_h)\|_{L^\infty(0,T;L^2(K))}^2 \\ &+ \sum_{E \in_h(\Omega_p)} h_E \|J_{E,n_{ep}}(U_h)\|_{L^\infty(0,T;L^2(E))}^2, \end{aligned} \quad (50)$$

$$\begin{aligned} \Theta_{K,pf}^2(U_h) := & \sum_{E \in_h(\Gamma_{pf})} h_E \|R_{E,pf,1}(U_h)\|_{L^\infty(0,T;L^2(E))}^2 \\ & + \sum_{E \in_h(\Gamma_{pf})} h_E \|R_{E,pf,2}(U_h)\|_{L^\infty(0,T;L^2(E))}^2 \\ & + \sum_{E \in_h(\Gamma_{pf})} h_E \|R_{E,pf,3}(U_h)\|_{L^\infty(0,T;L^2(E))}^2 \\ & + \sum_{E \in_h(\Gamma_{pf})} h_E \left\{ \sum_{j=1}^{d-1} \|R_{E,pf,4}(j)\|_{L^\infty(0,T;L^2(E))}^2 \right\}. \end{aligned} \quad (51)$$

The global residual error estimator is given by

$$\Theta(U_h) := \left[ \sum_{K \in_h} \Theta_K(U_h)^2 \right]^{1/2}. \quad (52)$$

Furthermore, denote the local approximation terms by

$$\zeta_K^2 := \zeta_{K,f}^2 + \zeta_{K,p}^2, \quad (53)$$

where

$$\begin{aligned} \zeta_{K,f}^2 := & h_K^2 \|R_{f,K}(U_h) - r_{f,K}(U_h)\|_{L^\infty(0,T;L^2(K))}^2 \\ & + h_K^2 \|R_{f,K}(U_h) - r_{f,K}(U_h)\|_{L^\infty(0,T;L^2(K))}^2 \text{ if } K \in_h^f, \\ \zeta_{K,p}^2 := & h_K^2 \|R_{p,K,1}(U_h) - r_{p,K,1}(U_h)\|_{L^\infty(0,T;L^2(K))}^2 \\ & + h_K^2 \|R_{p,K}(U_h) - r_{p,K}(U_h)\|_{L^\infty(0,T;L^2(K))}^2 \text{ if } K \in_h^p. \end{aligned} \quad (54)$$

The global approximation term is defined by

$$\zeta := \left[ \sum_{K \in_h} \zeta_K^2 \right]^{1/2}. \quad (55)$$

*Remark 6.* The residual character of each term on the right-hand sides of (48)–(51) is quite clear since if  $U_h$  would be the exact solution of (2)–(10), then they would vanish.

### 3.2. Analytical Tools

**3.2.1. Inverse Inequalities.** In order to derive the lower error bounds, we proceed similarly as in [29, 33] (see also [34]), by applying inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions. To this end, we recall some notation and introduce further preliminary results. Given  $K \in \mathcal{T}_h$ , and  $E \in (K)$ , we let  $b_K$  and  $b_E$  be the usual simplex-bubble and face-bubble functions, respectively (see (1.5) and (1.6) in [28]). In particular,  $b_K$  satisfies  $b_K \in \mathbb{P}^3(K)$ ,  $\text{supp}(b_K) \subseteq K$ ,  $b_K = 0$  on  $\partial K$ , and  $0 \leq b_K \leq 1$  on  $K$ . Similarly,  $b_E \in \mathbb{P}^2(K)$ ,  $\text{supp}(b_E) \subseteq \omega_E := \{K' \in \mathcal{T}_h : E \in (K')\}$ ,  $b_E = 0$  on  $\partial K \setminus E$  and  $0 \leq b_E \leq 1$  in  $\omega_E$ . We also recall from [35] that, given  $k \in \mathbb{N}$ , there exists an extension operator  $L : C(E) \rightarrow C(K)$  that satisfies  $L(p) \in \mathbb{P}^k(K)$

and  $L(p)|_E = p$ ,  $\forall p \in \mathbb{P}^k(E)$ . A corresponding vectorial version of  $L$ , that is, the componentwise application of  $L$ , is denoted by  $L$ . Additional properties of  $b_K$ ,  $b_E$ , and  $L$  are collected in the following lemma (see [35]):

**Lemma 7.** *Given  $k \in \mathbb{N}^*$ , there exist positive constants depending only on  $k$  and shape regularity of the triangulations (minimum angle condition), such that for each simplex  $K$  and  $E \in (K)$  there hold*

$$\begin{aligned} \|\phi\|_K &\leq \|\phi b_K^{1/2}\|_K \leq \|\phi\|_K, \quad \forall \phi \in \mathbb{P}^k(K), \\ |\phi b_K|_{I,K} &\leq h_K^{-1} \|\phi\|_K, \quad \forall \phi \in \mathbb{P}^k(K), \\ \|\psi\|_E &\leq \|b_E^{1/2} \psi\|_E \leq \|\psi\|_E, \quad \forall \psi \in \mathbb{P}^k(E), \\ \|L(\psi)\|_K + h_E |L(\psi)|_{I,K} &\leq h_E^{1/2} \|\psi\|_E \quad \forall \psi \in \mathbb{P}^k(E). \end{aligned} \quad (56)$$

**Lemma 8** (Continuous trace inequality). *There exists a positive constant  $\beta_1 > 0$  depending only on  $\sigma_0$  such that*

$$\|v\|_{\partial K}^2 \leq \beta_1 \|v\|_K \|v\|_{I,K}, \quad \forall K \in \mathcal{T}_h, \forall v \in [H^1(K)]^d. \quad (57)$$

**3.2.2. Clément Interpolation Operator.** In order to derive the upper error bounds, we introduce the Clément interpolation operator  $I_{Cl}^0 : H_0^1(\Omega) \rightarrow \mathcal{P}_c^b(\mathcal{T}_h)$  that approximates optimally nonsmooth functions by continuous piecewise linear functions:

$$\mathcal{P}_c^b(\mathcal{T}_h) := \{v \in C^0(\bar{\Omega}) : v|_K \in \mathbb{P}^1(K), \quad \forall K \in \mathcal{T}_h \text{ and } v = 0 \text{ on } \partial\Omega\}. \quad (58)$$

In addition, we will make use of a vector valued version of  $I_{Cl}^0$ , that is,

$I_{Cl}^0 : [H_0^1(\Omega)]^d \rightarrow [\mathcal{P}_c^b(\mathcal{T}_h)]^d$ , which is defined componentwise by  $I_{Cl}^0$ . The following lemma establishes the local approximation properties of  $I_{Cl}^0$  (and hence of  $I_{Cl}^0$ ), for a proof see ([36], Section 3).

**Lemma 9.** *There exist constants  $C_1, C_2 > 0$ , independent of  $h$ , such that for all  $v \in H_0^1(\Omega)$  there hold*

$$\begin{aligned} \|v - I_{Cl}^0(v)\|_K &\leq C_1 h_K \|v\|_{I,\Delta(K)}, \quad \forall K \in \mathcal{T}_h, \\ \|v - I_{Cl}^0(v)\|_E &\leq C_2 h_E^{1/2} \|v\|_{I,\Delta(E)}, \quad \forall E \in_h, \end{aligned} \quad (59)$$

where  $\Delta(K) := \cup\{K' \in \mathcal{T}_h : K' \cap K \neq \emptyset\}$  and  $\Delta(E) := \cup\{K' \in \mathcal{T}_h : K' \cap E \neq \emptyset\}$ .

### 3.2.3. Helmholtz Decomposition

**Lemma 10** ([7]). *There exists  $C_p > 0$  such that every  $v_p \in H(\cdot; \Omega_p)$  can be decomposed as  $v_p = w + \beta$ , where  $w \in [H^1(\Omega_p)]^d$ ,  $\beta \in H^1(\Omega_p)$ ,  $\int_{\Omega_p} \beta(x) dx = 0$  and*

$$\|w\|_{I,\Omega_p} + \|\beta\|_{I,\Omega_p} \leq C_p \|v_p\|_{H(\cdot; \Omega_p)}. \quad (60)$$

**3.3. Error Equation.** We set  $H = L^\infty(0, T; V_f) \times L^\infty(0, T; W_f) \times L^\infty(0, T; V_p) \times W^{1,\infty}(0, T; W_p) \times W^{1,\infty}(0, T; X_p) \times L^\infty(0, T; \Lambda)$  and  $H_h = L^\infty(0, T; V_{f,h}) \times L^\infty(0, T; W_{f,h}) \times L^\infty(0, T; V_{p,h}) \times W^{1,\infty}(0, T; W_{p,h}) \times W^{1,\infty}(0, T; X_{p,h}) \times L^\infty(0, T; \Lambda_h)$ .

Let  $U = (u_f, p_f, u_p, p_p, \eta_p, \lambda) \in H$  and  $W = (v_f, w_f, v_p, w_p, \xi_p, \mu) \in H$ . For  $t \in ]0, T]$ , we define the operator  $A$  by

$$\begin{aligned} A(U(t), W(t)) &:= a_f(u_f(t), v_f(t)) + a_p^d(u_p(t), v_p(t)) + b_p(v_p(t), p_p(t)) \\ &\quad + \alpha b_p(\xi_p(t), p_p(t)) + b_\Gamma(v_f(t), v_p(t), \xi_p(t), \lambda(t)) \\ &\quad + (s_0 \partial_t p_p(t), w_p(t))_{\Omega_p} - \alpha b_p(\partial_t \eta_p(t), w_p(t)) \\ &\quad - b_p(u_p(t), w_p(t)) - b_f(u_f(t), w_f(t)) \\ &\quad + b_\Gamma(u_f(t), u_p(t), \partial_t \eta_p(t); \mu), \\ F(W(t)) &:= (f_f, v_f(t))_{\Omega_f} + (f_p, \xi_p(t)) \\ &\quad + (q_f, w_f(t))_{\Omega_f} + (q_p, w_p(t))_{\Omega_p}. \end{aligned} \quad (61)$$

Then, the continuous problem (25)–(27) is equivalent to find  $U \in H$  such that for all  $t \in ]0, T]$ , we have:

$$A(U(t), W(t)) = F(W(t)), \quad \forall W \in H. \quad (62)$$

We define the discrete version by the same way: find  $U_h \in H_h$  such that for all  $t \in ]0, T]$ ,

$$A(U_h(t), W_h(t)) = F(W_h(t)), \quad \forall W_h \in H_h. \quad (63)$$

Since for all  $t \in ]0, T]$ , and  $W_h \in H_h$ ,  $A(U(t) - U_h(t), W_h(t)) = 0$ , then from (62), we obtain

$$\begin{aligned} A(U(t) - U_h(t), W(t)) &= A(U(t) - U_h(t), W(t) - W_h(t)) \\ &= A(U(t), W(t) - W_h(t)) - A(U_h(t), W(t) - W_h(t)) \\ &= F(W(t) - W_h(t)) - A(U_h(t), W(t) - W_h(t)) \\ &= (f_f, v_f(t) - v_{f,h}(t))_{\Omega_f} + (f_p, \xi_p(t) - \xi_{p,h}(t))_{\Omega_p} \\ &\quad + (q_f, w_f(t) - w_{f,h}(t))_{\Omega_f} + (q_p, w_p(t) - w_{p,h}(t))_{\Omega_p} \\ &\quad - A(U_h(t), W(t) - W_h(t)) = (f_f - f_{f,h}, v_f(t) - v_{f,h}(t))_{\Omega_f} \\ &\quad + (f_p - f_{p,h}, \xi_p(t) - \xi_{p,h}(t))_{\Omega_p} + (q_f - q_{f,h}, w_f(t) - w_{f,h}(t))_{\Omega_f} \\ &\quad + (q_p - q_{p,h}, w_p(t) - w_{p,h}(t))_{\Omega_p} + (f_{f,h}, v_f(t) - v_{f,h}(t))_{\Omega_f} \\ &\quad + (f_{p,h}, \xi_p(t) - \xi_{p,h}(t))_{\Omega_p} + (q_{f,h}, w_f(t) - w_{f,h}(t))_{\Omega_f} \\ &\quad + (q_{p,h}, w_p(t) - w_{p,h}(t))_{\Omega_p} - A(U_h(t), W(t) - W_h(t)). \end{aligned} \quad (64)$$

Now, we set,

$$A(U(t) - U_h(t), W(t)) = \sum_{K \in \mathcal{K}_h} A_K(U(t) - U_h(t), W(t)). \quad (65)$$

Then, from (64), we can write:

$$\begin{aligned} A_K(U(t) - U_h(t), W(t)) &:= (f_f - f_{f,h}, v_f(t) - v_{f,h}(t))_{\Omega_f \cap K} \\ &\quad + (f_p - f_{p,h}, \xi_p(t) - \xi_{p,h}(t))_{\Omega_p \cap K} + (q_f - q_{f,h}, w_f(t) - w_{f,h}(t))_{\Omega_f \cap K} \\ &\quad + (q_p - q_{p,h}, w_p(t) - w_{p,h}(t))_{\Omega_p \cap K} + (f_{f,h}, v_f(t) - v_{f,h}(t))_{\Omega_f \cap K} \\ &\quad + (f_{p,h}, \xi_p(t) - \xi_{p,h}(t))_{\Omega_p \cap K} + (q_{f,h}, w_f(t) - w_{f,h}(t))_{\Omega_f \cap K} \\ &\quad + (q_{p,h}, w_p(t) - w_{p,h}(t))_{\Omega_p \cap K} - A_K(U_h(t), W(t) - W_h(t)) \\ &= (R_{f,K}(U_h(t)) - r_{f,K}(U_h(t)), v_f(t) - v_{f,h}(t))_{\Omega_f \cap K} \\ &\quad + (R_{f,K}(U_h(t)) - r_{f,K}(U_h(t)), w_f(t) - w_{f,h}(t))_{\Omega_f \cap K} \\ &\quad + (R_{p,K,1}(U_h(t)) - r_{p,K,1}(U_h(t)), \xi_p(t) - \xi_{p,h}(t))_{\Omega_p \cap K} \\ &\quad + (R_{p,K}(U_h(t)) - r_{p,K}(U_h(t)), w_p(t) - w_{p,h}(t))_{\Omega_p \cap K} \\ &\quad + B_K(U_h(t), W(t) - W_h(t)), \end{aligned} \quad (66)$$

where

$$\begin{aligned} B_K(U_h(t), W(t) - W_h(t)) &= (f_{f,K}, v_f(t) - v_{f,h}(t))_{\Omega_f \cap K} \\ &\quad + (f_{p,K}, \xi_p(t) - \xi_{p,h}(t))_{\Omega_p \cap K} \\ &\quad + (q_{f,K}, w_f(t) - w_{f,h}(t))_{\Omega_f \cap K} \\ &\quad + (q_{p,K}, w_p(t) - w_{p,h}(t))_{\Omega_p \cap K} \\ &\quad - A_K(U_h(t), W(t) - W_h(t)). \end{aligned} \quad (67)$$

**3.4. Reliability of the A Posteriori Error Estimator.** The first main result is given by the following theorem:

**Theorem 11** (Upper Error Bound). *Let  $U = (u_f, p_f, u_p, p_p, \eta_p, \lambda) \in H$  be the exact solution and  $U_h = (u_{f,h}, p_{f,h}, u_{p,h}, p_{p,h}, \eta_{p,h}, \lambda_h) \in H_h$  be the finite element solution. Then, there exist a positive constant  $C_{rel}$  such that the error is bounded globally from above by:*

$$\|U - U_h\|_{H_h} \leq C_{rel} [\Theta(U_h) + \zeta]. \quad (68)$$

*Proof.* Let  $t \in ]0; T]$ . We consider the residue Equation (65), and we set  $W(t) = (v_f(t), w_f(t), v_p(t), w_p(t), \xi_p(t), \mu(t))$ . We take  $W_h(t) = (v_{f,h}(t), 0, v_{p,h}(t), 0, \xi_{p,h}(t), 0)$  with  $v_{f,h}(t) = I_{Cl}^0(v_f(t))$  and  $\xi_{p,h}(t) = I_{Cl}^0(\xi_p(t))$ . As  $v_p(t) \in H(\Omega_p)$ , then by Lemma 3.4,  $v_p(t)$  admits the decomposition  $v_p(t) = w_p(t) + \beta(t)$  where  $w_p(t) \in [H^1(\Omega_p)]^d$  and  $\beta(t) \in H^1(\Omega_p)$  with  $\int_{\Omega_p} \beta(t)(x) dx = 0$  and  $\|w_p\|_{L^\infty(0,T;[H^1(\Omega_p)]^d)} + \|\beta\|_{L^\infty(0,T;H^1(\Omega_p))} \leq C_p \|v_p\|_{L^\infty(0,T;V_p)}$ . We consider  $v_{p,h}(t) = w_{p,h}(t) + \beta_{p,h}(t)$  with  $w_{p,h}(t) = I_{Cl}(w_p(t))$  and  $\beta_{p,h}(t) = I_{Cl}(\beta_p(t))$ . Thus,  $v_p(t) - v_{p,h}(t)$

$= (w_p(t) - w_{p,h}(t)) + (\beta_p(t) - \beta_{p,h}(t))$ . Now, we recall:

$$\begin{aligned}
 B_K(U_h(t), W(t) - W_h(t)) &= \left( f_{f,K}, v_f(t) - v_{f,h}(t) \right)_{\Omega_f \cap K} \\
 &\quad + \left( f_{p,K}, \xi_p(t) - \xi_{p,h}(t) \right)_{\Omega_p \cap K} \\
 &\quad + \left( q_{f,K}, w_f(t) - w_{f,h}(t) \right)_{\Omega_f \cap K} \\
 &\quad + \left( q_{p,K}, w_p(t) - w_{p,h}(t) \right)_{\Omega_p \cap K} \\
 &\quad - A_K(U_h(t), W(t) - W_h(t)).
 \end{aligned} \tag{69}$$

Therefore, integrate by parts element by element we may write:

$$B_K(U_h(t), W(t) - W_h(t)) = \sum_{i=1}^{12} \eta_{i,K}, \tag{70}$$

where

$$\begin{aligned}
 \eta_{1,K} &:= (r_{f,K}(U_h(t)), v_f(t) - v_{f,h}(t))_{\Omega_f \cap K}, \\
 \eta_{2,K} &:= (r_{p,K,1}(U_h(t)), \xi_p(t) - \xi_{p,h}(t))_{\Omega_p \cap K}, \\
 \eta_{3,K} &:= -(r_{p,K,2}(U_h(t)), w_p(t) - w_{p,h}(t))_{\Omega_p \cap K}, \\
 \eta_{4,K} &:= -(r_{p,K,2}(U_h(t)), \beta_p(t) - \beta_{p,h}(t))_{\Omega_p \cap K}, \\
 \eta_{5,K} &:= (r_{f,K}(U_h(t)), w_f)_{\Omega_f \cap K}, \\
 \eta_{6,K} &:= (r_{p,K}(U_h(t)), w_p)_{\Omega_p \cap K}, \\
 \eta_{7,K} &:= - \sum_{E \in_h(\Omega_f \cap K)} (J_{E,n_{E,f}}(U_h(t)), v_f(t) - v_{f,h}(t))_E, \\
 \eta_{8,K} &:= - \sum_{E \in_h(\Omega_p \cap K)} (J_{E,n_{E,p}}(U_h(t)), \xi_p(t) - \xi_{p,h}(t))_E, \\
 \eta_{9,K} &:= \sum_{E \in_h(\Gamma_{fp})} (R_{E,pf,1}(U_h(t)), (v_f(t) - v_{f,h}(t)) \cdot n_{f,E})_E, \\
 \eta_{10,K} &:= \sum_{E \in_h(\Gamma_{fp})} (R_{E,pf,2}(U_h(t)), (v_f(t) - v_{f,h}(t)) \cdot n_{f,E})_E, \\
 \eta_{11,K} &:= \sum_{E \in_h(\Gamma_{fp})} (R_{E,pf,3}(U_h(t)), (\xi_p(t) - \xi_{p,h}(t)))_E, \\
 \eta_{12,K} &:= - \sum_{E \in_h(\Gamma_{fp})} \sum_{j=1}^{d-1} (R_{E,pf,4}(U_h(t))(j), v_f(t) - v_{f,h}(t) \cdot \tau_j)_E.
 \end{aligned} \tag{71}$$

Thus, from Equations (66) and (70), we obtain

$$\begin{aligned}
 A_K(U(t) - U_h(t), W(t)) &:= (R_{f,K}(U_h(t)) - r_{f,K}(U_h(t)), v_f(t) - v_{f,h}(t))_{\Omega_f \cap K} \\
 &\quad + (R_{f,K}(U_h(t)) - r_{f,K}(U_h(t)), w_f(t) - w_{f,h}(t))_{\Omega_f \cap K} \\
 &\quad + (R_{p,K,1}(U_h(t)) - r_{p,K,1}(U_h(t)), \xi_p(t) - \xi_{p,h}(t))_{\Omega_p \cap K} \\
 &\quad + (R_{p,K}(U_h(t)) - r_{p,K}(U_h(t)), w_p(t) - w_{p,h}(t))_{\Omega_p \cap K} \\
 &\quad + \sum_{i=1}^{12} \eta_{i,K}.
 \end{aligned} \tag{72}$$

Coercivity of operator  $A$  leads to inf-sup condition:

$$\|U - U_h\|_{H_h} \leq \sup_{W \in H} \frac{|A(U - U_h, W)|}{\|W\|_H}. \tag{73}$$

By consequently, the identities (65) and (72), inf-sup condition of operator  $A$  (73), Cauchy-Schwarz inequality, estimation of Lemma 10, and the approximation properties of Lemma 9 imply the required estimate and finish the proof.

**3.5. Efficiency of the A Posteriori Error Estimator.** For  $K \in_h$ , we set  $H_h(K) = L^\infty(0, T; V_{f,h}(K)) \times L^\infty(0, T; W_{f,h}(K)) \times L^\infty(0, T; V_{p,h}(K)) \times W^{1,\infty}(0, T; W_{p,h}(K)) \times W^{1,\infty}(0, T; X_{p,h}(K)) \times L^\infty(0, T; \Lambda_h(K))$ , where

$$\begin{aligned}
 V_{f,h}(K) &:= \{v_{f,h}|_K : v_{f,h} \in V_{f,h}\}, \\
 W_{f,h}(K) &:= \{w_{f,h}|_K : w_{f,h} \in W_{f,h}\}, \\
 V_{p,h}(K) &:= \{v_{p,h}|_K : v_{p,h} \in V_{p,h}\}, \\
 W_{p,h}(K) &:= \{w_{p,h}|_K : w_{p,h} \in W_{p,h}\}, \\
 X_{p,h}(K) &:= \{\xi_{p,h}|_K : \xi_{p,h} \in X_{p,h}\}, \\
 \Lambda_h(K) &:= \{\lambda_h|_K : \lambda_h \in \Lambda_h\}.
 \end{aligned} \tag{74}$$

The error estimator  $\Theta(U_h)$  is considered efficient if it satisfies the following theorem.

**Theorem 12 (Lower Error Bound).** *Let  $U = (u_f, p_f, u_p, p_p, \eta_p, \lambda) \in H$  be the exact solution and  $U_h = (u_{f,h}, p_{f,h}, u_{p,h}, p_{p,h}, \eta_{p,h}, \lambda_h) \in H_h$  be the finite element solution. Then, there exist a positive constant  $C_{eff}$  such that the error is bounded locally from below for all  $K \in_h$  by:*

$$\Theta_K(U_h) \leq C_{eff} \left[ \|U - U_h\|_{H_h(K)} + \sum_{K' \in \tilde{c}_K} \zeta_{K'} \right], \tag{75}$$

where  $\tilde{c}_K$  is a finite union of neighboring elements of  $K$  and  $\|\cdot\|_{H_h(K)}$  is the product norm.

*Proof.* The lower bound are proved using standard element-wise integration by parts, namely, error equation of Section

3.3 (i.e., identity (65) and Equation (72)) and some a inverse estimates of Lemma 7 (Cf. [37] for details).

#### 4. Discussion

In this paper, we have discussed a posteriori error estimates for a finite element approximation of the Stokes-Biot system. A residual type a posteriori error estimator is provided, that is both reliable and efficient.

There are many issues to be addressed in this area such as the other types of a posteriori error estimates and related implementation of the adaptive finite element methods.

Further, it is well known that an internal layer appears at the interface  $\Gamma_{fp}$  as the permeability tensor degenerates; in that case, anisotropic meshes have to be used in this layer (see for instance [10]). Hence, we intend to extend our results to such anisotropic meshes.

#### 5. Nomenclature

- (i)  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$  bounded domain
- (ii)  $\Omega_p$  : The poroelastic medium domain
- (iii)  $\Omega_f = \Omega \setminus \bar{\Omega}_d$
- (iv)  $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$
- (v)  $\Gamma_* = \partial\Omega_* \setminus \Gamma_{fp}$ ,  $* = f, p$
- (vi)  $n_f$  (resp.  $n_p$ ): The unit outward normal vector along  $\partial\Omega_f$  (resp.  $\partial\Omega_p$ )
- (vii)  $u_f$ : The fluid velocity in  $\Omega_f$
- (viii)  $p_f$ : The fluid pressure in  $\Omega_f$
- (ix)  $u_p, \eta_p$ : The fluid velocities in  $\Omega_p$
- (x)  $p_p$ : The fluid pressure in  $\Omega_p$
- (xi) In 2D, the curl of a scalar function  $w$  is given as usual by

$$\text{curl } w := \left( \frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1} \right)^T \quad (76)$$

- (xii) In 3D, the of a vector function  $w = (w_1, w_2, w_3)$  is given as usual by  $w := \nabla \times w$ , namely,

$$\text{curl } w := \left( \frac{\partial w_3}{\partial x_2} - \frac{\partial w_2}{\partial x_3}, \frac{\partial w_1}{\partial x_3} - \frac{\partial w_3}{\partial x_1}, \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2} \right) \quad (77)$$

- (xiii)  $\mathbb{P}^k$ : The space of polynomials of total degree not larger than  $k$
- (xiv)  $T_h$ : Triangulation of  $\Omega$

- (xv)  $T_h^*$ : The corresponding induced triangulation of  $\Omega_*$ ,  $* \in \{f, p\}$

- (xvi) For any  $K \in \mathcal{T}_h$ ,  $h_K$  is the diameter of  $K$ , and  $\rho_K = 2r_K$  is the diameter of the largest ball inscribed into  $K$

- (xvii)  $h := \max_{K \in \mathcal{T}_h} h_K$  and  $\sigma_h := \max_{K \in \mathcal{T}_h} (h_K / \rho_K)$

- (xviii)  $\varepsilon_h$ : The set of all the edges or faces of the triangulation

- (xix)  $\varepsilon(K)$ : The set of all the edges ( $N = 2$ ) or faces ( $N = 3$ ) of an element  $K$

- (xx)  $\varepsilon_h := \bigcup_{K \in \mathcal{T}_h} \varepsilon(K)$

- (xxi)  $\mathcal{N}(K)$ : The set of all the vertices of an element  $K$

- (xxii)  $\mathcal{N}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{N}(K)$

- (xxiii) For  $\mathcal{A} \subset \bar{\Omega}$ ,  $\mathcal{E}_h(\mathcal{A}) := \{E \in \varepsilon_h : E \subset \mathcal{A}\}$

- (xxiv) For  $E \in \varepsilon_h(\Omega_*)$ , we associate a unit vector  $n_{E,*}$  such that  $n_{E,*}$  is orthogonal to  $E$  and equals to the unit exterior normal vector to  $\partial\Omega_*$ ,  $* \in \{f, p\}$

- (xxv) For  $E \in \varepsilon_h(\Omega_*)$ ,  $[\phi]_E$  is the jump across  $E$  in the direction of  $n_{E,*}$

- (xxvi) In order to avoid excessive use of constants, the abbreviations  $x \leq y$  and  $x \sim y$  stand for  $x \leq c_1 y$  and  $c_1 x \leq y \leq c_2 x$ , respectively, with positive constants independent of  $x$ ,  $y$  or  $h$

- (xxvii)  $\partial_t = \partial / \partial t$

- (xxviii)  $\|\phi\|_{L^2(0,T;X)} := [\int_0^T \|\phi(s)\|_X^2 ds]^{1/2}$

- (xxix)  $\|\phi\|_{L^\infty(0,T;X)} := \sup_{0 < t \leq T} \|\phi(t)\|_X$

- (xxx)  $\|\phi\|_{W^{1,\infty}(0,T;X)} := \max \{ \|\phi\|_{L^\infty(0,T;X)}, \|\partial_t \phi\|_{L^\infty(0,T;X)} \}$

- (xxxix)  $\|\cdot\|_{H_h}$ : Product norm on  $H_h$

- (xxxix)  $\|\cdot\|_{H_h(K)}$ : Local product norm on  $H_h(K)$ .

#### Data Availability

There are no data underlying the findings in this paper to be shared.

#### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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