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Lorentzian manifolds with causal Killing vector field: causality and geodesic connectedness

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Abstract

We prove that a compact Lorentzian manifold $(\overline{M}, \overline{g})$ admitting a causal Killing vector field is totally vicious or it contains a compact achronal Killing horizon. In particular a compact spacetime which satisfies the null generic condition and admits a causal Killing vector field is totally vicious. If in addition, its universal Lorentzian covering is globally hyperbolic then it is geodesically connected. In the non-compact case, we prove that a chronological spacetime admitting a complete causal Killing vector field, a smooth spacelike partial Cauchy hypersurface S and satisfying the null generic condition is stably causal. If additionally S is compact then the spacetime is globally hyperbolic. We also determine the geodesic connectedness in this case.

Keywords Causality · Killing horizons · Causal Killing vector field

Mathematics Subject Classification 53C50 · 53C40

1 Introduction

Hopf–Rinow theorem is an important tool in Riemannian geometry. It gives an equivalence between Cauchy completeness, geodesic completeness and finite compactness (i.e., bounded sets have compact closure). It also guarantees the existence of at least one geodesic joining any two distinct points of a complete Riemannian manifold. This last property is known as geodesic connectedness.

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For semi-Riemannian manifolds the situation is quite different. Compact Riemannian manifolds are always complete and geodesically connected. However, there are examples showing compact spacetimes not geodesically complete and not geodesically connected, [4], even for globally hyperbolic spacetimes. The problem of geodesic connectedness in semi-Riemannian manifolds has been widely studied from very different viewpoints. This topic has applications in Physics, and it is challenging from both an analytical and a geometrical point of view. In [6] it is used pseudoconvexity and disprisonment to study geodesic connectedness. The variational approach is used in [8]. For stationary spacetime (with complete stationary vector field), it is proved that the spacetime is geodesically connected if it admits a complete (smooth, spacelike) Cauchy hypersurface [7] and a similar result was obtained in the case the Killing vector field is assumed lightlike [3]. It is proved that compact static spacetimes are totally vicious and geodesically connected. In fact the first claim holds even for compact conformally stationary spacetime; nevertheless, it is no longer true if we consider a causal Killing vector field, [25].

In the present paper, we study the causal structure and geodesic connectedness of spacetime admitting a causal Killing vector field.

In Sect. 2, we recall some notations, definitions and background tools on causality theory and geometry of null hypersurfaces and Killing horizons.

In Sect. 3, we consider the case of compact spacetimes. We prove that the spacetime is totally vicious or contains a compact achronal Killing horizon, Theorem 3.4. As a corollary, if it satisfies the null generic condition then it is totally vicious, and we study the geodesic connectivity, Corollary 3.6.

In Sect. 4, we consider the non-compact case. We prove that if the spacetime $(\overline{M}, \overline{g})$ is chronological and satisfies the null generic condition, it is a proper $\mathbb{R}$-manifold and so it is diffeomorphic to $\mathbb{R} \times \overline{M}/\mathbb{R}$, Theorem 4.11. If additionally $(\overline{M}, \overline{g})$ admits a smooth spacelike partial Cauchy hypersurface S then it is stably causal, Theorem 4.11. If moreover S is compact, it is globally hyperbolic, Theorem 4.16. Finally, in the case the Killing vector field is lightlike and S compact, we determine the geodesic connectedness of the spacetime.

2 Preliminaries

In all the paper $(\overline{M}, \overline{g})$ is a spacetime, that is, a connected, time-oriented Lorentzian manifold. Our convention for the signature of the Lorentz metric is $(-, +, ..., +)$.

2.1 Elements of causality theory and the causal hierarchy

We recall some definitions in causality theory, specially those which are used here. For general reference on causality see for instance [4, 20, 23].

The causality relations on $\overline{M}$ are defined as follows. If $p, q \in \overline{M}$, then $p \ll q$ (or $(p, q) \in I^+$) means there is a future-pointing timelike curve in $\overline{M}$ from p to q. The notation $p < q$ means there is a future-pointing causal curve in $\overline{M}$ from p to q. Evidently $p \ll q$ implies $p < q$. As usual, $p \leq q$ (or $(p, q) \in J^+$) means that either $p < q$ or $p = q$. For a subset A of $\overline{M}$, the subset

$$I^+(A) = \{q \in \overline{M} \,:\, \text{there is a } p \in A \text{ with } p \ll q\}$$

is called the chronological future of A, and

$$J^+(A) = \{q \in \overline{M} : \text{there is a } p \in A \text{ with } p \leq q\}$$

is called the causal future of A. Thus, $A \cup I^+(A) \subset J^+(A)$. For a single point, $I^+(p) = \{q : p \ll q\}$. Similarly for J^+. Dual to the preceding definitions are corresponding past versions. Thus

$$I^-(A) = \{q \in \overline{M} : \text{there is a } p \in A \text{ with } q \ll p\}$$

is the chronological past of A. In general, past definitions and proofs follow from the future versions (and vice versa) merely by reversing time orientation.

Definition 2.1 A point $p \in \overline{M}$ is a future endpoint of a future-directed causal curve $\gamma : I \longrightarrow \overline{M}$ if for every neighborhood U of p, there exists a point $t_0 \in I$ such that $\gamma(t) \in U$ for all $t > t_0$. A causal curve is future inextensible if it has no future endpoint.

Definition 2.2 A future inextensible causal curve $\gamma : I \longrightarrow \overline{M}$, is totally future imprisoned in the compact set C if there is $t_0 \in I$, such that for every $t > t_0$, with $t \in I$, $\gamma(t) \in C$, i.e., if it enters and remains in C. It is partially future imprisoned if for every $t_0 \in I$, there is $t > t_0$, with $t \in I$, such that $\gamma(t) \in C$, i.e., if it continually returns to it. The curve escapes to infinity in the future if it is not partially future imprisoned in any compact set.

A spacetime (M, g) satisfies the chronology (causal) condition at a point $p \in M$ provided there are not closed timelike (causal) curves through p. It satisfies the chronology (causal) condition in a subset A if it satisfies the chronology (causal) condition at each point $p \in A$. If $A = M$ we say that (M, g) satisfies the chronology (causal) condition. A spacetime is non-total future imprisoning if no future inextensible causal curve is totally future imprisoned in a compact set. A spacetime is non-partial future imprisoning if no future inextensible causal curve is partially future imprisoned in a compact set. Actually, Beem proved [5, Theorem 4] that a spacetime is non-total future imprisoning if and only if it is non-total past imprisoning; thus, in the non-total case one can simply speak of the non-total imprisoning property (condition N, in Beem's terminology [5]). The strong causality condition holds at $p \in \overline{M}$ provided that given any neighborhood U of p there is a neighborhood $V \subset U$ of p such that every causal curve segment with endpoints in V lies entirely in U. $\overline{M}$ is strongly causal if the strong causality condition holds at each $p \in \overline{M}$. The following new step on the causal ladder has also been established.

Definition 2.3 [19] A spacetime $(\overline{M}, \overline{g})$ is called feebly distinguishing if $(p, q) \in J^+$, $p \in \overline{I^+(q)}$ and $q \in \overline{I^-(p)}$ implies $p = q$.

A spacetime $(\overline{M}, \overline{g})$ is future-distinguishing at $p \in \overline{M}$ if $I^+(p) \neq I^+(q)$ for each $q \in \overline{M}$, with $q \neq p$. $\overline{M}$ is future-distinguishing if and only if it is future-distinguishing at every point. This property of being future-distinguishing is called future-distinction. A spacetime is stably causal if it cannot be made to contain closed causal curves by arbitrarily small perturbations of the metric. The condition of stable causality is equivalent to the existence of a time function on (M, g), that is to say, a continuous function on M strictly increasing along future-directed causal curves. There is one condition, related in some ways to the causality conditions below, which stands, nevertheless, outside the causal ladder.

Definition 2.4 A spacetime $(\overline{M}, \overline{g})$ is called reflecting if $I^+(q) \subset I^+(p) \Leftrightarrow I^-(p) \subset I^-(q)$ for all $p, q \in \overline{M}$.

A spacetime $(\overline{M}, \overline{g})$ is called causally continuous if it is reflecting and feebly distinguishing. Usually, causal continuity was defined as a spacetime being reflecting and distinguishing, see [20] . In [19] it is proved that the assumption can be relaxed to feeble distinction. Causal continuity is stronger than stable causality. A spacetime $(\overline{M}, \overline{g})$ is called causally simple if it is causal and $J^+(p), J^-(p)$ are closed sets for all $p \in \overline{M}$. Finally, $(\overline{M}, \overline{g})$ is called globally hyperbolic if it is causal and $J^+(p) \cap J^-(q)$ are compact sets for all $p, q \in \overline{M}$.

Now, we consider the case when the spacetime is non-chronological. In this case, the chronology violating set is $\mathcal{C} = \{x : x \ll x\}$, and is made by all the events through which there passes a closed timelike curve. The spacetime violates chronology if $\mathcal{C} \neq \emptyset$, that is, if there is a closed timelike curve, and it is totally vicious if $\mathcal{C} = \overline{M}$. Suppose $\mathcal{C} \neq \emptyset$, then $\mathcal{C}$ can split into equivalence classes according to Carter's equivalence relation $x \sim y \Leftrightarrow x \ll y$ and $y \ll x$. Two points belong to the same class if there is a closed timelike curve passing through them. The class of $x \in \mathcal{C}$ is denoted $[x]$. Note that $[x] = I^+(x) \cap I^-(x)$, thus $[x]$ is open. So the chronological violating set can be written $\mathcal{C} = \bigcup_\alpha \mathcal{C}_\alpha$, with $\mathcal{C}_\alpha$ its (open) connected components. The boundary of the component $\mathcal{C}_\alpha$ can be written $\partial\mathcal{C}_\alpha = \bigcup_k B_{\alpha k}$, with $B_{\alpha k}$ its (closed) connected components. Some authors have studied the compactness of the components of the chronological violating set's boundary in link with some energy condition, [17], or absence of null line, [21].

2.2 Geometry of null hypersurfaces

In this section, we review the basic elements of the theory of null hypersurfaces as introduced in [11] and complemented in [13]. Let $(\overline{M}, \overline{g})$ be a $(n+2)$-dimensional Lorentzian manifold and M a null hypersurface in $\overline{M}$. A *screen distribution* on M is a complementary bundle of $TM^\perp$ in TM. It is then a rank n non-degenerate distribution over M. In fact, there are infinitely many possibilities to choose such a distribution. Each of them is isomorphic to the factor vector bundle $TM/TM^\perp$. For a null hypersurface M equipped with a screen distribution $\mathcal{S}$, and a null vector field $\xi \in \mathfrak{X}(M)$, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, and a unique section N of $tr(TM)$ satisfying

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(N, W) = 0 \tag{1}$$

$\forall W \in \mathcal{S}$. Then, $T\overline{M}$ admits the splitting:

$$T\overline{M}|_M = TM \oplus tr(TM) = \{TM^\perp \oplus tr(TM)\} \oplus \mathcal{S}. \tag{2}$$

We call $tr(TM)$ a *(null) transverse vector bundle* along M.

A rigging for M is a vector field ζ defined on some open set containing M such that $\zeta_p \notin T_pM$ for each $p \in M$.

Given a rigging ζ of M in $(\overline{M}, \overline{g})$ let α denote the 1-form $\overline{g}$-metrically equivalent to ζ, i.e., $\alpha = \overline{g}(\zeta, \cdot)$. Take $\omega = i^*\alpha$, being $i : M \hookrightarrow \overline{M}$ the canonical inclusion. Next, consider the tensors

$$\breve{g} = \overline{g} + \alpha \otimes \alpha \quad \text{and} \quad \widetilde{g} = i^* \breve{g}. \tag{3}$$

It is easy to show that $\widetilde{g}$ defines a Riemannian metric on M. The *rigged vector field* of ζ is the $\widetilde{g}$-metrically equivalent vector field to the 1-form ω and it is denoted by ξ. In fact the rigged vector field ξ is the unique lightlike vector field in M such that $\overline{g}(\zeta, \xi) = 1$. Moreover, ξ is $\widetilde{g}$-unitary. A screen distribution on M is given by $\mathcal{S}^{\zeta} = TM \cap \zeta^{\perp}$. It is the $\widetilde{g}$-orthogonal subspace to ξ and the corresponding null transverse vector field to $\mathcal{S}^{\zeta}$ is

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi. \tag{4}$$

A null hypersurface M equipped with a rigging ζ is said to be normalized and is denoted (M, ζ) (the latter is called a normalization of the null hypersurface). A normalization (M, ζ) is said to be closed (resp. conformal) if the rigging ζ is closed, i.e., the 1-form α is closed (resp. ζ is a conformal vector field, i.e., there exists a function ρ on M such that $L_{\zeta}\overline{g} = 2\rho\overline{g}$). Finally, we will use the fact that $\overline{\nabla}_{\xi}\xi = -\tau(\xi)\xi$ for a one form τ in M, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$.

2.3 Killing horizon

The event horizon of a stationary and asymptotically flat spacetime with a black hole is a Killing horizon. This is known as the rigidity theorem, [15]. The study of globally hyperbolic spacetimes with a Killing horizon is important in the classification of the possible final states of gravitational collapse. We recall the ideas of Killing horizons as they was presented in [9].

Definition 2.5 [9] Let $(\overline{M}, \overline{g})$ be a spacetime. A connected smooth null hypersurface $M \subset \overline{M}$ is a Killing horizon if there exists a complete Killing vector field K with flow $\psi : \mathbb{R} \times \overline{M} \to \overline{M}$ such that

(1) $\psi_t(M) \subset M \; \forall \, t \in \mathbb{R}$,
(2) $g(K, K)_{|M} \equiv 0$,
(3) $K(p) \neq 0 \; \forall \, p \in M$.

The Killing vector field K is said to be adapted to the Killing horizon M.

There may exists more than one Killing field adapted to M. Every Killing horizon is totally geodesic. If K is a Killing vector field adapted to the Killing horizon, then the function $\kappa : M \to \mathbb{R}$ defined by $\overline{\nabla}_K K = \kappa K$ is called surface gravity. The integral curves of K contained in M are null pregeodesics whose images coincide with the null geodesic generators of M, and κ is constant along each one of them. A fundamental property, called the zero law of black hole thermodynamic, is that the surface gravity is in fact constant over the horizon in vacuum, or in electro-vacuum spacetime. A Killing horizon is called extremal if κ vanishes, and non-extremal otherwise.

3 Causal Killing vector field on compact spacetimes

In causality theory, it is proved that a compact spacetime admitting a timelike conformal vector field is totally vicious, [25]. This result is not true if the vector field is only causal. In [25] the author also proved that compact static spacetimes are causally geodesically connected. In this section, we prove that a compact spacetime admitting a causal Killing vector field satisfying the null generic condition is totally vicious. A consequence is that if the universal covering is globally hyperbolic, then the spacetime is geodesically connected.

The main result in this paper, Theorem 3.4, shows that in non-totally vicious compact spacetimes admitting a causal Killing vector field, compact Killing horizons necessarily appear. To prove it we need some basic results.

Lemma 3.1 [1] *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and ζ a timelike affine conformal Killing vector field (resp. a timelike projective vector field). For any null hypersurface M in $\overline{M}$, the normalized null hypersurface (M, ζ) satisfies*

$$\xi(\tau(\xi)) + 2(\tau(\xi))^2 = 0 \tag{5}$$

From Eq. (5), it follows that the function $\tau(\xi)$ vanishes identically on M if the rigged vector field ξ is complete. This is the case when M is a compact null hypersurface. Hence we get.

Corollary 3.2 [1] *Let $(\overline{M}, \overline{g})$ be a Lorentzian manifold and ζ a timelike affine conformal Killing vector field (resp. a timelike projective vector field). Let (M, ζ) be a normalized compact null hypersurface, then the rigged vector field ξ is $\overline{g}$-geodesic that is $\tau(\xi) = 0$.*

We recall some basic definitions from dynamical system, see [18].

Definition 3.3 Let M be a manifold, X a complete vector field on M and ϕ its flow. Let $\gamma : \mathbb{R} \longrightarrow M$ be an integral curve of X. The sets

$$\omega(\gamma) = \{p \in M : \gamma(t_n) \to p; t_n \to \infty\}$$

and

$$\alpha(\gamma) = \{p \in M : \gamma(t_n) \to p; t_n \to -\infty\}$$

are called respectively the ω-limit set and the α-limit set of the orbit γ. A point p is called positively recurrent if $p \in \omega(\gamma_p)$ and it is called negatively recurrent if $p \in \alpha(\gamma_p)$, where γ_p is the unique integral curve of X through p. A subset $A \subset M$ is invariant if $\phi_t(A) \subset A, \forall t \in \mathbb{R}$. It is known that for any integral curve γ, $\omega(\gamma)$ and $\alpha(\gamma)$ are closed (probably empty) invariant subsets. A closed, non-empty, invariant subset $A \subset M$ is a minimal set if it contains no proper, closed, non-empty, invariant subset.

Now we can prove the main theorem.

Theorem 3.4 *Let $(\overline{M}, \overline{g})$ be a compact spacetime admitting a causal Killing vector field K on $\overline{M}$. Then, $(\overline{M}, \overline{g})$ is totally vicious or it contains a compact achronal Killing horizon M. If additionally $(\overline{M}, \overline{g})$ admits a timelike projective vector field (resp. a timelike affine conformal Killing vector field) then the Killing horizon is extremal.*

Proof Suppose $(\overline{M}, \overline{g})$ is not totally vicious. Let $\mathcal{C}$ denote the chronological violating set (which is open). Note that it is not empty since any compact spacetime admit a closed timelike curve. Let $F = \overline{M} \setminus \mathcal{C}$ be the set of points at which the chronological condition is satisfied. As $(\overline{M}, \overline{g})$ is not totally vicious, F is not empty and is moreover compact. Take $p \in F$ and let γ_p be the maximal integral curve of K through p and let $\phi : \mathbb{R} \times \overline{M} \to \overline{M}$ be the flow of K. We prove that γ_p is entirely contained in F. Suppose there exists $t_0 \in \mathbb{R}$ such that $\gamma_p(t_0) = q \in \mathcal{C}$ then $\phi_{t_0}(p) = q$ and $\phi_{-t_0}(q) = p$. Since $q \in \mathcal{C}$ there is a closed timelike curve α through q and then $\phi_{-t_0}\circ\alpha$ is a closed timelike curve through p since ϕ_{-t_0} is an isometry. The contradiction follows from the fact that $p \notin \mathcal{C}$. So γ_p is an inextensible causal curve totally imprisoned in the compact set F on which the chronological condition holds. From [18, Proposition 3.2 and 3.4 (ii)], $\omega(\gamma_p) \neq \emptyset$, achronal and contained in F. Take $q \in \omega(\gamma_p)$ then as $\omega(\gamma_p)$ is an invariant subset of the flow of K, we have $\gamma_q \subset \omega(\gamma_p)$. It follows that γ_q is an achronal inextensible causal curve hence a null line. Since $q \notin \mathcal{C}$, $I^+(q) \neq \overline{M}$ so that $\partial I^+(\gamma_q) \neq \emptyset$. With this and being γ_q a null line, we can follow an argument as in the proof of [12, Theorem IV.1], to get that $\partial I^+(\gamma_q)$ is a C^0 past null hypersurface.

We claim that each connected component of $\partial I^+(\gamma_q)$ is a compact Killing Horizon. We have $\phi_t(\partial I^+(\gamma_q)) = \partial(\phi_t(I^+(\gamma_q)))$ since ϕ_t is a diffeomorphism. As ϕ_t is an isometry, $\partial(\phi_t(I^+(\gamma_q))) = \partial I^+(\phi_t(\gamma_q))$. But $\phi_t(\gamma_q) = \gamma_q$ so that $\phi_t(\partial I^+(\gamma_q)) = \partial I^+(\gamma_q)$. So $\partial I^+(\gamma_q)$ is sent into itself by the flow of K. Since $\partial I^+(\gamma_q)$ is achronal K must be lightlike on $\partial I^+(\gamma_q)$ and $\partial I^+(\gamma_q)$ is generated by K. From [22, Theorem 24], $\partial I^+(\gamma_q)$ is a smooth null hypersurface. Hence the connected components of $\partial I^+(\gamma_q)$ are compact Killing horizons.

Now we show that if additionally $(\overline{M}, \overline{g})$ admits a timelike projective vector field ζ, then the Killing horizon M is extremal. Let κ be the surface gravity of the Killing horizon, then along the horizon, it holds $\overline{\nabla}_K K = \kappa K$. Now consider the normalized null hypersurface (M, ζ) and call ξ its associated rigged vector field. Then, there exists $f \in C^\infty(M)$ such that $K = f\xi$ on M. It follows that

$$\overline{\nabla}_K K = \overline{\nabla}_{f\xi}(f\xi) = f(\xi(f)\xi + f\overline{\nabla}_\xi \xi).$$

From Corollary 3.2, $\overline{\nabla}_\xi \xi = 0$ hence

$$\overline{\nabla}_K K = f(\xi(f))\xi = \kappa K.$$

It follows that

$$\xi(f) = \kappa. \tag{6}$$

As M is a Killing horizon, κ is constant along integral curves of K so $K(\kappa) = f\xi(\kappa) = 0$, thus $\xi(\kappa) = 0$ as f is nowhere vanishing. From Eq. (6), we get

$$\xi(\xi(f)) = \xi(\kappa) = 0. \tag{7}$$

As M is compact f is bounded. Let γ be a maximal integral curve of ξ. Then, from Eq. (7), $f\circ\gamma$ is an affine function defined on the hole real line $\mathbb{R}$ (ξ is complete as M is compact) which is bounded so it is constant. It follows that f is constant along each integral curve of ξ which means that $\xi(f) = 0$. Using Eq. (6) again we get $\kappa = 0$. So M is an extremal Killing horizon. □

Definition 3.5 An inextensible lightlike geodesic γ of the spacetime $(\overline{M}, \overline{g})$ satisfies the generic condition if at some $x = \gamma(t_0)$, the tangent vector $\gamma'(t_0)$ to the curve is a generic vector, that is, the Jacobi operator

$$R_{(\cdot)\gamma'(t_0)}\gamma'(t_0) \,:\, \gamma'(t_0)^\perp \to T_{\gamma(t_0)}\overline{M}$$

is non-proportional to $\gamma'(t_0)$. A spacetime satisfies the null generic condition if every inextensible lightlike geodesic satisfies the generic condition. See [4, Proposition 2.11].

Let us recall the following classical result of Avez [2] and Seifert [26]. In any globally hyperbolic spacetime, each two causally related points p, q can be joined by a causal geodesic, with length equal to the time-separation between p and q.

Corollary 3.6 *Let $(\overline{M}, \overline{g})$ be a compact spacetime admitting a causal Killing vector field and satisfying the null generic condition. Then, it is totally vicious.*

Moreover if its universal Lorentzian covering is globally hyperbolic then it is geodesically connected.

Proof Suppose $(\overline{M}, \overline{g})$ is not totally vicious. Theorem 3.4 implies it contains a compact Killing horizon M, so each generator of M is an inextensible lightlike geodesic in $\overline{M}$ because M is closed. Using that M is totally geodesic and the Gauss–Codazzi equations, we obtain $\overline{g}(R_{UV}W, \xi) = 0$ for any $U, V, W \in \mathfrak{X}(M)$, so the generic condition is violated. Contradiction.

For the second assertion, let $\Pi : (M^*, g^*) \to (\overline{M}, \overline{g})$ be the universal Lorentzian covering of $(\overline{M}, \overline{g})$. Since $(\overline{M}, \overline{g})$ is totally vicious there exists a timelike curve α from any two points $p, q \in \overline{M}$. The lifting curve $\overline{\alpha}$ in the universal covering (M^*, g^*) is a timelike curve from a point in $\Pi^{-1}(p)$ to a point in $\Pi^{-1}(q)$ and using that this is globally hyperbolic, Avez-Seifert result yields a timelike geodesic $\overline{\gamma}$ connecting the endpoints of $\overline{\alpha}$. Thus, the required geodesic is $\gamma = \Pi \circ \overline{\gamma}$. □

Example 3.7 Lorentzian Berger spheres do not admit compact null hypersurfaces, [13, Example 3.12], so they are totally vicious. This is a particular case of a well known result asserting that a compact Lorentzian manifold admitting a timelike conformal vector field is totally vicious, [25].

4 Causal Killing vector field on non-compact spacetimes

In semi-Riemannian geometry, the problem of topological splitting appears extensively. This can be accompanied by an interesting splitting of the metric g. A result of Harris shows that a chronological spacetime which admits a complete timelike conformal vector field splits topologically, [14]. Also, in [16], the authors consider the global metric decomposition of such spacetimes. Finally, in [10], the topological splitting in the case the vector field is causal has been considered.

In this section, we study the geodesic connectedness of non-compact spacetimes with a complete causal Killing vector field. We prove that they split topologically under the null generic condition. The existence of a partial Cauchy hypersurface gives a splitting too. We get a global decomposition of the metric and the spacetime is in fact stably causal. This global decomposition is a Key of the geodesic connectedness of the spacetime, [3].

We recall some facts on Lie group action as appeared in [24].

Definition 4.1 Let $(\mathbb{R}, +)$ be the real line with the structure of an additive group. An $\mathbb{R}$-action on a manifold $\overline{M}$, is a homomorphism f of $(\mathbb{R}, +)$ into *Diff*$(\overline{M})$, such that the map $\mathbb{R} \times \overline{M} \to \overline{M}$ with $(t, p) \longmapsto f(t)(p) =: tp$ is smooth. Such a manifold $\overline{M}$, together with a fixed $\mathbb{R}$-action f, is called a $\mathbb{R}$-manifold.

We use the following standard characterization of the group action.

For any set $U \subset \overline{M}$, and any $t \in \mathbb{R}$, we write $tU := \{tp : p \in U\}$, and $\mathbb{R}p := \{tp : t \in \mathbb{R}\}$ for the orbit of $p \in \overline{M}$. $\mathbb{R}_p$ denotes the isotropy group at p, i.e., $\mathbb{R}_p = \{t \in \mathbb{R} : tp = p\}$. The action f is free if $\mathbb{R}_p = 0$ for every $p \in \overline{M}$. For any sets $U, V \subset \overline{M}$, we define $((U, V)) := \{t \in \mathbb{R} : tU \cap V \neq \emptyset\}$. The map $\Pi : \overline{M} \to \overline{M}/\mathbb{R}$ is the standard projection, the quotient space $\overline{M}/\mathbb{R}$ furnished with the quotient topology. It is well known that Π is an open map.

Definition 4.2 If U and V are subsets of $\overline{M}$, U is thin relative to V if $((U, V))$ has compact closure in $\mathbb{R}$. If U is thin relative to itself, then we say that U is thin.

Note that in the special case that the group is $\mathbb{R}$, $((U, V))$ relatively compact in $\mathbb{R}$ means it is bounded. If U is thin relative to V, then V is thin relative to U, and so we often say that U and V are relatively thin.

Definition 4.3 We say that $\overline{M}$ is a Cartan $\mathbb{R}$-manifold if every $p \in \overline{M}$ has a thin neighborhood.

Proposition 4.4 *If $\overline{M}$ is a Cartan $\mathbb{R}$-manifold , then each orbit is closed in $\overline{M}$ (so that $\overline{M}/\mathbb{R}$ is a T_1 topological space) and each isotropy group is compact.*

Remark 4.5 If $\overline{M}$ is a Cartan $\mathbb{R}$-manifold, $\overline{M}/\mathbb{R}$ may not be Hausdorff even if $\mathbb{R}$ is acting freely on $\overline{M}$.

Definition 4.6 A subset $S \subset \overline{M}$ is called small if each point $p \in \overline{M}$ has a neighborhood which is thin relative to S.

Definition 4.7 We say that $\overline{M}$ is a proper $\mathbb{R}$-manifold if every point $p \in \overline{M}$ has a small neighborhood.

Proposition 4.8 *If $\overline{M}$ is a proper $\mathbb{R}$-manifold, then it is a Cartan $\mathbb{R}$-manifold.*

The following lemma gives a characterization of when $\overline{M}/\mathbb{R}$ is Hausdorff. It is well known, but we provide a proof for completeness.

Lemma 4.9 *Let $\overline{M}$ be a $\mathbb{R}$-space. The set $R = \{(p, tp) \ / \ p \in \overline{M}, t \in \mathbb{R}\}$ is closed in $\overline{M} \times \overline{M}$ if and only if $\overline{M}/\mathbb{R}$ is Hausdorff.*

Proof Observe that $\Pi \times \Pi(R) = \Delta$ is the diagonal in $\overline{M}/\mathbb{R} \times \overline{M}/\mathbb{R}$. Being Π surjective, and using the $\mathbb{R}$-space structure of $\overline{M}$, we can see that

$$\Pi \times \Pi(R^c) = (\Pi \times \Pi(R))^c = \Delta^c.$$

Then, use Π is an open map.

Conversely, let $(p_n, t_n p_n) \in R$ be a sequence with limit $(p, q) \in \overline{M} \times \overline{M}$. If $\Pi(p) = \Pi(q)$, we are done. Otherwise, let U be a neighborhood of $\Pi(p)$ and V a neighborhood of $\Pi(q)$ with $U \cap V = \emptyset$. Using Π is continuous, for large n, $\Pi(t_n p_n) = \Pi(p_n) \in U$ since $p_n \to p$, and $\Pi(t_n p_n) \in V$ since $t_n p_n \to q$. Thus, $\Pi(t_n p_n) \in U \cap V$, contradiction. ☐

The following provides equivalent conditions for an $\mathbb{R}$-manifold to be proper.

Theorem 4.10 *Let $\overline{M}$ be an $\mathbb{R}$-manifold. Then, the following items are equivalent.*

1. *For all $p, q \in \overline{M}$ there exist relatively thin neighborhoods $S, T \subset \overline{M}$ with $p \in S$ and $q \in T$.*
2. *$\overline{M}$ is a Cartan $\mathbb{R}$-manifold and $\overline{M}/\mathbb{R}$ is Hausdorff.*
3. *$\overline{M}$ is a proper $\mathbb{R}$-manifold.*
4. *If $K \subset \overline{M}$ is compact, $((K, K))$ is compact.*

A chronological spacetime which admits a complete timelike conformal vector field splits diffeomorphically as $\mathbb{R} \times Q$ being Q the space of the integral curves of the conformal vector field, [14]. In [10], the authors prove that under the strong causality condition, if the complete conformal vector field is assumed to be only causal, then the splitting still holds. In the next theorem, we prove that strong causality can be replaced by chronology and the null generic condition.

Theorem 4.11 *Let $(\overline{M}, \overline{g})$ be a chronological spacetime which satisfies the null generic condition and admits a complete causal Killing vector field. Then, $(\overline{M}, \overline{g})$ is a proper $\mathbb{R}$-space diffeomorphic to $\mathbb{R} \times \overline{M}/\mathbb{R}$.*

Proof Let K be the complete causal Killing vector field. We claim that for any integral curve γ_p of K, it holds $I^-(\gamma_p) = \overline{M}$ and $I^+(\gamma_p) = \overline{M}$. Suppose the opposite. Then, without loss of generality, we can suppose $I^+(\gamma_p) \neq \overline{M}$. In this case $\partial I^+(\gamma_p) \neq \emptyset$. Let $\phi : \mathbb{R} \times \overline{M} \to \overline{M}$ be the flow of K. Using that ϕ_t are isometries and γ_p is ϕ-invariant, we have $\phi_t(\partial I^+(\gamma_p)) = \partial I^+(\gamma_p)$. So $\partial I^+(\gamma_p)$ is invariant by the flow of K. Take $q \in \partial I^+(\gamma_p)$ then $\gamma_q \subset \partial I^+(\gamma_p)$ since both γ_q and $\partial I^+(\gamma_p)$ are ϕ-invariant. Using that $\partial I^+(\gamma_p)$ is achronal, γ_q is an inextensible achronal causal curve, thus a null line. Being γ_q achronal, we have $\gamma_q \subset \partial I^+(\gamma_q)$ and hence $\partial I^+(\gamma_q) \neq \emptyset$. With these γ_q and $\partial I^+(\gamma_q)$, we can follow the same argument as in the proof of Theorem 3.4, to get that $\partial I^+(\gamma_q)$ is a smooth null hypersurface. Hence the connected components of $\partial I^+(\gamma_q)$ are (topologically closed) Killing horizons. Following the same argument as in the proof of Corollary 3.6, the null generic condition is violated. Contradiction.

It follows that for any integral curve γ_p of K, it holds $I^-(\gamma_p) = \overline{M}$ and $I^+(\gamma_p) = \overline{M}$. Now, we prove first that $\overline{M}$ is a Cartan $\mathbb{R}$-manifold with the action defined by the flow of K. Let $p \in \overline{M}$. If p does not have a thin neighborhood, then there exist sequences $\{t_n\} \subset \mathbb{R}$ and $\{p_n\} \subset \overline{M}$ such that both p_n and $t_n p_n$ converge to p and $|t_n| \to +\infty$. Without loss of generality, we assume that $t_n \to +\infty$. Observe that $p \in I^-(\gamma_p) = \overline{M}$. So there exist $q = t_0 p \in \gamma_p$, and $t_0 \in \mathbb{R}$ such that $p \in I^-(q)$. Note that t_0 is a positive number, otherwise the chronological condition will not hold at p. Since $p \ll t_0 p$, there exist open neighborhoods $U \in t_0 p$ and $V \in p$ such that $p_1 \ll p_2$ whenever $p_1 \in V$ and $p_2 \in U$. For n large enough, $t_n > t_0, t_n p_n \in V$ and $t_0 p_n \in U$, so $t_0 p_n < t_n p_n \ll t_0 p_n$, contradicting chronology. So $\overline{M}$ is a Cartan $\mathbb{R}$-space and, as a consequence, the action is free.

Now we show that $\overline{M}$ is a proper $\mathbb{R}$-manifold. It suffices to show that $\overline{M}/\mathbb{R}$ is Hausdorff. Suppose the opposite, then by Lemma 4.9, there exist sequences $\{t_n\} \subset \mathbb{R}$ and $\{p_n\} \subset \overline{M}$ such that $p_n \to p$ and $t_n p_n \to q$, while $q \notin \mathbb{R}p$. The sequence $\{t_n\}$ cannot be bounded, so, again without loss of generality, we can suppose $t_n \to +\infty$. Fix any $r \in I^+(q)$ and let $t > 0$. Since $q \in I^-(r)$ and $I^-(r)$ is open, for n large enough we have $t_n > t$ and $t_n p_n \in I^-(r)$. It follows that $tp_n < t_n p_n \ll r$, and hence $tp \in \overline{I^-(r)}$ $\forall t > 0$ and consequently $\forall t \in \mathbb{R}$. We conclude that $\mathbb{R}p \subset \overline{I^-(r)}$ and then $I^-(\gamma_p) \subset I^-(r)$. As $I^-(\gamma_p) = \overline{M}$ it follows that $I^-(r) = \overline{M}$ contradicting chronology.

We have that $\overline{M}$ is a rank one vector bundle over $\overline{M}/\mathbb{R}$, but being $\overline{M}$ time oriented, it is diffeomorphic to $\mathbb{R} \times \overline{M}/\mathbb{R}$. □

Let us recall that a partial Cauchy hypersurface is an acausal edgeless (and hence closed) set.

Theorem 4.12 *Suppose $(\overline{M}, \overline{g})$ is a chronological spacetime which admits a complete causal Killing vector field K, and possesses a partial Cauchy hypersurface $A \subset \overline{M}$. If the null generic condition holds, then each orbit of the flow of K intersects A exactly once. Moreover, $\overline{M}$ is homeomorphic to $\mathbb{R} \times A$. In particular, if A is a smooth, closed and acausal spacelike hypersurface, then the flow restricted to $\mathbb{R} \times A$ is actually a diffeomorphism onto $\overline{M}$.*

Proof Let $\phi : \mathbb{R} \times A \longrightarrow \overline{M}$ be the restriction of the flow of the complete causal Killing vector field. It is continuous (smooth if A is smooth), since A is a C^0 hypersurface, and one-to-one since A is acausal. By invariance of domain, ϕ is a homeomorphism onto an open subset $O \subset \overline{M}$. The first and second claims follow showing that $O \equiv \overline{M}$. Since the latter set is open and $\overline{M}$ is connected, we have to show that O is closed. To this end, consider a sequence (t_k, x_k) in $\mathbb{R} \times A$ and $p \in \overline{M}$ such that $\phi_{t_k}(x_k) \to p$. Assume first that $\{t_k\}$ is unbounded. We may assume, up to passing to a subsequence, that $t_k \to +\infty$. If $t_k \to -\infty$, the argument is analogous. Note that given any integral curve γ_p of K, it holds $I^-(\gamma_p) = \overline{M}$ and $I^+(\gamma_p) = \overline{M}$ (see the proof of Theorem 4.11). So $\phi_s(p) \in I^-(x_1)$ for some $s \in \mathbb{R}$ (x_1 being the first term in the sequence (x_k)). But then

$$\phi_{t_k+s}(x_k) = \phi_s(\phi_{t_k}(x_k)) \to \phi_s(p),$$

so for large enough k we have $t_k + s > 0$ and $\phi_{t_k+s}(x_k) \in I^-(x_1)$. Hence,

$$x_k \ll \phi_{t_k+s}(x_k) \ll x_1,$$

which contradicts the achronality of A. Therefore, $\{t_k\}$ must be bounded. But in that case, up to passing to a subsequence, we may assume that it converges, say, $t_k \to t_0$. Let $x_0 := \phi_{-t_0}(p)$. Then

$$x_k = \phi(-t_k, \phi_{t_k}(x_k)) \to x_0.$$

Since A is edgeless, it is closed, so we conclude that $x_0 \in A$, and $\phi(t_0, x_0) = p$, which shows that O is closed, as desired. □

Although the following result is well known, see [3], we include the proof because it clarifies the remark below. If $\overline{M} = \mathbb{R} \times S$, we identify

$$U = (\tau, X) \in T_z\overline{M} = \mathbb{R} \times T_xS$$

and $z = (t, x) \in \overline{M}$.

Proposition 4.13 *Let $(\overline{M}, \overline{g})$ be a globally hyperbolic spacetime admitting a complete causal Killing vector field K. Then, there exists a Riemannian manifold (S, h), a* *differentiable vector field $Z \in \mathfrak{X}(S)$ and a differentiable nonnegative function $f \in C^\infty(S)$ such that $\overline{M} = \mathbb{R} \times S$and*

$$\overline{g}(U, U') = h(X, X') + h(Z, X)\tau' + h(Z, X')\tau - f\tau\tau'. \tag{8}$$

Furthermore, if K is timelike then $f(x) > 0$ for all $x \in S$. If K is lightlike then $f \equiv 0$, and Z is non-vanishing.

Proof Since $\overline{M}$ is a globally hyperbolic spacetime, it admits a spacelike Cauchy hypersurface S which becomes a Riemannian manifold when endowed with the induced metric $h = \overline{g}_{|S}$. Let us consider the flow of K

$$\phi : \mathbb{R} \times S \longrightarrow \overline{M}.$$

Since K is causal, its integral curves are also causal. So, each point of $\overline{M}$ is crossed by one integral curve of K, which crosses S at exactly one point, otherwise, it is easy to construct a nearby inextensible timelike curve crossing S more than once, using an argument similar to that in [23, Lemma 30 (1) p. 416]. Therefore, ϕ is a diffeomorphism. The map $f(x) = -g(K(z), K(z))$ and the orthogonal projection $Z(x)$ of $K(z)$ on T_xS does not depend on t. The metric expression (8) follows from the decomposition $U = \tau K + X \equiv (\tau, X)$. Furthermore, if K is timelike, then f is clearly strictly positive. If K is lightlike, then $f \equiv 0$ and Z is non-vanishing since $K(z)$ cannot be orthogonal to T_xS. □

Remark 4.14 In the above proposition, the global hyperbolicity of $\overline{M}$ is only used to ensure the existence of the spacelike Cauchy hypersurface S and to prove that the map ϕ is a diffeomorphism. In the following theorem, we prove that global hyperbolicity can be replaced by alternative hypotheses.

Recall that a partial Cauchy hypersurface is an acausal edgeless (and hence closed) set, and a time function is a continuous function which is strictly increasing on future-directed causal curves.

Theorem 4.15 *Let $(\overline{M}, \overline{g})$ be a chronological spacetime admitting a complete causal Killing vector field K and possesses a smooth spacelike partial Cauchy hypersurface. If the null generic condition holds, then there exists a Riemannian manifold (S, h), a differentiable vector field $Z \in \mathfrak{X}(S)$ and a differentiable nonnegative function $f \in C^\infty(S)$ such that $\overline{M} = \mathbb{R} \times S$and*

$$\overline{g}(U, U') = h(X, X') + h(Z, X)\tau' + h(Z, X')\tau - f\tau\tau'. \tag{9}$$

Furthermore, if K is timelike then $f(x) > 0$ for all $x \in S$. If K is lightlike then $f \equiv 0$, and Z is non-vanishing.

Moreover, the first projection $t : \overline{M} \longrightarrow \mathbb{R}$ is a time function and so $(\overline{M}, \overline{g})$ is stably causal.

Proof Taking into account Theorem 4.12 and Remark 4.14, we need only to show that the first projection $t : \overline{M} \longrightarrow \mathbb{R}$ is a time function.

Suppose we have a time orientation such that the causal Killing vector field is future directed. Let $\gamma(s) = (t(s), x(s))$ be a future-directed timelike curve. Then, it satisfies the inequality

$$h(\dot{x}, \dot{x}) + 2h(Z, \dot{x})\dot{t} - f\dot{t}^2 < 0$$

which is equivalent to

$$h(\dot{x}, \dot{x}) + h(Z, \dot{x})\dot{t} + \dot{t}(h(Z, \dot{x}) - f\dot{t}) < 0. \tag{10}$$

First note that $\dot{t}(s) = 0$ implies $h(\dot{x}, \dot{x}) < 0$ which is not possible. Moreover, using that $K \equiv \partial_t$ is future directed, we have

$$\overline{g}(\gamma', K) = h(Z, \dot{x}) - f\dot{t} < 0. \tag{11}$$

If $\dot{t}(s) < 0$, using f is nonnegative and Eq. (11) we get $h(Z, \dot{x}) < 0$. But in this case, the first term of Eq. (10) is nonnegative whereas the second and the third one are positive. Contradiction.

It follows that $t : \overline{M} \longrightarrow \mathbb{R}$ is strictly increasing along future-directed timelike curves and by continuity it is increasing along future-directed causal curve.

Now, let $\gamma(s) = (t(s), x(s))$ be a future-directed causal curve. Then, it satisfies the inequality:

$$h(\dot{x}, \dot{x}) + h(Z, \dot{x})\dot{t} + \dot{t}(h(Z, \dot{x}) - f\dot{t}) \leq 0. \tag{12}$$

As t is increasing along γ, we have $\dot{t}(s) \geq 0$. Moreover, from Eq. (12), $\dot{t}(s) = 0$ implies $h(\dot{x}(s), \dot{x}(s)) \leq 0$. Hence $\dot{t}(s) = 0$ and $\dot{x}(s) = 0$, which is not possible since γ is causal. So, t is continuous and strictly increasing along future-directed causal curves. It follows that $t : \overline{M} \longrightarrow \mathbb{R}$ is a time function and then $(\overline{M}, \overline{g})$ is also stably causal. □

Now we prove the following theorem about geodesic connectedness of spacetime admitting a causal Killing vector field.

Theorem 4.16 *Let $(\overline{M}, \overline{g})$ be a chronological spacetime admitting a complete causal Killing vector field K and a smooth compact spacelike partial Cauchy hypersurface S. If the null generic condition holds, then $(\overline{M}, \overline{g})$ is globally hyperbolic with compact Cauchy hypersurface S. Moreover if K is lightlike then given two points $p, q \in \overline{M}$, the following statements are equivalent:*

(1) *p and q are geodesically connected in $\overline{M}$.*
(2) *p and q can be connected by a C^1 curve α on $\overline{M}$ such that $\overline{g}(\alpha', K(\alpha))$ is constant.*

Proof We will prove that $(\overline{M}, \overline{g})$ is globally hyperbolic and then point (1) and (2) will follow from [3, Theorem 1.2].

From Theorem 4.15, $(\overline{M}, \overline{g})$ is strongly causal. Let $T \in \mathbb{R}$. Assume $\gamma : \mathbb{R} \longrightarrow \overline{M}$ is an inextensible timelike curve and assume without lost of generality that γ is future directed with $t(0) < T$, but γ does not intersect $\sigma_T = S \times \{T\}$. As t is strictly increasing along γ we can infer that $\gamma([0, \infty)) \subset S \times [t(0), T]$. As S is compact, γ is future imprisoned in a compact set, in contradiction with strong causality. The proof for $t(0) > T$ works completely analogous. □

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