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PRINCIPAL EIGENVALUE FOR QUASILINEAR COOPERATIVE ELLIPTIC SYSTEMS

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Abstract. We study the first eigenvalue of a λ-dependent cooperative elliptic system involving two quasilinear operators. By variational arguments, we find an expression for the limit of this eigenvalue as $\lambda \to -\infty$, which improves and extends (for gradient quasilinear systems) a result proved by Álvarez Caudevilla-López Gómez [4] and Dancer [9]. We apply this result to deduce the existence of strictly principal eigenvalues (i.e., whose eigenfunctions have both components positive) of a weighted system and extend the results proved in Cuesta-Ramos Quoirin [7] for the scalar case.

1. Introduction and statement of the main results

Let Ω be a bounded domain of $\mathbb{R}^N$ and α, β, p, q be constants such that

$$\alpha \geq 0, \beta \geq 0, p > 1, q > 1 \quad \text{and} \quad \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

In this article, we study the eigenvalue system

$$\begin{cases} -\Delta_p u - \lambda a(x)|u|^{p-2}u - b(x)|u|^{\alpha-1}u|v|^{\beta}v = \mu|u|^{p-2}u & \text{in} \quad \Omega, \\ -\Delta_q v - \lambda d(x)|v|^{q-2}v - b(x)|u|^{\alpha}u|v|^{\beta-1}v = \mu|v|^{q-2}v & \text{in} \quad \Omega, \\ u = v = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u \stackrel{\text{def}}{=} div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and λ is a real parameter. The coefficients a, d are allowed to change sign in Ω, while b is assumed to be non-negative in Ω, so the above system is cooperative.

The motivation for considering (1.1) is twofold. Firstly, we are interested in studying its first eigencurve, which is a λ-dependent curve set as the first

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eigenvalue of (1.1). The study of the behavior of this curve goes back to [13] in the linear scalar case and allows one to deduce the existence of positive solutions (on both components) for the system

$$\begin{cases} -\Delta_p u - b(x)|u|^{\alpha-1}u|v|^{\beta}v = \lambda a(x)|u|^{p-2}u & \text{in} \quad \Omega, \\ -\Delta_q v - b(x)|u|^{\alpha}u|v|^{\beta-1}v = \lambda d(x)|v|^{q-2}v & \text{in} \quad \Omega, \\ u = v = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

It is important to note that (1.2) has "semitrivial" solutions of the form $(u, 0)$ and $(0, v)$, and consequently $\lambda_{1,p}(a)$ and $\lambda_{1,q}(d)$ are "semitrivial" principal eigenvalues of (1.2). Here $\lambda_{1,p}(m)$ is the positive principal eigenvalue of

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega).$$

In view of this fact, let us agree to say that an eigenvalue of (1.1) or (1.2) is **strictly principal** if it associated to an eigenfunction (u, v) such that $u, v > 0$. Principal eigenvalues of quasilinear systems such as (1.1) and (1.2) have been investigated by several authors over the last years (see, e.g., [10, 16, 17]).

In [7], the authors have investigated the scalar version of (1.2), which reads

$$(P_m) \qquad \begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

They proved that the existence of principal eigenvalues for this equation depends on the value

$$\alpha(V, m) \stackrel{\text{def}}{=} \inf\Big\{ \int_\Omega (|\nabla u|^p + V(x)|u|^p) : u \in W_0^{1,p}(\Omega), \int_\Omega |u|^p = 1, \int_\Omega m(x)|u|^p = 0 \Big\}.$$

Roughly speaking, the positivity of $\alpha(V, m)$ is a necessary and sufficient condition for (P_m) to admit a principal eigenvalue. In addition, it is shown that if $m \geq 0$ and $\Omega_0^m \stackrel{\text{def}}{=} \Omega \setminus \operatorname{supp} m$ is sufficiently regular, then $\alpha(V, m)$ is the first eigenvalue of

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda |u|^{p-2}u & \text{in} \quad \Omega_0^m, \\ u = 0 & \text{on} \quad \partial\Omega_0^m. \end{cases}$$

This result, for a general second-order uniformly elliptic and linear operator, but under stronger assumptions on m and Ω_0^m, has been established in [14].

The second motivation for studying (1.1) comes from [2, 4, 5, 9], where the main issue addressed is the asymptotic behavior of the first eigenvalue of the linear cooperative system

$$\begin{cases} (L_1+\lambda a)u - bv = \tau u & \text{in} \quad \Omega, \\ (L_2+\lambda d)v - cu = \tau v & \text{in} \quad \Omega, \\ u = v = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

Here $L_k u \stackrel{\text{def}}{=} -\partial_i(\alpha_{ij}^k \partial_j u) + \beta_i^k \partial_i u + \gamma^k$, $k=1,2$, are two uniformly strongly elliptic operators; i.e., $\alpha_{ij}^k(x)\eta_i\eta_j \geq \sigma|\eta|^2$ for some $\sigma > 0$, $k=1,2$, and every $x \in \Omega$ and $\eta \in \mathbb{R}^2$. Moreover, $\alpha_{ij}^k, \beta_i^k, \gamma^k, a, b, c, d \in L^\infty(\Omega)$, $a, d \geq 0$, and $b, c > 0$ almost everywhere in Ω. This system is important in population dynamics models where the species are cooperative and the environment is nonhomogeneous in the space variable, as discussed in [3, 4, 9].

In [9], the author improves the main result of [4]. He assumes that Ω can be decomposed in two parts A_1, Z_1 such that A_1 is open and

$$(H_D) \qquad \begin{cases} \Omega = \overline{A}_1 \cup Z_1, \quad \overline{Z}_1 = \Omega \setminus \overline{A}_1 \\ a \equiv 0 \text{ on } \overline{A}_1, \quad a > 0 \ a.e. \text{ on } Z_1 \\ H_0^1(A_1) = \{u \in H_0^1(\Omega);\ u = 0 \ a.e. \text{ on } \Omega \setminus \overline{A}_1\}. \end{cases}$$

The latter assumption is a regularity condition on A_1, which is discussed for instance in [8, 11]. Moreover, A_1 has only finitely many components. A similar decomposition is assumed involving A_2, the set where d vanishes. Let $\tau(\lambda)$ denote the first eigenvalue of (1.3). The author shows that $\lim_{\lambda\to\infty} \tau(\lambda) = \hat{\tau}$, where $\hat{\tau}$ is the first eigenvalue of

$$\begin{cases} L_1 u - bPv = \tau u, \\ L_2 v - cPu = \tau v, \\ u \in H_0^1(A_1), v \in H_0^1(A_2). \end{cases} \tag{1.4}$$

Here $P\psi \stackrel{\text{def}}{=} \chi_{A_1\cap A_2}\psi$, where χ is the characteristic function. Moreover,

$$\hat{\tau} \leq \min\{\hat{\lambda}_1(A_1), \hat{\lambda}_1(A_2)\},$$

where $\hat{\lambda}_1(A_k)$ is the principal eigenvalue of $L_k u = \lambda u$, $u \in H_0^1(A_k)$, $k=1,2$. An alternative proof of this result by a Γ-convergence approach and under simpler assumptions has been provided by [2].

Given a measurable function w defined on Ω, we set $\Omega_0^w \stackrel{\text{def}}{=} \Omega \setminus \operatorname{supp} w$, where $\operatorname{supp} w$ denotes the support of w in the measurable sense.

One of our purposes here is to simplify as much as possible the above decomposition condition on Ω. To this aim, we shall assume a regularity condition on the sets where a and d vanish. More precisely, we use the notion of p-stability, which we recall now.

An open set $\mathrm{E} \subset \mathbb{R}^N$ is said to be *p-stable* (in the capacity sense) if, for every $u \in W^{1,p}(\mathbb{R}^N)$,

$$u = 0 \textit{ quasi-everywhere } \text{ in } \mathbb{R}^N \setminus \overline{\mathrm{E}} \implies u = 0 \textit{ quasi-everywhere } \text{ in } \mathbb{R}^N \setminus \mathrm{E}. \tag{1.5}$$

Let E be an open subset of Ω. Recall the following characterization of $W_0^{1,p}(\mathrm{E})$ (cf. [12, Theorem 3.3.42]):

$$u \in W_0^{1,p}(\mathrm{E}) \Longleftrightarrow u \in W_0^{1,p}(\Omega) \text{ and } \tilde{u} = 0 \textit{ quasi-everywhere on } \Omega \setminus \mathrm{E},$$

where $\tilde{u}$ denotes the unique *quasi-continuous* representative of u. From (1.5), it follows that if E is a p-stable subset of Ω then

$$u \in W_0^{1,p}(\mathrm{E}) \Longleftrightarrow u \in W_0^{1,p}(\Omega) \text{ and } \tilde{u} = 0 \textit{ quasi-everywhere on } \Omega \setminus \overline{\mathrm{E}}. \tag{1.6}$$

Therefore, if $a \geq 0$ and Ω_0^a is p-stable then we can deduce that $u \in W_0^{1,p}(\Omega_0^a)$ whenever $\int_\Omega a(x)|u|^p = 0$ (as shown in the proof of [7, Proposition 11]). Note also that the third assumption in (H_D) is equivalent to (1.6). Let us remark that any Lipschitzian domain (more generally, any domain satisfying the uniform exterior cone property) is a p-stable set. We refer to [6, 12] for other characterizations of p-stable sets and more details on this subject.

In contrast with [2, 4, 9], our method is solely variational, and, as a consequence, we only deal with the gradient version of (1.3). However, our approach works in a quasilinear setting and applies also if a or d change sign.

The system (1.1) is a particular case of

$$\begin{cases} -\Delta_p u - \lambda a(x)|u|^{p-2}u - b(x)|u|^{\alpha-1}u|v|^{\beta}v = \mu|u|^{p-2}u & \text{in } \Omega \\ -\Delta_q v - \lambda d(x)|v|^{q-2}v - c(x)|u|^{\alpha}u|v|^{\beta-1}v = \mu|v|^{q-2}v & \text{in } \Omega \\ u = v = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.7}$$

which reduces to (1.3) when $\alpha = \beta = 0$, $p = q = 2$ and $L_1 = L_2 = -\Delta$. Actually, (1.7) is a gradient-type system if and only if $b = c$, in which case it reduces to (1.1).

We assume throughout this work that the weights a, b, d satisfy the following conditions:

(H1) $a \in L^r(\Omega)$ where $\begin{cases} r > N/p & \text{if } 1 < p \leq N, \\ r > 1 & \text{if } p > N. \end{cases}$

(H2) $d \in L^s(\Omega)$ where $\begin{cases} s > N/q & \text{if } 1 < q \leq N, \\ r > 1 & \text{if } q > N. \end{cases}$

(H3) $b(x) \in L^r(\Omega) \cap L^s(\Omega)$ and $b(x) \geq 0$ in Ω.

We refer to $(u,v) \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)\setminus\{(0,0)\}$ as a (weak) eigenfunction of (1.1) associated to the eigenvalue μ provided

$$\int_\Omega \Big(|\nabla u|^{p-2}\nabla u\nabla\phi - \lambda a(x)|u|^{p-2}u\phi - b(x)|u|^{\alpha-1}|v|^{\beta+1}u\phi\Big) = \mu\int_\Omega |u|^{p-2}u\phi$$

and

$$\int_\Omega \Big(|\nabla v|^{q-2}\nabla v\nabla\psi - \lambda d(x)|v|^{q-2}v\psi - b(x)|v|^{\beta-1}|u|^{\alpha+1}v\psi\Big) = \mu\int_\Omega |v|^{q-2}v\psi$$

for every $(\phi,\psi)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. These equalities are equivalent to

$$\nabla J_\lambda(u,v) = \mu\nabla I(u,v),$$

where

$$J_\lambda(u,v) \stackrel{\text{def}}{=} \frac{\alpha+1}{p}\int_\Omega(|\nabla u|^p - \lambda a(x)|u|^p) + \frac{\beta+1}{q}\int_\Omega(|\nabla v|^q - \lambda d(x)|v|^q) - \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1},$$

and

$$I(u,v) \stackrel{\text{def}}{=} \int_\Omega\left(\frac{\alpha+1}{p}|u|^p + \frac{\beta+1}{q}|v|^q\right). \tag{1.8}$$

These functionals are respectively weakly lower semi-continuous and weakly continuous on $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ whenever $(H1)-(H3)$ are satisfied. For convenience, we also set

$$A(u) \stackrel{\text{def}}{=} \int_\Omega a(x)|u|^p, \quad D(v) \stackrel{\text{def}}{=} \int_\Omega d(x)|v|^q, \quad \text{and}$$

$$B(u,v) \stackrel{\text{def}}{=} \int_\Omega b(x)|u|^{\alpha+1}|v|^{\beta+1},$$

for $u\in W_0^{1,p}(\Omega)$ and $v\in W_0^{1,q}(\Omega)$. Let

$$\mathcal{M} \stackrel{\text{def}}{=} \{(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega) : I(u,v)=1\}, \tag{1.9}$$

which is a C^1 submanifold of $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$. By the Lagrange multipliers rule, critical points of J_λ constrained to $\mathcal{M}$ correspond to eigenfunctions of (1.1).

We state now our main results.

Theorem 1.1. *Under* $(H1)$, $(H2)$ *and* $(H3)$, *the system* (1.1) *has a first eigenvalue, which is given by*

$$\mu_1(\lambda) \stackrel{\text{def}}{=} \inf_{(u,v)\in\mathcal{M}} J_\lambda(u,v). \tag{1.10}$$

Moreover, $\mu_1(\lambda)$ *is the only strictly principal eigenvalue of* (1.1)*, it is simple (i.e., given two eigenfunctions* (u, v), (u_0, v_0) *associated to* $\mu_1(\lambda)$*, there is a positive constant* k *such that* $u = ku_0$ *and* $v = k^{p/q}v_0$)*, and the following properties hold:*

(1) $\sup_{\lambda\in\mathbb{R}} \mu_1(\lambda) = \theta$*, where*

$$\theta = \theta(a, b, d) \stackrel{\text{def}}{=} \inf\left\{J_0(u, v) : I(u, v) = 1, \frac{\alpha+1}{p}A(u) + \frac{\beta+1}{q}D(v) = 0\right\}.$$

(2) *If* $a, d \geq 0$*, then* $\lim_{\lambda\to-\infty} \mu_1(\lambda) = \theta$*. If, in addition,* Ω_0^a *and* Ω_0^d *are non-empty and p-stable then* θ *is the first eigenvalue of*

$$\begin{cases} -\Delta_p u - b(x)|u|^{\alpha-1}u|v|^\beta v = \nu|u|^{p-2}u, \\ -\Delta_q v - b(x)|u|^\alpha u|v|^{\beta-1}v = \nu|v|^{q-2}v, \\ u \in W_0^{1,p}(\Omega_0^a), v \in W_0^{1,p}(\Omega_0^d), \end{cases} \tag{1.11}$$

and $\theta \leq \min\{\lambda_{1,p}(\Omega_0^a), \lambda_{1,q}(\Omega_0^d)\}$*, where* $\lambda_{1,p}(\tilde{\Omega})$ *denotes the first eigenvalue of* $-\Delta_p$ *in* $\tilde{\Omega}$*. Moreover, equality holds if* $\Omega_0^a \cap \Omega_0^d = \emptyset$.

Note that the second statement in the above theorem extends to (1.1) the result proved in [2, 4, 9] for the linear self-adjoint case, without any decomposition assumption related to the zero sets of a and d.

Using the properties of $\mu_1(\lambda)$, we derive a condition that guarantees the existence of strictly principal eigenvalues of (1.2) (since these ones are precisely the zeros of μ_1).

Theorem 1.2. *Assume* $(H1)$, $(H2)$ *and* $(H3)$.

(1) *If* $a, d \geq 0$*, then there exists a strictly principal eigenvalue of* (1.2) *if and only if* $\theta > 0$*. In this case the strictly principal eigenvalue is unique and is characterized by*

$$\lambda_1 = \lambda_1(a, b, d) = \inf_{\mathcal{M}^+_{a,d}} J_0(u, v), \tag{1.12}$$

where

$$\mathcal{M}^+_{a,d} \stackrel{\text{def}}{=} \{(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) : \frac{\alpha+1}{p}A(u) + \frac{\beta+1}{q}D(v) = 1\}.$$

(2) *If either* $a^- \not\equiv 0$ *or* $d^- \not\equiv 0$ *then there exists a strictly principal eigenvalue of* (1.2) *if and only if* $\theta \geq 0$*. More precisely*

(a) *If* $\theta > 0$*, then* (1.2) *admits exactly two strictly principal eigenvalues* $\lambda_{-1} < \lambda_1$*, with* λ_1 *given by* (1.12) *and*

$$\lambda_{-1} = \lambda_{-1}(a, b, d) = -\inf_{\mathcal{M}^-_{a,d}} J_0(u, v) \tag{1.13}$$

where

$$\mathcal{M}_{a,d}^{-} \stackrel{\text{def}}{=} \{(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) : \frac{\alpha+1}{p} A(u) + \frac{\beta+1}{q} D(v) = -1\}.$$

(*b*) *If* $\theta = 0$, *then* (1.2) *has a unique strictly principal eigenvalue* λ_1 *given by*

$$\lambda_1 = \lambda_1(a,b,d) = \inf_{\mathcal{M}_{a,d}^{+}} J_0(u,v) = -\inf_{\mathcal{M}_{a,d}^{-}} J_0(u,v).$$

These infima are not achieved. Moreover, any function (u,v) *in* $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *satisfying*

$$u, v \not\equiv 0 \text{ and } \quad \frac{\alpha+1}{p} A(u) + \frac{\beta+1}{q} D(v) = 0 \tag{1.14}$$

is an eigenfunction associated to λ_1.

The above theorem extends to (1.3) the result proved in [7, Theorem 7], for the scalar case.

The Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r$ and the usual norm of $W_0^{1,p}(\Omega)$ by $\|\cdot\|$. The weak convergence is denoted by $\rightharpoonup$. The positive and negative part of u are defined by $u^{\pm} := \max\{\pm u, 0\}$.

2. Proof of the results

We start by minimizing J_λ on $\mathcal{M}$. For this purpose, we need the following inequality, which is similar to [7, Lemma 2].

Lemma 2.1. *Let* $(\omega_1, \omega_2) \in L^r(\Omega) \times L^s(\Omega)$, *with* r, s *satisfying (H1) and (H2) respectively. If* $\omega_1, \omega_2 > 0$ *on* Ω *then there exist three positive constants* C_1, C_2, C_3 *such that*

$$\int_\Omega (|\nabla u|^p + |\nabla v|^q) \le C_1 J_\lambda(u,v) + C_2 \int_\Omega \omega_1 |u|^p + C_3 \int_\Omega \omega_2 |v|^q \tag{2.1}$$

for every $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Proof. By Hölder's inequality, we have

$$|A(u)| \le \|a\|_r \|u\|_{pr'}^p, \quad |D(v)| \le \|d\|_s \|v\|_{qs'}^q,$$

$$\begin{aligned} B(u,v) &\le \int_\Omega b(x) \Big[\frac{\alpha+1}{p} |u|^p + \frac{\beta+1}{p} |v|^q\Big] \\ &\le \frac{\alpha+1}{p} \|b\|_r \|u\|_{pr'}^p + \frac{\beta+1}{q} \|b\|_s \|v\|_{qs'}^q. \end{aligned}$$

On the other hand, according to the proof of [7, Lemma 2], for every $\varepsilon > 0$ there exist $M_\varepsilon, M'_\varepsilon > 0$ such that

$$\|u\|_{pr'}^p \leq \varepsilon \int_\Omega |\nabla u|^p + M_\varepsilon \int_\Omega \omega_1 |u|^p, \quad \|v\|_{qs'}^q \leq \varepsilon \int_\Omega |\nabla v|^q + M'_\varepsilon \int_\Omega \omega_2 |v|^q. \tag{2.2}$$

Thus, we have

$$\begin{aligned}
&\frac{\alpha+1}{p} \int_\Omega |\nabla u|^p + \frac{\beta+1}{q} \int |\nabla v|^q \\
&= J_\lambda(u,v) + \lambda \left[\frac{\alpha+1}{p} A(u) + \frac{\beta+1}{q} D(v) \right] + B(u,v) \\
&\leq J_\lambda(u,v) + \frac{\alpha+1}{p} (|\lambda| \|a\|_r + \|b\|_r) \|u\|_{pr'}^p + \frac{\beta+1}{q} (|\lambda| \|d\|_s + \|b\|_s) \|v\|_{qs'}^q.
\end{aligned}$$

We set $k_1 = \frac{\alpha+1}{p}(|\lambda|\|a\|_r + \|b\|_r)$, $k_2 = \frac{\beta+1}{q}(|\lambda|\|d\|_r + \|b\|_s)$ and $k = \max(k_1, k_2)$. From (2.2), one gets

$$\begin{aligned}
&\frac{\alpha+1}{p} \int_\Omega |\nabla u|^p + \frac{\beta+1}{q} \int_\Omega |\nabla v|^q \\
&\leq J_\lambda(u,v) + \varepsilon k \int_\Omega (|\nabla u|^p + |\nabla v|^q) + k M_\varepsilon \int_\Omega \omega_1 |u|^p + k M'_\varepsilon \int_\Omega \omega_2 |v|^q.
\end{aligned}$$

Let $\varepsilon > 0$ be such that $k_3 = \min\{\frac{\alpha+1}{p} - \varepsilon k, \frac{\beta+1}{q} - \varepsilon k\} > 0$. Then

$$0 \leq k_3 \int_\Omega (|\nabla u|^p + |\nabla v|^q) \leq J_\lambda(u,v) + k M_\varepsilon \int_\Omega \omega_1 |u|^p + k M'_\varepsilon \int_\Omega \omega_2 |v|^q,$$

and the result follows. □

We can now follow the standard minimization procedure.

Proposition 2.2. *Assume* $(H1)$, $(H2)$ *and* $(H3)$. *Then* $\mu_1(\lambda)$ *defined in* (1.10) *is the least eigenvalue of* (1.1) *and is a strictly principal eigenvalue.*

Proof. Let us first show that J_λ is bounded below on $\mathcal{M}$. Indeed, taking $\omega_1 = \omega_2 \equiv 1$ in (2.1), we have

$$0 \leq \int_\Omega (|\nabla u|^p + |\nabla v|^q) \leq C_1 J_\lambda(u,v) + C \int_\Omega (|u|^p + |v|^q), \tag{2.3}$$

where $C = \max(C_2, C_3)$. Thus,

$$C_1 J_\lambda(u,v) + C \geq 0 \quad \text{for every} \quad (u,v) \in \mathcal{M},$$

so that J_λ is bounded below on $\mathcal{M}$. In addition, any sequence (u_n, v_n) that minimizes J_λ on $\mathcal{M}$ is bounded in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Thus there exists

$(u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that, up to a subsequence, $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, and $(u_n, v_n) \to (u_0, v_0)$ in $L^p(\Omega) \times L^q(\Omega)$ and in $L^{pr'}(\Omega) \times L^{qs'}(\Omega)$. Hence

$$J_\lambda(u_0, v_0) \le \lim_{n\to+\infty} J_\lambda(u_n, v_n) = \mu_1(\lambda) \text{ and } (u_0, v_0) \in \mathcal{M}.$$

Consequently, $J_\lambda(u_0, v_0) = \mu_1(\lambda)$. By the Lagrange multipliers rule, $\mu_1(\lambda)$ is an eigenvalue for (1.1). Moreover $J_\lambda(|u|, |v|) = J_\lambda(u, v)$ for any (u, v), so that $\mu_1(\lambda)$ admits a componentwise nonnegative eigenfunction. From the Harnack inequality (cf. [15]), we deduce that $u_0, v_0 > 0$ in Ω; i.e., $\mu_1(\lambda)$ is a strictly principal eigenvalue. □

The following lemma shows that $\mu_1(\lambda)$ is the only strictly principal eigenvalue of (1.1) and is simple, in the sense that there is a unique eigenfunction (u_λ, v_λ) associated to $\mu_1(\lambda)$ such that $u_\lambda, v_\lambda > 0$ and $I(u_\lambda, v_\lambda) = 1$. Such (u_λ, v_λ) is said to be I-normalized. Note that up to now the condition $b(x) \ge 0$ was not employed yet.

Lemma 2.3. *Assume* $(H1)$, $(H2)$ *and* $(H3)$.

(1) *Let* $(u, v), (u_0, v_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *be two eigenfunctions of* (1.1) *associated to* $\mu_1(\lambda)$. *Assume also that* $u, v \ge 0$ *and* $u_0, v_0 > 0$ *in* Ω. *Then there exists a positive constant* k *such that* $u = ku_0$ *and* $v = k^{p/q}v_0$.

(2) $\mu_1(\lambda)$ *is the only strictly principal eigenvalue of* (1.1). *Moreover, there is exactly one eigenfunction* (u_λ, v_λ) *associated to* $\mu_1(\lambda)$ *such that* $u_\lambda, v_\lambda > 0$ *and* $I(u_\lambda, v_\lambda) = 1$.

Proof. (1) Let us set

$$L(\xi, \phi) \overset{\text{def}}{=} |\nabla \xi|^p + (p-1)\frac{\xi^p}{\phi^p}|\nabla \phi|^p - p\frac{\xi^{p-1}}{\phi^{p-1}}|\nabla \phi|^{p-2}\nabla \phi \nabla \xi,$$

and

$$R(\xi, \phi) \overset{\text{def}}{=} |\nabla \xi|^p - |\nabla \phi|^{p-2}\nabla\Big(\frac{\xi^p}{\phi^{p-1}}\Big)\nabla \phi.$$

for any $\xi \ge 0$ and $\phi > 0$ two almost everywhere differentiable functions. By Picone's identity (see [1]), $0 \le L = R$ and, if $\frac{\xi}{\phi} \in W_{loc}^{1,1}(\Omega)$, then $L(\xi, \phi) = 0$ if and only if ξ and ϕ are colinear (the regularity condition $\frac{\xi}{\phi} \in W_{loc}^{1,1}(\Omega)$ is needed to infer that $\frac{\xi}{\phi}$ is constant if $\nabla(\frac{\xi}{\phi}) = 0$). We apply Picone's identity to u and $u_0 + \varepsilon$, as well as to v and $v_0 + \varepsilon$, for any $\epsilon > 0$. By integration we find

$$0 \le \int_\Omega L(u, u_0 + \varepsilon) = \int_\Omega R(u, u_0 + \varepsilon)$$

$$= \int_\Omega \Big[|\nabla u|^p - |\nabla u_0|^{p-2}\nabla\Big(\frac{u^p}{(u_0+\varepsilon)^{p-1}}\Big)\nabla u_0\Big]$$
$$= \mu \int_\Omega |u|^p + \lambda \int_\Omega a(x)|u|^p + \int_\Omega b(x)u^{\alpha+1}v^{\beta+1} - \mu \int_\Omega u_0^{p-1}\frac{u^p}{(u_0+\varepsilon)^{p-1}}$$
$$- \lambda \int_\Omega a(x)u_0^{p-1}\frac{u^p}{(u_0+\varepsilon)^{p-1}} - \int_\Omega b(x)u_0^\alpha v_0^{\beta+1}\frac{u^p}{(u_0+\varepsilon)^{p-1}}.$$

We let $\varepsilon \to 0$ to get

$$0 \le \int_\Omega L(u,u_0) \le \int_\Omega b(x)[u^{\alpha+1}v^{\beta+1} - u_0^{\alpha-p+1}v_0^{\beta+1}u^p]. \tag{2.4}$$

Similarly, one gets

$$0 \le \int_\Omega L(v,v_0) \le \int_\Omega b(x)[u^{\alpha+1}v^{\beta+1} - u_0^{\alpha+1}v_0^{\beta-q+1}v^q]. \tag{2.5}$$

Since $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, multiplying (2.4) by $\frac{\alpha+1}{p}$ and (2.5) by $\frac{\beta+1}{q}$ and adding up, we get

$$0 \le \frac{\alpha+1}{p}L(u,u_0) + \frac{\beta+1}{q}L(v,v_0) \le \int_\Omega b(x)Q(u,u_0,v,v_0), \tag{2.6}$$

where

$$Q(u,u_0,v,v_0) = u^{\alpha+1}v^{\beta+1} - \frac{\alpha+1}{p}u_0^{\alpha-p+1}v_0^{\beta+1}u^p - \frac{\beta+1}{q}u_0^{\alpha+1}v_0^{\beta-q+1}v^q.$$

Arguing as in [1], we write

$$u^{\alpha+1}v^{\beta+1} = \Big(u^{\alpha+1}\frac{v_0^{\theta_2}}{u_0^{\theta_1}}\Big)\Big(v^{\beta+1}\frac{u_0^{\theta_1}}{v_0^{\theta_2}}\Big),$$

with $\theta_1 = \frac{(\alpha+1)(\beta+1)}{q}$ and $\theta_2 = \frac{(\alpha+1)(\beta+1)}{p}$. By Young's inequality, one has

$$\begin{aligned} u^{\alpha+1}v^{\beta+1} &\le \frac{\alpha+1}{p}\Big(u^{\alpha+1}\frac{v_0^{\theta_2}}{u_0^{\theta_1}}\Big)^{\frac{p}{\alpha+1}} + \frac{\beta+1}{q}\Big(v^{\beta+1}\frac{u_0^{\theta_1}}{v_0^{\theta_2}}\Big)^{\frac{q}{\beta+1}} \\ &= \frac{\alpha+1}{p}u^p u_0^{\alpha-p+1}v_0^{\beta+1} + \frac{\beta+1}{q}v^q u_0^{\alpha+1}v_0^{\beta-q+1}; \end{aligned} \tag{2.7}$$

i.e., $Q(u,u_0,v,v_0) \le 0$. Since $b(x) \ge 0$, from (2.6), we get $L(u,u_0) = L(v,v_0) = 0$. Hence $u = ku_0$ and $v = \delta v_0$, for some positive constants k, δ. Finally, we have

$$\begin{aligned} k^{p-1}(-\Delta_p u_0) &= -\Delta_p u \\ &= \lambda a(x)k^{p-1}u_0^{p-1} + b(x)k^\alpha\delta^{\beta+1}u_0^\alpha v_0^{\beta+1} + \mu_1(\lambda)k^{p-1}u_0^{p-1}. \end{aligned}$$

Dividing this equation by k^{p-1} and using the equation satisfied by u_0, we find $k^{\alpha-p+1}\delta^{\beta+1} = 1$. Similarly, from the equations satisfied by v and v_0, we get $k^{\alpha+1}\delta^{\beta-q+1} = 1$, and thus $\delta = k^{\frac{p}{q}}$.

(2) Let (u_λ, v_λ) be an eigenfunction associated to $\mu_1(\lambda)$ with $u_\lambda, v_\lambda \geq 0$, and let (u, v) be associated to some eigenvalue ν of (1.1), with $u, v > 0$. Normalizing it, we can assume that $I(u, v) = 1$. It is easily seen that $\nu \geq \mu_1(\lambda)$. Arguing as before, one has

$$0 \leq \int_\Omega L(u_\lambda, u) \leq (\mu_1(\lambda) - \nu)\int_\Omega |u_\lambda|^p + \int_\Omega b(x)[u_\lambda^{\alpha+1}v_\lambda^{\beta+1} - u^{\alpha-p+1}v^{\beta+1}u_\lambda^p]$$

$$0 \leq \int_\Omega L(v_\lambda, v) \leq (\mu_1(\lambda) - \nu)\int_\Omega |v_\lambda|^q + \int_\Omega b(x)[u_\lambda^{\alpha+1}v_\lambda^{\beta+1} - u^{\alpha+1}v^{\beta-q+1}v_\lambda^q],$$

and

$$0 \leq (\mu_1(\lambda) - \nu)\int_\Omega \Big(\frac{\alpha+1}{p}|u_\lambda|^p + \frac{\beta+1}{q}|v_\lambda|^q\Big) + \int_\Omega b(x)Q(u, u_\lambda, v, v_\lambda).$$

Since $\nu \geq \mu_1(\lambda)$, we find once again $L(u_\lambda, u) = L(v_\lambda, v) = 0$, so $u_\lambda = ku$ and $v_\lambda = k^{\frac{p}{q}}v$. But as $I(u, v) = I(u_\lambda, v_\lambda) = 1$, it follows that $k = 1$ and consequently $\nu = \mu_1(\lambda)$ and $(u, v) = (u_\lambda, v_\lambda)$. □

Next, we give some properties of $\mu_1(\lambda)$ as a function of λ. These ones are simple adaptations of [7, Proposition 6]. In particular, we complete the proof of Theorem 1.1.

Proposition 2.4. (1) $\mu_1 : \mathbb{R} \to \mathbb{R}$ *is concave and differentiable. Moreover,*

$$\mu_1'(\lambda) = -\Big(\frac{\alpha+1}{p}A(u_\lambda) + \frac{\beta+1}{q}D(v_\lambda)\Big) \quad \text{for every } \lambda \in \mathbb{R}. \tag{2.8}$$

Here (u_λ, v_λ) *is the* I*-normalized eigenfunction associated to* $\mu_1(\lambda)$ *with* $u_\lambda, v_\lambda > 0$.

(2) *If either* $a^\pm \not\equiv 0$ *or* $d^\pm \not\equiv 0$, *then* $\lim_{\lambda\to\pm\infty} \mu_1(\lambda) = -\infty$.

(3) *If* $a, d \geq 0$, *then* μ_1 *is strictly decreasing.*

(4) $\sup_{\lambda\in\mathbb{R}} \mu_1(\lambda) = \theta$. *Moreover, if* $a, d > 0$ *on* Ω, *then* $\theta = \infty$. *Otherwise,* θ *is achieved by some* $(\xi_0, \zeta_0) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *such that* $\xi_0, \zeta_0 \geq 0$.

(5) *If* $a, d \geq 0$ *and* Ω_0^a, Ω_0^d *are p-stable, then* θ *is the first eigenvalue of* (1.11) *and* $\theta \leq \min\{\lambda_{1,p}(\Omega_0^a), \lambda_{1,q}(\Omega_0^d)\}$. *Moreover, equality holds if* $\Omega_0^a \cap \Omega_0^d = \emptyset$.

Proof. (1) The concavity of μ_1 follows from the linearity of $J_\lambda(u, v)$ as a function of λ. In particular μ_1 is continuous. Now let $\lambda_n \to \lambda$ and

$(u_n, v_n), (u_\lambda, v_\lambda)$ be the I-normalized and componentwise positive eigenfunctions associated to $\mu_1(\lambda_n), \mu_1(\lambda)$, respectively. Since the weights $\lambda_n a, \lambda_n d$ are uniformly bounded in $L^r(\Omega)$ and $L^s(\Omega)$ respectively, we apply Lemma 2.1 with $\omega_1 = \omega_2 \equiv 1$ to get

$$0 \leq \int_\Omega (|\nabla u_n|^p + |\nabla v_n|^q) \leq C_1 J_{\lambda_n}(u_n, v_n) + C_2 = C_1 \mu_1(\lambda_n) + C_2.$$

So we infer that (u_n, v_n) is a bounded sequence in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Hence there exists $(\overline{u}, \overline{v})$ such that, up to a subsequence, $u_n \rightharpoonup \overline{u}$ in $W_0^{1,p}(\Omega)$, $v_n \rightharpoonup \overline{v}$ in $W_0^{1,q},(\Omega)$, $u_n \to \overline{u}$ in $L^{pr'}(\Omega)$ and $v_n \to \overline{v}$ in $L^{ps'}(\Omega)$. Then $I(\overline{u}, \overline{v}) = 1$ and from

$$\mu_1(\lambda) = \lim_{n \to +\infty} \mu_1(\lambda_n) \geq J_\lambda(\overline{u}, \overline{v}) \geq \mu_1(\lambda)$$

we derive that $(\overline{u}, \overline{v}) = (u_\lambda, v_\lambda)$. Furthermore,

$$\begin{aligned} \mu_1(\lambda_n) = J_{\lambda_n}(u_n, v_n) &= J_\lambda(u_n, v_n) + (\lambda - \lambda_n)\Big(\frac{\alpha+1}{p} A(u_n) + \frac{\beta+1}{q} D(v_n)\Big) \\ &\geq \mu_1(\lambda) + (\lambda - \lambda_n)\Big(\frac{\alpha+1}{p} A(u_n) + \frac{\beta+1}{q} D(v_n)\Big) \end{aligned}$$

and replacing λ (respectively φ) by λ_n (respectively φ_n) in this inequality we have, for $\lambda_n > \lambda$,

$$\begin{aligned} -\Big(\frac{\alpha+1}{p} A(u_n) + \frac{\beta+1}{q} D(v_n)\Big) &\leq \frac{\mu_1(\lambda_n) - \mu_1(\lambda)}{\lambda_n - \lambda} \\ &\leq -\Big(\frac{\alpha+1}{p} A(u_\lambda) + \frac{\beta+1}{q} D(v_\lambda)\Big). \end{aligned}$$

Passing to the limit as $n \to \infty$ we get (2.8). A similar argument holds if $\lambda_n < \lambda$.

(2) Since $a^+ \not\equiv 0$ (respectively $d^+ \not\equiv 0$) there exists a function $\xi \in W_0^{1,p}(\Omega)$ (respectively $\zeta \in W_0^{1,q}(\Omega)$) such that $A(\xi) > 0$ and $I(\xi, 0) = 1$ (respectively $D(\zeta) > 0$ and $I(0, \zeta) = 1$). Then, for every $\lambda > 0$ we have

$$\mu_1(\lambda) \leq \frac{\alpha+1}{p}\Big(\int_\Omega |\nabla \xi|^p - \lambda A(\xi)\Big) \longrightarrow -\infty \text{ as } \lambda \to +\infty.$$

$$\text{(respectively } \mu_1(\lambda) \leq \frac{\beta+1}{q}\Big(\int_\Omega |\nabla \zeta|^q - \lambda D(\zeta)\Big) \longrightarrow -\infty \text{ as } \lambda \to +\infty).$$

A similar proof holds if either $a^- \not\equiv 0$ or $d^- \not\equiv 0$.

(3) This result follows clearly from the fact that $\frac{\alpha+1}{p} A(u_\lambda) + \frac{\beta+1}{q} D(v_\lambda) > 0$ for every $\lambda \in \mathbb{R}$. Indeed, if $\lambda_1 < \lambda_2$, then $\mu_1(\lambda_1) = J_{\lambda_1}(u_{\lambda_1}, v_{\lambda_1}) > J_{\lambda_2}(u_{\lambda_1}, v_{\lambda_1}) \geq \mu_1(\lambda_2)$.

(4) Let us prove that $\sup_{\lambda\in\mathbb{R}} \mu_1(\lambda) = \theta$. First of all, it is clear that $\theta \geq \mu_1(\lambda)$ for every $\lambda \in \mathbb{R}$ and that $\theta = +\infty$ if and only if $a, d > 0$ on Ω. We distinguish two cases:

(a) $a, d \geq 0$. In this case we know that $\mu_1(\lambda)$ is strictly decreasing so that

$$\sup \mu_1(\lambda) = \lim_{\lambda\to-\infty} \mu(\lambda). \tag{2.9}$$

Let $\lambda_n \to -\infty$ when $n \to \infty$ and let (u_n, v_n) be the I-normalized eigenfunction associated to $\mu_1(\lambda_n)$. From (2.3), we have

$$\int_\Omega (|\nabla u_n|^p + |\nabla v_n|^q) \leq C_1\mu_1(\lambda_n) + C$$

for every $\lambda_n \leq 0$. Thus (u_n, v_n) is bounded in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, and there exists $(u, v) \in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ such that, up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v)$ in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$, and $(u_n, v_n) \to (u, v)$ in $L^{pr'}(\Omega)\times L^{qs'}(\Omega)$. Thus $I(u, v) = 1$ and

$$\theta \geq \lim_{n\to\infty} \mu_1(\lambda_n) \geq J_0(u, v) - \lim_{n\to-\infty} \lambda_n\Big(\frac{\alpha+1}{p}A(u_n) + \frac{\beta+1}{q}D(v_n)\Big). \tag{2.10}$$

If $a, d > 0$ on Ω, then

$$\frac{\alpha+1}{p}A(u_n) + \frac{\beta+1}{q}D(v_n) \to \frac{\alpha+1}{p}A(u) + \frac{\beta+1}{q}D(v) > 0,$$

so, from (2.10), $\lim_{\lambda\to-\infty} \mu_1(\lambda) = +\infty = \theta$. Otherwise $\theta < +\infty$, so that μ_1 is bounded from above and we conclude from (2.10) that

$$\frac{\alpha+1}{p}A(u) + \frac{\beta+1}{q}D(v) = 0.$$

Therefore, (u, v) is admissible in the definition of θ and, still from (2.10), we get

$$\theta \geq \lim_{n\to\infty} \mu_1(\lambda_n) \geq J_0(u, v) \geq \theta$$

and the result follows.

(b) $a^- \not\equiv 0$ or $d^- \not\equiv 0$. In this case it follows from (1) and (2) that μ_1 is bounded from above. Then $\sup_{\lambda\in\mathbb{R}} \mu_1(\lambda)$ is achieved at some λ_0 that satisfies

$$0 = \mu_1'(\lambda_0) = -\Big(\frac{\alpha+1}{p}A(u_{\lambda_0}) + \frac{\beta+1}{q}D(v_{\lambda_0})\Big).$$

We conclude that $(u_{\lambda_0}, v_{\lambda_0})$ is admissible in the definition of θ, and thus

$$\theta \leq J_0(u_{\lambda_0}, v_{\lambda_0}) = \mu_1(\lambda_0) = \sup_{\lambda\in\mathbb{R}} \mu_1(\lambda).$$

Then the conclusion follows.

It remains to show that θ is achieved whenever it is finite. Let (u_n, v_n) be a minimizing sequence for θ. Lemma 2.1 with $\omega_1 \equiv \omega_2 \equiv 1$ shows that (u_n, v_n) is bounded in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. Up to a subsequence, $(u_n, v_n) \rightharpoonup (u_0, v_0)$ in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, and $(u_n, v_n) \to (u_0, v_0)$ in $L^{pr'}(\Omega) \times L^{qs'}(\Omega)$. A standard minimization argument shows that θ is achieved by (u_0, v_0).

(5) Let θ be achieved by (u_0, v_0); i.e.,

$$J_0(u_0, v_0) = \theta, \quad I(u_0, v_0) = 1 \quad \text{and} \quad \frac{\alpha+1}{p} A(u_0) + \frac{\beta+1}{q} D(v_0) = 0.$$

Since $a, d \geq 0$ almost everywhere in Ω, it follows that $A(u_0) = D(v_0) = 0$. From the p-stability assumption on Ω_0^a and Ω_0^d, we infer that $u_0 \in W_0^{1,p}(\Omega_0^a)$ and $v_0 \in W_0^{1,q}(\Omega_0^d)$. Thus

$$\theta = \min\{\tilde{J}_0(u, v) : (u, v) \in W_0^{1,p}(\Omega_0^a) \times W_0^{1,q}(\Omega_0^d), \tilde{I}(u, v) = 1\}, \tag{2.11}$$

where $\tilde{J}_0, \tilde{I}$ are the restrictions of J_0, I to $W_0^{1,p}(\Omega_0^a) \times W_0^{1,q}(\Omega_0^d)$, respectively. By the Lagrange multipliers rule, $((u_0, v_0), \theta)$ solves (1.11). Moreover, by (2.11), θ is the first eigenvalue of (1.11). As $\lambda_{1,p}(\Omega_0^a)$ and $\lambda_{1,q}(\Omega_0^d)$ are eigenvalues of (1.11), we obtain $\theta \leq \min\{\lambda_{1,p}(\Omega_0^a), \lambda_{1,q}(\Omega_0^d)\}$. Finally, if $\Omega_0^a \cap \Omega_0^d = \emptyset$ then (1.11) reduces to the non-coupled system

$$\begin{cases} -\Delta_p u = \nu |u|^{p-2} u, \\ -\Delta_q v = \nu |v|^{q-2} v, \\ u \in W_0^{1,p}(\Omega_0^a), v \in W_0^{1,p}(\Omega_0^d), \end{cases} \tag{2.12}$$

whose first eigenvalue is clearly $\min\{\lambda_{1,p}(\Omega_0^a), \lambda_{1,q}(\Omega_0^d)\}$. $\square$

We are now in position to prove Theorem 1.2.

Proof of Theorem 1.2. (1) By (3) and (4) of Proposition 2.4, it is clear that $\theta > 0$ is a necessary and sufficient condition for the curve $\mu_1(\lambda)$ to vanish at some unique value λ_1. It follows that

$$J_0(u, v) \geq \lambda_1 \Big(\frac{\alpha+1}{p} A(u) + \frac{\beta+1}{q} D(v) \Big)$$

for every u, v such that $I(u, v) = 1$. Moreover, equality holds for $(u, v) = (u_{\lambda_1}, v_{\lambda_1})$. From (2.8), we know that $\frac{\alpha+1}{p} A(u_{\lambda_1}) + \frac{\beta+1}{q} D(v_{\lambda_1}) > 0$, so we can set

$$\tilde{u} \stackrel{\text{def}}{=} \frac{u_{\lambda_1}}{\big(\frac{\alpha+1}{p} A(u_{\lambda_1}) + \frac{\beta+1}{q} D(v_{\lambda_1})\big)^{\frac{1}{p}}} \quad \text{and} \quad \tilde{v} \stackrel{\text{def}}{=} \frac{v_{\lambda_1}}{\big(\frac{\alpha+1}{p} A(u_{\lambda_1}) + \frac{\beta+1}{q} D(v_{\lambda_1})\big)^{\frac{1}{q}}}.$$

Thus, $J_0(\tilde{u}, \tilde{v}) = \lambda_1$ and consequently (1.12) is proved.

(2) By Proposition 2.4-(2), $\mu_1(\lambda)$ is concave and differentiable and $\mu_1(\lambda) \to -\infty$ when $\lambda \to \pm\infty$. Thus, if $\theta > 0$ then $\mu_1(\tilde{\lambda}) = 0$ for some $\tilde{\lambda} \in \mathbb{R}$ which provides a strictly principal eigenvalue of (1.2). Conversely, if (1.2) has a strictly principal eigenvalue, say λ, then $0 = \mu_1(\lambda) \leq \theta$.

(a) If $\theta > 0$, then $\mu_1(\lambda)$ vanishes twice, let us say at $\lambda_{-1} < \lambda_1$. We claim that $\mu_1'(\lambda_{-1}) > 0$. Indeed, assume that

$$\mu_1'(\lambda_{-1}) = -\Big[\frac{\alpha+1}{p} A(u_{-1}) + \frac{\beta+1}{q} D(v_{-1})\Big] = 0,$$

where (u_{-1}, v_{-1}) is the I-normalized eigenfunction associated to λ_{-1}. Then (u_{-1}, v_{-1}) is an admissible couple in the definition of θ and we have

$$\theta \leq J_0(u_{-1}, v_{-1}) = \mu_1(\lambda_{-1}) = 0,$$

a contradiction. Thus, we have shown that

$$\frac{\alpha+1}{p} A(u_{-1}) + \frac{\beta+1}{q} D(v_{-1}) < 0.$$

After a renormalization we can assume that

$$\frac{\alpha+1}{p} A(u_{-1}) + \frac{\beta+1}{q} D(v_{-1}) = -1.$$

Then formula (1.13) is directly deduced from the definition of $\mu_1(\lambda)$. A similar argument holds for λ_1 and we omit it.

(b) If $\theta = 0$, then $\mu_1(\lambda_0) = 0$ at some λ_0 providing a strictly principal eigenvalue of 1.2. Moreover, λ_0 is unique and is a global maximum point of μ, so

$$\mu_1'(\lambda_0) = -\Big[\frac{\alpha+1}{p} A(u_{\lambda_0}) + \frac{\beta+1}{q} D(v_{\lambda_0})\Big] = 0.$$

Let us prove that $\lambda_0 = \inf_{\mathcal{M}^+_{a,d}} J_0$. Take $(u, v) \in \mathcal{M}^+_{a,d}$ and assume that $u, v \geq 0$ by replacing u, v by $|u|, |v|$ if necessary. Picone's identity applied to $u_T \stackrel{\text{def}}{=} \min\{u, T\}$ and $u_{\lambda_0} + \varepsilon$ yields

$$\begin{aligned} 0 \leq \int_\Omega L(u_T, u_{\lambda_0} + \varepsilon) &= \int_\Omega R(u_T, u_{\lambda_0} + \varepsilon) \\ &= \int_\Omega \Big[|\nabla u_T|^p - |\nabla u_{\lambda_0}|^{p-2} \nabla\Big(\frac{u_T^p}{(u_{\lambda_0} + \varepsilon)^{p-1}}\Big) \nabla u_{\lambda_0}\Big] \\ &= \int_\Omega |\nabla u_T|^p - \lambda_0 \int_\Omega a(x) \Big(\frac{u_T^p}{(u_{\lambda_0} + \varepsilon)^{p-1}}\Big) u_{\lambda_0}^{p-1} \\ &\quad - \int_\Omega b(x) \Big(\frac{u_T^p}{(u_{\lambda_0} + \varepsilon)^{p-1}}\Big) u_{\lambda_0}^{\alpha} v_{\lambda_0}^{\beta+1}. \end{aligned}$$

In a similar way, applying Picone's identity to $v_T = \min\{v, T\}$ and $v_{\lambda_0} + \varepsilon$, one gets

$$\begin{aligned}0 \leq \int_\Omega L(v_T, v_{\lambda_0} + \varepsilon) &= \int_\Omega R(v_T, v_{\lambda_0} + \varepsilon) \\ &= \int_\Omega \Big[|\nabla v_T|^q - |\nabla v_{\lambda_0}|^{q-2}\nabla\Big(\frac{v_T^q}{(v_{\lambda_0}+\varepsilon)^{q-1}}\Big)\nabla v_{\lambda_0}\Big] \\ &= \int_\Omega |\nabla v_T|^q - \lambda_0 \int_\Omega d(x)\Big(\frac{v_T^q}{(v_{\lambda_0}+\varepsilon)^{q-1}}\Big)v_{\lambda_0}^{q-1} \\ &\quad - \int_\Omega b(x)\Big(\frac{v_T^q}{(v_{\lambda_0}+\varepsilon)^{q-1}}\Big)u_{\lambda_0}^{\alpha+1}v_{\lambda_0}^{\beta}.\end{aligned}$$

Now we let $\varepsilon \to 0$ and $T \to \infty$ to get

$$0 \leq \int_\Omega L(u, u_{\lambda_0}) \leq \int_\Omega |\nabla u|^p - \lambda_0 A(u) - \int_\Omega b(x)u^p u_{\lambda_0}^{\alpha-p+1}v_{\lambda_0}^{\beta+1},$$

$$0 \leq \int_\Omega L(v, v_{\lambda_0}) \leq \int_\Omega |\nabla v|^p - \lambda_0 D(v) - \int_\Omega b(x)v^q u_{\lambda_0}^{\alpha+1}v_{\lambda_0}^{\beta-q+1},$$

and consequently,

$$\begin{aligned}&\frac{\alpha+1}{p}\int_\Omega |\nabla u|^p + \frac{\beta+1}{q}\int_\Omega |\nabla v|^q - \lambda_0\Big[\frac{\alpha+1}{p}A(u) + \frac{\beta+1}{q}D(v)\Big] \\ &\quad - \int_\Omega b(x)\Big[\frac{\alpha+1}{p}u^p u_{\lambda_0}^{\alpha-p+1}v_{\lambda_0}^{\beta+1} + \frac{\beta+1}{q}v^q u_{\lambda_0}^{\alpha+1}v_{\lambda_0}^{\beta-q+1}\Big] \geq 0.\end{aligned}$$

Hence, using (2.7) (where we replace u_0, v_0 by $u_{\lambda_0}, v_{\lambda_0}$ respectively), we deduce that $J_0(u,v) \geq \lambda_0$ for every $(u,v) \in \mathcal{M}_{a,d}^+$. Consider now the sequence $(u_n, v_n) \in \mathcal{M}_{a,d}^+$ given by

$$\begin{aligned}u_n &= \frac{u_{\lambda_0} + \frac{\psi}{n}}{\Big(\frac{\alpha+1}{p}A(u_{\lambda_0} + \frac{\psi}{n}) + \frac{\beta+1}{q}D(v_{\lambda_0} + \frac{\eta}{n})\Big)^{\frac{1}{p}}} \\ v_n &= \frac{v_{\lambda_0} + \frac{\eta}{n}}{\Big(\frac{\alpha+1}{p}A(u_{\lambda_0} + \frac{\psi}{n}) + \frac{\beta+1}{q}D(v_{\lambda_0} + \frac{\eta}{n})\Big)^{\frac{1}{q}}},\end{aligned} \tag{2.13}$$

where $\psi, \eta \in C_0^\infty(\Omega)$ are positive such that

$$\frac{\alpha+1}{p}A(\psi) + \frac{\beta+1}{q}D(\eta) > 0, \quad \int_\Omega a(x)u_{\lambda_0}^{p-1}\psi > 0, \quad \text{and} \quad \int_\Omega d(x)v_{\lambda_0}^{q-1}\eta > 0.$$

One can easily see that

$$\frac{\alpha+1}{p}A\Big(u_{\lambda_0}+\frac{\psi}{n}\Big)+\frac{\beta+1}{q}D\Big(v_{\lambda_0}+\frac{\eta}{n}\Big)>0$$

for n large enough. Moreover, by the mean value theorem, for such n we can find $0<t_n,s_n,x_n,y_n<\frac{1}{n}$ such that

$$J_0\Big(u_{\lambda_0}+\frac{\psi}{n},v_{\lambda_0}+\frac{\eta}{n}\Big)=\frac{1}{n}\Big[\Big\langle\frac{\partial J_0}{\partial u}(u_{\lambda_0}+t_n\psi),\psi\Big\rangle+\Big\langle\frac{\partial J_0}{\partial v}(v_{\lambda_0}+s_n\eta),\eta\Big\rangle\Big]$$

and

$$\frac{\alpha+1}{p}A\Big(u_{\lambda_0}+\frac{\psi}{n}\Big)+\frac{\beta+1}{q}D\Big(v_{\lambda_0}+\frac{\eta}{n}\Big)$$
$$=\frac{1}{n}\Big[(\alpha+1)\int_\Omega a(x)|u_{\lambda_0}+x_n\psi|^{p-1}\psi+(\beta+1)\int_\Omega d(x)|v_{\lambda_0}+y_n\eta|^{q-1}\eta\Big].$$

Hence,

$$J_0(u_n,v_n)=\frac{J_0(u_{\lambda_0}+\frac{\psi}{n},v_{\lambda_0}+\frac{\eta}{n})}{\frac{\alpha+1}{p}A(u_{\lambda_0}+\frac{\psi}{n})+\frac{\beta+1}{q}D(v_{\lambda_0}+\frac{\eta}{n})}\longrightarrow\lambda_0.$$

Since $(u_{\lambda_0},v_{\lambda_0})$ satisfies (1.14) it is clear that λ_0 is not achieved. By repeating the above argument for $(u,v)\in\mathcal{M}^-_{a,d}$ and $\psi,\eta<0$ such that

$$\frac{\alpha+1}{p}A(\psi)+\frac{\beta+1}{q}D(\eta)<0,\quad\int_\Omega a(x)u_{\lambda_0}^{p-1}\psi<0,\quad\text{and}\quad\int_\Omega d(x)v_{\lambda_0}^{q-1}\eta<0,$$

we can show that $\lambda_0=\inf_{\mathcal{M}^-_{a,b}}J_0$. Finally if (u,v) is such that

$$J_0(u,v)=\frac{\alpha+1}{p}A(u)+\frac{\beta+1}{q}D(v)=0,$$

after I-normalization, one has $I(u,v)=1$ and

$$\sup_{\lambda\in\mathbb{R}}\mu_1(\lambda)=0=J_0(u,v)=J_{\lambda_0}(u,v)\geq\mu_1(\lambda_0)=0,$$

so (u,v) achieves $\mu_1(\lambda_0)$. Then $u=cu_{\lambda_0}$ and $v=c^{\frac{p}{q}}v_{\lambda_0}$ for some positive constant c and the result follows. We put $\lambda_1=\lambda_0$. □

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