

# A Note on Relations between Malcev Superalgebras and Bol Superalgebras

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## Abstract

In this paper we prove that any Malcev superalgebra has a Lie supertriple system structure and a Bol superalgebra structure.

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## 1 Introduction

A vector space  $V$  equipped with a trilinear operation  $[\cdot, \cdot, \cdot]$  is called a Lie triple system if

$$[a, a, b] = 0,$$

$$[a, b, c] + [b, c, a] + [c, a, b] = 0,$$

$$[x, y, [a, b, c]] = [[x, y, a], b, c] + [a, [x, y, b], c] + [a, b, [x, y, c]]$$

for all  $x, y, a, b, c \in V$ . A (left) Bol algebra  $(V, [\cdot, \cdot, \cdot], [\cdot, \cdot])$  is a Lie triple system  $(V, [\cdot, \cdot, \cdot])$  with an additional bilinear skew-symmetric operation  $[\cdot, \cdot]$  satisfying

$$[a, b, [c, d]] = [[a, b, c], d] + [c, [a, b, d]] + [c, d, [a, b]] + [[a, b], [c, d]].$$

Bol algebras were introduced in differential geometry to study smooth Bol loops [10],[13], [14].

If  $(L, [, ,])$  is a Lie algebra, then  $(L, [, ,])$  is a Lie triple system where  $[x, y, z] = [[x, y], z]$  [6],[7],[9]. Another construction of Lie triple system from binary algebras is the one from Malcev algebras found by Loos [8]. This result by Loos is extended to Bol algebras setting. Precisely, P. O. Mikheev proved that any Malcev algebra has a Bol algebra structure [10].

In [1], a Malcev superalgebra is introduced while a Lie supertriple system is introduced in [16]. The reader may also refer to [11] for the introduction of Lie triple (Lie-Yamaguti) superalgebras where it is observed that these superalgebras generalize Lie supertriple systems. Note that Lie-Yamaguti superalgebras are examples of binary-ternary superalgebras. Another class of binary-ternary superalgebras is the one of Akinis superalgebras. They are recently introduced in [4] where the study of enveloping superalgebras for this class of algebras and the proof of an analogous of the PBW Theorem, are given. The class of Akinis superalgebras generalizes the one of Lie superalgebras and akivis algebras. As Lie-Yamaguti superalgebras, Bol superalgebras introduced in [12] may also be viewed as some generalization of Lie supertriple systems. In this work, we more specifically focus on the  $\mathbb{Z}_2$ -graded version of Malcev and Bol algebras, called Malcev and Bol superalgebras respectively and we prove a  $\mathbb{Z}_2$ -graded version of Loos and Mikheev result [8], [10] for Malcev superalgebras, Lie supertriple systems and Bol superalgebras .

A description of the rest of this paper is as follows.

The section 2 deals with some basics on superalgebras. In section 3, we recall the definition of Lie supertriple systems and prove some construction theorems. In particular, any Lie superalgebra has a Lie supertriple system structure (Proposition 3.2). This result is mentioned in [16] as well in [17] to give examples of Lie supertriple systems. We also prove a  $\mathbb{Z}_2$ -graded version of a Loos's construction of Lie triple systems from Malcev algebras i.e. any Malcev superalgebra has a Lie supertriple system structure (Theorem 3.5). We then prove that any Malcev superalgebra has a Bol superalgebra structure (Proposition 3.7). This result is a  $\mathbb{Z}_2$ -graded version of the one of P. O. Mikheev found in [10].

In this paper,  $\mathbb{K}$  is an algebraically closed field of characteristic 0.

## 2 Some basics on superalgebras

In this section we recall useful notions on Lie superalgebras as well as the one of Malcev superalgebras [1],[2]. Let  $M$  be a linear superspace over  $\mathbb{K}$  that is a  $\mathbb{Z}_2$ -graded linear space with a direct sum  $M = M_0 \oplus M_1$ . The elements of  $M_j$ ,  $j \in \mathbb{Z}_2$ , are said to be homogeneous of parity  $j$ . The parity of a homogeneous element  $x$  is denoted by  $\bar{x}$ . In the sequel, we will denote by  $\mathcal{H}(M)$  the set of all homogeneous elements of  $M$ . An algebra  $(A, [, ,])$  is called a superalgebra if the underlying vector space is  $\mathbb{Z}_2$ -graded i.e  $A = A_0 \oplus A_1$  and if furthermore  $[A_i, A_j] \subseteq A_{i+j}$  for all  $i, j$  in  $\mathbb{Z}_2$ ,  $i + j$  calculated modulo 2.

**Definition 2.1.** A Lie superalgebra is the superalgebra  $(A = A_0 \oplus A_1, [,])$  satisfying the super skew-symmetry and the super Jacobi identities that is

$$[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x] \tag{1}$$

$$[[x, y], z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[[y, z], x] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[[z, x], y] = 0 \tag{2}$$

for all  $x, y, z \in \mathcal{H}(A)$ . In term of the super Jacobian

$$SJ(x, y, z) = [[x, y], z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[[y, z], x] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[[z, x], y] \tag{3}$$

the super Jacobi identity is written  $SJ(x, y, z) = 0$

for all  $x, y, z \in \mathcal{H}(A)$ .

Another class of superalgebras that is of interest in this paper is the one of Malcev superalgebras.

**Definition 2.2.** [1][2] A superalgebra  $(M = M_0 \oplus M_1, [,])$  is called a Malcev superalgebra if it satisfies the following super identities:

$$\begin{aligned} [x, y] &= -(-1)^{\bar{x}\bar{y}}[y, x] \text{ (super skew-symmetry),} \\ [[x, y], z], t - [x, [[y, z], t]] - (-1)^{\bar{y}(\bar{z}+\bar{t})}[[x, [z, t]], y] - (-1)^{\bar{t}(\bar{y}+\bar{z})}[[x, t], y], z \\ &= (-1)^{\bar{y}\bar{z}}[[x, z], [y, t]] \text{ (super Malcev identity)} \end{aligned} \tag{4}$$

for  $x, y, z, t \in \mathcal{H}(M)$ .

It is observed in [5] that the superanalogues of some well-known identities in any Malcev algebra, hold in Malcev superalgebras, in particular, the following identities hold

$$\begin{aligned} SJ(x, y, [u, v]) &= (-1)^{\bar{u}(\bar{x}+\bar{y})}[u, SJ(x, y, v)] + [SJ(x, y, u), v] \\ &\quad - 2(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}SJ(u, v, [x, y]) \end{aligned} \tag{5}$$

$$\begin{aligned} &SJ(x, y, [t, z]) + (-1)^{\bar{x}\bar{y}+\bar{t}(\bar{x}+\bar{y})}SJ(t, y, [x, z]) \\ &= (-1)^{\bar{t}\bar{z}}[SJ(x, y, z), t] + (-1)^{\bar{x}(\bar{y}+\bar{z}+\bar{t})+\bar{t}\bar{y}}[SJ(t, y, z), x] \end{aligned} \tag{6}$$

for all  $x, y, u, v \in \mathcal{H}(M)$  where  $M$  is any Malcev superalgebra.

We note that, given a multilinear identity  $f$  we can construct a superidentity by the so-called superization rule, see [15]. Simply said, this rule states that whenever the order of two elements  $x, y$  is changed the coefficient  $(-1)^{\bar{x}\bar{y}}$  is added.

### 3 Ternary and binary-ternary superalgebras

In this section, we prove our main results. These results could be seen as a  $\mathbb{Z}_2$ -graded generalization of the result found by Loos [8] and the one found by P. O. Mikheev [10]. We finish the section by giving an example of a 4-dimensional Bol superalgebra. First we have the following definition.

**Definition 3.1.** (1) A *supertriple system* is a couple  $(S = S_0 \oplus S_1, [, , ])$  consisting of a  $\mathbb{Z}_2$ -graded  $\mathbb{K}$ -vector space  $S = S_0 \oplus S_1$  and a  $\mathbb{K}$ -trilinear map  $[, , ]$  satisfying  $[S_i, S_j, S_k] \subseteq S_{i+j+k}$  for all  $i, j, k$  in  $\mathbb{Z}_2$ ,  $i + j + k$  calculated modulo 2 such that for all  $x, y, z, \in \mathcal{H}(S)$ :

$$\begin{aligned} [x, y, z] &= -(-1)^{\bar{x}\bar{y}}[y, x, z] \text{ (left super skew-symmetry)} \\ [x, y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, x] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, x, y] &= 0 \\ &\text{(super ternary Jacobi identity)} \end{aligned}$$

(2) A *Lie supertriple system* [16],[17] is a supertriple system  $(S = S_0 \oplus S_1, [, , ])$  such that the super ternary Nambu identity

$$\begin{aligned} [x, y, [u, v, w]] &= [[x, y, u], v, w] + (-1)^{\bar{u}(\bar{x}+\bar{y})}[u, [x, y, v], w] \\ &\quad + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}[u, v, [x, y, w]] \end{aligned} \quad (7)$$

holds for all  $x, y, u, v, w \in \mathcal{H}(S)$ .

The following result is a  $\mathbb{Z}_2$ -graded generalization of the construction of Lie triple systems from Lie algebras [6],[7],[9]. Such result is mentioned in [16] as well in [17] to give examples of Lie supertriple systems.

**Proposition 3.2.** Let  $(A = A_0 \oplus A_1, [, , ])$  be a Lie superalgebra. Define on  $A$  the ternary operation

$$[x, y, z] = [[x, y], z] \quad (8)$$

for all  $x, y, z \in \mathcal{H}(A)$ . Then  $(A = A_0 \oplus A_1, [, , ])$  is a Lie supertriple system.

**Proof.** We have  $[A_i, A_j, A_k] \subseteq A_{i+j+k}$ . Next the left super skew-symmetry of  $[, , ]$  and the super ternary Jacobi identity follow from the super skew-symmetry of  $[, ]$  and the super Jacobi identity. The super ternary Nambu identity is proved as follows

$$\begin{aligned} &[x, y, [u, v, w]] \\ &= [[x, y], [[u, v], w]] \text{ by (8)} \\ &= -(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}[[[u, v], w], [x, y]] \text{ (by the super skew-symmetry of [, ] )} \\ &= -(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})} \{ -(-1)^{(\bar{u}+\bar{v})(\bar{x}+\bar{y}+\bar{w})}[[w, [x, y]], [u, v]] \\ &\quad -(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}[[[x, y], [u, v]], w] \} \text{ ( by (2) )} \\ &= (-1)^{\bar{w}(\bar{x}+\bar{y})+\bar{w}(\bar{u}+\bar{v})}[[w, [x, y]], [u, v]] + [[x, y], [u, v], w] \\ &= (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}[[u, v], [[x, y], w]] - (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}[[[u, v], [x, y]], w] \\ &\text{(by the super skew-symmetry of [, ] )} \end{aligned} \quad (9)$$

$$\begin{aligned}
 &= (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} [[u, v], [[x, y], w]] \\
 &- (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} \{ -(-1)^{\bar{u}(\bar{x}+\bar{y}+\bar{v})} [[[v, [x, y]], u], w] \\
 &- (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} [[[[x, y], u], v], w] \} \text{ ( by (2) )} \\
 &= [[[[x, y], u], v], w] + (-1)^{\bar{u}(\bar{x}+\bar{y})} [u, [[x, y], v]], w] \\
 &+ (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} [[u, v], [[x, y], w]] \\
 &\text{(rearranging terms and using the super skew-symmetry of [ , ] )} \\
 &= [[x, y, u], v, w] + (-1)^{\bar{u}(\bar{x}+\bar{y})} [u, [x, y, v], w] \\
 &+ (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} [u, v, [x, y, w]] \text{ (by (8) ).}
 \end{aligned}$$

Therefore  $(A = A_0 \oplus A_1, [ , , ])$  is a Lie supertriple system. □

In any Malcev superalgebra  $M$ , consider the ternary operation defined by

$$[x, y, z] := 2[[x, y], z] - (-1)^{\bar{x}(\bar{y}+\bar{z})} [[y, z], x] - (-1)^{\bar{z}(\bar{x}+\bar{y})} [[z, x], y] \quad (10)$$

From (10) it clearly follows that  $[ , , ]$  can also be written as

$$[x, y, z] = -SJ(x, y, z) + 3[[x, y], z] \quad (11)$$

for all  $x, y, z \in \mathcal{H}(M)$ . Then we have:

**Lemma 3.3.** *Let  $(M = M_0 \oplus M_1, [ , , ])$  be a Malcev superalgebra. Then*

$$\begin{aligned}
 [x, y, [u, v]] &= (-1)^{\bar{u}(\bar{x}+\bar{y})} [u, [x, y, v]] + [[x, y, u], v] \\
 &- (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} SJ(u, v, [x, y])
 \end{aligned} \quad (12)$$

holds for all  $x, y, u, v \in \mathcal{H}(M)$ .

**Proof.** By (5), we have

$$\begin{aligned}
 -SJ(x, y, [u, v]) &= (-1)^{\bar{u}(\bar{x}+\bar{y})} [u, -SJ(x, y, v)] + [-SJ(x, y, u), v] \\
 &+ 3(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} SJ(u, v, [x, y]) \\
 &- (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} SJ(u, v, [x, y])
 \end{aligned}$$

i.e.

$$\begin{aligned}
 -SJ(x, y, [u, v]) &= (-1)^{\bar{u}(\bar{x}+\bar{y})} [u, -SJ(x, y, v)] + [-SJ(x, y, u), v] \\
 &+ 3(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} [[u, v], [x, y]] + 3(-1)^{\bar{v}(\bar{u}+\bar{x}+\bar{y})} [[v, [x, y]], u] \\
 &+ 3[[x, y], u], v] - (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} SJ(u, v, [x, y]).
 \end{aligned}$$

Therefore by the super skew-symmetry of  $[ , ]$ , we have

$$\begin{aligned}
 -SJ(x, y, [u, v]) + 3[[x, y], [u, v]] &= (-1)^{\bar{u}(\bar{x}+\bar{y})} [u, -SJ(x, y, v) + 3[[x, y], v]] \\
 + [-SJ(x, y, u) + 3[[x, y], u], v] &- (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})} SJ(u, v, [x, y]) \text{ and thus we} \\
 \text{get (12) by (11).} & \quad \square
 \end{aligned}$$

We have also the following useful result.

**Lemma 3.4.** *Let  $(M = M_0 \oplus M_1, [, ,])$  be a Malcev superalgebra. Then*

$$\begin{aligned}
& N(u, v, x, y, w) \\
&= -2(-1)^{(\bar{u}+\bar{v})(\bar{x}+\bar{y})}[SJ(u, v, [x, y]), w] - 2(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([u, v], w, [x, y]) \\
&+ (-1)^{(\bar{v}+\bar{w})(\bar{x}+\bar{y}+\bar{u})}[SJ(v, w, [x, y]), u] + (-1)^{\bar{u}(\bar{v}+\bar{w})+(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([v, w], u, [x, y]) \\
&+ (-1)^{\bar{w}(\bar{u}+\bar{v})+(\bar{u}+\bar{w})(\bar{x}+\bar{y})}[SJ(w, u, [x, y]), v] \\
&+ (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})+\bar{w}(\bar{u}+\bar{v})}SJ([w, u], v, [x, y]) = 0
\end{aligned} \tag{13}$$

for all  $x, y, u, v, w \in \mathcal{H}(M)$ .

**Proof.** Observe that by (6), we have:

$$\begin{aligned}
& SJ(u, [x, y], [v, w]) + (-1)^{\bar{v}(\bar{x}+\bar{y})+\bar{u}(\bar{v}+\bar{x}+\bar{y})}SJ(v, [x, y], [u, w]) \\
&= (-1)^{\bar{v}\bar{w}}[SJ(u, [x, y], w), v] + (-1)^{\bar{v}(\bar{x}+\bar{y})+\bar{u}(\bar{v}+\bar{w}+\bar{x}+\bar{y})}[SJ(v, [x, y], w), u]
\end{aligned}$$

i.e.

$$\begin{aligned}
& (-1)^{\bar{w}(\bar{u}+\bar{v})+(\bar{x}+\bar{y})(\bar{u}+\bar{w})}[SJ(w, u, [x, y]), v] + (-1)^{\bar{w}(\bar{u}+\bar{v})+(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([w, u], v, [x, y]) \\
&= (-1)^{\bar{u}(\bar{v}+\bar{w})+(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([v, w], u, [x, y]) + (-1)^{(\bar{v}+\bar{w})(\bar{x}+\bar{y}+\bar{u})}SJ(v, w, [x, y]), u]
\end{aligned}$$

and therefore we have

$$\begin{aligned}
& N(u, v, x, y, w) \\
&= -2(-1)^{(\bar{u}+\bar{v})(\bar{x}+\bar{y})}[SJ(u, v, [x, y]), w] - 2(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([u, v], w, [x, y]) \\
&+ 2(-1)^{(\bar{v}+\bar{w})(\bar{x}+\bar{y}+\bar{u})}[SJ(v, w, [x, y]), u] + 2(-1)^{\bar{u}(\bar{v}+\bar{w})+(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([v, w], u, [x, y]) \\
&= -2(-1)^{\bar{w}(\bar{x}+\bar{y}+\bar{u}+\bar{v})}SJ(w, [x, y], [u, v]) - 2(-1)^{\bar{u}(\bar{x}+\bar{y})+\bar{v}\bar{w}}SJ(u, [x, y], [w, v]) \\
&+ 2(-1)^{\bar{w}(\bar{x}+\bar{y}+\bar{u}+\bar{v})+\bar{u}\bar{v}}[SJ(w, [x, y], v), u] + 2(-1)^{\bar{u}(\bar{x}+\bar{y})}[SJ(u, [x, y], v), w] \\
&= -2(-1)^{\bar{w}(\bar{x}+\bar{y}+\bar{u}+\bar{v})}\{SJ(w, [x, y], [u, v]) + (-1)^{\bar{w}[x, y]+\bar{u}(\bar{w}+[x, y])}SJ(u, [x, y], [w, v])\} \\
&- (-1)^{\bar{u}\bar{v}}[SJ(w, [x, y], v), u] - (-1)^{\bar{w}([x, y]+\bar{u}+\bar{v})+\bar{u}[x, y]}[SJ(u, [x, y], v), w] = 0 \text{ by (6)}. \square
\end{aligned}$$

One of the main result of this paper is the following. It is a  $\mathbb{Z}_2$ -graded generalization of the construction of Lie triple systems from Malcev algebras [8].

**Theorem 3.5.** *Let  $(M = M_0 \oplus M_1, [, ,])$  be a Malcev superalgebra. Define on  $M$  the ternary operation by (10) then  $(M = M_0 \oplus M_1, [, ,])$  is a Lie supertriple system.*

**Proof.** We have  $[M_i, M_j, M_k] \subseteq M_{i+j+k}$  and the left super skew-symmetry of  $[, ,]$  is quite obvious. The super ternary Jacobi identity is proved as follows

$$\begin{aligned}
& [x, y, z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[y, z, x] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[z, x, y] \\
&= -SJ(x, y, z) + 3[[x, y], z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}(-SJ(y, z, x) + 3[[y, z], x]) \\
&+ (-1)^{\bar{z}(\bar{x}+\bar{y})}(-SJ(z, x, y) + 3[[z, x], y]) \text{ (by (11))} \\
&= -SJ(x, y, z) - (-1)^{\bar{x}(\bar{y}+\bar{z})}SJ(y, z, x) - (-1)^{\bar{z}(\bar{x}+\bar{y})}SJ(z, x, y) + 3SJ(x, y, z) \\
&= -3SJ(x, y, z) + 3SJ(x, y, z) = 0.
\end{aligned}$$

Thus  $(M = M_0 \oplus M_1, [, ,])$  is a supertriple system. To conclude the result, we proceed to prove the super ternary Nambu identity (7) for  $(M = M_0 \oplus M_1, [, ,])$ . We have:

$$\begin{aligned}
 & [x, y, [u, v, w]] \\
 &= 2[x, y, [[u, v], w]] - (-1)^{\bar{u}(\bar{v}+\bar{w})}[x, y, [[v, w], u]] - (-1)^{\bar{w}(\bar{u}+\bar{v})}[x, y, [[w, u], v]] \text{ (by (10) )} \\
 &= 2((-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}[[u, v], [x, y, w]] + [[x, y, [u, v]], w] + \\
 &\quad - (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([u, v], w, [x, y])) - (-1)^{\bar{u}(\bar{v}+\bar{w})}((-1)^{(\bar{x}+\bar{y})(\bar{v}+\bar{w})}[[v, w], [x, y, u]] + \\
 &\quad + [[x, y, [v, w]], u] - (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([v, w], u, [x, y])) \\
 &\quad - (-1)^{\bar{w}(\bar{u}+\bar{v})}((-1)^{(\bar{u}+\bar{w})(\bar{x}+\bar{y})}[[w, u], [x, y, v]] + [[x, y, [w, u]], v] \\
 &\quad - (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([w, u], v, [x, y])) \text{ (by (12))} \\
 &= 2(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}[[u, v], [x, y, w]] + 2(-1)^{\bar{u}(\bar{x}+\bar{y})}[[u, [x, y, v]], w] \\
 &\quad + 2[[x, y, u], v, w] - 2(-1)^{(\bar{u}+\bar{v})(\bar{x}+\bar{y})}[SJ(u, v, [x, y]), w] \\
 &\quad - 2(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([u, v], w, [x, y]) - (-1)^{(\bar{x}+\bar{y}+\bar{u})(\bar{v}+\bar{w})}[[v, w], [x, y, u]] \\
 &\quad - (-1)^{\bar{v}(\bar{x}+\bar{y}+\bar{u})(\bar{v}+\bar{w})}[[v, [x, y, w]], u] - (-1)^{\bar{u}(\bar{v}+\bar{w})}[[x, y, v], w, u] \\
 &\quad + (-1)^{(\bar{v}+\bar{w})(\bar{x}+\bar{y}+\bar{u})}[SJ(v, w, [x, y]), u] + (-1)^{\bar{u}(\bar{v}+\bar{w})+(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([v, w], u, [x, y]) \\
 &\quad - (-1)^{\bar{w}(\bar{u}+\bar{v})+(\bar{u}+\bar{w})(\bar{x}+\bar{y})}[[w, u], [x, y, v]] \\
 &\quad - (-1)^{\bar{w}(\bar{x}+\bar{y}+\bar{u}+\bar{v})}[[w, [x, y, u]], v] - (-1)^{\bar{w}(\bar{u}+\bar{v})}[[x, y, w], u, v] \\
 &\quad + (-1)^{\bar{w}(\bar{u}+\bar{v})+(\bar{u}+\bar{w})(\bar{x}+\bar{y})}[SJ(w, u, [x, y]), v] \\
 &\quad + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})+\bar{w}(\bar{u}+\bar{v})}SJ([w, u], v, [x, y]) \text{ ( again by (12) )} \\
 &= \{2[[x, y, u], v, w] - (-1)^{(\bar{x}+\bar{y}+\bar{u})(\bar{v}+\bar{w})}[[v, w], [x, y, u]] \\
 &\quad - (-1)^{\bar{w}(\bar{x}+\bar{y}+\bar{u}+\bar{v})}[[w, [x, y, u]], v]\} + (-1)^{\bar{u}(\bar{x}+\bar{y})}\{2[[u, [x, y, v]], w] \\
 &\quad - (-1)^{\bar{u}(\bar{x}+\bar{y}+\bar{v}+\bar{w})}[[x, y, v], w, u] - (-1)^{\bar{w}(\bar{x}+\bar{y}+\bar{u}+\bar{v})}[[w, u], [x, y, v]]\} \\
 &\quad (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}\{2[[u, v], [x, y, w]] - (-1)^{\bar{u}(\bar{x}+\bar{y}+\bar{v}+\bar{w})}[[v, [x, y, w]], u] \\
 &\quad - (-1)^{(\bar{u}+\bar{v})(\bar{x}+\bar{y}+\bar{w})}[[x, y, w], u, v]\} - 2(-1)^{(\bar{u}+\bar{v})(\bar{x}+\bar{y})}[SJ(u, v, [x, y]), w] \\
 &\quad - 2(-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([u, v], w, [x, y]) + (-1)^{(\bar{v}+\bar{w})(\bar{x}+\bar{y}+\bar{u})}[SJ(v, w, [x, y]), u] \\
 &\quad + (-1)^{\bar{u}(\bar{v}+\bar{w})+(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})}SJ([v, w], u, [x, y]) \\
 &\quad + (-1)^{\bar{w}(\bar{u}+\bar{v})+(\bar{u}+\bar{w})(\bar{x}+\bar{y})}[SJ(w, u, [x, y]), v] \\
 &\quad + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v}+\bar{w})+\bar{w}(\bar{u}+\bar{v})}SJ([w, u], v, [x, y]) \\
 &= [[x, y, u], v, w] + (-1)^{\bar{u}(\bar{x}+\bar{y})}[u, [x, y, v], w] \\
 &\quad + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}[u, v, [x, y, w]] + N(u, v, x, y, w).
 \end{aligned}$$

Hence, the conclusion holds by (13). □

As Lie-Yamaguti superalgebras, another generalization of Lie supertriple systems is the category of Bol superalgebras. They are also a  $\mathbb{Z}_2$ -graded generalization of the category of Bol algebras which are introduced in differential geometry to study smooth Bol loops [10],[13], [14].

**Definition 3.6.** [12] *A Bol superalgebra is a  $\mathbb{Z}_2$ -graded vector space*

$B = B_0 \oplus B_1$  with a binary operation  $[\cdot, \cdot]$  satisfying  $[B_i, B_j] \subseteq B_{i+j}$  and a ternary operation  $\{\cdot, \cdot, \cdot\}$  satisfying  $\{B_i, B_j, B_k\} \subseteq B_{i+j+k}$  ( $i, j, k \in \mathbb{Z}_2$ ,  $i + j$  and  $i + j + k$  calculated modulo 2) such that:

1.  $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$ ,
2.  $\{x, y, z\} = -(-1)^{\bar{x}\bar{y}}\{y, x, z\}$ ,
3.  $\{x, y, z\} + (-1)^{\bar{x}(\bar{y}+\bar{z})}\{y, z, x\} + (-1)^{\bar{z}(\bar{x}+\bar{y})}\{z, x, y\} = 0$ ,
4.  $\{x, y, \{u, v, w\}\} = \{\{x, y, u\}, v, w\} + (-1)^{\bar{u}(\bar{x}+\bar{y})}\{u, \{x, y, v\}, w\} + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}\{u, v, \{x, y, w\}\}$ ,
5.  $\{x, y, [u, v]\} = (-1)^{\bar{u}(\bar{x}+\bar{y})}[u, \{x, y, v\}] + [\{x, y, u\}, v] + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}\{u, v, [x, y]\} + [[x, y], [u, v]]$ ,

for all  $x, y, z, u, v, w \in \mathcal{H}(B)$ .

**Remark:** A Bol superalgebra is a Lie supertriple system  $(B = B_0 \oplus B_1, \{\cdot, \cdot, \cdot\})$  with a super skew-symmetric binary map  $[\cdot, \cdot]$  satisfying  $[B_i, B_j] \subseteq B_{i+j}$  for all  $i, j$  in  $\mathbb{Z}_2$ ,  $i + j$  calculated modulo 2 such that

$$\begin{aligned} \{x, y, [u, v]\} &= (-1)^{\bar{u}(\bar{x}+\bar{y})}[u, \{x, y, v\}] + [\{x, y, u\}, v] \\ &\quad + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}\{u, v, [x, y]\} + [[x, y], [u, v]] \end{aligned} \quad (14)$$

for all  $x, y, u, v \in \mathcal{H}(B)$ .

Now, we give a  $\mathbb{Z}_2$ -graded generalization of the construction of Bol algebras from Malcev algebras [10].

**Proposition 3.7.** Let  $(M = M_0 \oplus M_1, [\cdot, \cdot])$  be a Malcev superalgebra. Define on  $M$  the ternary operation

$$\{x, y, z\} := \frac{1}{3}[x, y, z] \quad (15)$$

where  $[\cdot, \cdot]$  is defined as in (11) for all  $x, y, z \in \mathcal{H}(M)$ . Then  $(M = M_0 \oplus M_1, [\cdot, \cdot], \{\cdot, \cdot, \cdot\})$  is a Bol superalgebra.

**Proof.** We have  $[M_i, M_j] \subseteq M_{i+j}$ . Next the definition (15) and Theorem 3.5 imply that  $(M = M_0 \oplus M_1, \{\cdot, \cdot, \cdot\})$  is a Lie supertriple system. Finally, we are done if we prove (14) for  $(M = M_0 \oplus M_1, [\cdot, \cdot], \{\cdot, \cdot, \cdot\})$ . Note that by (11), we have

$$-SJ(u, v, [x, y]) = [u, v, [x, y]] - 3[[u, v], [x, y]]$$

and therefore by the super skew-symmetry of  $[\cdot, \cdot]$ , (12) takes the form

$$\begin{aligned} [x, y, [u, v]] &= (-1)^{\bar{u}(\bar{x}+\bar{y})}[u, [x, y, v]] + [[x, y, u], v] \\ &\quad + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}[u, v, [x, y]] + 3[[x, y], [u, v]] \end{aligned} \quad (16)$$

Multiplying each member of (16) by  $\frac{1}{3}$  and using (15), we have

$$\begin{aligned} \{x, y, [u, v]\} &= (-1)^{\bar{u}(\bar{x}+\bar{y})}[u, \{x, y, v\}] + [\{x, y, u\}, v] \\ &\quad + (-1)^{(\bar{x}+\bar{y})(\bar{u}+\bar{v})}\{u, v, [x, y]\} + [[x, y], [u, v]] \end{aligned}$$

which is (14) and therefore  $(M = M_0 \oplus M_1, [, ], \{, \}, \cdot)$  is a Bol superalgebra.  $\square$

From this result, we have the following example of Bol superalgebras.

**Example 3.8.** Let  $(L^2(3, 1), [, ], \{, \}, \cdot)$  be a non-Lie Malcev superalgebra [3] defined with respect to a basis  $(e_1, e_2, e_3, e_4)$  where  $L^2(3, 1)_{\bar{0}} = \text{span}(e_1, e_2, e_3)$ ,  $L^2(3, 1)_{\bar{1}} = \text{span}(e_4)$  with the nonzero products given by  $[e_1, e_3] = e_1$ ,  $[e_2, e_3] = e_1 + e_2$ ,  $[e_3, e_4] = e_4$ ,  $[e_4, e_4] = e_1$ . According to Proposition 3.7,  $(L^2(3, 1), [, ], \{, \}, \cdot)$  is a 4-dimensional Bol superalgebra with  $L^2(3, 1)_{\bar{0}} = \text{span}(e_1, e_2, e_3)$ ,  $L^2(3, 1)_{\bar{1}} = \text{span}(e_4)$ , where the nonzero products are given by  $[e_1, e_3] = e_1$ ,  $[e_2, e_3] = e_1 + e_2$ ,  $[e_3, e_4] = e_4$ ,  $[e_4, e_4] = e_1$  and  $\{e_1, e_3, e_3\} = e_1$ ,  $\{e_2, e_3, e_3\} = 2e_1 + e_2$ ,  $\{e_3, e_4, e_3\} = -e_4$ .  $\square$

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