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Residual-based a posteriori error estimates for a nonconforming finite element discretization of the Stokes–Darcy coupled problem: isotropic discretization

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Abstract In this paper we develop an a posteriori error analysis of a nonconforming mixed finite element method for the coupling of fluid flow with porous media flow. The approach utilizes the same nonconforming Crouzeix–Raviart element discretization on the entire domain (Rui and Zhang, Comput Methods Appl Mech Eng 198:2692–2699, 2009). The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. It is proven that the a posteriori error estimate provided in this paper is both reliable and efficient.

Keywords Mixed finite elements · A posteriori error analysis · Stokes equation · Darcy equation · Nonconforming method

Mathematics Subject Classification 74S05 · 74S10 · 74S15 · 74S20 · 74S25 · 74S30

Contents

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1 Introduction

There are many serious problems currently facing the world in which the coupling between groundwater and surface water is important. These include questions such as predicting how pollution spreads into streams, lakes and rivers, affecting the water supply. This coupling is also important in technological applications involving filtration. We refer to the nice overview [21] and the references therein for its physical background, modeling, and standard numerical methods. One important issue in the modeling of the coupled Darcy–Stokes flow is the treatment of the interface condition, where the Stokes fluid meets the porous medium. In this paper, we only consider the so-called Beavers–Joseph–Saffman condition, which was experimentally derived by Beavers and Joseph in [6], modified by Saffman in [45], and later mathematically justified in [32–34,41].

There are two popular formulations of the coupled Darcy–Stokes flow, namely the primal formulation or the mixed formulation in the Darcy region, see [2,23,26,27,37,42,43,52] for some mathematical analysis. The authors in [52] studied two different mixed formulations: the first one imposes the weak continuity of the normal component of the velocity field on the interface, by using a Lagrange multiplier; while the second one imposes the strong continuity in the functional space. Later on we call these two mixed formulations, the weakly coupled formulation and the strongly coupled formulation respectively. The weakly coupled formulation gives more freedom in the choice of the discretization in the Stokes side and the Darcy side separately. The works in [23,26,27,44,52] are based on the weakly coupled formulation. Researches on the strongly coupled formulation have been focused on the development of an unified discretization, that is, the Stokes side and the Darcy side are discretized using the same finite element. This approach simplifies the numerical implementation, only if the unified discretization is not significantly more complicated than the commonly used discretizations for the Darcy and the Stokes problems. In [2], a conforming, unified finite element has been proposed for the strongly coupled mixed formulation. Superconvergence analysis of the finite element methods for the Stokes–Darcy system was studied in [12]. Other less restrictive discretizations as the non-conforming unified approach [37,44] or the discontinuous Galerkin (DG) approach have been proposed in [35,42,43]. Due to its discontinuous nature, some (DG) discretizations for the coupled Darcy–Stokes problem may break the strong coupling in the discrete level [42,43], as they impose the normal continuity across the interface via interior penalties.

A posteriori error estimators are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. They are essential to design adaptive mesh refinement algorithms which equi-distribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuska and Rheinboldt [4], adaptive finite element methods based on a posteriori error estimates have been extensively investigated.

In [44], a nonconforming, unified finite element has been proposed for the strongly coupled mixed formulation. The authors have used the Crouzeiz–Raviart nonconforming element for the approximation of the velocity and piecewise constant functions for the approximation of the pressure in both region; they further have added penalizing terms corresponding to the jumps over the edges/faces of the piecewise velocities. An a priori error analysis is performed with some numerical tests confirming the convergence rates.

A posteriori error estimations have been well-established for both the mixed formulation of the Darcy flow [8,9,38], and the Stokes flow [1,5,11,19,22,31,40,46,50,51]. However, only few works exist for the coupled Darcy–Stokes problem, see for instance [3,13,18,25]. The paper [13] concerns the strongly coupled mixed formulation where a $H(\mathrm{div})$ conforming finite element method has been used, [3,23] concern the weakly coupled mixed formulation while [18] uses the primal formulation on the Darcy side and [25] concerns a fully-mixed formulation. To our best knowledge, there is no a posteriori error estimation for the strongly coupled mixed formulation for the coupled Darcy–Stokes problem where a nonconforming finite element method is used. Here we develop such a posteriori error analysis of a nonconforming mixed finite element method, that is a slight variant of the one proposed in [44]. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. We further prove that our a posteriori error estimator is both reliable and efficient. The difference between our paper and the references [3,13,18,25] is that our discretization is nonconforming in both the Stokes domain and Darcy domain. As a result, additional terms are included in the error estimators that measure the non-conformity of the method. In order to treat appropriately this non-conformity, we further need a special Helmholtz decomposition (Theorem 3.1), a regularity result (Theorem 3.2) and an estimate of the non-conforming error (Theorem 3.3).

The paper is organized as follows. Some preliminaries and notation are given in Sect. 2. Because of the complexity of the problem in hand (Stokes–Darcy equations using Crouzeix–Raviart element) a special section (Sect. 3) dedicated to technical results is incorporated to facilitate the process of obtaining reliability result. Hence we prove a kind of Helmholtz decomposition of elements of the natural velocity space $\mathbf{H}$ introduced below, an estimate of the non conformity error by introducing an adapted Oswald interpolant which preserves the continuity of the normal trace through the interface and a regularity result in Ω_d of the solution of our Stokes-Darcy problem since one essential difficulty in choosing the unified discretisation is that, the Stokes side velocity is in H^1 while the Darcy side velocity is only in $H(\text{div})$. In Sect. 4, the a posteriori error estimates are derived.

2 Preliminaries and notation

2.1 Model problem

We consider the model of a flow in a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3), consisting of a porous medium domain Ω_d, where the flow is a Darcy flow, and an open region $\Omega_s = \Omega \setminus \overline{\Omega}_d$, where the flow is governed by the Stokes equations. The two regions are separated by an interface $\Gamma_I = \partial\Omega_d \cap \partial\Omega_s$. Let $\Gamma_l = \partial\Omega_l \setminus \Gamma_I$, $l = s, d$. Each interface and boundary is assumed to be polygonal ($N = 2$) or polyhedral ($N = 3$). We denote by $\mathbf{n}_s$ (resp. $\mathbf{n}_d$) the unit outward normal vector along $\partial\Omega_s$ (resp. $\partial\Omega_d$). Note that on the interface Γ_I, we have $\mathbf{n}_s = -\mathbf{n}_d$. Figure 1 gives a schematic representation of the geometry.

For any function v defined in Ω, since its restriction to Ω_s or to Ω_d could play a different mathematical roles (for instance their traces on Γ_I), we will set $v_s = v_{|\Omega_s}$ and $v_d = v_{|\Omega_d}$.

In Ω, we denote by $\mathbf{u}$ the fluid velocity and by p the pressure. The motion of the fluid in Ω_s is described by the Stokes equations

$$\begin{cases} -2\mu \text{div}\mathbf{D}(\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega_s, \\ \text{div}\mathbf{u} = g & \text{in } \Omega_s, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_s, \end{cases} \tag{1}$$

while in the porous medium Ω_d, by Darcy's law

$$\begin{cases} \mu \mathbf{K}^{-1}\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega_d, \\ \text{div}\mathbf{u} = g & \text{in } \Omega_d, \\ \mathbf{u}\cdot\mathbf{n}_d = 0 & \text{on } \Gamma_d. \end{cases} \tag{2}$$

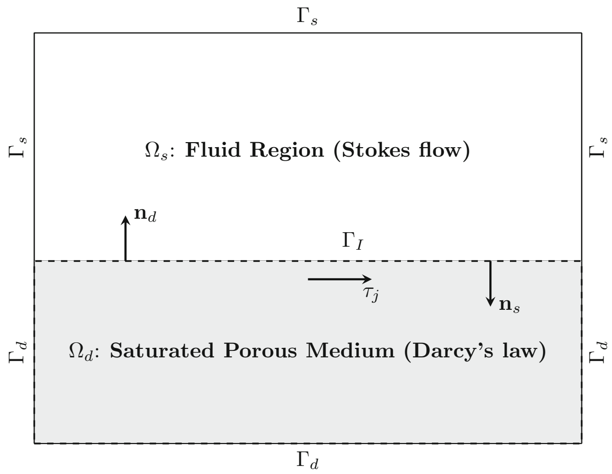

Fig. 1 A sketch of the geometry of the problem

Here, $\mu > 0$ is the fluid viscosity, $\mathbf{D}$ the deformation rate tensor defined by

$$\mathbf{D}(\psi)_{ij} := \frac{1}{2}\left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right), \quad 1 \leqslant i, j \leqslant N,$$

and $\mathbf{K}$ a symmetric and uniformly positive definite tensor representing the rock permeability and satisfying, for some constants $0 < K_* \leqslant K^* < +\infty$,

$$K_* \xi^T \xi \leqslant \xi^T \mathbf{K}(x) \xi \leqslant K^* \xi^T \xi, \quad \forall x \in \Omega_d, \quad \xi \in \mathbb{R}^N.$$

$\mathbf{f} \in [L^2(\Omega)]^N$ is a term related to body forces and $g \in L^2(\Omega)$ a source or sink term satisfying the compatibility condition

$$\int_\Omega g(x)dx = 0.$$

Finally we consider the following interface conditions on Γ_I:

$$\mathbf{u}_s \cdot \mathbf{n}_s + \mathbf{u}_d \cdot \mathbf{n}_d = 0, \tag{3}$$

$$p_s - 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_s) \cdot \mathbf{n}_s = p_d, \tag{4}$$

$$\frac{\sqrt{\kappa_j}}{\alpha_1} 2\mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_s) \cdot \tau_j = -\mathbf{u}_s \cdot \tau_j, \quad j = 1, \ldots, N-1. \tag{5}$$

Here, Eq. (3) represents mass conservation, Eq. (4) the balance of normal forces, and Eq. (5) the Beavers–Joseph–Saffman conditions. Moreover, $\{\tau_j\}_{j=1,\ldots,N-1}$ denotes an orthonormal system of tangent vectors on Γ_I, $\kappa_j = \tau_j \cdot \mathbf{K} \cdot \tau_j$, and α_1 is a parameter determined by experimental evidence.

Equations (1)–(5) consist of the model of the coupled Stokes and Darcy flows problem that we will study below.

2.2 Weak formulation

We begin this subsection by introducing some useful notations.

If W is a bounded domain of $\mathbb{R}^N$ and m is a non negative integer, the Sobolev space $H^m(W) = W^{m,2}(W)$ is defined in the usual way with the usual norm $\| \cdot \|_{m,W}$ and seminorm $| \cdot |_{m,W}$. In particular, $H^0(W) = L^2(W)$ and we write $\| \cdot \|_W$ for $\| \cdot \|_{0,W}$. Similarly we denote by $(\cdot, \cdot)_W$ the $L^2(W)$ $[L^2(W)]^N$ or $[L^2(W)]^{N\times N}$ inner product. For shortness if W is equal to Ω, we will drop the index Ω, while for any $m \geq 0$, $\| \cdot \|_{m,l}=\| \cdot \|_{m,\Omega_l}$, $| \cdot |_{m,l} = | \cdot |_{m,\Omega_l}$ and $(.,.)_l = (\cdot,\cdot)_{\Omega_l}$, for $l = s, d$. The space $H_0^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Let $[H^m(\Omega)]^N$ be the space of vector valued functions $\mathbf{v} = (v_1, \ldots, v_N)$ with components v_i in $H^m(\Omega)$. The norm and the seminorm on $[H^m(\Omega)]^N$ are given by

$$\| \mathbf{v} \|_{m,\Omega} := \left(\sum_{i=0}^{N} \| v_i \|_{m,\Omega}^2 \right)^{1/2} \quad \text{and} \quad |\mathbf{v}|_{m,\Omega} := \left(\sum_{i=0}^{N} |v_i|_{m,\Omega}^2 \right)^{1/2}. \tag{6}$$

For a connected open subset of the boundary $\Gamma \subset \partial\Omega_s \cup \partial\Omega_d$, we write $\langle .,. \rangle_\Gamma$ for the $L^2(\Gamma)$ inner product, that is, for scalar valued functions $\lambda, \eta \in L^2(\Gamma)$, one defines

$$\langle \lambda, \eta \rangle_\Gamma := \int_\Gamma \lambda(s)\eta(s)ds \tag{7}$$

We also define the special vector-valued functions space

$$\mathbf{H}(\mathrm{div}, \Omega) := \{\mathbf{v} \in [L^2(\Omega)]^N : \mathrm{div}\mathbf{v} \in L^2(\Omega)\} \tag{8}$$

To give the variational formulation of our coupled problem we define the following two spaces for the velocity and the pressure:

$$\mathbf{H} := \{\mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega) : \mathbf{v}_s \in [H^1(\Omega_s)]^N, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_s \text{ and } \mathbf{v} \cdot \mathbf{n}_d = 0 \text{ on } \Gamma_d\}$$

equipped with the norm

$$\| \mathbf{v} \|_{\mathbf{H}} := (|\mathbf{v}|_{1,s}^2 + \| \mathbf{v} \|_d^2 + \| \mathrm{div}\mathbf{v} \|_d^2)^{1/2}, \tag{9}$$

and

$$Q = L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q(x)dx = 0 \right\}. \tag{10}$$

Note that the vector valued functions in $\mathbf{H}$ have (weakly) continuous normal components on Γ_I (consequence of Theorem I.2.5 of [28, p. 27]).

Let us further introduce two bilinear forms

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := 2\mu(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_s + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} \langle \mathbf{u}_s \cdot \tau_j, \mathbf{v}_s \cdot \tau_j \rangle_{\Gamma_I} + \mu(\mathbf{K}^{-1}\mathbf{u}, \mathbf{v})_d,$$

$$\mathbf{b}(\mathbf{v}, q) := -\int_\Omega q \,\mathrm{div}\mathbf{v}$$

and two linear forms

$$L(\mathbf{v}) := (\mathbf{f}, \mathbf{v})_\Omega \text{ and } G(q) := -(g, q)_\Omega.$$

The weak formulation of the coupled problem (1)–(5) can be stated as follows [52]: find $(\mathbf{u}, p) \in \mathbf{H} \times Q$ such that

$$\begin{cases} \mathbf{a}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, p) = L(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{H}, \\ \mathbf{b}(\mathbf{u}, q) = G(q), & \forall q \in Q. \end{cases} \tag{11}$$

Note that if g is of mean zero, (11) directly implies that (1), (2) and (3) hold (the differential equations being understood in the distributional sense), while the interface conditions (4) and (5) are imposed in a weak sense.

This problem has a unique solution as proved in [52, Section 3].

Theorem 2.1 *If* $\mathbf{f} \in [L^2(\Omega)]^N$ *and* $g \in L^2_0(\Omega)$*, there exists a unique solution* $(\mathbf{u}, p) \in \mathbf{H} \times Q$ *to the problem* (11).

2.3 Finite element discretization

In this subsection, we will use a variant of the nonconforming Crouzeix–Raviart piecewise linear finite element approximation for the velocity and piecewise constant approximation for the pressure. The existence and uniqueness of a finite element solution of the corresponding discrete problem is proved like in [44].

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of triangulations of Ω with nondegenerate elements (i.e. triangles for $N = 2$ and tetrahedrons for $N = 3$). For any $T \in \mathcal{T}_h$, we denote by h_T the diameter of T and ρ_T the diameter of the largest ball inscribed into T and set

$$h = \max_{T\in\mathcal{T}_h} h_T, \quad \text{and} \quad \sigma_h = \max_{T\in\mathcal{T}_h} \frac{h_T}{\rho_T} \tag{12}$$

We assume that the family of triangulations is regular, in the sense that there exists $\sigma_0 > 0$ such that $\sigma_h \leqslant \sigma_0$, for all $h > 0$. We also assume that the triangulation is conform with respect to the partition of Ω into Ω_s and Ω_d, namely each $T \in \mathcal{T}_h$ is either in Ω_s or in Ω_d. Let $\mathcal{T}_h^s$ and $\mathcal{T}_h^d$ be the corresponding induced triangulations of Ω_s and Ω_d. For any $T \in \mathcal{T}_h$, we denote by $\mathcal{E}(T)$ (resp. $\mathcal{N}(T)$) the set of its edges ($N = 2$) or faces ($N = 3$) (resp. vertices) and set $\mathcal{E}_h = \bigcup_{T\in\mathcal{T}_h} \mathcal{E}(T)$, $\mathcal{N}_h = \bigcup_{T\in\mathcal{T}_h} \mathcal{N}(T)$. For $\mathcal{A} \subset \overline{\Omega}$ we define

$$\mathcal{E}_h(\mathcal{A}) = \{E \in \mathcal{E}_h : E \subset \mathcal{A}\}.$$

Notice that $\mathcal{E}_h$ can be split up in the form

$$\mathcal{E}_h = \mathcal{E}_h(\Omega_s^+) \cup \mathcal{E}_h(\Omega_d) \cup \mathcal{E}_h(\partial\Omega_d) \tag{13}$$

where $\Omega_s^+ = \Omega_s \cup \Gamma_s$. Note that $\mathcal{E}_h(\Gamma_I)$ is included in $\mathcal{E}_h(\partial\Omega_d)$.

With every edges $E \in \mathcal{E}_h$, we associate a unit vector $\mathbf{n}_E$ such that $\mathbf{n}_E$ is orthogonal to E and equals to the unit exterior normal vector to $\partial\Omega$ if $E \subset \partial\Omega$. For any $E \in \mathcal{E}_h$ and any piecewise continuous function φ, we denote by $[\varphi]_E$ its jump across E in the direction of $\mathbf{n}_E$:

$$[\varphi]_E(x) := \begin{cases} \lim\limits_{t\to 0+} \varphi(x + t\mathbf{n}_E) - \lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for an interior edge/face } E, \\ -\lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for a boundary edge/face } E \end{cases}$$

Based on the above notation, we introduce a variant of the nonconforming Crouzeix–Raviart piecewise linear finite element space (larger than the space $\mathbf{H}_h$ used in [44])

$$\begin{aligned}\mathbf{H}_h := \{\mathbf{v}_h : \mathbf{v}_{h|T} \in [\mathbb{P}^1(T)]^N \ \forall T \in \mathcal{T}_h, ([\mathbf{v}_h]_E, \mathbf{1})_E = 0 \ \forall E \in \mathcal{E}_h(\Omega_s^+),\\ ([\mathbf{v}_h \cdot \mathbf{n}_E]_E, 1)_E = 0 \quad \forall E \in \mathcal{E}_h(\Omega_d) \cup \mathcal{E}_h(\partial\Omega_d)\}\end{aligned}$$

and piecewise constant function space

$$Q_h := \{q_h \in L^2_0(\Omega) : q_{h|T} \in \mathbb{P}^0(T) \ \forall T \in \mathcal{T}_h\},$$

where $\mathbb{P}^m(T)$ is the space of the restrictions to T of all polynomials of degree less than or equal to m. The space Q_h is equipped with the norm $\| \cdot \|$ while the norm on $\mathbf{H}_h$ will be specified later on. The choice of $\mathbf{H}_h$ is more natural than the one introduced in [44] since the space $\mathbf{H}_h$ approximates only $H(\text{div}, \Omega_d)$ and not $[H^1(\Omega_d)]^N$, while our a posteriori analysis is only valid in this larger space.

Let us introduce the discrete divergence operator $\text{div}_h \in \mathcal{L}(\mathbf{H}_h; Q_h) \cap \mathcal{L}(\mathbf{H}; Q)$ by

$$(\text{div}_h \mathbf{v}_h)_{|T} = \text{div}(\mathbf{v}_{h|T}), \quad \forall T \in \mathcal{T}_h. \tag{14}$$

Then, we can introduce two bilinear forms

$$\mathbf{a}_h(\mathbf{u}, \mathbf{v}) := 2\mu \sum_{T \in \mathcal{T}_h^s} (\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_T + \sum_{j=1}^{N-1} \frac{\mu \alpha_1}{\sqrt{\kappa_j}} \langle \mathbf{u}_s \cdot \tau_j, \mathbf{v}_s \cdot \tau_j \rangle_{\Gamma_I}$$
$$+ \mu (\mathbf{K}^{-1} \mathbf{u}, \mathbf{v})_{\Omega_d}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H} + \mathbf{H}_h$$

and

$$\mathbf{b}_h(\mathbf{v}, q) := -(q, \text{div}_h \mathbf{v})_\Omega, \quad \forall \mathbf{v} \in \mathbf{H} + \mathbf{H}_h, \quad \forall q \in Q_h.$$

Then the finite element discretization of (11) is to find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times Q_h$ such that

$$\begin{cases} \mathbf{a}_h(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{b}_h(\mathbf{v}_h, p_h) + \mathbf{J}(\mathbf{u}_h, \mathbf{v}_h) = L(\mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{H}_h, \\ \mathbf{b}_h(\mathbf{u}_h, q_h) = G(q_h), & \forall q_h \in Q_h. \end{cases} \tag{15}$$

This is the natural discretization of the weak formulation (11) except that the penalizing term $\mathbf{J}(\mathbf{u}_h, \mathbf{v}_h)$ is added. This bilinear form $\mathbf{J}(.,.)$ is defined by following the decomposition (13) of $\mathcal{E}_h$:

$$\mathbf{J}(\mathbf{u}, \mathbf{v}) = \mathbf{J}_{\Omega_s^+}(\mathbf{u}, \mathbf{v}) + \mathbf{J}_{\Omega_d}(\mathbf{u}, \mathbf{v}) + \mathbf{J}_{\partial\Omega_d}(\mathbf{u}, \mathbf{v}) \tag{16}$$

where

$$\mathbf{J}_{\Omega_s^+}(\mathbf{u}, \mathbf{v}) := (1 + 2\mu) \sum_{E \in \mathcal{E}_h(\Omega_s^+)} h_E^{-1} \int_E [\mathbf{u}]_E . [\mathbf{v}]_E ds,$$
$$\mathbf{J}_{\Omega_d}(\mathbf{u}, \mathbf{v}) := \sum_{E \in \mathcal{E}_h(\Omega_d)} h_E^{-1} \int_E [\mathbf{u}]_E . [\mathbf{v}]_E ds, \quad \text{and}$$
$$\mathbf{J}_{\partial\Omega_d}(\mathbf{u}, \mathbf{v}) := \sum_{E \in \mathcal{E}_h(\partial\Omega_d)} h_E^{-1} \int_E [\mathbf{u} \cdot \mathbf{n}_E]_E [\mathbf{v} \cdot \mathbf{n}_E]_E ds.$$

Here, h_E is the length ($N = 2$) or diameter ($N = 3$) of E. Note that each element of $\mathcal{E}_h$ only contributes with one jump term in $\mathbf{J}(\mathbf{u}, \mathbf{v})$.

We are now able to define the norm on $\mathbf{H}_h$ (see [44]):

$$\| \mathbf{v} \|_h := \left(\sum_{T \in \mathcal{T}_h^s} |\mathbf{v}|_{1,T}^2 + \sum_{j=1}^{N-1} \langle \mathbf{v}_s \cdot \tau_j, \mathbf{v}_s \cdot \tau_j \rangle_{\Gamma_I} + \| \mathbf{v} \|_d^2 + \| \text{div}_h \mathbf{v} \|_d^2 + \mathbf{J}(\mathbf{v}, \mathbf{v}) \right)^{1/2}.$$

Note that the difference between problem (15) and the formulation (4.2) from [44] relies on the use of a larger space $\mathbf{H}_h$. This choice does not affect the existence result of problem (15) since $\mathbf{a}_h$ remains coercive in $Z_h = \{\mathbf{v}_h \in \mathbf{H}_h : \mathbf{b}_h(\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h\}$ (see the proof of Lemmas 4.2 and 4.3 of [44]). Similarly a priori error estimates can be proved as in Theorem 5.4 of [44]. In summary the following results hold.

Theorem 2.2 *There exists a unique solution* $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times Q_h$ *to problem* (15) *and if the solution* $(\mathbf{u}, p) \in \mathbf{H} \times Q$ *of the continuous problem* (11) *is smooth enough, then we have*

$$\| \mathbf{u} - \mathbf{u}_h \|_h + \| p - p_h \| \lesssim h(|\mathbf{u}_s|_{2,s} + |\mathbf{u}_d|_{2,d} + |p_s|_{1,s} + |p_d|_{1,d}). \tag{17}$$

Here and below, in order to avoid excessive use of constants, the abbreviation $x \lesssim y$ stand for $x \leqslant cy$, with c a positive constant independent of x, y and $\mathcal{T}_h$.

For $\mathbf{v} \in \mathbf{H} \cap \mathbf{H}_h$ and for $q \in Q_h$, we can subtract (15) to (11) to obtain the Galerkin orthogonality relation:

$$\begin{aligned} &2\mu \sum_{T \in \mathcal{T}_h^s} (\mathbf{D}(\mathbf{e}), \mathbf{D}(\mathbf{v}))_T + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} \langle \mathbf{e}_s \cdot \tau_j, \mathbf{v}_s \cdot \tau_j \rangle_{\Gamma_I} + \mu(\mathbf{K}^{-1}\mathbf{e}, \mathbf{v})_d - (\varepsilon, \mathrm{div}_h \mathbf{v})_\Omega \\ &-(q, \mathrm{div}_h \mathbf{e})_\Omega - \mathbf{J}(\mathbf{u}_h, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{H} \cap \mathbf{H}_h, \quad \forall q \in Q_h, \end{aligned} \tag{18}$$

where here and below, the errors in the velocity and in the pressure are respectively defined by

$$\mathbf{e} = \mathbf{u} - \mathbf{u}_h \quad \text{and} \quad \varepsilon = p - p_h.$$

3 Some technical results

Our a posteriori analysis requires some analytical results that are proved or recalled. The first one concerns a sort of Helmholtz decomposition of elements of $\mathbf{H}$. Recall first that if $N = 3$,

$$H_0(\mathrm{curl}, \Omega_d) = \{\psi \in L^2(\Omega_d)^3 : \mathrm{curl}\psi \in L^2(\Omega_d)^3 \text{ and } \psi \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega_d\}.$$

Theorem 3.1 *Any* $\mathbf{v} \in \mathbf{H}$ *admits the Helmholtz type decomposition*

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \tag{19}$$

where $\mathbf{v}_0, \mathbf{v}_1 \in \mathbf{H}$ *but satisfying* $\mathbf{v}_0 \in H^1(\Omega)^N$,

$$\mathbf{v}_1 = \begin{cases} \mathbf{0} & \text{in } \Omega_s, \\ \mathrm{curl}\psi & \text{in } \Omega_d, \end{cases} \tag{20}$$

where $\psi \in H_0^1(\Omega_d)$ *if* $N = 2$, *while* $\psi \in H^1(\Omega_d)^3 \cap H_0(\mathrm{curl}, \Omega_d)$ *if* $N = 3$, *with the estimate*

$$\|\mathbf{v}_0\|_{1,\Omega} + \|\psi\|_{1,\Omega_d} \lesssim \|\mathbf{v}\|_{\mathbf{H}}. \tag{21}$$

Proof Fix $\mathbf{v} \in \mathbf{H}$. As $\gamma_s \mathbf{v} \in \tilde{H}^{\frac{1}{2}}(\Gamma_I)$, according to Theorem 1.5.2.3 of [30] if $N = 2$ and Section 2 of [29] if $N = 3$ we can consider a lifting $R\mathbf{v} \in H^1(\Omega_d)^N$ such that

$$\begin{cases} \gamma_d R\mathbf{v} = \gamma_s \mathbf{v} & \text{on } \Gamma_I, \\ \gamma_d R\mathbf{v} = 0 & \text{on } \Gamma_d, \end{cases} \tag{22}$$

that will satisfy

$$\|R\mathbf{v}\|_{1,\Omega} \lesssim \|\gamma_s \mathbf{v}\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_I)} \lesssim \|\mathbf{v}\|_{\mathbf{H}}, \tag{23}$$

where γ_l is the trace operator in $H^1(\Omega_l)^N$, $l = s$ or d. Hence the function $\mathbf{v} - R\mathbf{v}$ will belong to $H(\mathrm{div}, \Omega_d)$ and

$$(\mathbf{v} - R\mathbf{v}) \cdot \mathbf{n}_d = 0 \quad \text{on } \partial\Omega_d.$$

Therefore by Corollary I.2.4 of [28, Page 24], there exists a unique $\mathbf{w}_0 \in H_0^1(\Omega_d)^N$ such that

$$\operatorname{div}\mathbf{w}_0 = \operatorname{div}(\mathbf{v} - R\mathbf{v}) \quad \text{in } \Omega_d,$$

with the estimate

$$\|\mathbf{w}_0\|_{1,\Omega_d} \lesssim \|\operatorname{div}(\mathbf{v} - R\mathbf{v})\|_{\Omega_d}. \tag{24}$$

Hence this property of $\mathbf{w}_0$ and the estimates (23) and (24) show that $\mathbf{w} = \mathbf{w}_0 + R\mathbf{v}$ belongs to $H^1(\Omega_d)^N$ with the properties

$$\begin{cases} \operatorname{div}\mathbf{w} = \operatorname{div}\mathbf{v} & \text{in } \Omega_d, \\ \gamma_d\mathbf{w} = \gamma_s\mathbf{v} & \text{on } \Gamma_I, \\ \gamma_d\mathbf{w} = 0 & \text{on } \Gamma_d, \end{cases} \tag{25}$$

and the estimate

$$\|\mathbf{w}\|_{1,\Omega_d} \lesssim \|\mathbf{v}\|_{\mathbf{H}}. \tag{26}$$

In a second step we notice that $\mathbf{v} - \mathbf{w}$ belongs to $H_0(\operatorname{div}, \Omega_d)$ and is divergence free (i.e., $\operatorname{div}(\mathbf{v} - \mathbf{w}) = 0$ in Ω_d), hence if $N = 2$, by Corollary I.3.1 of [28, Page 40], there exists a unique $\psi \in H_0^1(\Omega_d)$ such that

$$\operatorname{curl}\psi = \mathbf{v} - \mathbf{w} \quad \text{in } \Omega_d, \tag{27}$$

with the estimate

$$\|\psi\|_{1,\Omega_d} \lesssim \|\mathbf{v} - \mathbf{w}\|_{\Omega_d}. \tag{28}$$

By (26), we then get

$$\|\psi\|_{1,\Omega_d} \lesssim \|\mathbf{v}\|_{\mathbf{H}}. \tag{29}$$

Let us show similar results if $N = 3$. We first use Theorem I.3.6 of [28, Page 48] to get $\psi_0 \in H(\operatorname{div}, \Omega_d) \cap H_0(\operatorname{curl}, \Omega_d)$ (divergence free) such that

$$\operatorname{curl}\psi_0 = \mathbf{v} - \mathbf{w} \quad \text{in } \Omega_d, \tag{30}$$

with the estimate

$$\|\psi_0\|_{H_0(\operatorname{curl},\Omega_d)} \lesssim \|\mathbf{v} - \mathbf{w}\|_{\Omega_d}. \tag{31}$$

Then we take advantage of Theorem 1.1 of [15] that yields

$$\psi_0 = \psi + \nabla\Phi \quad \text{in } \Omega_d,$$

with $\psi \in H^1(\Omega_d)^3 \cap H_0(\operatorname{curl}, \Omega_d)$ and $\Phi \in H_0^1(\Omega_d)$, with the estimate

$$\|\psi\|_{1,\Omega_d} + \|\Phi\|_{1,\Omega_d} \lesssim \|\psi_0\|_{H_0(\operatorname{curl},\Omega_d)}.$$

This estimate with (31) leads to (28) and hence to (29), while the identity (27) still holds since $\operatorname{curl}\psi = \operatorname{curl}\psi_0$.

Now if we come back to (27), we have shown that

$$\mathbf{v} = \mathbf{w} + \operatorname{curl}\psi \quad \text{in } \Omega_d, \tag{32}$$

and by the fact that $\psi \in H_0^1(\Omega_d)$ if $N = 2$ and $\psi \in H_0(\operatorname{curl}, \Omega_d)$ if $N = 3$, we also have

$$\operatorname{curl}\psi \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_I. \tag{33}$$

Now defining $\mathbf{v}_1$ by (20) and $\mathbf{v}_0$ by

$$\mathbf{v}_0 = \begin{cases} \mathbf{v} & \text{in } \Omega_s, \\ \mathbf{w} & \text{in } \Omega_d, \end{cases} \tag{34}$$

we get the splitting (19) due to (32). By (25), we deduce that $\mathbf{v}_0$ belongs to $H_0^1(\Omega)^N \cap \mathbf{H}$, while (33) and the regularity of ψ yield $\mathbf{v}_1 \in \mathbf{H}$. Finally the estimate (21) follows from (26) and (29). □

The second result that we need is a regularity result for the solution $(\mathbf{u}, p)$ of (11).

Theorem 3.2 *Let* $(\mathbf{u}, p) \in \mathbf{H} \times Q$ *be the unique solution of* (11). *If* $\mathbf{f} \in H(\text{curl}, \Omega_d)$ *and* $\mathbf{K} \in [C^{0,1}(\bar{\Omega}_d)]^{N\times N}$, *then there exists* $\epsilon > 0$ *such that*

$$\mathbf{u} \in [H^{\frac{1}{2}+\epsilon}(\Omega_d)]^N.$$

Proof As (2) implies that

$$\mu\mathbf{K}^{-1}\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega_d,$$

as well as

$$\text{div}\mathbf{u} = g \quad \text{in } \Omega_d,$$

with $g \in L^2(\Omega_d)$, we deduce that $\mathbf{v} := \mathbf{K}^{-1}\mathbf{u}$ satisfies

$$\text{curl}\mathbf{v} \in L^2(\Omega_d), \quad \text{if } N = 2, \text{ and curl}\mathbf{v} \in [L^2(\Omega_d)]^3, \quad \text{if } N = 3,$$

and

$$\text{div}(\mathbf{K}\mathbf{v}) \in L^2(\Omega_d),$$

and the boundary conditions

$$(\mathbf{K}\mathbf{v}) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_d,$$
$$\mathbf{v} \in [\tilde{H}^{\frac{1}{2}}(\Gamma_I)]^N.$$

Hence using the lifting $R\mathbf{v} \in H^1(\Omega_d)^N$ like in Theorem 3.1, we deduce that $\mathbf{v}_0 = \mathbf{v} - R\mathbf{v}$ satisfies

$$\text{curl}\mathbf{v}_0 \in L^2(\Omega_d), \text{ if } N = 2, \text{ and curl}\mathbf{v}_0 \in [L^2(\Omega_d)]^3, \quad \text{if } N = 3,$$

and

$$\text{div}(\mathbf{K}\mathbf{v}_0) \in L^2(\Omega_d),$$

and the boundary conditions

$$(\mathbf{K}\mathbf{v}_0) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega_d.$$

Hence the arguments from Theorem 1.1 of [15] (see also Theorem 3.5 of [16]) yield that

$$\mathbf{v}_0 = \psi + \nabla\Phi \quad \text{in } \Omega_d,$$

with $\psi \in H^1(\Omega_d)^N \cap H_0(\text{curl}, \Omega_d)$ and $\Phi \in H_0^1(\Omega_d)$, such that

$$\text{div}(\mathbf{K}\nabla\Phi) \in L^2(\Omega_d).$$

Since such a Φ belongs to $H^{\frac{3}{2}+\epsilon}(\Omega_d)$ for some $\epsilon > 0$ (see for instance [20]), the result follows. □

Note that the regularity of $\mathbf{u} \in [H^{\frac{1}{2}+\epsilon}(\Omega_d)]^N$, with $\epsilon > 0$ allows to give a meaning to $\mathbf{J}(\mathbf{u}, \mathbf{w})$ for any $\mathbf{w} \in \mathbf{H} \cup \mathbf{H}_h$ and hence to show that $\mathbf{J}(\mathbf{u}, \mathbf{w}) = 0$ for any $\mathbf{w} \in \mathbf{H} \cup \mathbf{H}_h$.

Let us finish this section by an estimation of the non conformity error.

Theorem 3.3 *For any* $\mathbf{u}_h \in \mathbf{H}_h$ *we have*

$$\inf_{\mathbf{w}_h \in \mathbf{H} \cap \mathbf{H}_h} \|\mathbf{u}_h - \mathbf{w}_h\|_h^2 \lesssim \mathbf{J}(\mathbf{u}_h, \mathbf{u}_h). \tag{35}$$

Proof It suffices to built an appropriated element $\mathbf{w}_h$ of $\mathbf{H} \cap \mathbf{H}_h$ such that:

$$\|\mathbf{u}_h - \mathbf{w}_h\|_h^2 \lesssim \mathbf{J}(\mathbf{u}_h, \mathbf{u}_h). \tag{36}$$

This element is a kind of Oswald interpolant $\tilde{I}_{Os}\mathbf{u}_h$ of $\mathbf{u}_h$, cf. [36].

Let us explain our construction in 2D, the idea being the same in 3D. Actually we take

$$\tilde{I}_{Os}\mathbf{u}_{s,h} = \sum_{x \in \mathcal{N}_h \cap \bar{\Omega}_s} \alpha_x^{(s)} \lambda_x \quad \text{and} \quad \tilde{I}_{Os}\mathbf{u}_{d,h} = \sum_{x \in \mathcal{N}_h \cap \bar{\Omega}_d} \alpha_x^{(d)} \lambda_x, \tag{37}$$

where λ_x is the standard basis element of

$$\mathcal{P}_1(\mathcal{T}_h) := \{v \in C(\overline{\Omega}) : v_{|T} \in \mathbb{P}^1(T), \quad \forall T \in \mathcal{T}_h\}$$

associated with x and $\alpha_x^{(l)} \in \mathbb{R}^N$, $l = s$ or d will be appropriately chosen. The idea is that we only need the continuity of the normal trace through Γ_I. More precisely, we distinguish different cases:

1. If $x \in \Omega_l$ with $l = s$ or d, then we take the standard Oswald definition, namely
$$\alpha_x^{(l)} = \sum_{T \in \mathcal{T}_h : x \in \mathcal{N}(T)} \mathbf{u}_{h|T}(x). \tag{38}$$
2. If $x \in \bar{\Gamma}_s$, we need that
$$\alpha_x^{(s)} = 0. \tag{39}$$
3. If $x \in \Gamma_I \backslash \bar{\Gamma}_s$, we distinguish two cases:

3a. Γ_I is flat near x, which means that a tangential vector $\mathbf{t}_x$ and a normal vector $\mathbf{n}_x$ to Γ_I exist at x. In that case, we take
$$\alpha_x^{(s)} \cdot \mathbf{n}_x = \alpha_x^{(d)} \cdot \mathbf{n}_x = \sum_{T \in \mathcal{T}_h : x \in \mathcal{N}(T)} (\mathbf{u}_{h|T} \cdot \mathbf{n}_x)(x), \tag{40}$$
$$\alpha_x^{(l)} \cdot \mathbf{t}_x = \sum_{T \in \mathcal{T}_h \cap \bar{\Omega}_l : x \in \mathcal{N}(T)} (\mathbf{u}_{h|T} \cdot \mathbf{t}_x)(x), \quad l = s, d. \tag{41}$$

3b. Γ_I is not regular at x, which means that no tangential vector and no normal vector to Γ_I exist at x. In that case, we define $\alpha_x^{(l)}$ by (38).

4. If $x \in \Gamma_d$, then in the same spirit than in the point 3, we distinguish two cases:

4a. Γ_d is flat near x, which means that a tangential vector $\mathbf{t}_x$ and a normal vector $\mathbf{n}_x$ to Γ_d exist at x. In that case, we take
$$\alpha_x^{(d)} \cdot \mathbf{n}_x = 0, \tag{42}$$
$$\alpha_x^{(d)} \cdot \mathbf{t}_x = \sum_{T \in \mathcal{T}_h \cap \bar{\Omega}_d : x \in \mathcal{N}(T)} (\mathbf{u}_{h|T} \cdot \mathbf{t}_x)(x). \tag{43}$$

4b. Γ_d is not regular at x, and in that case, we take

$$\alpha_x^{(d)} = 0. \tag{44}$$

5. If $x \in \Gamma_I \cap \bar{\Gamma}_d$, $\alpha_x^{(s)}$ was already defined in point 2, while for $\alpha_x^{(d)}$, we need that

$$\alpha_x^{(d)} \cdot \mathbf{n}_I = 0, \tag{45}$$

while we take

$$\alpha_x^{(d)} \cdot \mathbf{t}_I = \sum_{T \in \mathcal{T}_h \cap \bar{\Omega}_d : x \in \mathcal{N}(T)} (\mathbf{u}_{h|T} \cdot \mathbf{t}_I)(x), \tag{46}$$

where $\mathbf{n}_I$ (resp. $\mathbf{t}_I$) is the normal (resp.) tangential vector on the edge having x as vertex and included into Γ_I.
With this definition we show in Lemma 3.1 below that for all $T \in \mathcal{T}_h \subset \Omega_l$, $l = s$ or d and $x \in \mathcal{N}(T)$, one has the estimate:

$$\begin{aligned} |\mathbf{u}_{l,h|T}(x) - \alpha_x^{(l)}| \lesssim &\sum_{E \in \mathcal{E}_h(\Omega_s) \cup \mathcal{E}_h(\Omega_d) : x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h]_E\|_E \\ &+ \sum_{E \in \mathcal{E}_h(\Gamma_I) \cup \mathcal{E}_h(\Gamma_d) : x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h \cdot \mathbf{n}_E]_E\|_E. \end{aligned} \tag{47}$$

If this estimate holds, then for any $T \in \mathcal{T}_h \subset \Omega_l$, $l = s$ or d since we have

$$\|\nabla(\mathbf{u}_h - \tilde{I}_{Os}\mathbf{u}_h)\|_T \lesssim \sum_{x \in \mathcal{N}(T)} |\mathbf{u}_{h|T}(x) - \alpha_x^{(l)}|,$$
$$\|\mathbf{u}_h - \tilde{I}_{Os}\mathbf{u}_h\|_T \lesssim h_T \sum_{x \in \mathcal{N}(T)} |\mathbf{u}_{h|T}(x) - \alpha_x^{(l)}|,$$

we will get

$$\begin{aligned} \|\mathbf{u}_h - \tilde{I}_{Os}\mathbf{u}_h\|_{1,T} \lesssim \sum_{x \in \mathcal{N}(T)} \Bigg(&\sum_{E \in \mathcal{E}_h(\Omega_s) \cup \mathcal{E}_h(\Omega_d) : x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h]_E\|_E \\ &+ \sum_{E \in \mathcal{E}_h(\Gamma_I) \cup \mathcal{E}_h(\Gamma_d) : x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h \cdot \mathbf{n}_E]_E\|_E \Bigg). \end{aligned} \tag{48}$$

Similarly for an edge $E' \subset \Gamma_I$, we have

$$\|\mathbf{u}_{s,h} - \tilde{I}_{Os}\mathbf{u}_{s,h}\|_{E'} \lesssim h_E^{\frac{1}{2}} \sum_{x \in \mathcal{N}(E')} |\mathbf{u}_{s,h|T}(x) - \alpha_x^{(s)}|,$$

where T is the unique triangle included into Ω_s having E' as edge. Hence by the estimate (47) we deduce that

$$\begin{aligned} \|\mathbf{u}_{s,h} - \tilde{I}_{Os}\mathbf{u}_{s,h}\|_{E'} \lesssim \sum_{x \in \mathcal{N}(E')} \Bigg(&\sum_{E \in \mathcal{E}_h(\Omega_s) \cup \mathcal{E}_h(\Omega_d) : x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h]_E\|_E \\ &+ \sum_{E \in \mathcal{E}_h(\Gamma_I) \cup \mathcal{E}_h(\Gamma_d) : x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h \cdot \mathbf{n}_E]_E\|_E \Bigg). \end{aligned} \tag{49}$$

The estimates (48) and (49) directly lead to (36). □

Lemma 3.1 *For all $T \in \mathcal{T}_h \subset \Omega_l$, $l = s$ or d and $x \in \mathcal{N}(T)$, the estimate* (47) *holds.*

Proof We distinguish different cases corresponding to the previous points 1 to 5 above. First if x enters in the setting of points 1 and 2, the estimate (47) is proved in Theorem 2.2 of [36]. In the case 3a, we notice that by Theorem 2.2 of [36], we have

$$|\mathbf{u}_{l,h|T}(x) \cdot \mathbf{n}_x - \alpha_x^{(l)} \cdot \mathbf{n}_x| \lesssim \sum_{E \in \mathcal{E}_h(\Omega_s) \cup \mathcal{E}_h(\Omega_d): x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h \cdot \mathbf{n}_x]_E\|_E.$$

Since in that case $\mathbf{n}_x$ corresponds to $\mathbf{n}_I$, we obtain that

$$|\mathbf{u}_{l,h|T}(x) \cdot \mathbf{n}_x - \alpha_x^{(l)} \cdot \mathbf{n}_x| \lesssim \sum_{E \in \mathcal{E}_h(\Omega_s) \cup \mathcal{E}_h(\Omega_d): x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h]_E\|_E + \sum_{E \in \mathcal{E}_h(\Gamma_I): x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h \cdot \mathbf{n}_E]_E\|_E. \tag{50}$$

For the tangential component, we notice that $\alpha_x^{(l)} \cdot \mathbf{t}_x$ is the mean of the values of $\mathbf{u}_{l,h} \cdot \mathbf{t}_x$ in Ω_l around x, hence by applying Lemma 2.2 of [36] we get

$$|\mathbf{u}_{l,h|T}(x) \cdot \mathbf{t}_x - \alpha_x^{(l)} \cdot \mathbf{t}_x| \lesssim \sum_{j=1}^{N_x^{(l)}-1} |\mathbf{u}_{l,h|T_{j+1}}(x) \cdot \mathbf{t}_x - \mathbf{u}_{l,h|T_j}(x) \cdot \mathbf{t}_x|,$$

where T_j, $j = 1, \ldots, N_x^{(l)}$ are the triangles of the triangulation included into Ω_l and having x as node. This yields

$$|\mathbf{u}_{l,h|T}(x) \cdot \mathbf{t}_x - \alpha_x^{(l)} \cdot \mathbf{t}_x| \lesssim \sum_{j=1}^{N_x^{(l)}-1} |\mathbf{u}_{l,h|T_{j+1}}(x) - \mathbf{u}_{l,h|T_j}(x)|,$$

and since by the usual equivalence of norms in finite dimension, we have for any edge E and any vertex $x \in \mathcal{N}(E)$

$$|p(x)| \lesssim h_E^{-\frac{1}{2}} \|p\|_E, \quad \forall p \in \mathbb{P}^1(E), \tag{51}$$

we deduce that

$$|\mathbf{u}_{l,h|T}(x) \cdot \mathbf{t}_x - \alpha_x^{(l)} \cdot \mathbf{t}_x| \lesssim \sum_{E \in \mathcal{E}_h(\Omega_l): x \in E} h_E^{-\frac{1}{2}} \|[\mathbf{u}_h]_E\|_E.$$

This estimate and (50) lead to (47) in the case 3a.

In the case 3b, if we denote by T_j, $j = 1, \ldots, N_x$ the set of triangles having x as node, we can always assume that $T_j \subset \bar{\Omega}_d$, if $j = 1, \ldots, N_x^{(d)}$ with $N_x^{(l)} < N_x$. Hence using again Lemma 2.2 of [36] we get

$$|\mathbf{u}_{l,h|T}(x) - \alpha_x^{(l)}| \lesssim \sum_{j=1}^{N_x-1} |\mathbf{u}_{h|T_{j+1}}(x) - \mathbf{u}_{h|T_j}(x)|,$$

and by distinguishing the triangles included in Ω_s and Ω_d, we can write

$$|\mathbf{u}_{l,h|T}(x)-\alpha_x^{(l)}| \lesssim \sum_{j=1}^{N_x^{(d)}-1} |\mathbf{u}_{h|T_{j+1}}(x) - \mathbf{u}_{h|T_j}(x)| + \sum_{j=N_x^{(d)}+1}^{N_x^{(d)}-1} |\mathbf{u}_{h|T_{j+1}}(x) - \mathbf{u}_{h|T_j}(x)| + |\mathbf{u}_{h|T_{N_x^{(d)}+1}}(x) - \mathbf{u}_{h|T_{N_x^{(d)}}}(x)|. \quad (52)$$

The two first terms of the right-hand side are clearly estimated since they correspond to jumps inside Ω_d or Ω_s. For the last term, as Γ_I is piecewise polygonal, let us denote by $\mathbf{n}_1$, $\mathbf{n}_2$ the two normal vectors of Γ_I at x, or more properly the two normal vectors of the two edges E_1, E_2 of the triangulation included in Γ_I that have x as vertex. Suppose that E_2 is the edge between $T_{N_x^{(d)}}$ and $T_{N_x^{(d)}+1}$ In the case 3b, these two vectors form an basis of $\mathbb{R}^2$ and therefore

$$|\mathbf{u}_{h|T_{N_x^{(d)}+1}}(x) - \mathbf{u}_{h|T_{N_x^{(d)}}}(x)| \lesssim |\mathbf{u}_{h|T_{N_x^{(d)}+1}}(x)\cdot\mathbf{n}_1 - \mathbf{u}_{h|T_{N_x^{(d)}}}(x)\cdot\mathbf{n}_1| + |\mathbf{u}_{h|T_{N_x^{(d)}+1}}(x)\cdot\mathbf{n}_2 - \mathbf{u}_{h|T_{N_x^{(d)}}}(x)\cdot\mathbf{n}_2|. \quad (53)$$

The second term is convenient since it corresponds to the normal jump of $\mathbf{u}_h$ through E_2. For the first term, we make a complete tour around x, namely we write

$$\mathbf{u}_{h|T_{N_x^{(d)}+1}}(x)\cdot\mathbf{n}_1 - \mathbf{u}_{h|T_{N_x^{(d)}}}(x)\cdot\mathbf{n}_1 = \sum_{j=N_x^{(d)}}^{N_x-1} (\mathbf{u}_{h|T_{j+1}}(x)\cdot\mathbf{n}_1 - \mathbf{u}_{h|T_j}(x)\cdot\mathbf{n}_1) + \mathbf{u}_{h|T_{N_x}}(x)\cdot\mathbf{n}_1 - \mathbf{u}_{h|T_1}(x)\cdot\mathbf{n}_1 + \sum_{j=1}^{N_x^{(d)}-1} (\mathbf{u}_{h|T_{j+1}}(x)\cdot\mathbf{n}_1 - \mathbf{u}_{h|T_j}(x)\cdot\mathbf{n}_1).$$

This identity implies that

$$|\mathbf{u}_{h|T_{N_x^{(d)}+1}}(x)\cdot\mathbf{n}_1 - \mathbf{u}_{h|T_{N_x^{(d)}}}(x)\cdot\mathbf{n}_1| \lesssim \sum_{j=1}^{N_x^{(d)}-1} |\mathbf{u}_{h|T_{j+1}}(x) - \mathbf{u}_{h|T_j}(x)| + \sum_{j=N_x^{(d)}}^{N_x-1} |\mathbf{u}_{h|T_{j+1}}(x) - \mathbf{u}_{h|T_j}(x)| + |\mathbf{u}_{h|T_{N_x}}(x)\cdot\mathbf{n}_1 - \mathbf{u}_{h|T_1}(x)\cdot\mathbf{n}_1|.$$

Using this estimate in (53) leads to

$$|\mathbf{u}_{h|T_{N_x^{(d)}+1}}(x) - \mathbf{u}_{h|T_{N_x^{(d)}}}(x)| \lesssim \sum_{j=1}^{N_x^{(d)}-1} |\mathbf{u}_{h|T_{j+1}}(x) - \mathbf{u}_{h|T_j}(x)| + \sum_{j=N_x^{(d)}}^{N_x-1} |\mathbf{u}_{h|T_{j+1}}(x) - \mathbf{u}_{h|T_j}(x)| + |\mathbf{u}_{h|T_{N_x}}(x)\cdot\mathbf{n}_1 - \mathbf{u}_{h|T_1}(x)\cdot\mathbf{n}_1| + |\mathbf{u}_{h|T_{N_x^{(d)}+1}}(x)\cdot\mathbf{n}_2 - \mathbf{u}_{h|T_{N_x^{(d)}}}(x)\cdot\mathbf{n}_2|. \quad (54)$$

Coming back to (52) and using (54) we have shown that

$$
\begin{aligned}
|\mathbf{u}_{l,h|T}(x) - \alpha_x^{(l)}| \lesssim & \sum_{j=1}^{N_x^{(d)}-1} |\mathbf{u}_{h|T_{j+1}}(x) - \mathbf{u}_{h|T_j}(x)| \\
& + \sum_{j=N_x^{(d)}}^{N_x-1} |\mathbf{u}_{h|T_{j+1}}(x) - \mathbf{u}_{h|T_j}(x)| \\
& + |\mathbf{u}_{h|T_{N_x}}(x) \cdot \mathbf{n}_1 - \mathbf{u}_{h|T_1}(x) \cdot \mathbf{n}_1| \\
& + |\mathbf{u}_{h|T_{N_x^{(d)}+1}}(x) \cdot \mathbf{n}_2 - \mathbf{u}_{h|T_{N_x^{(d)}}}(x) \cdot \mathbf{n}_2|.
\end{aligned} \tag{55}
$$

As before by using the estimate (51) we arrive at (47) in the case 3b.

The proof of (47) in the cases 4 and 5 is made in the same spirit and is left to the reader. □

4 Error estimators

In order to solve the Stokes–Darcy coupled problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriated indicators. In this section, we first define the local and global indicators and then the lower and upper error bounds are derived (see Sects. 4.2 and 4.3).

4.1 Residual error estimators

The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier, and that involve the data at hand. To this end denote the exact element residuals by

$$
\mathbf{R}_{s,T} = \mathbf{f} + 2\mu \operatorname{div} \mathbf{D}(\mathbf{u}_h) - \nabla p_h \quad \text{in } T \in \mathcal{T}_h^s, \tag{56}
$$

$$
\mathbf{R}_{d,T} = \mathbf{f} - \mu \mathbf{K}^{-1} \mathbf{u}_h - \nabla p_h \quad \text{in } T \in \mathcal{T}_h^d. \tag{57}
$$

As it is common, these exact residuals are replaced by some finite-dimensional approximation called approximate element residual $\mathbf{r}_{l,T}$, $l = s, d$,

$$
\mathbf{r}_{l,T} \in [\mathbb{P}^k(T)]^N \quad \text{on } T \in \mathcal{T}_h^l.
$$

This approximation is here achieved by projecting $\mathbf{f}$ on the space of piecewise constant functions in Ω_s and piecewise $\mathbb{P}_1$ functions in Ω_d, more precisely for all $T \in \mathcal{T}_h^s$, we take

$$
\mathbf{f}_T = \frac{1}{|T|} \int_T \mathbf{f}(x)\, dx,
$$

while for all $T \in \mathcal{T}_h^d$, we take $\mathbf{f}_T$ as the unique element of $[\mathbb{P}^1(T)]^N$ such that

$$
\int_T \mathbf{f}_T(x) \cdot \mathbf{q}(x)\, dx = \int_T \mathbf{f}(x) \cdot \mathbf{q}(x)\, dx, \quad \forall \mathbf{q} \in [\mathbb{P}^1(T)]^N.
$$

Finally the global function $\mathbf{f}_h$ is defined by

$$
\mathbf{f}_h = \mathbf{f}_T \quad \text{in } T, \quad \forall T \in \mathcal{T}_h.
$$

Hence

$$\mathbf{r}_{s,T} = \mathbf{f}_T + 2\mu \mathrm{div}\mathbf{D}(\mathbf{u}_h) - \nabla p_h \quad \text{in } T \in \mathcal{T}_h^s, \tag{58}$$

$$\mathbf{r}_{d,T} = \mathbf{f}_T - \mu \mathbf{K}^{-1}\mathbf{u}_h - \nabla p_h \quad \text{in } T \in \mathcal{T}_h^d, \tag{59}$$

Next, introduce the gradient jump in normal direction by

$$\mathbf{J}_{E,\mathbf{n}_E} := \begin{cases} [(2\mu\mathbf{D}(\mathbf{u}_h) - p_h\mathbf{I}) \cdot \mathbf{n}_E]_E & \text{for an interior edge/face } E, \\ \mathbf{0} & \text{for a boundary edge/face } E. \end{cases}$$

where $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{N\times N}$.

Definition 4.1 (*Residual error estimator*) The residual error estimator is locally defined by:

$$\Theta_T := \left(\sum_{i=1}^{9} \Theta_{i,T}^2 \right)^{1/2}, \quad \text{for each } T \in \mathcal{T}_h. \tag{60}$$

where

$$\Theta_{1,T}^2 := h_T^2 \parallel \mathbf{r}_{l,T} \parallel_T^2, \quad \text{if } T \in \mathcal{T}_h^l, l = s \text{ or } d,$$

$$\Theta_{2,T}^2 := \begin{cases} h_T^2 \parallel \mathrm{curl}(\mathbf{f}_h - \mu\mathbf{K}^{-1}\mathbf{u}_h) \parallel_T^2, & \text{if } T \in \mathcal{T}_h^d, \\ 0 & \text{if } T \in \mathcal{T}_h^s, \end{cases}$$

$$\Theta_{3,T}^2 := \parallel g - \mathrm{div}\mathbf{u}_h \parallel_T^2,$$

$$\Theta_{4,T}^2 := \sum_{E\in\mathcal{E}_h(\partial T\cap\bar{\Gamma}_I)} h_E \left\{ \sum_{j=1}^{N-1} \left\| \mathbf{u}_{s,h} \cdot \tau_j + \frac{\sqrt{\kappa_j}}{\alpha_1} 2\mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \tau_j \right\|_E^2 \right\},$$

$$\Theta_{5,T}^2 := \sum_{E\in\mathcal{E}_h(\partial T\cap\bar{\Gamma}_I)} h_E \parallel p_{d,h} - p_{s,h} + 2\mu\mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \mathbf{n}_s \parallel_E^2,$$

$$\Theta_{6,T}^2 := \begin{cases} \displaystyle\sum_{E\in\mathcal{E}_h(\partial T\cap\bar{\Omega}_s)} h_E \parallel \mathbf{J}_{E,\mathbf{n}_E} \parallel_E^2, & \text{if } T \in \mathcal{T}_h^s, \\ \displaystyle\sum_{E\in\mathcal{E}_h(\partial T\cap\Omega_d)} h_E \parallel [p_h]_E \parallel_E^2 & \text{if } T \in \mathcal{T}_h^d, \end{cases}$$

$$\Theta_{7,T}^2 := \sum_{E\in\mathcal{E}_h(\partial T\cap\Omega_d)} h_E^{-1} \parallel [\mathbf{u}_h]_E \parallel_E^2,$$

$$\Theta_{8,T}^2 := \sum_{E\in\mathcal{E}_h(\partial T\cap\partial\Omega_d)} h_E^{-1} \parallel [\mathbf{u}_h \cdot \mathbf{n}_E]_E \parallel_E^2,$$

$$\Theta_{9,T}^2 := \sum_{E\in\mathcal{E}_h(\partial T\cap\Omega_s^+)} h_E^{-1}(1 + 2\mu) \parallel [\mathbf{u}_h]_E \parallel_E^2,$$

with $\mathbf{u}_{l,h} := \mathbf{u}_{h|\Omega_l}$, and $p_{l,h} := p_{h|\Omega_l}$, $l = s, d$.

The global residual error estimator is given by:

$$\Theta := \left(\sum_{T\in\mathcal{T}_h} \Theta_T^2 \right)^{1/2}. \tag{61}$$

Furthermore denote the local and global approximation terms by

$$\zeta_T := \begin{cases} h_T \parallel \mathbf{f} - \mathbf{f}_h \parallel_T, & \forall T \in \mathcal{T}_h^s, \\ h_T(\parallel \mathbf{f} - \mathbf{f}_h \parallel_T + \|\mathrm{curl}(\mathbf{f} - \mathbf{f}_h)\|_T), & \forall T \in \mathcal{T}_h^d, \end{cases}$$

and

$$\zeta := \left(\sum_{T \in \mathcal{T}_h} \zeta_T^2 \right)^{1/2}. \tag{62}$$

The residual character of each term on the right-hand sides of (60) is quite clear since if $(\mathbf{u}_h, p_h)$ would be the exact solution of (11), then they would vanish.

4.2 Efficiency of the a posteriori error estimator

In order to derive the local lower bounds, we proceed similarly as in [9,10] (see also [24]), by applying inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions. To this end, we recall some notation and introduce further preliminary results. Given $T \in \mathcal{T}_h$, and $E \in \mathcal{E}(T)$, we let b_T and b_E be the usual simplex-bubble and face-bubble functions respectively (see (1.5) and (1.6) in [48]). In particular, b_T satisfies $b_T \in \mathbb{P}^3(T)$, $supp(b_T) \subseteq T$, $b_T = 0$ sur ∂T, and $0 \leq b_T \leq 1$ on T. Similarly, $b_E \in \mathbb{P}^2(T)$, $supp(b_E) \subseteq \omega_E := \{T' \in \mathcal{T}_h : E \in \mathcal{E}(T')\}$, $b_E = 0$ on $\partial T \setminus E$ and $0 \leq b_E \leq 1$ in ω_E. We also recall from [47] that, given $k \in \mathbb{N}$, there exists an extension operator $L : C(E) \longrightarrow C(T)$ that satisfies $L(p) \in \mathbb{P}^k(T)$ and $L(p)_{|E} = p, \forall p \in \mathbb{P}^k(E)$. A corresponding vectorial version of L, that is, the componentwise application of L, is denoted by $\mathbf{L}$. Additional properties of b_T, b_E and L are collected in the following lemma (see [47])

Lemma 4.1 *Given $k \in \mathbb{N}^*$, there exist positive constants depending only on k and shape-regularity of the triangulations (minimum angle condition), such that for each simplex T and $E \in \mathcal{E}(T)$ there hold*

$$\| q \|_T \lesssim \| q b_T^{1/2} \|_T \lesssim \| q \|_T, \quad \forall q \in \mathbb{P}^k(T) \tag{63}$$

$$|q b_T|_{1,T} \lesssim h_T^{-1} \| q \|_T, \quad \forall q \in \mathbb{P}^k(T) \tag{64}$$

$$\| p \|_E \lesssim \| b_E^{1/2} p \|_E \lesssim \| p \|_E, \quad \forall p \in \mathbb{P}^k(E) \tag{65}$$

$$\| L(p) \|_T + h_E |L(p)|_{1,T} \lesssim h_E^{1/2} \| p \|_E \quad \forall p \in \mathbb{P}^k(E) \tag{66}$$

To prove local efficiency for $\omega \subset \Omega$, let us denote by

$$\begin{aligned} \| \mathbf{v} \|_{h,\omega}^2 = & \sum_{T \subset \bar{\omega} \cap \bar{\Omega}_s} |\mathbf{v}|_{1,T}^2 \\ & + \sum_{T \subset \bar{\omega} \cap \bar{\Omega}_d} (\| \mathbf{v} \|_T^2 + \| \operatorname{div}_h \mathbf{v} \|_T^2) \\ & + \|\mathbf{v}_s \times \mathbf{n}\|_{\Gamma_I \cap \bar{\omega}}^2 + \sum_{T \subset \bar{\omega}} \mathbf{J}_T(\mathbf{v}, \mathbf{v}), \end{aligned}$$

where

$$\begin{aligned} \mathbf{J}_T(\mathbf{v}, \mathbf{v}) = (1 + 2\mu) & \sum_{E \in \mathcal{E}_h(\Omega_s^+) \cap \mathcal{E}(T)} h_E^{-1} \|[\mathbf{v}]_E\|_E^2 \\ & + \sum_{E \in \mathcal{E}_h(\Omega_d) \cap \mathcal{E}(T)} h_E^{-1} \|[\mathbf{v}]_E\|_E^2 + \sum_{E \in \mathcal{E}_h(\partial \Omega_d) \cap \mathcal{E}(T)} h_E^{-1} \|[\mathbf{v} \cdot \mathbf{n}_E]_E\|_E^2. \end{aligned}$$

The main result of this subsection can be stated as follows.

Theorem 4.1 *Under the assumptions of Theorem 3.2, the following lower error bound holds:*

$$\Theta_T \lesssim \| \mathbf{e} \|_{h,\tilde{\omega}_T} + \| \varepsilon \|_{\tilde{\omega}_T} + \sum_{T' \subset \tilde{\omega}_T} \zeta_{T'}, \tag{67}$$

where $\tilde{\omega}_T$ is a finite union of neighboring elements of T.

Proof We bound each term of the residual separately. Since by Theorem 3.2 the jump of $\mathbf{u}$ is zero through all the edges of Ω_d, hence for all $i = 7, 8$ or 9, we clearly have

$$\Theta_{i,T}^2 \lesssim \mathbf{J}_T(\mathbf{u}_h, \mathbf{u}_h) = \mathbf{J}_T(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - \mathbf{u}) \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{h,T}. \tag{68}$$

.

Hence it remains to estimate the local indicators for $i \leq 6$.

(1) Element residual in Ω_s. Set $\mathbf{w}_T := \mathbf{r}_{s,T} b_T \in [H_0^1(T)]^N$ and consider

$$\int_T \mathbf{r}_{s,T} \cdot \mathbf{w}_T = \int_T (\mathbf{f}_h + 2\mu \mathrm{div} \mathbf{D}(\mathbf{u}_h) - \nabla p_h) \cdot \mathbf{w}_T \tag{69}$$

Introduce $\mathbf{f}$ and use the weak formulation (11) to get

$$\begin{aligned}\int_T \mathbf{r}_{s,T} \cdot \mathbf{w}_T &= \int_T (\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T \\ &\quad + \int_T (2\mu \mathbf{D}(\mathbf{u}) : \nabla \mathbf{w}_T - p \mathrm{div} \mathbf{w}_T) \\ &\quad + \int_T (2\mu \mathrm{div} \mathbf{D}(\mathbf{u}_h) - \nabla p_h) \cdot \mathbf{w}_T.\end{aligned}$$

Integrating by parts in this last term we get

$$\int_T \mathbf{r}_{s,T} \cdot \mathbf{w}_T = \int_T (\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T + 2\mu \int_T \mathbf{D}(\mathbf{e}) : \nabla(\mathbf{w}_T) - \int_T \varepsilon \mathrm{div} \mathbf{w}_T.$$

Cauchy–Schwarz inequality implies that

$$\int_T \mathbf{r}_{s,T} \cdot \mathbf{w}_T \lesssim \| \mathbf{f} - \mathbf{f}_h \|_T \| \mathbf{w}_T \|_T + (2\mu |\mathbf{e}|_{1,T} + \| \varepsilon \|_T) |\mathbf{w}_T|_{1,T}.$$

The inverse inequalities (63), (64) and the obvious relation $\| \mathbf{w}_T \|_T \leqslant \| \mathbf{r}_{s,T} \|_T$ imply

$$\| \mathbf{r}_{s,T} \|_T^2 \lesssim (\| \mathbf{f} - \mathbf{f}_h \|_T + h_T^{-1} |\mathbf{e}|_{1,T} + h_T^{-1} \| \varepsilon \|_T) \| \mathbf{r}_{s,T} \|_T,$$

or equivalently

$$\Theta_{1,T} \lesssim h_T \| \mathbf{f} - \mathbf{f}_h \|_T + |\mathbf{e}|_{1,T} + \| \varepsilon \|_T . \tag{70}$$

(2) Element residual in Ω_d. Set $\mathbf{w}_T := \mathbf{r}_{d,T} b_T \in [H_0^1(T)]^N$, use (11) and integrate by parts to obtain

$$\begin{aligned}\int_T \mathbf{r}_{d,T} \cdot \mathbf{w}_T &= \int_T (\mathbf{f}_h - \mu \mathbf{K}^{-1} \mathbf{u}_h - \nabla p_h) \cdot \mathbf{w}_T \\ &= \int_T (\mathbf{f}_h - \mu \mathbf{K}^{-1} \mathbf{u}_h - \nabla p_h) \cdot \mathbf{w}_T \\ &\quad + \int_T (\mu \mathbf{K}^{-1} \mathbf{u} - \mathbf{f}) \cdot \mathbf{w}_T - p \mathrm{div} \mathbf{w}_T) \\ &= \int_T (\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T + \int_T (\mu \mathbf{K}^{-1} \mathbf{e} \cdot \mathbf{w}_T - \varepsilon \mathrm{div} \mathbf{w}_T).\end{aligned}$$

As before Cauchy–Schwarz inequality and the inverse inequalities (63) and (64) lead to

$$\Theta_{1,T} \lesssim h_T \parallel \mathbf{f} - \mathbf{f}_h \parallel_T + \parallel \mathbf{K}^{-1}\mathbf{e} \parallel_T + \parallel \varepsilon \parallel_T . \tag{71}$$

(3) Curl element residual in Ω_d.
For $T \in \mathcal{T}_h^d$, we set $C_T = \text{curl}(\mathbf{f}_h - \mu\mathbf{K}^{-1}\mathbf{u}_h)$ and $\mathbf{w}_T = C_T b_T$. Hence we notice that $\text{curl}\mathbf{w}_T$ belongs to $\mathbf{H}$ and is divergence free, therefore by (11), we have

$$\mathbf{a}(\mathbf{u}, \text{curl}\mathbf{w}_T) = (\mathbf{f}, \text{curl}\mathbf{w}_T),$$

or equivalently

$$\int_T (\mu\mathbf{K}^{-1}\mathbf{u} - \mathbf{f}) \cdot \text{curl}\mathbf{w}_T = 0. \tag{72}$$

But by Green's formula we may write

$$\int_T C_T \cdot \mathbf{w}_T = \int_T \text{curl}(\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T + \int_T (\mathbf{f} - \mu\mathbf{K}^{-1}\mathbf{u}_h) \cdot \text{curl}\mathbf{w}_T,$$

and by using (72) we deduce that

$$\int_T C_T \mathbf{w}_T = \int_T \text{curl}(\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T + \int_T \mu\mathbf{K}^{-1}(\mathbf{u} - \mathbf{u}_h) \cdot \text{curl}\mathbf{w}_T.$$

By Cauchy–Schwarz inequality we obtain

$$\int_T C_T \cdot \mathbf{w}_T \leq \|\text{curl}(\mathbf{f}_h - \mathbf{f})\|_T \|\mathbf{w}_T\|_T + \parallel \mathbf{K}^{-1}\mathbf{e} \parallel_T \|\text{curl}\mathbf{w}_T\|_T.$$

Again the inverse inequalities (63) and (64) allows to get

$$\Theta_{2,T} \lesssim \parallel \mathbf{K}^{-1}\mathbf{e} \parallel_T + h_T \|\text{curl}(\mathbf{f}_h - \mathbf{f})\|_T. \tag{73}$$

(4) Divergence element residual in Ω. We directly see that

$$g - \text{div}\mathbf{u}_h = \text{div}\mathbf{u} - \text{div}\mathbf{u}_h = \text{div}\mathbf{e},$$

hence by Cauchy–Schwarz inequality we conclude

$$\Theta_{3,T} = \parallel g - \text{div}\mathbf{u}_h \parallel_T \leqslant \parallel \text{div}\mathbf{e} \parallel_T . \tag{74}$$

(5) Interface elements on Γ_I.
To estimate $\Theta_{4,T}$ and $\Theta_{5,T}$, we fix an edge E included in Γ_I and for a constant r_E fixed later on and a unit vector $\mathbf{N}$, we consider

$$\mathbf{w}_E = r_E b_E \mathbf{N},$$

that clearly belongs to $\mathbf{H}$. Hence the weak formulation (11) yields

$$\mathbf{a}(\mathbf{u}, \mathbf{w}_E) + \mathbf{b}(\mathbf{w}_E, p) = (\mathbf{f}, \mathbf{w}_E)_{\omega_E},$$

that is equivalent to

$$\int_{T_s} (2\mu\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w}_E) - p\text{div}\mathbf{w}_E) + \int_{T_d} (\mu\mathbf{K}^{-1}\mathbf{u} \cdot \mathbf{w}_E - p\text{div}\mathbf{w}_E)$$
$$+ \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} (\mathbf{u}_s \cdot \tau_j, \mathbf{w}_{E,s} \cdot \tau_j)_E = (\mathbf{f}, \mathbf{w}_E)_{\omega_E}, \tag{75}$$

where T_s (resp. T_d) is the unique triangle/tetrahedron included in $\bar{\Omega}_s$ (resp. $\bar{\Omega}_d$) having E as edge/face. On the other hand, integrating by parts in T_s and in T_d yields

$$\begin{aligned}
&\int_{T_s}(2\mu\mathbf{D}(\mathbf{u}_h):\mathbf{D}(\mathbf{w}_E)-p_h\mathrm{div}\mathbf{w}_E)+\int_{T_d}(\mu\mathbf{K}^{-1}\mathbf{u}_h\cdot\mathbf{w}_E-p_h\mathrm{div}\mathbf{w}_E)\\
&\quad+\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}(\mathbf{u}_{s,h}\cdot\tau_j,\mathbf{w}_{E,s}\cdot\tau_j)_E\\
&=-\int_{T_s}(2\mu\mathrm{div}\mathbf{D}(\mathbf{u}_h)-\nabla p_h)\cdot\mathbf{w}_E+\int_{T_d}(\mu\mathbf{K}^{-1}\mathbf{u}_h\cdot\mathbf{w}_E+\nabla p_h)\cdot\mathbf{w}_E\\
&\quad+\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}(\mathbf{u}_{s,h}\cdot\tau_j,\mathbf{w}_{E,s}\cdot\tau_j)_E-\int_E([p_h]_E\mathbf{w}_E\cdot\mathbf{n}_E-2\mu(D(\mathbf{u}_{s,h}\mathbf{n}_E)\cdot\mathbf{w}_E).
\end{aligned}$$

Subtracting this identity to (75) we find

$$\begin{aligned}
&\int_E([p_h]_E\mathbf{w}_E\cdot\mathbf{n}_E-2\mu(\mathbf{D}(\mathbf{u}_{s,h}\mathbf{n}_E)\cdot\mathbf{w}_E)-\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}(\mathbf{u}_{s,h}\cdot\tau_j,\mathbf{w}_{E,s}\cdot\tau_j)_E\\
&\quad=\int_{T_s}(2\mu\mathbf{D}(\mathbf{e}):\mathbf{D}(\mathbf{w}_E)-\varepsilon\mathrm{div}\mathbf{w}_E)+\int_{T_d}(\mu\mathbf{K}^{-1}\mathbf{e}\cdot\mathbf{w}_E-\varepsilon\mathrm{div}\mathbf{w}_E)\\
&\qquad+\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}(\mathbf{e}_s\cdot\tau_j,\mathbf{w}_{E,s}\cdot\tau_j)_E-\int_{T_s}(\mathbf{f}+2\mu\mathrm{div}\mathbf{D}(\mathbf{u}_h)\\
&\qquad-\nabla p_h)\cdot\mathbf{w}_E-\int_{T_d}(\mathbf{f}-\mu\mathbf{K}^{-1}\mathbf{u}_h\cdot\mathbf{w}_E-\nabla p_h)\cdot\mathbf{w}_E.
\end{aligned}$$

In that last terms introducing the element residual $\mathbf{r}_{l,T}$, we arrive at

$$\begin{aligned}
&\int_E([p_h]_E\mathbf{w}_E\cdot\mathbf{n}_E-2\mu(\mathbf{D}(\mathbf{u}_{s,h})\mathbf{n}_E\cdot\mathbf{w}_E)-\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}(\mathbf{u}_{s,h}\cdot\tau_j,\mathbf{w}_{E,s}\cdot\tau_j)_E\\
&\quad=\int_{T_s}(2\mu\mathbf{D}(\mathbf{e}):\mathbf{D}(\mathbf{w}_E)-\varepsilon\mathrm{div}\mathbf{w}_E)+\int_{T_d}(\mu\mathbf{K}^{-1}\mathbf{e}\cdot\mathbf{w}_E-\varepsilon\mathrm{div}\mathbf{w}_E)\\
&\qquad+\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}(\mathbf{e}_s\cdot\tau_j,\mathbf{w}_{E,s}\cdot\tau_j)_E-\int_{T_s}(\mathbf{f}-\mathbf{f}_h+\mathbf{r}_{s,T})\cdot\mathbf{w}_E\\
&\qquad-\int_{T_d}(\mathbf{f}-\mathbf{f}_h+\mathbf{r}_{d,T})\cdot\mathbf{w}_E.
\end{aligned}\tag{76}$$

- To estimate $\Theta_{4,T}$, for each $j=1,\ldots,N-1$, we take $r_E=\mathbf{u}_h\cdot\tau_j+\frac{\sqrt{\kappa_j}}{\alpha_1}2\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_h)\cdot\tau_j$ and $\mathbf{N}=\tau_j$. With this choice, the identity (76) and the inverse inequality (65) yield

$$\begin{aligned}
\|r_E\|_E^2\lesssim&\int_{T_s}(2\mu\mathbf{D}(\mathbf{e}):\mathbf{D}(\mathbf{w}_E)-\varepsilon\mathrm{div}\mathbf{w}_E)+\int_{T_d}(\mu\mathbf{K}^{-1}\mathbf{e}\cdot\mathbf{w}_E-\varepsilon\mathrm{div}\mathbf{w}_E)\\
&+\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}(\mathbf{e}_s\cdot\tau_j,\mathbf{w}_{E,s}\cdot\tau_j)_E\\
&-\int_{T_s}(\mathbf{f}-\mathbf{f}_h+\mathbf{r}_{s,T})\cdot\mathbf{w}_E-\int_{T_d}(\mathbf{f}-\mathbf{f}_h+\mathbf{r}_{d,T})\cdot\mathbf{w}_E.
\end{aligned}$$

Hence Cauchy–Schwarz inequality, the inverse inequalities (66) and the estimates (70) and (71) lead to

$$h_E^{\frac{1}{2}}\|\mathbf{u}_h\cdot\tau_j+\frac{\sqrt{\kappa_j}}{\alpha_1}2\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_h)\cdot\tau_j\|_E\lesssim|\mathbf{e}|_{h,\omega_E}+\|\varepsilon\|_{h,\omega_E}+\sum_{T'\subset\omega_E}\zeta_{T'},\quad (77)$$

with $\omega_E=T_s\cup T_d$.

- To estimate $\Theta_{5,T}$, we take $r_E=p_{d,h}-p_{s,h}+2\mu\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\mathbf{n}_s$ and $\mathbf{N}=\mathbf{n}_s$. As before the identity (76), the inverse inequalities (65) and (66) and the estimates (70) and (71) lead to

$$h_E^{\frac{1}{2}}\|p_{d,h}-p_{s,h}+2\mu\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\mathbf{n}_s\|_E\lesssim|\mathbf{e}|_{h,\omega_E}+\|\varepsilon\|_{h,\omega_E}+\sum_{T'\subset\omega_E}\zeta_{T'}.\quad (78)$$

(6) Normal jump in Ω_s. For each edge/face $E\in\mathcal{E}_h(\Omega_s)$, we consider $\omega_E=T_1\cup T_2$. As $\mathbf{J}_{E,\mathbf{n}_E}\in[\mathbb{P}^0(E)]^N$ we set

$$\mathbf{w}_E:=-\mathbf{J}_{E,\mathbf{n}_E}b_E\in[H_0^1(\omega_E)]^N.$$

First the weak formulation (11) yields

$$\mathbf{a}(\mathbf{u},\mathbf{w}_E)+\mathbf{b}(\mathbf{w}_E,p)=(\mathbf{f},\mathbf{w}_E)_{\omega_E},$$

that is equivalent to

$$\int_{\omega_E}\mathbf{f}\cdot\mathbf{w}_E=\int_{\omega_E}(2\mu\mathbf{D}(\mathbf{u})-p\mathbf{I}):D(\mathbf{w}_E)+\int_{\partial\omega_E}(p\mathbf{I}-2\mu\mathbf{D}(\mathbf{u}))\mathbf{n}_E\cdot\mathbf{w}_E.$$

By elementwise partial integration we further have

$$-\int_E\mathbf{J}_{E,\mathbf{n}_E}\cdot\mathbf{w}_E=\int_{\omega_E}(2\mu\mathbf{D}(\mathbf{u}_h)-p_h\mathbf{I}):D(\mathbf{w}_E)-\sum_{i=1}^{2}\int_{T_i}(-2\mu\mathrm{div}\mathbf{D}(\mathbf{u}_h)+\nabla p_h)\cdot\mathbf{w}_E.$$

Hence by the previous identity (79) we get

$$-\int_E\mathbf{J}_{E,\mathbf{n}_E}\cdot\mathbf{w}_E=\sum_{i=1}^{2}\int_{T_i}(\mathbf{f}-(-2\mu\mathrm{div}\mathbf{D}(\mathbf{u}_h)+\nabla p_h))\cdot\mathbf{w}_E-\int_{\omega_E}(2\mu\mathbf{D}(\mathbf{e})-\varepsilon\mathbf{I}):D(\mathbf{w}_E)$$

We introduce the approximation $\mathbf{f}_h$ of $\mathbf{f}$, use the Cauchy–Schwarz inequality and the inverse inequalities (65) and (66) to get

$$\|\mathbf{J}_{E,\mathbf{n}_E}\|_E\lesssim h_E^{1/2}\left(\sum_{i=1}^{2}(\|\mathbf{f}-\mathbf{f}_h\|_{T_i}+\|\mathbf{r}_{s,T_i}\|_{T_i})\right)+h_E^{-1/2}(|\mathbf{e}|_{1,\omega_E}+\|\varepsilon\|_{\omega_E})$$

The previous bound (70) of $\Theta_{1,T}$ and the obvious estimate $h_E\leqslant h_T$ imply that

$$h_E^{1/2}\|\mathbf{J}_{E,\mathbf{n}_E}\|_E\lesssim|\mathbf{e}|_{1,\omega_E}+\|\varepsilon\|_{\omega_E}+\sum_{T'\subset\omega_E}h_{T'}\|\mathbf{f}-\mathbf{f}_h\|_{T'}.\quad (79)$$

(7) Pressure jump in Ω_d. For each edge/face $E \in \mathcal{E}_h(\Omega_d)$, we consider $\omega_E = T_1 \cup T_2$. As $[p_h]_E \in \mathbb{P}^0(E)$ we set

$$\mathbf{w}_E := [p_h]_E b_E \mathbf{n}_E \in [H_0^1(\omega_E)]^N.$$

First we notice that as $p \in H^1(\omega_E)$ we have by Green formula

$$\int_{\omega_E} (\nabla p \cdot \mathbf{w}_E + p \operatorname{div} \mathbf{w}_E) = 0.$$

Again by elementwise partial integration we further have

$$\int_E [p_h]_E \mathbf{w}_E.\mathbf{n}_E = \sum_{i=1}^2 \int_{T_i} (\nabla p_h \cdot \mathbf{w}_E + p_h \operatorname{div} \mathbf{w}_E).$$

Taking the difference of these two identities we obtain

$$\int_E [p_h]_E \mathbf{w}_E.\mathbf{n}_E = \sum_{i=1}^2 \int_{T_i} (\nabla (p_h - p) \cdot \mathbf{w}_E + (p_h - p) \operatorname{div} \mathbf{w}_E).$$

Recalling that $\nabla p = \mathbf{f} - \mu \mathbf{K}^{-1}\mathbf{u}$ and introducing the term $\mathbf{f}_h - \mu \mathbf{K}^{-1}\mathbf{u}_h$, we find

$$\int_E [p_h]_E \mathbf{w}_E.\mathbf{n}_E = \sum_{i=1}^2 \int_{T_i} (\nabla p_h - \mathbf{f} + \mu \mathbf{K}^{-1}\mathbf{u}) \cdot \mathbf{w}_E + (p_h - p) \operatorname{div} \mathbf{w}_E) \tag{80}$$

$$= \sum_{i=1}^2 \int_{T_i} (\nabla p_h - \mathbf{f}_h + \mu \mathbf{K}^{-1}\mathbf{u}_h) \cdot \mathbf{w}_E + (p_h - p) \operatorname{div} \mathbf{w}_E) \tag{81}$$

$$+ \sum_{i=1}^2 \int_{T_i} (\mathbf{f}_h - \mathbf{f} + \mu \mathbf{K}^{-1}(\mathbf{u} - \mathbf{u}_h)) \cdot \mathbf{w}_E. \tag{82}$$

Cauchy–Schwarz inequality and inverse inequalities lead to

$$\|[p_h]_E\|_E \lesssim \sum_{i=1}^2 \|\mathbf{r}_{d,T_i}\|_{T_i} h_E^{\frac{1}{2}} + \|p_h - p\|_{T_i} h_E^{-\frac{1}{2}} \tag{83}$$

$$+ h_E^{\frac{1}{2}} \sum_{i=1}^2 (\|\mathbf{f} - \mathbf{f}_h\|_{T_i} + \|\mathbf{K}^{-1}(\mathbf{u} - \mathbf{u}_h)\|_{T_i}). \tag{84}$$

By (71), we deduce that

$$h_E^{\frac{1}{2}} \|[p_h]_E\|_E \lesssim \sum_{T' \subset \omega_E} h_{T'} \|\mathbf{f} - \mathbf{f}_h\|_{T'} + \|\varepsilon\|_{\omega_E} + \|\mathbf{K}^{-1}\mathbf{e}\|_{\omega_E}. \tag{85}$$

The estimates (68), (70), (71), (73), (74), (77), (78), (79) and (85) provide the desired local lower error bound. □

4.3 Reliability of the a posteriori error estimator

The a posteriori error estimator Θ is consider reliable if it also satisfies

$$\| \mathbf{e} \|_h + \| \varepsilon \| \lesssim \Theta + \zeta. \tag{86}$$

In this subsection, we shall prove this estimate. But before, we specify some analytical tools. For detailed proof we refer to [48].

Lemma 4.2 (Continuous trace inequality) *There exists a positive constant $\beta_1 > 0$ depending only on σ_0 such that*

$$\| \mathbf{v} \|_{\partial T}^2 \leqslant \beta_1 \| \mathbf{v} \|_T \times \| \mathbf{v} \|_{1,T}, \quad \forall T \in \mathcal{T}_h, \quad \forall \mathbf{v} \in [H^1(T)]^N. \tag{87}$$

Proof Based on [49, Lemma 3.2] and a scaling argument. □

Lemma 4.3 (Inverse inequality) *Let $k \in \mathbb{N}$. Then there exists a positive constant $\beta_2 > 0$ depending only on k and on σ_0 such that*

$$\forall T \in \mathcal{T}_h, \quad \forall \mathbf{v} \in [\mathbb{P}^k(T)]^N, \quad \| \mathbf{v} \|_{\partial T} \leqslant \beta_2 h_T^{-1/2} \| \mathbf{v} \|_T \tag{88}$$

Proof See [48, Lemma 3.3, Page 59]. □

Further, we introduce the Clément interpolation operator $\mathrm{I}_{\mathrm{Cl}}^0 : H_0^1(\Omega) \longrightarrow \mathcal{P}_c^b(\mathcal{T}_h)$ that approximates optimally non-smooth functions by continuous piecewise linear functions:

$$\mathcal{P}_c^b(\mathcal{T}_h) := \{v \in C^0(\overline{\Omega}) : v_{|T} \in \mathbb{P}^1(T), \quad \forall T \in \mathcal{T}_h \text{ and } v = 0 \text{ on } \partial\Omega\}$$

In addition, we will make use of a vector valued version of $\mathrm{I}_{\mathrm{Cl}}^0$, that is, $\mathbf{I}_{\mathrm{Cl}}^0 : [H_0^1(\Omega)]^N \longrightarrow [\mathcal{P}_c^b(\mathcal{T}_h)]^N$, which is defined componentwise by $\mathrm{I}_{\mathrm{Cl}}^0$. The following lemma establishes the local approximation properties of $\mathrm{I}_{\mathrm{Cl}}^0$ (and hence of $\mathbf{I}_{\mathrm{Cl}}^0$), for a proof see [14, Section 3].

Lemma 4.4 *There exist constants $C_1, C_2 > 0$, independent of h, such that for all $v \in H_0^1(\Omega)$ there hold*

$$\| v - I_{Cl}^0(v) \|_T \leq C_1 h_T \| v \|_{1,\Delta(T)} \quad \forall T \in \mathcal{T}_h, \ \textit{and} \tag{89}$$

$$\| v - I_{Cl}^0(v) \|_E \leq C_2 h_E^{1/2} \| v \|_{1,\Delta(E)} \quad \forall E \in \mathcal{E}_h, \tag{90}$$

where $\Delta(T) := \cup\{T' \in \mathcal{T}_h : T' \cap T \neq \emptyset\}$ and $\Delta(E) := \cup\{T' \in \mathcal{T}_h : T' \cap E \neq \emptyset\}$.

We set $\mathbf{X} := \mathbf{H} \times Q$ and $\mathbf{X}_h := \mathbf{H}_h \times Q_h$ and define on $\mathbf{X}$, the continuous bilinear form $\mathbb{B}$ by:

$$\mathbb{B}(\mathbf{U}, \mathbf{W}) := \mathbf{a}(\mathbf{u}, \mathbf{w}) + \mathbf{b}(\mathbf{u}, q) + \mathbf{b}(\mathbf{w}, p), \quad \text{for } \mathbf{U} = (\mathbf{u}, p) \quad \text{and for } \mathbf{W} = (\mathbf{w}, q).$$

We also define on the discrete space $\mathbf{X}_h$, the form

$$\mathbb{B}_h(\mathbf{U}_h, \mathbf{W}_h) := \mathbf{a}_h(\mathbf{u}_h, \mathbf{w}_h) + \mathbf{b}_h(\mathbf{u}_h, q_h) + \mathbf{b}_h(\mathbf{w}_h, p_h) + \mathbf{J}(\mathbf{u}_h, \mathbf{w}_h)$$

The spaces $\mathbf{X}$ and $\mathbf{X}_h$ are equipped with the product-norms

$$|||(\mathbf{u}, p)||| = \|\mathbf{u}\|_{\mathbf{H}} + \|p\| \quad \text{and} \quad |||(\mathbf{u}_h, p_h)|||_h = \|\mathbf{u}_h\|_h + \|p_h\| \text{ respectively.}$$

Let us start with the following result.

Lemma 4.5 *Let the assumptions of Theorem 3.2 be satisfied. Then for all $\mathbf{W} = (\mathbf{v}, q) \in \mathbf{H} \times Q$, we have the estimate:*

$$\mathbb{B}_h(\mathbf{U}-\mathbf{U}_h,\mathbf{W}) \lesssim (\Theta_1+\zeta)\times |||\mathbf{W}|||_h, \tag{91}$$

where the estimator Θ_1 *is defined by*

$$\Theta_1 := \left\{\sum_{T\in\mathcal{T}_h}\left(\sum_{i=1}^{6}\Theta_{i,T}^2\right)\right\}^{1/2}. \tag{92}$$

Proof Let $\mathbf{W}=(\mathbf{v},q)\in\mathbf{H}\times Q$. By Theorem 3.1 $\mathbf{v}$ admits the decomposition (19) with $\mathbf{v}_0,\mathbf{v}_1\in\mathbf{H}$ and satisfying the properties stated in Theorem 3.1. Then we take $\mathbf{W}_h=(\mathbf{v}_h,0)\in\mathbf{H}_h\times Q_h$ with $\mathbf{v}_h=\mathbf{v}_{0,h}+\mathbf{v}_{1,h}$, where $\mathbf{v}_{0,h}=\mathbf{I}^0_{\mathrm{Cl}}\mathbf{v}_0$ and

$$\mathbf{v}_{1,h}=\begin{cases}\mathbf{0} & \text{in } \Omega_s,\\ \mathrm{curl}\mathbf{I}^0_{\mathrm{Cl}}\psi & \text{in } \Omega_d,\end{cases} \tag{93}$$

where in 2D, $\mathbf{I}^0_{\mathrm{Cl}}\psi$ is the standard Clément interpolant of ψ, while in 3D, we take the vectorial Clément interpolant from [7] (see also Definition 3.5 of [39]) that satisfies the same estimate as the standard one (see [7]). Note that $\mathbf{v}_{0,h}$ belongs to $\mathbf{H}_h\cap H^1_0(\Omega)^N$ while $\mathbf{v}_{1,h}$ simply belongs to $\mathbf{H}_h\cap\mathbf{H}$ ($\mathbf{I}^0_{\mathrm{Cl}}\psi$ being in $H^1_0(\Omega_d)$ if $N=2$ and $\mathbf{I}^0_{\mathrm{Cl}}\psi\in H^1(\Omega_d)^3\cap H_0(\mathrm{curl},\Omega_d)$ if $N=3$, its curl belongs to $H_0(\mathrm{div},\Omega_d)$, hence $\mathbf{v}_{1,h}$, its extension by zero in Ω_s, stays in $H_0(\mathrm{div},\Omega)$). With these definitions and noticing that $\mathrm{div}(\mathbf{v}-\mathbf{v}_h)=\mathrm{div}(\mathbf{v}_0-\mathbf{v}_{0,h})$ and that $\mathbf{J}(\mathbf{u}_h,\mathbf{v}_h)=0$, we may write

$$\begin{aligned}
\mathbb{B}_h(\mathbf{U}-\mathbf{U}_h,\mathbf{W}) &= \mathbb{B}_h(\mathbf{U}-\mathbf{U}_h,\mathbf{W}-\mathbf{W}_h)\\
&= \mathbf{a}_h(\mathbf{u}-\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h)+\mathbf{b}_h(\mathbf{v}-\mathbf{v}_h,p-p_h)+\mathbf{b}_h(\mathbf{u}-\mathbf{u}_h,q)\\
&= \mathbf{a}_h(\mathbf{u},\mathbf{v}-\mathbf{v}_h)+\mathbf{b}_h(\mathbf{v}-\mathbf{v}_h,p)+\mathbf{b}_h(\mathbf{u},q)\\
&\quad -[\mathbf{a}_h(\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h)+\mathbf{b}_h(\mathbf{v}-\mathbf{v}_h,p_h)+\mathbf{b}_h(\mathbf{u}_h,q)]\\
&= (\mathbf{f},\mathbf{v}-\mathbf{v}_h)_\Omega-(g,q)_\Omega\\
&\quad -[\mathbf{a}_h(\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h)+\mathbf{b}_h(\mathbf{v}-\mathbf{v}_h,p_h)+\mathbf{b}_h(\mathbf{u}_h,q)]\\
&= \sum_{T\in\mathcal{T}_h}\{(\mathbf{f},\mathbf{v}-\mathbf{v}_h)_T-(g,q)_T\}\\
&\quad -2\mu\sum_{T\in\mathcal{T}_h^s}(\mathbf{D}(\mathbf{u}_h),\mathbf{D}(\mathbf{v}-\mathbf{v}_h))_T\\
&\quad -\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}\langle\mathbf{u}_h\cdot\tau_j,(\mathbf{v}-\mathbf{v}_h)\cdot\tau_j\rangle_{\Gamma_I}\\
&\quad -\sum_{T\in\mathcal{T}_h^d}(\mu\mathbf{K}^{-1}\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h)_T\\
&\quad +\sum_{T\in\mathcal{T}_h}[(p_h,\mathrm{div}(\mathbf{v}_0-\mathbf{v}_{0,h}))_T+(q,\mathrm{div}\mathbf{u}_h)_T].
\end{aligned}$$

Integrate by parts element by element to obtain:

$$\begin{aligned}
\mathbb{B}_h(\mathbf{U}-\mathbf{U}_h,\mathbf{W}) = \sum_{T\in\mathcal{T}_h^s}\Big[&2\mu(\mathrm{div}\mathbf{D}(\mathbf{u}_h),\mathbf{v}-\mathbf{v}_h)_T-2\mu(\mathbf{D}(\mathbf{u}_h)\mathbf{n},\mathbf{v}-\mathbf{v}_h)_{\partial T}\\
&+(\mathbf{f},\mathbf{v}-\mathbf{v}_h)_T-(g,q)_T-(\nabla p_h,\mathbf{v}_0-\mathbf{v}_{0,h})_T\\
&+(p_h,(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n})_{\partial T}+(q,\mathrm{div}\mathbf{u}_h)_T\Big]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{T\in\mathcal{T}_h^d}[-(\mu\mathbf{K}^{-1}\mathbf{u}_h,\mathbf{v}-\mathbf{v}_h)_T-(\nabla p_h,\mathbf{v}_0-\mathbf{v}_{0,h})_T\\
&+(p_h,(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n})_{\partial T}+(q,\mathrm{div}\mathbf{u}_h)_T+(\mathbf{f},\mathbf{v}-\mathbf{v}_h)_T-(g,q)_T]\\
&-\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}\langle\mathbf{u}_h\cdot\tau_j,(\mathbf{v}-\mathbf{v}_h)\cdot\tau_j\rangle_{\Gamma_I}.
\end{aligned}
$$

We now introduce the approximation $\mathbf{f}_h$ of $\mathbf{f}$ for appropriated terms and add boundary (resp. internal) terms that appear on the same edge (resp. at the same element) and reminding that $\mathbf{v}=\mathbf{v}_0$ and $\mathbf{v}_h=\mathbf{v}_{0,h}$ in Ω_s, we obtain

$$
\begin{aligned}
\mathbb{B}_h(\mathbf{U}-\mathbf{U}_h,\mathbf{W}) = &\sum_{T\in\mathcal{T}_h^s}[(\mathbf{f}_h+2\mu\mathrm{div}\mathbf{D}(\mathbf{u}_h)-\nabla p_h,\mathbf{v}-\mathbf{v}_h)_T\\
&+(\mathbf{f}-\mathbf{f}_h,\mathbf{v}-\mathbf{v}_h)_T+(q,\mathrm{div}\mathbf{u}_h-g)_T]\\
&+\sum_{T\in\mathcal{T}_h^d}[(\mathbf{f}_h-\mu\mathbf{K}^{-1}\mathbf{u}_h-\nabla p_h,\mathbf{v}_0-\mathbf{v}_{0,h})_T+(q,\mathrm{div}\mathbf{u}_h-g)_T]\\
&+\sum_{T\in\mathcal{T}_h^d}[(\mathbf{f}-\mathbf{f}_h,\mathbf{v}_0-\mathbf{v}_{0,h})_T+(\mathbf{f}-\mu\mathbf{K}^{-1}\mathbf{u}_h-\nabla p_h,\mathbf{v}_1-\mathbf{v}_{1,h})_T]\\
&-\sum_{E\in\mathcal{E}_h(\Omega_s^+)}(\mathbf{J}_{E,\mathbf{n}_E},\mathbf{v}_0-\mathbf{v}_{0,h})_E+\sum_{E\in\mathcal{E}_h(\Omega_d)}([p_h]_E,(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n}_E)_E\\
&-\sum_{E\in\mathcal{E}_h(\Gamma_I)}\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}\left(\mathbf{u}_{s,h}\cdot\tau_j+2\frac{\sqrt{\kappa_j}}{\alpha_1}\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\tau_j,(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\tau_j\right)_E\\
&+\sum_{E\in\mathcal{E}_h(\Gamma_I)}(p_{s,h}-p_{d,h}-2\mu\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\mathbf{n}_s),(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n}_s)_E. \quad (94)
\end{aligned}
$$

Now for a triangle $T\in\mathcal{T}_h^d$, we recall that

$$\mathbf{v}_1-\mathbf{v}_{1,h}=\mathrm{curl}(\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi) \quad \text{in } T,$$

and use Green's formula to get

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h^d}(\mathbf{f}-\mu\mathbf{K}^{-1}\mathbf{u}_h,\mathbf{v}_1-\mathbf{v}_{1,h})_T &= \sum_{T\in\mathcal{T}_h^d}(\mathbf{f}-\mu\mathbf{K}^{-1}\mathbf{u}_h,\mathrm{curl}(\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi))_T\\
&=\sum_{T\in\mathcal{T}_h^d}[(\mathrm{curl}(\mathbf{f}-\mu\mathbf{K}^{-1}\mathbf{u}_h),\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi)_T\\
&\quad+((\mathbf{f}-\mu\mathbf{K}^{-1}\mathbf{u}_h)\times\mathbf{n},\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi)_{\partial T}].
\end{aligned}
$$

Hence using that $\mathbf{f}$ is in $H(\mathrm{curl},\Omega_d)$ we deduce that

$$
\begin{aligned}
\sum_{T\in\mathcal{T}_h^d}(\mathbf{f}-\mu\mathbf{K}^{-1}\mathbf{u}_h,\mathbf{v}_1-\mathbf{v}_{1,h})_T &= \sum_{T\in\mathcal{T}_h^d}[(\mathrm{curl}(\mathbf{f}_h-\mu\mathbf{K}^{-1}\mathbf{u}_h),\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi)_T\\
&\quad+(\mathrm{curl}(\mathbf{f}-\mathbf{f}_h),\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi)_T]\\
&\quad-\sum_{E\in\mathcal{E}_h(\Omega_d)}([\mu\mathbf{K}^{-1}\mathbf{u}_h\times\mathbf{n}_E],\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi)_E.
\end{aligned}
$$

This identity in (94) leads to

$$\begin{aligned}\mathbb{B}_h(\mathbf{U}-\mathbf{U}_h,\mathbf{W}) = &\sum_{T\in\mathcal{T}_h^s}[(\mathbf{f}_h+2\mu\mathrm{div}\mathbf{D}(\mathbf{u}_h)-\nabla p_h,\mathbf{v}-\mathbf{v}_h)_T\\ &+(\mathbf{f}-\mathbf{f}_h,\mathbf{v}-\mathbf{v}_h)_T+(q,\mathrm{div}\mathbf{u}_h-g)_T]\\ &+\sum_{T\in\mathcal{T}_h^d}[(\mathbf{f}_h-\mu\mathbf{K}^{-1}\mathbf{u}_h-\nabla p_h,\mathbf{v}_0-\mathbf{v}_{0,h})_T+(q,\mathrm{div}\mathbf{u}_h-g)_T]\\ &+\sum_{T\in\mathcal{T}_h^d}(\mathbf{f}-\mathbf{f}_h,\mathbf{v}_0-\mathbf{v}_{0,h})_T+\sum_{T\in\mathcal{T}_h^d}[(\mathrm{curl}(\mathbf{f}_h-\mu\mathbf{K}^{-1}\mathbf{u}_h),\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi)_T\\ &+(\mathrm{curl}(\mathbf{f}-\mathbf{f}_h),\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi)_T]-\sum_{E\in\mathcal{E}_h(\Omega_d)}([\mu\mathbf{K}^{-1}\mathbf{u}_h\times\mathbf{n}_E]_E,\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi)_E\\ &-\sum_{E\in\mathcal{E}_h(\Omega_s^+)}(\mathbf{J}_{E,\mathbf{n}_E},\mathbf{v}_0-\mathbf{v}_{0,h})_E+\sum_{E\in\mathcal{E}_h(\Omega_d)}([p_h]_E,(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n}_E)_E\\ &-\sum_{E\in\mathcal{E}_h(\Gamma_I)}\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}\left(\mathbf{u}_{s,h}\cdot\tau_j+2\frac{\sqrt{\kappa_j}}{\alpha_1}\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\tau_j,(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\tau_j\right)_E\\ &+\sum_{E\in\mathcal{E}_h(\Gamma_I)}(p_{s,h}-p_{d,h}-2\mu\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\mathbf{n}_s),(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n}_s)_E.\end{aligned}$$

Cauchy–Schwarz inequality leads to

$$\begin{aligned}\mathbb{B}_h(\mathbf{U}-\mathbf{U}_h,\mathbf{W}) \le &\sum_{T\in\mathcal{T}_h^s}[\|\mathbf{f}_h+2\mu\mathrm{div}\mathbf{D}(\mathbf{u}_h)-\nabla p_h\|_T\|\mathbf{v}-\mathbf{v}_h\|_T\\ &+\|\mathbf{f}-\mathbf{f}_h\|_T\|\mathbf{v}-\mathbf{v}_h\|_T+\|q\|_T\|\mathrm{div}\mathbf{u}_h-g\|_T]\\ &+\sum_{T\in\mathcal{T}_h^d}[\|\mathbf{f}_h-\mu\mathbf{K}^{-1}\mathbf{u}_h-\nabla p_h\|_T\|\mathbf{v}_0-\mathbf{v}_{0,h}\|_T+\|q\|_T\|\mathrm{div}\mathbf{u}_h-g\|_T]\\ &+\sum_{T\in\mathcal{T}_h^d}\|\mathbf{f}-\mathbf{f}_h\|_T\|\mathbf{v}_0-\mathbf{v}_{0,h}\|_T\\ &+\sum_{T\in\mathcal{T}_h^d}[\|\mathrm{curl}(\mathbf{f}_h-\mu\mathbf{K}^{-1}\mathbf{u}_h)\|_T\|\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi\|_T\\ &+\|\mathrm{curl}(\mathbf{f}-\mathbf{f}_h)\|_T\|\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi\|_T]\\ &+\sum_{E\in\mathcal{E}_h(\Omega_d)}\|[\mu\mathbf{K}^{-1}\mathbf{u}_h\times\mathbf{n}_E]_E\|_E\|\psi-\mathbf{I}_{\mathrm{Cl}}^0\psi\|_E\\ &+\sum_{E\in\mathcal{E}_h(\Omega_s^+)}\|\mathbf{J}_{E,\mathbf{n}_E}\|_E\|\mathbf{v}_0-\mathbf{v}_{0,h}\|_E\\ &+\sum_{E\in\mathcal{E}_h(\Omega_d)}\|[p_h]_E\|_E\|(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n}_E\|_E\\ &+\sum_{E\in\mathcal{E}_h(\Gamma_I)}\sum_{j=1}^{N-1}\frac{\mu\alpha_1}{\sqrt{\kappa_j}}\left\|\mathbf{u}_{s,h}\cdot\tau_j+2\frac{\sqrt{\kappa_j}}{\alpha_1}\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\tau_j\right\|_E\\ &\times\|(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\tau_j\|_E\\ &+\sum_{E\in\mathcal{E}_h(\Gamma_I)}\|p_{s,h}-p_{d,h}-2\mu\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\mathbf{n}_s\|_E\|(\mathbf{v}_0-\mathbf{v}_{0,h})\cdot\mathbf{n}_s\|_E.\end{aligned}$$

The approximation properties of Lemma 4.4 imply the required estimate and finish the proof. □

The second result of this subsection is given by the following lemma:

Lemma 4.6 *Under the assumptions of Theorem 3.2, the following estimation holds:*

$$\| \mathbf{e} \|_h + \| \varepsilon \| \lesssim \Theta_1 + \zeta + \inf_{\mathbf{W}_h \in \mathbf{H} \cap \mathbf{H}_h \times Q_h} |||\mathbf{U}_h - \mathbf{W}_h|||_h, \tag{95}$$

where Θ_1 *is given by* (92).

Proof For an arbitrary $\mathbf{W}_h \in \mathbf{H} \cap \mathbf{H}_h \times Q_h$, the inf-sup condition of $\mathbb{B}$ on $\mathbf{H} \times Q$ leads to

$$|||\mathbf{U} - \mathbf{W}_h|||_h \lesssim \sup_{\mathbf{W} \in \mathbf{H} \times Q} \frac{\mathbb{B}(\mathbf{U} - \mathbf{W}_h, \mathbf{W})}{|||\mathbf{W}|||_h}$$

hence

$$|||\mathbf{U} - \mathbf{W}_h|||_h \lesssim \sup_{\mathbf{W} \in \mathbf{H} \times Q} \left\{ \frac{\mathbb{B}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{W}) + \mathbb{B}_h(\mathbf{U}_h - \mathbf{W}_h, \mathbf{W})}{|||\mathbf{W}|||_h} \right\}. \tag{96}$$

Combining the estimates (91) and (96), it comes:

$$|||\mathbf{U} - \mathbf{W}_h|||_h \lesssim \Theta_1 + \zeta + \sup_{\mathbf{W} \in \mathbf{H} \times Q} \frac{\mathbb{B}_h(\mathbf{U}_h - \mathbf{W}_h, \mathbf{W})}{|||\mathbf{W}|||_h}. \tag{97}$$

The continuity of the operator $\mathbb{B}_h$ implies that

$$|||\mathbf{U} - \mathbf{W}_h|||_h \lesssim \Theta_1 + \zeta + |||\mathbf{U}_h - \mathbf{W}_h|||_h. \tag{98}$$

Thus, by the triangular inequality we deduce that

$$|||\mathbf{U} - \mathbf{U}_h|||_h \lesssim \Theta_1 + \zeta + |||\mathbf{U}_h - \mathbf{W}_h|||_h, \quad \forall \mathbf{W}_h \in \mathbf{H} \cap \mathbf{H}_h \times Q_h, \tag{99}$$

or equivalently

$$|||\mathbf{U} - \mathbf{U}_h|||_h \lesssim \Theta_1 + \zeta + \inf_{\mathbf{W}_h \in \mathbf{H} \cap \mathbf{H}_h \times Q_h} |||\mathbf{U}_h - \mathbf{W}_h|||_h. \tag{100}$$

Thus, this lemma holds. □

Combining (35) and (95), we have the main result of this subsection:

Theorem 4.2 *Under the assumptions of Theorem 3.2, the a posteriori error estimator* Θ *satisfies* (86).

5 Summary

In this paper we have discussed a posteriori error estimates for a finite element approximation of the Stokes–Darcy system. A residual type a posteriori error estimator is provided, that is both reliable and efficient. Many issues remain to be addressed in this area, let us mention other types of a posteriori error estimators or implementation and convergence analysis of adaptive finite element methods. Further it is well known that an internal layer appears at the interface Γ_I as the permeability tensor degenerates, in that case anisotropic meshes have to be used in this layer (see for instance [17]). Hence we intend to extend our results to such anisotropic meshes.

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