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STOCHASTIC CHAOS IN A CLASS OF FOKKER-PLANCK EQUATIONS DERIVED FROM POPULATION DYNAMICS

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Abstract

In this work, we obtain results by using a physical potential $\Phi_m(x, y)$ whose parameters have biological significance [5] to explain the interaction between two species in population dynamics. In using the

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degenerated parameters [8], this potential is reduced to the form of $\Phi_m(x, y)$, where m is the coupling constant. Consequently, we study the effect of this constant on the potential $\Phi_m(x, y)$. The deterministic chaos results are obtained in Figures 2 and 3. An interesting result of our theoretical model resides in the fact that after many manifestations of the deterministic chaos, the physical potential $\Phi_m(x, y)$ remained unchanged above a critical value $m = 2$. This situation corresponds without any doubt to the Hopf bifurcation in the nonlinear system, where the stationary effect changes to the unstable to stable and leads to a limit cycle. Then we studied the manifestations of the stochastic chaos by considering the transformed potential $\hat{\Phi}_{m,\varepsilon}(x, y)$, where ε is the noise intensity. In such a case the combined effect of the noise and the coupling constant, gives results as illustrated in Figures 4 and 5. The second model [5] leads to a potential $\Phi_{m,l}(x, y)$ with two coupling constants (m, l) which indicate that the use of degenerated parameter is strictly forbidden. The results obtained show the chaotic behavior of the potential $\Phi_{m,l}(x, y)$ for the arbitraries values of coupling constants (m, l), Figures 6 and 7. The stochastic manifestations are also shown by the transformed potential $\hat{\Phi}_{m,l,\varepsilon}(x, y)$, Figures 8 and 9.

1. Introduction

The study of manifestations of chaos in a class of Fokker-Plank equations is by now an interesting subject of research regarding mathematic but also because of so many applications in Physics, Chemistry and Biology [4].

When the coefficient of the Fokker-Planck equation depends explicitly of time, it has been proving that the spectrum of Floquet for this system presents a transition on these statistical properties.

However, Millonas and Reichl [10] recently shown that not only a whole class of Fokker-Planck equations with time-independent coefficients exhibits such transition, but also how this transition can be related to the dynamical properties of certain Hamiltonian equations of motion.

In general case, it is worthy of note that, since few years, the new nonlinear physics that takes on, have as goal, fascinating phenomenon such as turbulences, physico-chemical and biological oscillations. The chaotic appearances of these phenomenons are now replaced by an unexpected unification concept of strange attractors and Lyapunov exponent.

In this work, we show the manifestation of the stochastic chaos by considering a family of Fokker-Planck equations in two dimensions. The potential $\Phi(x, y)$ will be obtained by generalizing the Hutchinson equations [5] for a model of interaction between two species in population dynamics [3, 6, 12].

2. Stochastic Chaos in a Dissipative System

2.1. Diffusion process in $\mathfrak{R}^n$ with a potential $\Phi(\vec{x})$ independent of the time and in the fluctuating environment

Let us consider a diffusion process described by a set of coupled stochastic differential equations [1]:

$$dx_i = -\partial_i \Phi(\vec{x}) dt + \sqrt{\varepsilon} dW_i(t); \quad i = 1, 2, ..., n, \tag{2.1}$$

where $\Phi(\vec{x})$, with $\vec{x} \in \mathfrak{R}^n$, is the potential bounded from below, $W_i(t)$ are uncorrelated Wiener process and ε is a diffusion coefficient. Note that (2.1) describes a purely classical diffusion process and leads to a purely classical Fokker-Planck equation [13].

Although not the only possible physical basis for this equation, one might think of (2.1) as a mechanical system with potential proportional to $\Phi(\vec{x})$, subjected to very strong friction in a fluctuating environment.

2.2. Fokker-Planck equation

The evolution of probability density $P(\vec{x}, t)$ on $\mathfrak{R}^n$ is described by the Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \frac{\varepsilon}{2} \Delta P + \vec{\nabla}(P \vec{\nabla} \Phi). \tag{2.2}$$

2.3. Transformation of the Fokker-Planck equation to a Schrödinger equation

Equation (2.2) can be readily written in a form of *eigenvalue equation*. For this purpose, using *the time separation ansatz*, such that

$$P(\vec{x}, t) = \Psi_\lambda(\vec{x})e^{-\frac{\lambda t}{\varepsilon}}, \tag{2.3}$$

we successively obtain [10]:

$$\begin{cases} \dfrac{\partial P}{\partial t} = -\dfrac{\lambda}{\varepsilon}\Psi_\lambda e^{-\frac{\lambda t}{\varepsilon}}, \\ \dfrac{1}{2}\varepsilon\Delta P = \dfrac{1}{2}\varepsilon e^{-\frac{\lambda t}{\varepsilon}}\Delta\Psi_\lambda, \\ \vec{\nabla}(P\vec{\nabla}\Phi) = e^{-\frac{\lambda t}{\varepsilon}} \end{cases} \tag{2.4}$$

by substituting (2.4) into (2.2), one obtains:

$$L\Psi_\lambda = -\lambda\Psi_\lambda. \tag{2.5}$$

The operator L is defined by the formula

$$L = \frac{1}{2}\varepsilon^2\Delta + \varepsilon\vec{\nabla}\Phi\vec{\nabla} + \varepsilon\Delta\Phi. \tag{2.6}$$

Let us consider now the following *eigenfunction*:

$$\Psi_\lambda = \varphi_\lambda e^{-\Phi/\varepsilon}, \tag{2.7}$$

we obtain in order

$$\begin{cases} \dfrac{1}{2}\varepsilon^2\Delta\Psi_\lambda = e^{-\frac{\Phi}{\varepsilon}}\left[\dfrac{1}{2}\varepsilon^{2\Delta\varphi_\lambda - \varepsilon\vec{\nabla}\Phi\vec{\nabla}\varphi_\lambda - \frac{1}{2}\varepsilon\varphi_\lambda\Delta\Phi + \frac{1}{2}\varphi_\lambda(\vec{\nabla}\Phi)^2}\right], \\ \varepsilon\vec{\nabla}\Phi\vec{\nabla}\Psi_\lambda = e^{-\frac{\Phi}{\varepsilon}}[-\varphi_\lambda(\vec{\nabla}\Phi)^2 + \varepsilon\vec{\nabla}\Phi\vec{\nabla}\varphi_\lambda], \\ \varepsilon\Delta\Phi\Psi_\lambda = \varepsilon e^{-\frac{\Phi}{\varepsilon}}\Delta\Phi\varphi_\lambda. \end{cases} \tag{2.8}$$

By substituting (2.8) in (2.5), we obtain after some algebra:

$$L\Psi_\lambda = e^{-\frac{\Phi}{\varepsilon}}\left[\left[\frac{1}{2}\varepsilon^2\Delta - \frac{1}{2}(\vec{\nabla}\Phi)^2 + \frac{1}{2}\varepsilon\Delta\Phi\right]\varphi_\lambda = -\lambda\varphi_\lambda e^{-\frac{\Phi}{\varepsilon}}\right]. \quad (2.9)$$

Multiplying (2.9) by $e^{-\frac{\Phi}{\varepsilon}}$ and by written $\zeta = e^{-\frac{\Phi}{\varepsilon}} L e^{-\frac{\Phi}{\varepsilon}}$, we obtain an eigenvalue equation:

$$\zeta\varphi_\lambda = \lambda\varphi_\lambda \quad (2.10)$$

with the operator

$$\zeta = -\frac{1}{2}\varepsilon^2\Delta + \hat{\Phi}, \quad (2.11)$$

ζ is a Hermitian Schrödinger type equation with a transformed potential $\hat{\Phi}$ defined by:

$$\hat{\Phi} = \frac{1}{2}(\vec{\nabla}\Phi)^2 + \frac{1}{2}\varepsilon\Delta\Phi. \quad (2.12)$$

Consequently, the problem of solving the Fokker-Planck (2.2), has been reduced to the problem defined by the Schrödinger equation (2.10).

3. Generalization of the Volterra Equations

3.1. Volterra classical model

The classical model of Volterra [16], for competition between two species with densities N_1 and N_2 living in the same environment is:

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 W(N_1/\theta_1) - \alpha_1 N_1 N_2, \\ \dfrac{dN_2}{dt} = r_2 N_2 W(N_2/\theta_2) - \alpha_2 N_2 N_1. \end{cases} \quad (3.1)$$

In (3.1), the first terms represent the intrinsic growth terms of the given species, while the second terms represent competition terms. Consequently, r_1 and r_2 are the intrinsic growth rates, α_1 and α_2 are the coefficients of competition, θ_1 and θ_2 are the saturation values for populations N_1 and N_2 when there is not competition. The function $W(N_i/\theta_i)$ is the well-known saturation inducing function introduced in mathematical ecology by Verhulst [6, 3, 12], viz:

$$W(N_i/\theta_i) = 1 - (N_i/\theta_i). \tag{3.2}$$

We observe that in the classical model of Volterra [13], there are binary interaction terms. However, it is well known by experiences that competition coefficient α_1 and α_2 can vary depending of the environment. That is why the social phenomenon has been introduced by Hutchinson [5] in the classical model of Volterra [13].

3.2. Hutchinson models

3.2.1. Tertiary model of interaction

Hutchinson firstly considers the simplest case by assuming that the effect of a given species upon another species is determined not only by constants α_1 and α_2 but by a factor $\beta_1 N_2$ or $\beta_2 N_1$ proportional to the total number of species in competition. Then the competition equations are as follows:

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 W(N_1/\theta_1) - \beta_1 N_1 N_2^2, \\ \dfrac{dN_2}{dt} = r_2 N_2 W(N_2/\theta_2) - \beta_2 N_2 N_1^2. \end{cases} \tag{3.3}$$

We now observe that the interaction terms are tertiary (3.3) can best be considered by drawing the isoclines:

$$\begin{cases} \dfrac{dN_1}{dN_2} = 0. \quad \text{Thus} \quad \theta_1 - N_1 - \gamma_1 N_2^2 = 0, \\ \dfrac{dN_2}{dN_1} = 0. \quad \text{Thus} \quad \theta_2 - N_2 - \gamma_2 N_1^2 = 0. \end{cases} \tag{3.4}$$

By written $\gamma_1 = \dfrac{\beta_i \theta_i}{r_i}$, equations (3.4) are obtained by using the explicit form (3.2) of the inducing function. Thus for the positive value of the coefficient, the isocline $\frac{dN_1}{dN_2} = 0$ cuts the N_1 axis at a point $N_1 = \theta_1$ and N_2 axis at a point $N_2 = \sqrt{\frac{\theta_1}{\gamma_1}}$. The isocline $\frac{dN_2}{dN_1} = 0$ cuts the N_2 axis at a point $N_1 = \theta_1$ and N_1 axis at a point $N_1 = \sqrt{\frac{\theta_2}{\gamma_2}}$.

With reasonably large and commensurate values of θ_1 and θ_2 and moderated values of the coefficients γ_1 and γ_2, we obtain:

$$\sqrt{\frac{\theta_1}{\gamma_1}} < \theta_2 \text{ or } \gamma_1 > \frac{\theta_1}{\theta_2^2}, \text{ and similarly:}$$

$$\sqrt{\frac{\theta_2}{\gamma_2}} < \theta_1 \text{ or } \gamma_2 > \frac{\theta_2}{\theta_1^2}.$$

These conditions show that the isoclines intersect at a saddle in the entirely positive quadrant. The quadrant is divided into two fields in one of which one species wins in competition, in the other species. The effect of introducing the social factor is therefore to convert a competitive process which can begin as if α_1 or α_2 or both are less than the ratios of the relevant saturation values into one in which both coefficients greatly exceed the ratio.

3.2.2. Binary and tertiary coupling model of interactions

Hutchinson [5] also studied a more general pairs of equation (3.3) such that:

$$\begin{cases} \frac{dN_1}{dt} = r_1 N_1 W(N_1/\theta_1) - \beta_1 N_1 N_2 - \gamma_1 N_1 N_2^2, \\ \frac{dN_2}{dt} = r_2 N_2 W(N_2/\theta_2) - \beta_2 N_2 N_1 - \gamma_2 N_2 N_1^2. \end{cases} \tag{3.5}$$

This form can be considered as containing the first two terms of power series:

$$\beta_1 N_2,\ \gamma_1 N_2^2\ ...,$$

$$\beta_3 N_2,\ \gamma_2 N_1^2\ ...$$

as approximations to more general competition functions $F_1(N_2)$ and $F_2(N_1)$.

Equations (3.5) with positive coefficient of competition can be used to express various intermediate situations between equations (3.1) and (3.3).

3.2.3. Others models

Cunningham [2] studied the equation of competition which is the generalization of Volterra and Hutchinson equations thus:

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 W(N_1/\theta_1) - N_1 F_1(N_2), \\ \dfrac{dN_2}{dt} = r_2 N_2 W(N_2/\theta_2) - N_2 F_2(N_1). \end{cases} \tag{3.6}$$

These equations represent the prey-predator model or of symbiosis. Cunningham studied in detail all the singular point of this model of competition, but did not find the general agreement for the existence of periodical solutions.

Furthermore, Utz and Waltman [14] give the sufficient conditions for the existence of periodical solutions of Cunningham [2].

4. Physical Potential for the Stochastic Chaos

4.1. The potential $\Phi_m(x,\ y)$ with one coupling constant

The study of the generalization of the Volterra equations shows that, we actually have various two coupled deterministic differential equations in $\Re_+^2$. We can generate a set of physical potentials in two variables $\Phi(x,\ y)$ and check whether the potential conditions are satisfied or not, viz:

$$\frac{\partial^2 \Phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}. \tag{4.1}$$

The advantage of our theoretical approach is in the fact that the parameters of $\Phi(x, y)$ obtained will have well defined physical interpretations.

We are going to introduce in this work a supplementary generalization, by considering in $\mathfrak{R}_+^2$ a generalized inducing function of Verhulst's type [3, 6], viz:

$$W\left(\frac{N}{\theta}\right) = 1 - \left(\frac{N}{\theta}\right)^n, \tag{4.2}$$

where n is a positive parameter having a biological significance. See Figure 1 and Table 1.

Table 1. Main characteristics of the generalized logistic model [3]

N	$r(0)$	$r(\theta)$	$r'(0)$	$r'(\theta)$	Name
$n > 1$	r_0	0	0	$\frac{-nr_0}{\theta}$	Generalized logistic
$n = 1$	r_0	0	$\frac{-nr_0}{\theta}$	$\frac{-nr_0}{\theta}$	Logistic
$0 < n < 1$	r_0	0	∞	$\frac{-nr_0}{\theta}$	Generalized logistic
$n \to 0$	∞	0	∞	0	Gompertz
$n = 0$	0	0	0	0	Constant

In this work, we consider competition equations (3.3) of Hutchinson [5] by using the inducing function (4.2) and we obtain:

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 \left(1 - \dfrac{N_1^n}{\theta_1^n}\right) - \beta_1 N_1 N_2^2, \\ \dfrac{dN_2}{dt} = r_2 N_2 \left(1 - \dfrac{N_2^n}{\theta_2^n}\right) - \beta_2 N_1 N_1^2. \end{cases} \tag{4.3}$$

Let us introduce variables $x = N_1$, $y = N_2$ to simplify the written, and, new coefficients define as follows:

$$\alpha_i = \frac{r_i}{\theta_i^n}, \quad m_i = \frac{\beta\theta_i^n}{n}, \quad k_i = \theta_i^n. \tag{4.4}$$

Then (4.3) reads in the following form:

$$\begin{cases} \dfrac{dx}{dt} = \alpha_1 x(k_1 - x^n - m_1 y^2), \\ \dfrac{dy}{dt} = \alpha_2 y(k_2 - y^n - m_2 x^2). \end{cases} \tag{4.5}$$

Consequently, the potential $\Phi(x, y)$ is readily defined by the following equations:

$$\begin{cases} \dfrac{dx}{dt} = \alpha_1 x(k_1 - x^n - m_1 y^2) = -\dfrac{\partial\Phi}{\partial x}, \\ \dfrac{dy}{dt} = \alpha_2 y(k_2 - y^n - m_2 x^2) = -\dfrac{\partial\Phi}{\partial y} \end{cases} \tag{4.6}$$

by integrating (4.6) according to x, we have

$$\Phi(x, y) = \alpha_1\left(k_1\frac{x^2}{2} - \frac{x^{n+2}}{n+2}m_1 - \frac{x^2y^2}{2}\right) + C. \tag{4.7}$$

Now, by using the second term of (4.5), we have the following equality:

$$\frac{\partial\Phi}{\partial y} = \frac{\partial C}{\partial y} - \gamma_1 m_1 x^2 y = \alpha_2 y(k_2 - y^n - m_2 x^2)$$

hence

$$C = \frac{\alpha_1 m_1}{2}x^2y^2 + \frac{\alpha_2 k_2}{2}y^2 - \frac{\alpha}{n+2}y^{n+2} - \frac{\alpha_2 m_2}{2}x^2y^2, \tag{4.8}$$

substituting C by its value in (4.7), we obtain the desired result, viz:

$$\Phi(x, y) = -\frac{1}{2}(\alpha_1 k_1 x^2 + \alpha_2 k_2 y^2) + \frac{1}{n+2}(\alpha_1 x^{n+2} + \alpha_2 y^{n+2}) + \frac{\alpha_2 m_2}{2}x^2y^2. \tag{4.9}$$

Using the potential condition defined by (4.1), we get:

$$\alpha_1 m_1 = \alpha_2 m_2 \Rightarrow \gamma_1 = \gamma_2. \tag{4.10}$$

This result is checked by equaling the partial derivative of $\Phi(x, y)$ according to x, with the first equation of (4.6).

4.2. Remark

(a) In the expression of $\Phi(x, y)$ the mutual interaction parameters m_1 and m_2 only appear in the coupled term $x^2 y^2$. If we consider the degenerated parameters case proposed by Leung [8] by written:

$$m_1 = m_2 = m; \quad \alpha_1 = \alpha_2 = \alpha; \quad k_1 = k_2 = k, \tag{4.11}$$

the potential $\Phi_m(x, y)$ with subscript m is written as follows:

$$\Phi_m(x, y) = -\frac{1}{2}\alpha k(x^2 + y^2) + \frac{\alpha}{n+2}(x^{n+2} + y^{n+2}) + \frac{1}{2}\alpha m x^2 y^2. \tag{4.12}$$

(b) If the system is without mutual interaction $(m = 0)$, then the equation is analytically integrable for some n values.

If $m \neq 0$, then the system moves in chaotic way when m increases. Considering the higher change of behavior, main illustrations are given in Figures 2 and 3.

(c) If $n = 1$ (logistic model), then the corresponding potential is written as:

$$\Phi_m(x, y) = -\frac{1}{2}\alpha k(x^2 + y^2) + \frac{\alpha}{3}(x^3 + y^3) + \frac{1}{2}\alpha m x^2 y^2. \tag{4.13}$$

If $n = 2$, then the corresponding potential is:

$$\Phi_m(x, y) = -\frac{1}{2}\alpha k(x^2 + y^2) + \frac{\alpha}{4}(x^4 + y^4) + \frac{1}{2}\alpha m x^2 y^2. \tag{4.14}$$

It is interesting to notice that for variables $(x, y) \in R_+^2$ and with the redefinition of the potential parameter (4.14) $\Phi_m(x, y)$ is identical to the potential $\Phi_{\xi}(I_1, I_2)$ used by Lett [7] to study the problem of First Passage Time in a double modes laser viz:

$$\Phi_{\xi}(I_1, I_2) = -\frac{1}{2}\alpha(I_1 + I_2) + \frac{1}{4}(I_1 + I_2) + \frac{1}{2}\xi I_1 I_2. \qquad (4.15)$$

By considering an extension of the domain of variables, thus $(x, y) \in R^2$, the potential $\Phi(x, y)$ also allows to study others physical phenomena. The result of our biological model has been obtained by considering the domain $(x, y) \in R_+^2$.

4.3. The study of projected potential $\Phi(x, 0)$

By projecting $\Phi_m(x, y)$ in the plane $(y = 0)$, we can write $\Phi_m(x, y) = \Phi(x, 0)$. Then we obtain:

$$\Phi(x, 0) = -\frac{1}{2}\alpha k x^2\left(1 - \frac{2x^2}{k(n+2)}\right). \qquad (4.16)$$

It is useful to observe that the physical potential (4.16), $\Phi(x, 0)$ with $x \in R^2$ gives for $n = 1$ and $n = 2$ two classical potential used in the study of Stochastic Resonance (SR) [11] and of the Mean First Passage Time (MFPT) in a nonlinear system influenced by the noise [9].

4.4. Expression of the transformed potential $\hat{\Phi}_{m,\varepsilon}(x, y)$

According to (2.12), the transformed potential $\hat{\Phi}_{m,\varepsilon}(x, y)$ of the potential $\Phi_m(x, y)$ is writing as:

$$\hat{\Phi}_{m,\varepsilon}(x, y) = -\frac{1}{2}\varepsilon\Delta\Phi_m(x, y) + \frac{1}{2}(\vec{\nabla}\Phi_m(x, y))^2. \qquad (4.17)$$

Components of $\vec{\nabla}\Phi_m(x, y)$ are written according to (4.12) as:

$$\nabla\Phi_m(x, y) = [-\alpha x(k - x^n - my^2); \quad -\alpha y(k - y^n - mx^2)]. \qquad (4.18)$$

Furthermore, we have:

$$\frac{\partial^2 \Phi_m}{\partial x^2} = -\alpha(k - (1+n)x^n - my^2),$$

$$\frac{\partial^2 \Phi_m}{\partial y^2} = -\alpha(k - mx^2 - (1+n)y^n) \quad (4.19)$$

that give for $\frac{\partial^2 \Phi_m}{\partial x^2} + \frac{\partial^2 \Phi_m}{\partial y^2}$ the following relation:

$$\Delta\Phi_m = -\alpha(2k - (1+n)(x^n + y^n) - m(x^n + y^n)), \quad (4.20)$$

by using equations (4.18) and (4.19), we obtain

$$\hat{\Phi}_{m,\varepsilon}(x, y) = -\frac{1}{2}\varepsilon \cdot \alpha[-2k + (1+n)(x^n + y^n) + m(x^2 + y^2)]$$
$$+ \frac{1}{2}\alpha^2[x^2(k - x^n - my^2)^2 + y^2(k - mx^2 - y^n)^2]. \quad (4.21)$$

(a) If $n = 1$, then we have

$$\hat{\Phi}_{m,\varepsilon}(x, y) = -\frac{1}{2}\varepsilon\alpha \cdot [-2k + 2(x + y) + m(x^2 + y^2)]$$
$$+ \frac{1}{2}\alpha^2[x^2(k - x - my^2)^2 + y^2(k - mx^2 - y)^2]. \quad (4.22)$$

(b) If $n = 2$, then we have

$$\hat{\Phi}_{m,\varepsilon}(x, y) = -\frac{1}{2}\varepsilon\alpha \cdot [-2k + 2(m+3)(x^2 + y^2)]$$
$$+ \frac{1}{2}\alpha^2[x^2(k - x - my^2)^2 + y^2(k - mx^2 - y^2)^2]. \quad (4.23)$$

By considering the noise, the chaotic behavior of this potential is illustrated in Figures 4 and 5.

4.5. The potential $\Phi_{m,l}(x, y)$ with two coupling constants

We start from equation (3.5) that we generalize by considering as previously a generalized inducing function of Verhulst type; we obtain a two dimensional system, viz:

$$\begin{cases} \dfrac{dN_1}{dt} = r_1 N_1 \left(1 - \dfrac{N_1^n}{\theta_1^n} - \beta_1 N_2 - \gamma_1 N_2^2 \right), \\ \dfrac{dN_2}{dt} = r_2 N_2 \left(1 - \dfrac{N_2^n}{\theta_2^n} - \beta_2 N_1 - \gamma_2 N_1^2 \right). \end{cases} \tag{4.24}$$

By introducing variables $x = N_1$ and $y = N_2$ and by introducing new coefficients defined as:

$$\alpha_i = \frac{r_i}{\theta_i^n}; \quad k_i = \theta_i^n, \quad m_i = \frac{\gamma_i \theta_i^n}{r_i}; \quad l_i = \frac{\beta_i \theta_i^n}{r_i}. \tag{4.25}$$

Equations (4.24) are writing in the following form:

$$\begin{cases} \dfrac{dx}{dt} = \alpha_1 x(k_1 - x^n - l_1 y - m_1 y^2), \\ \dfrac{dy}{dt} = \alpha_2 y(k_2 - y^n - l_2 x - m_2 x^2). \end{cases} \tag{4.26}$$

To find the potential $\Phi(x, y)$, we use the following conditions:

$$\dot{x} = -\frac{\partial \Phi}{\partial x}; \quad \dot{y} = -\frac{\partial \Phi}{\partial y},$$

thus:

$$\begin{cases} -\dfrac{\partial \Phi}{dx} = \alpha_1 x(k_1 - x^2 - l_1 y - m_1 y^2), \\ -\dfrac{\partial \Phi}{\partial y} = \alpha_2 y(k_2 - y^n - l_2 x - m_2 x^2). \end{cases} \tag{4.27}$$

By integrating the first equation, we obtain

$$\Phi(x, y) = \alpha_1 \left(k_1 \frac{x^2}{2} - \frac{x^{n+2}}{n+2} - \frac{l_1}{2} y x^2 - m_1 \frac{x^2 y^2}{2} \right) + C. \tag{4.28}$$

The second equation helps to determine the function C.

Then

$$\frac{\partial C}{\partial y} = \frac{\alpha_1 l_1}{2} x^2 + \alpha_1 m_1 x^2 y - \frac{\partial \Phi}{\partial y}.$$

By integration, we have:

$$C = \frac{\alpha_1 l_1}{2} x^2 y + \frac{\alpha_1 m_1}{2} x^2 y^2 + \frac{\alpha_2 k_2}{2} y^2$$

$$- \frac{\alpha_2 l_2}{2} xy^2 - \frac{\alpha_2}{n+2} y^{n+2} - \frac{\alpha_2 m_2}{2} x^2 y^2. \quad (4.29)$$

By substituting (4.29) into (4.28), we obtain:

$$\Phi(x, y) = -\frac{1}{2}(\alpha_1 k_1 x^2 + \alpha_2 k_2 y^2)$$

$$+ \frac{1}{n+2}(\alpha_1 x^{n+2} + \alpha_2 y^{n+2}) + \frac{\alpha_2 l_2}{2} xy^2 + \frac{\alpha_2 m_2}{2} x^2 y^2. \quad (4.30)$$

Now, let us find whether the potential condition (4.1) is verified; we get:

$$\frac{\partial^2 \Phi}{\partial x \partial y} = 2\alpha_2 m_2 xy + \alpha_2 l_2 y,$$

$$\frac{\partial^2 \Phi}{\partial y \partial x} = 2\alpha_2 m_2 xy + \alpha_2 l_2 y. \quad (4.31)$$

Consequently, the potential condition is therefore verified with the parameters (α_2, m_2, l_2). However, due to the linear terms, the condition of the potential is not reduced to a relation between the constant of the model, but implies variables (x, y); see (4.31).

Consequently, let us find the potential $\Phi(x, y)$ by integrating the second equation of the system (4.30). We obtain the following result:

$$\Phi(x, y) = -\frac{1}{2}(\alpha_1 k_1 x^2 + \alpha_2 k_2 y^2)$$

$$+ \frac{1}{n+2}(\alpha_1 x^{n+2} + \alpha_2 y^{n+2}) + \frac{\alpha_1 l_1}{2} yx^2 + \frac{\alpha_1 m_1}{2} x^2 y^2. \quad (4.32)$$

Then the potential condition (4.1) is also verified; we get:

$$\frac{\partial^2\Phi}{\partial x\partial y} = 2\alpha_1 m_1 xy + \alpha_1 l_1 x,$$

$$\frac{\partial^2\Phi}{\partial y\partial x} = 2\alpha_1 m_1 xy + \alpha_1 l_1 x. \tag{4.33}$$

We observe that the coupling terms did not have the same mathematical structure in the expressions (4.30) and (4.32) of the physical potential $\Phi_{m,l}(x, y)$ with two coupling constants (m, l) such as $(m = m_1)$ and $(l = l_1)$ as in the (4.32) whereas $(m = m_2)$ and $(l = l_2)$ as in equation (4.30). This raises up the fundamental problem of finding the subsidiary condition allowing to pass from relation (4.30) to relation (4.32) and reciprocally.

After those conditions of $\Phi(x, y)$, the following relation is obtained:

$$\alpha_1 l_1 x + \alpha_1 m_1 xy = \alpha_2 l_2 y + \alpha_2 m_2 xy, \tag{4.34}$$

we observe that regarding to condition (4.34) the use of degenerated parameters [8] becomes impossible.

4.6. Expression of the transformed potential $\hat{\Phi}_{m,l,\varepsilon}(x, y)$

By following the preceding method, we obtain for the transformed potential $\hat{\Phi}(x, y)$ the following relation:

$$\hat{\Phi}_{m,l,\varepsilon}(x, y) = -\frac{1}{2}\varepsilon[-(\alpha_1 k_1 + \alpha_2 k_2) + (n+1)(\alpha_1 x^n + \alpha_2 y^n)$$

$$+ (\alpha_1 m_1 y^2 + \alpha_2 m_2 x^2) + (\alpha_1 l_1 y + \alpha_2 l_2 x)]$$

$$+ \frac{1}{2}[\alpha_1^2 x^2 (k_1 - x^n - l_1 y - m_1 y^2)^2$$

$$+ \alpha_2^2 y^2 (k_2 - y^n - l_2 x - m_2 x^2)^2]. \tag{4.35}$$

As previously mentioned with various chaotic behaviors, we give in Figures 8 and 9, two characteristics examples for the physical potential (4.30) and two characteristics examples for the transformed potential (4.35) on the figures enclosed in the text.

5. Conclusion

Chaotic phenomena abound in the sciences, they can be found in nearly all branches of nonlinear modeling. In mechanics for instance where two degrees of freedom play a part, or alternatively, if we have a nonlinear oscillator with external parametric forcing or noise. Other examples can be found in chemistry and theoretical biology, where interactions between various components, chemical elements or population densities, give rise to nonlinear equations. However, although chaotic dynamics had been known to exist for a long time, its importance for broad variety of applications began to be widely appreciated only within the last decade. The field continues to develop rapidly in many directions, and implications continue to grow. In particular, Millonas and Reichl [10] consider the manifestation of chaos in the diffusion process on R^n described by the set of coupled stochastic differential equations with a potential $\Phi(\vec{x})$. For the probability density $\rho(\vec{x})$, the problem of solving the Fokker-Planck equation (2.2) has been reduced to the problem defined by a Schrödinger type equation (2.10) with a transformed potential $\hat{\Phi}(\vec{x})$ (2.12).

In order to obtain chaos, the system needed to be at least two dimensional, and in the previous work [10] the potential $\Phi(\vec{x})$ is chosen for convenience only. the aim is to focus attention in the effects produced by changes in the noise intensity ε. In this work, we report the results obtained by using the physical potential $\Phi_m(x, y)$ with one coupling constant, and the physical potential $\Phi_{m,l}(x, y)$ with two coupling constants. These potentials which parameters have biological significances are derived through the generalization of kinetic equation proposed by Hutchinson [5] to explain the interactions between two species in population dynamics.

As the system of equations is hardly nonlinear, we distinguish this work into two cases:

- The deterministic chaos obtained by changing the values of the parameters in the physical potentials.
- The stochastic chaos obtained by changing simultaneously, the values of the parameters and the noise intensity.

It is interesting to observe that results obtained through our theoretical models are spectacular as illustrated in Figures 2 and 9.

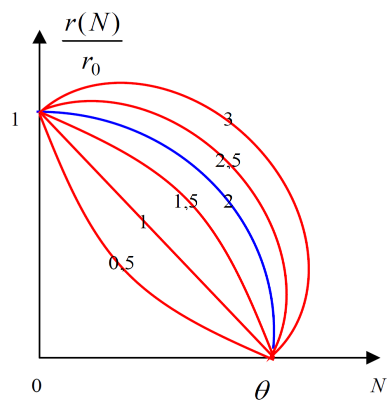

Figure 1. Behavior of the inducing function $\frac{r(N)}{r_0}$.

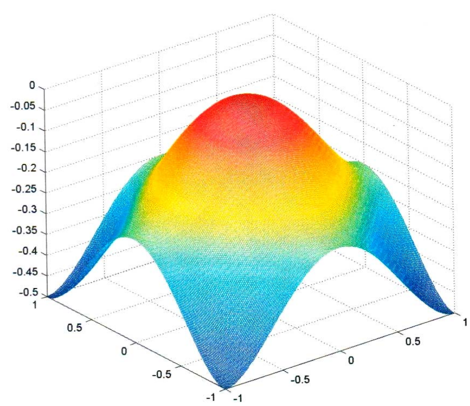

Figure 2. Deterministic chaos in the potential $\Phi_m(x, y)$. Values of the parameters $\alpha = 1$, $k = 1$, $m = 0$.

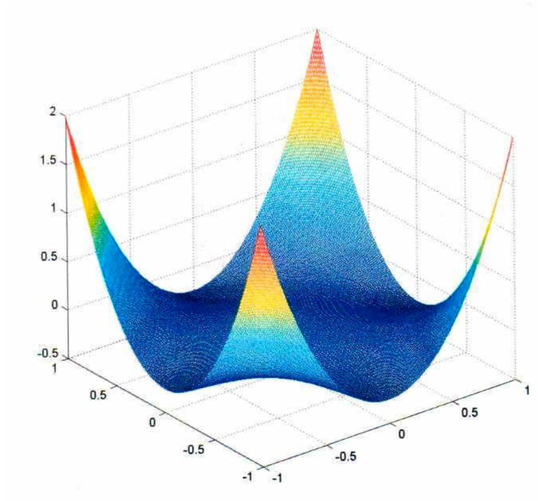

Figure 3. Deterministic chaos in the potential $\Phi_m(x, y)$. Values of the parameters $\alpha = 1$, $k = 1$, $m = 5$.

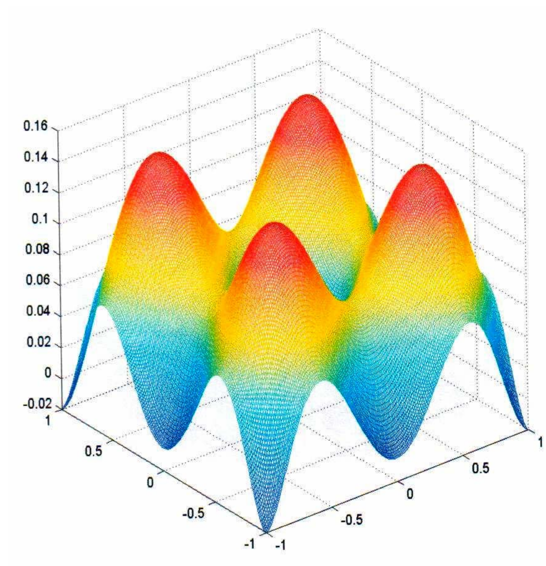

Figure 4. Stochastic chaos in the potential $\Phi_{m,\varepsilon}(x, y)$. Values of the parameters $\alpha = 1$, $k = 1$, $m = 0$, $\varepsilon = 0.01$.

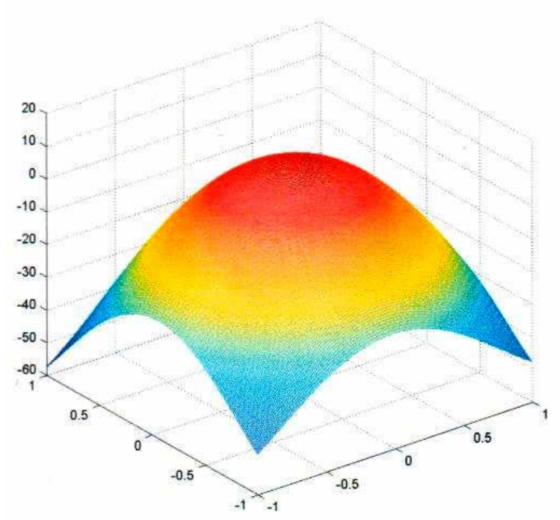

Figure 5. Stochastic chaos in the potential $\Phi_{m,\varepsilon}(x, y)$. Values of the parameters $\alpha = 1$, $k = 1$, $m = 0.5$, $\varepsilon = 1$.

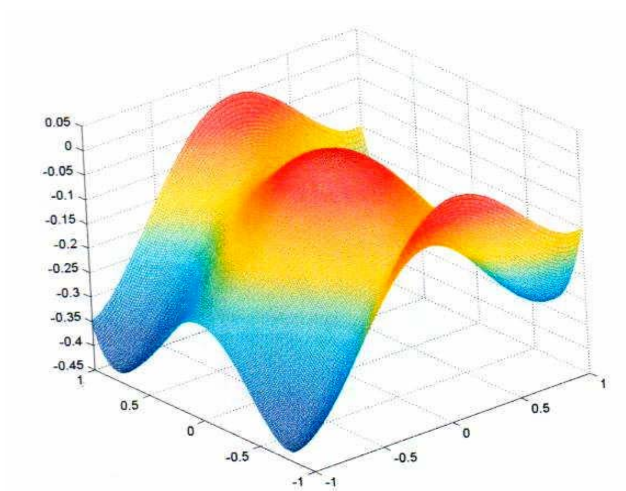

Figure 6. Deterministic chaos in the potential $\Phi_{m,l}(x, y)$. Values of the parameters $\alpha_1 = 1$, $\alpha_2 = 2$, $k_1 = 1$, $k_2 = 0.5$, $m_2 = 0$, $l_2 = 0.1$.

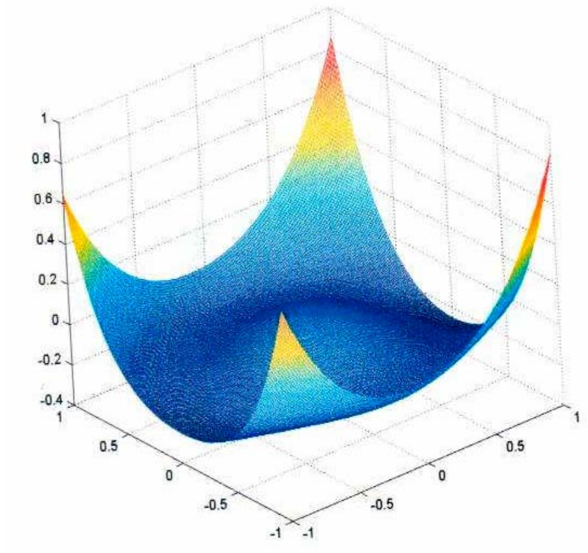

Figure 7. Deterministic chaos in the potential $\Phi_{m,l}(x, y)$. Values of the parameters $\alpha_1 = 1$, $\alpha_2 = 2$, $k_1 = 1$, $k_2 = 0.5$, $m_2 = 1$, $l_2 = 0.1$.

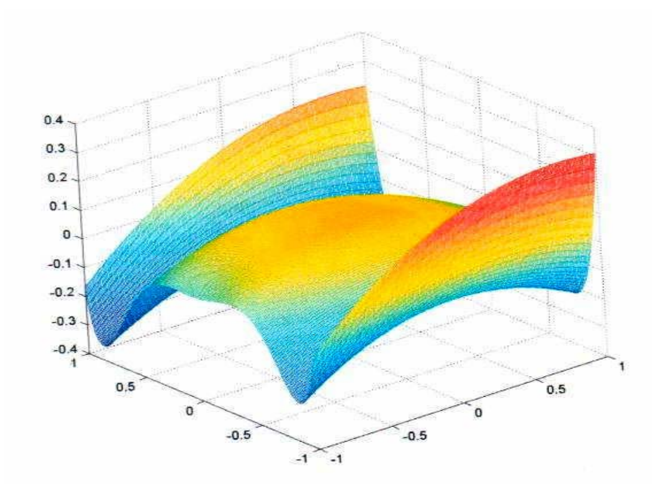

Figure 8. Stochastic chaos in the potential $\Phi_{m,l,\varepsilon}(x, y)$. Values of the parameters $\alpha_1 = 1$, $\alpha_2 = 2$, $k_1 = 1$, $k_2 = 0.5$, $m_1 = 2$, $m_2 = 0$, $l_1 = 1$, $l_2 = 0.1$, $\varepsilon = 0.1$.

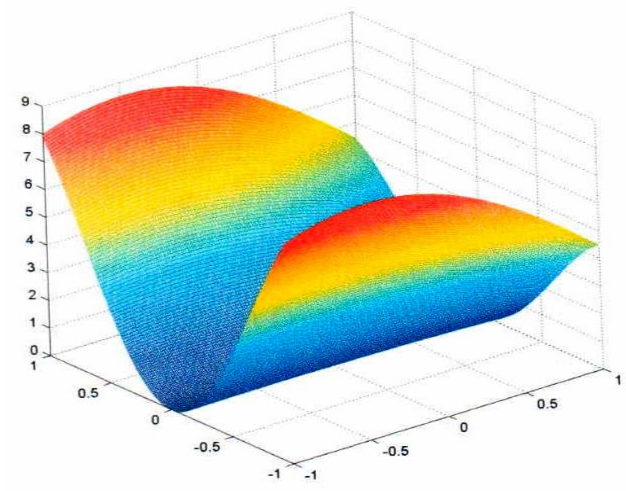

Figure 9. Stochastic chaos in the potential $\Phi_{m, l, \varepsilon}(x, y)$. Values of the parameters $\alpha_1 = 2$, $\alpha_2 = 1$, $k_1 = 1$, $k_2 = 5$, $m_1 = 1$, $m_2 = 0.5$, $l_1 = 2$, $l_2 = 0.5$, $\varepsilon = 0.01$.

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