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Weighted asymmetric problems for an indefinite elliptic operator

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Dedicated to Jean-Pierre Gossez, on the occasion of his 65th birthday

Abstract. We investigate two asymmetric eigenvalue type problems for the linear operator $-\Delta + V$, and, more generally, for its homogeneous quasilinear version $-\Delta_p + V$. The main novelty appears when this operator is indefinite and non-coercive, due to the possible change of structure in its weighted spectrum. We introduce a *first non-trivial asymmetric eigencurve* for $-\Delta_p + V$, whose first zero provides the *first non-principal* eigenvalue and allows us to build up a *first non-trivial* curve in the Fučik spectrum with weights. We partially extend the corresponding results already known when $V \equiv 0$ and obtain some new features in the indefinite case.

1. Introduction

Let V, m be smooth functions defined in a bounded domain $\Omega \subset \mathbb{R}^N$. It is known that the problem

$$\begin{cases} -\Delta u + V(x)u = \lambda m(x)u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

has no principal eigenvalue (i.e., an eigenvalue whose eigenfunctions are sign-constant) when V has a negative part sufficiently large and m changes sign or vanishes in some part of Ω. This non-existence result was proved in [**1**] (see also [**8, 10**]) and later on extended to the quasilinear problem

$$(P_m) \qquad \begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

in [**4, 5**]. Here $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator, with $p > 1$. Lately, an optimal condition was established in [**7**] in order to guarantee the existence of principal eigenvalues of (P_m). In sum, let

$$(1.1) \qquad \alpha(V, m) := \inf\{E_V(u); \|u\|_p = 1, M(u) = 0\} \in \overline{\mathbb{R}},$$

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where

$$E_V(u) := \int_\Omega (|\nabla u|^p + V(x)|u|^p)\,dx \quad \text{and} \quad M(u) := \int_\Omega m(x)|u|^p\,dx,$$

for $u \in W_0^{1,p}(\Omega)$. If m is sign-changing then

$$\lambda_{\pm 1}(V,m) := \pm \inf_{M(u)=\pm 1} E_V(u) \tag{1.2}$$

are (the only) principal eigenvalues of (P_m) if, and only if, $\alpha(V,m) \geq 0$. Furthermore, if $\alpha(V,m) < 0$ then $\lambda_{\pm 1}(V,m) = -\infty$.

In the present article, we carry on the investigation of weighted eigenvalue problems for $-\Delta_p + V$. We are now concerned with the asymmetric problem

$$(AP) \quad \begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ and V, m, n satisfy an integrability condition that will be specified later. This problem is closely related to (P_m), for it can be regarded as an extension of this one. Indeed, (AP) reduces to (P_m) when $m \equiv n$. Note that every principal eigenvalue of (P_m) (and (P_n)) is a principal eigenvalue of (AP).

We are also interested in the problem

$$(FP) \quad \begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = Am(x)(u^+)^{p-1} - Bn(x)(u^-)^{p-1} & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $A, B \in \mathbb{R}$, which in its turns extends (AP). Let us recall that

$$\Sigma = \Sigma(m,n) := \{\ (A,B) \in \mathbb{R}^2; (FP) \text{ has a nontrivial solution }\}$$

is the so-called Fučik spectrum of (FP).

In [**2**], the authors studied (AP) and (FP) for $V \equiv 0$, providing a mountain-pass characterization of the first positive non-principal eigenvalue of (AP) and using this one to construct a first non-trivial curve in Σ. More precisely, let

$$M_{m,n} := \left\{ u \in W_0^{1,p}(\Omega); \int_\Omega \left[m(u^+)^p + n(u^-)^p \right]\,dx = 1 \right\}.$$

and

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], M_{m,n});\ \gamma(0) = \varphi_1(m),\ \ \gamma(1) = -\varphi_1(n)\},$$

where $\varphi_1(m) \in M_{m,n}$ is the positive eigenfunction associated to

$$\lambda_1(m) := \min_{M(u)=1} \int_\Omega |\nabla u|^p\,dx.$$

The first positive non-principal eigenvalue of (AP) when $V \equiv 0$ is then characterized by

$$c(m,n) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \int_\Omega |\nabla(\gamma(t))|^p\,dx.$$

Thereafter, it is shown that

$$\mathcal{C} := \{(\alpha(s), \beta(s)) = (c(m,sn), sc(m,sn)); s > 0\}$$

is the *first curve* in

$$\Sigma^+ = \Sigma \cap (]\lambda_1(m), \infty[\times]\lambda_1(n), \infty[), \tag{1.3}$$

in the sense that, for every $s > 0$, $(\alpha(s), \beta(s))$ is the first intersection of Σ^+ with the line $\beta = s\alpha$. Furthermore, the authors show that $\mathcal{C}$ is a continuous, decreasing and hyperbolic like curve, which in some cases is asymptotic to the lines $\lambda_1(m)\times\mathbb{R}$ and $\mathbb{R}\times\lambda_1(n)$.

The behavior of the operator $-\Delta_p+V$ on $W_0^{1,p}(\Omega)$ depends essentially on the sign of its first eigenvalue $\lambda_1(V,1)$, in the following way:

(1) if $\lambda_1(V,1)>0$ then $E_V(u)\geq C_V\|u\|^p$, for some constant $C_V>0$ and every $u\in W_0^{1,p}(\Omega)$, so that E_V is coercive.
(2) if $\lambda_1(V,1)=0$ then E_V is non-coercive, but still bounded from below.
(3) if $\lambda_1(V,1)<0$ then E_V is unbounded from below.

In this sense, $-\Delta_p+V$ brings some new features (in comparison with $-\Delta_p$) only when $\lambda_1(V,1)\leq 0$.

Under suitable conditions on V, m and n, we aim at extending the results of [**2**] on $c(m,n)$ and $\mathcal{C}$. In particular, we are specially interested in the cases where either (P_m) or (P_n) has no principal eigenvalue, so that the spectrum structure of (AP) and (FP) may change considerably.

We mean by *first non-principal* (or *non-trivial*) eigenvalue of (AP) its least eigenvalue larger than $\max\{\lambda_1(V,m),\lambda_1(V,n)\}$, if this maximum is finite. Otherwise, we restrict ourselves to the case $m,n\geq 0$, so that the first non-principal eigenvalue of (AP) is actually its first one. Whenever it exists, the first non-principal eigenvalue of (AP) will be denoted by $c(m,n)$.

This article is organized as follows: in Section 2, we recall and collect some preliminary results on (P_m) and (AP), respectively. In Section 3, we define a mountain-pass level which provides a non-principal eigenvalue of (AP) when $\lambda_1(V,m)$ and $\lambda_1(V,n)$ are finite (Th.3.6). Furthermore, we show that this mountain-pass value is actually equal to $c(m,n)$ under some additional conditions (Th.3.9). In Section 4, with the help of a first non-principal eigencurve, a characterization of $c(m,n)$ is obtained under more general conditions. In particular, we deal with the cases where either $\lambda_1(V,m)=-\infty$ or $\lambda_1(V,n)=-\infty$. (Th.4.5 and Th.4.8). Finally, in Section 5, we use the first non-principal eigencurve approach to find a first non-trivial curve in the Fučik spectrum of (FP) (Prop. 5.1, Prop. 5.4 and Prop. 5.6).

The Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r$ and the usual norm of $W_0^{1,p}(\Omega)$ by $\|\cdot\|$. The weak convergence is denoted by $\rightharpoonup$. The positive and negative part of u are defined by $u^\pm:=\max\{\pm u,0\}$. The support of u is denoted by supp u and is always understood in the measurable sense. If A is a measurable subset of $\mathbb{R}^N$, $|A|$ stands for its Lebesgue measure. The ball with center x and radius $R>0$ is denoted by $B(x,R)$.

2. Preliminaries

We recall now from [**7**] the main result on the principal eigenvalues of (P_m). Let

$$(H1)\qquad \begin{cases} r>N/p & \text{if}\quad 1<p\leq N,\\ r=1 & \text{if}\quad p>N.\end{cases}$$

Under this assumption, $W_0^{1,p}(\Omega)$ is compactly imbedded in $\begin{cases} L^{pr'}(\Omega) & \text{if } p\leq N\\ C(\overline{\Omega}) & \text{if } p>N\end{cases}$, where r' is the Hölder conjugate of r. Consequently, E_V and M are respectively weakly lower semi-continuous and weakly continuous on $W_0^{1,p}(\Omega)$ whenever $V,m\in L^r(\Omega)$.

Theorem 2.1. [**7**] *Let $V, m \in L^r(\Omega)$ with r satisfying $(H1)$ and $m^+ \not\equiv 0$.*

(1) *If $m \geq 0$ then there exists a principal eigenvalue of (P_m) if and only if $\alpha(V,m) > 0$, where $\alpha(V,m)$ is defined in (1.1). In this case there is an unique principal eigenvalue, which is characterized by*

$$\lambda_1(V,m) = \min_{\mathcal{M}^+} E_V, \tag{2.1}$$

where $\mathcal{M}^+ := \{u \in W_0^{1,p}(\Omega); M(u) = 1\}$.

(2) *If m changes sign then there exists a principal eigenvalue of (P_m) if and only if $\alpha(V,m) \geq 0$. More precisely :*

  (a) *if $\alpha(V,m) > 0$ then (P_m) admits exactly two principal eigenvalues $\lambda_{-1}(V,m) < \lambda_1(V,m)$, with $\lambda_1(V,m)$ characterized as in (2.1) and*

$$\lambda_{-1}(V,m) = -\min_{\mathcal{M}^-} E_V \tag{2.2}$$

  where $\mathcal{M}^- := \{u \in W_0^{1,p}(\Omega); M(u) = -1\}$.

  (b) *If $\alpha(V,m) = 0$ then (P_m) has an unique principal eigenvalue $\lambda_1(V,m)$ given by*

$$\lambda_1(V,m) = \inf_{\mathcal{M}^+} E_V = -\inf_{\mathcal{M}^-} E_V.$$

  These infima are not achieved. Moreover any function $u \not\equiv 0$ in $W_0^{1,p}(\Omega)$ satisfying

$$E_V(u) = M(u) = 0 \tag{2.3}$$

  is an eigenfunction associated to $\lambda_1(V,m)$.

Furthermore, every principal eigenvalue of (P_m) is simple and isolated.

It should be noted that (P_m) may indeed have no principal eigenvalue, for instance, when $V \equiv V_0 < -\alpha(0,m)$.

The proof of Theorem 2.1 relies on the properties of the map

$$\mu_1(\lambda) := \lambda_1(V - \lambda m, 1), \tag{2.4}$$

which can be referred to as the *(weighted) principal eigencurve* (see [**4**]) of the operator

$$(-\Delta_p + V)u := -\Delta_p u + V|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega).$$

In other words, for every fixed λ, $\mu_1(\lambda)$ is the only principal eigenvalue of the problem

$$\begin{cases} -\Delta_p u + (V - \lambda m)|u|^{p-2}u = \mu|u|^{p-2}u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

It is easily seen that the zeros of μ_1 are the principal eigenvalues of (P_m), whose existence can be derived from the following result:

Proposition 2.2. [**7**]

(1) *μ_1 is concave, differentiable and $\mu_1'(\lambda) = -M(\varphi_\lambda)$, where φ_λ realizes $\mu_1(\lambda)$.*

(2) *If $m^\pm \not\equiv 0$ then $\lim_{\lambda \to \pm\infty} \mu_1(\lambda) = -\infty$.*

(3) *If $m \geq 0$ then μ_1 is strictly decreasing.*

(4) *$\alpha(V,m) = \sup_{\lambda \in \mathbb{R}} \mu_1(\lambda)$.*

For the sake of simplicity, we use the following notation throughout this work:

- Since V will not vary, $\lambda_{\pm1}(V,m)$, $\lambda_2(V,m)$ and $\alpha(V,m)$ will be denoted respectively by $\lambda_{\pm1}(m)$, $\lambda_2(m)$ and $\alpha(m)$. The positive eigenfunction associated to $\lambda_{\pm1}(m)$ and normalized by $M(\varphi_{\pm1}(m)) = \pm1$ will be denoted by $\varphi_{\pm1}(m)$. By simplicity of $\lambda_{\pm1}(m)$, $\varphi_{\pm1}(m)$ are uniquely determined. If $m \equiv 1$, we denote $\varphi_1(m)$ by φ_1.
- In view of its dependence on m, it will be convenient to denote by $\mu_1(\lambda, m)$ the eigenvalue defined in (2.4).

Furthermore, we assume that $\Omega \setminus \operatorname{supp} m$ and $\Omega \setminus \operatorname{supp} n$ are p-stable (see (4.6)) if $m, n \geq 0$. From this condition, it follows that $\lambda_{\pm1}(m)$, defined in (1.2), is an eigenvalue of (P_m) if, and only if, it is finite (cf. [**7**, Prop. 12]). In particular, there holds $\lambda_1(m) = -\infty$ if either $\alpha(m) \leq 0$ and $m \geq 0$ or $\alpha(m) < 0$ and m changes sign.

It is worth pointing out that even when (P_m) and (P_n) have both principal eigenvalues, (AP) and (FP) may exhibit striking differences with the case $V \equiv 0$. For instance, if $\lambda_1(1) < 0$, one may easily find m, n sign-changing and such that $\lambda_{-1}(m) < \lambda_1(m) < \lambda_{-1}(n) < \lambda_1(n)$. In this case, one cannot exclude the existence of eigenvalues of (AP) between $\lambda_1(m)$ and $\lambda_{-1}(n)$.

Let us agree to say that $\lambda_1(m)$ and $\lambda_1(n)$ are *ordered* if

$$\max\{\lambda_{-1}(m), \lambda_{-1}(n)\} < \min\{\lambda_1(m), \lambda_1(n)\}. \tag{2.5}$$

Otherwise, we say that $\lambda_1(m)$ and $\lambda_1(n)$ are *non ordered*. Note that the ordered case holds for instance when $\lambda_1(1) \geq 0$ or $m, n \geq 0$, while the non ordered one comprises the cases where $\lambda_1(m) = -\infty$ and/or $\lambda_1(n) = -\infty$. In Rem.4.7, we give an example of V, m, n such that $\lambda_1(m)$ and $\lambda_1(n)$ are finite and non ordered and we show that (AP) may possess an eigenvalue between $\lambda_1(m)$ and $\lambda_{-1}(n)$.

Below we depict some examples of principal eigencurves in the ordered and non-ordered cases. Here $\lambda_1(1) < 0$ and m, n are sign-changing.

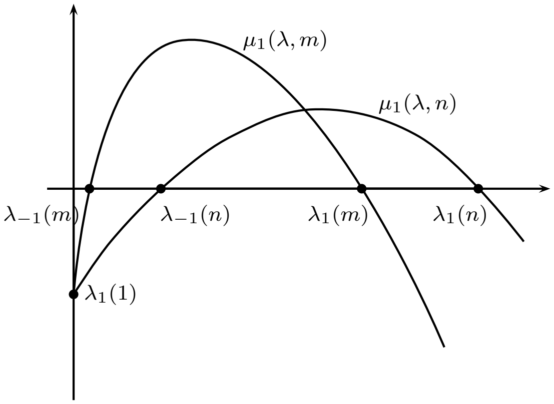

Figure 1: The ordered case

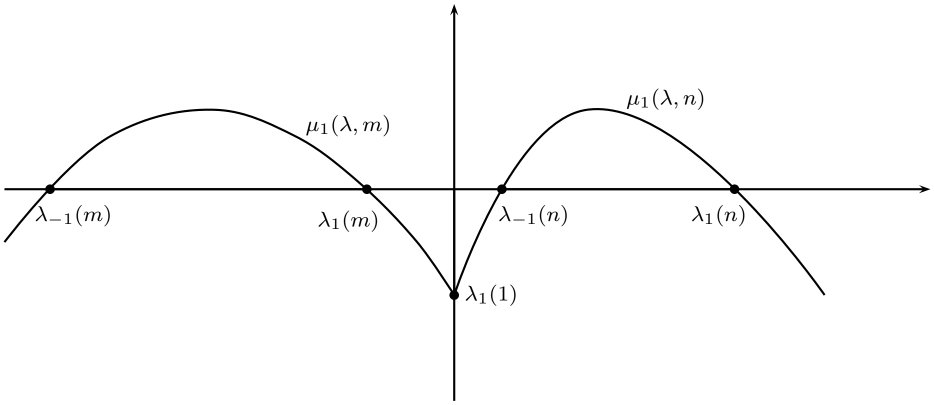

Figure 2: The non-ordered case $(\lambda_1(m), \lambda_1(n) > -\infty)$

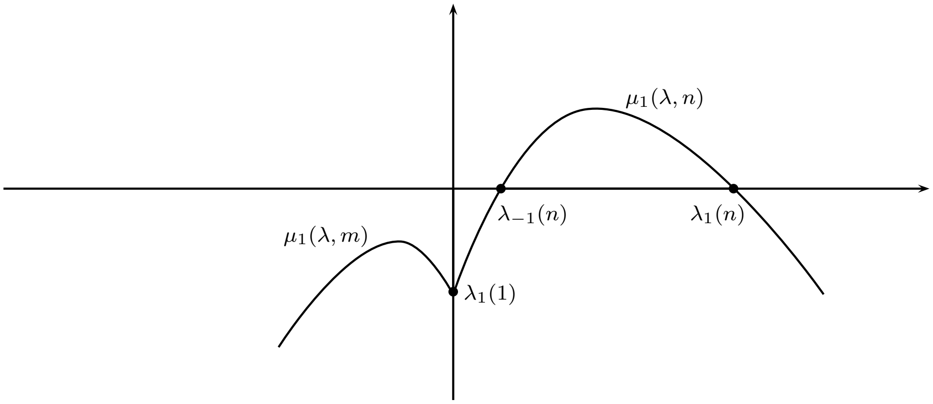

Figure 3: The non-ordered case $(\lambda_1(n) > \lambda_1(m) = -\infty)$

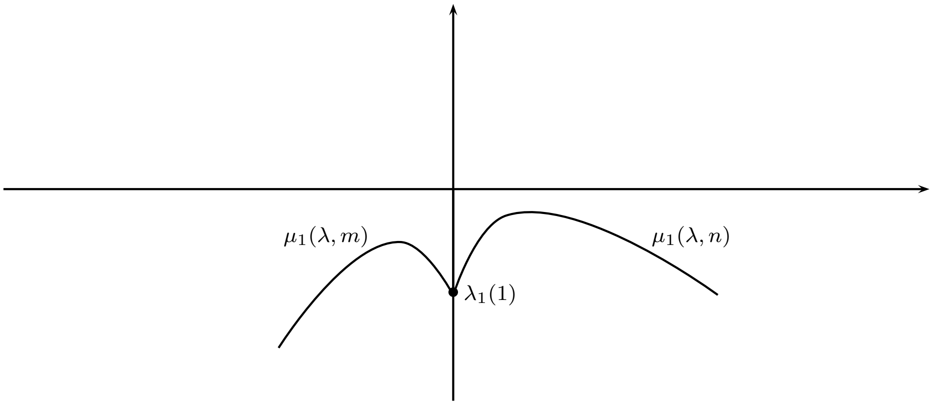

Figure 4: The non-ordered case ($\lambda_1(n) = \lambda_1(m) = -\infty$)

In order to deal with (AP) and (FP), let us set

$$N(u) := \int_\Omega n|u|^p \, dx \quad \text{and} \quad I_{m,n}(u) := \int_\Omega [m(u^+)^p + n(u^-)^p] \, dx$$

for $u \in W_0^{1,p}(\Omega)$, and

$$\mathcal{S}_{m,n} := \{u \in W_0^{1,p}(\Omega);\ I_{m,n}(u) = 1\}.$$

It is clear that $I_{m,n}$ is a C^1 functional on $W_0^{1,p}(\Omega)$ and, consequently, $\mathcal{S}_{m,n}$ is a C^1 manifold. We denote by $\tilde{E}_V$ the restriction of E_V to $\mathcal{S}_{m,n}$.

Any non-trivial solution of (AP) (resp. (FP)) is understood in the weak sense, viz, $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$E'_V(u) = \lambda I'_{m,n}(u) \text{ (resp. } E'_V(u) = AM'(u^+) - BN'(u^-)).$$

Remark 2.3. In contrast with the case $V \equiv 0$, sign-changing solutions of (AP) and (FP) may satisfy

$$M(u^+) = 0 \quad \text{or} \quad N(u^-) = 0.$$

Let us give a trivial example of this situation:
Choose $m, n_0 \in L^r(\Omega)$ such that $m^+, n_0^+ \not\equiv 0$, and let u be a solution of

$$-\Delta_p u = \lambda_0[m(u^+)^{p-1} - n_0(u^-)^{p-1}], \quad u \in W_0^{1,p}(\Omega),$$

where λ_0 is the first non-trivial eigenvalue of this problem. We set $V = \lambda_0 n_0 \mathbf{1}_{\Omega^-}$, where $\Omega^- = \{x \in \Omega;\ u(x) < 0\}$. So, if $n \in L^r(\Omega)$ vanishes in Ω^-, then $N(u^-) = 0$. Moreover, u solves

$$-\Delta_p u + V|u|^{p-2} = \lambda_0 m(u^+)^{p-1} - Bn(u^-)^{p-1}, \quad u \in W_0^{1,p}(\Omega),$$

for every $B \in \mathbb{R}$. In particular, $\lambda_0 \times \mathbb{R} \subset \Sigma^+$.

The above phenomenon may occur only when $\alpha(m) \leq 0$ or $\alpha(n) \leq 0$, as shown in the next lemma:

Lemma 2.4. *Assume that* $\alpha(m), \alpha(n) > 0$ *and* λ *is an eigenvalue of* (AP). *Then*

$$\lambda > \max\{\lambda_1(m), \lambda_1(n)\} \Longleftrightarrow M(u^+), N(u^-) > 0$$

and

$$\lambda < \min\{\lambda_{-1}(m), \lambda_{-1}(n)\} \Longleftrightarrow M(u^+), N(u^-) < 0$$

In particular, whenever it exists, $c(m,n)$ *is the least eigenvalue whose eigenfunctions satisfy* $M(u^+) > 0$ *and* $N(u^-) > 0$.

Proof. Let (λ, u) be an eigenpair of (AP). Then

$$E_V(u^+) = \lambda M(u^+) \quad \text{and} \quad E_V(u^-) = \lambda N(u^-). \tag{2.6}$$

If $\lambda > \max\{\lambda_1(m), \lambda_1(n)\}$ and $M(u^+) = 0$ then, after L^p normalization, u^+ is an admissible function for $\alpha(m)$, and consequently $\alpha(m) \leq E_V(u^+) \leq 0$, which is a contradiction. Moreover, if $M(u^+) < 0$ then

$$-\lambda = -\frac{E_V(u^+)}{M(u^+)} \geq -\lambda_{-1}(m),$$

so that $\lambda \leq \lambda_{-1}(m) < \lambda_1(m)$, which is contrary to the choice of λ. Thus $M(u^+) > 0$. In a similar way, we show that $N(u^-) > 0$. Finally, if $M(u^+), N(u^-) > 0$ then (2.6) clearly implies that $\lambda > \max\{\lambda_1(m), \lambda_1(n)\}$. The second equivalence can be proved in a similar way. □

3. The mountain-pass level

In this section, we assume that

$$\alpha(m), \alpha(n) > 0,$$

so that (AP) has at least two principal eigenvalues. We aim at showing that $\tilde{E}_V$ has the mountain-pass geometry. Let

$$\beta(m,n) := \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} E_V(\gamma(t)), \tag{3.1}$$

where

$$\Gamma_0 = \Gamma_0(m,n) := \{\gamma \in C([0,1], \mathcal{S}_{m,n}) : \gamma(0) = \varphi_1(m), \gamma(1) = -\varphi_1(n)\}.$$

We will prove that $\beta(m,n) > \max\{\lambda_1(m), \lambda_1(n)\}$ and that $\beta(m,n)$ is a non-principal eigenvalue of (AP) when $\alpha(m), \alpha(n) > 0$. Furthermore, we will investigate whether $\beta(m,n)$ is the first non-principal eigenvalue of (AP).

Before proceeding, we need to recall some auxiliary results:

Lemma 3.1. [**2**] *If* $v_k \in W_0^{1,p}(\Omega)$ *with* $v_k \not\equiv 0$, $v_k \geq 0$, $|v_k > 0| \to 0$ *and* $n_k \to n$ *in* $L^r(\Omega)$ *then*

$$\frac{\int_\Omega n_k v_k^p \, dx}{\|v_k\|^p} \to 0.$$

Lemma 3.2. [**7**]

(1) *Let* $\omega \in L^r(\Omega)$, *with* r *satisfying (H1) and let* $B \subset L^r(\Omega)$ *be a bounded set. If* $\omega > 0$ *on* Ω *then there exists two positive constants* C_1, C_2 *such that*

$$\|u\|^p \leq C_1 E_V(u) + C_2 \int_\Omega \omega |u|^p \, dx$$

for every $V \in B$ *and every* $u \in W_0^{1,p}(\Omega)$.

(2) *Assume that* $\alpha(m) > 0$. *If* $V_k \rightharpoonup V, m_k \rightharpoonup m$ *in* $L^r(\Omega)$ *and* u_k *is a sequence such that* $E_{V_k}(u_k)$ *and* $M(u_k)$ *are bounded then* u_k *is bounded.*

Corollary 3.3. *Let* $m, m_k \in L^r(\Omega)$ *be such that* $\alpha(m) > 0$, $m_k^+ \not\equiv 0$, *and* $m_k \to m, m_k^+ \to 0$ *in* $L^r(\Omega)$. *Then* $\lambda_1(m_k) \to \infty$.

Proof. Let $\varphi_k > 0$ achieve $\lambda_1(m_k)$. We have

$$1 = \int_\Omega m_k \varphi_k^p \, dx \leq \int_\Omega m_k^+ \varphi_k^p \, dx \leq \|m_k^+\|_r \|\varphi_k\|_{pr'}^p,$$

so that $\|\varphi_k\|_{pr'} \to \infty$. By Lemma 3.2, if $\lambda_1(m_k) = E_V(\varphi_k)$ is bounded then φ_k is bounded, and we get a contradiction. Thus $\lambda_1(m_k) \to \infty$. □

Let us show that $\varphi_1(m)$ and $-\varphi_1(n)$ minimize $\tilde{E}_V$ locally and strictly.

Lemma 3.4. *If* $\alpha(m), \alpha(n) > 0$, *then* $\varphi_1(m)$ *and* $-\varphi_1(n)$ *are strict local minimizers of* $\tilde{E}_V$, *with corresponding critical values* $\lambda_1(m)$ *and* $\lambda_1(n)$, *respectively. Moreover, either* $\varphi_1(m)$ *or* $-\varphi_1(n)$ *is a global minimizer of* $\tilde{E}_V$.

Proof. It is easily seen that $\min\{\lambda_1(m), \lambda_1(n)\}$ is the global minimum of $\tilde{E}_V$:

$$E_V(u) \geq \begin{cases} \lambda_1(m) M(u^+) & \text{if } \ M(u^+) > 0. \\ 0 & \text{if } \ M(u^+) = 0. \\ \lambda_{-1}(m) M(u^+) \geq \lambda_1(m) M(u^+) & \text{if } \ M(u^+) < 0. \end{cases}$$

Similar inequalities hold for $N(u^-)$ and $\lambda_1(n)$. Thus

$$E_V(u) \geq \min\{\lambda_1(m), \lambda_1(n)\}$$

for every $u \in \mathcal{S}_{m,n}$, and equality holds for either $u = \varphi_1(m)$ or $u = -\varphi_1(n)$. We show now that these ones are strict minimizers. Assume by contradiction the existence of a sequence $u_k \in \mathcal{S}_{m,n}$ such that

$$u_k \neq \varphi_1(m), \quad u_k \to \varphi_1(m), \quad \text{and} \quad \tilde{E}_V(u_k) \leq \lambda_1(m).$$

We claim that u_k changes sign for k large enough. Indeed, since $u_k \to \varphi_1(m)$, u_k must be positive somewhere. If $u_k \geq 0$, then $u_k \in \mathcal{M}^+$, so that $\tilde{E}_V(u_k) \geq \lambda_1(m)$. Thus u_k realizes $\lambda_1(m)$, which is simple. Hence $u_k = \varphi_1(m)$, contrary to our assumption. Therefore u_k changes sign for k sufficiently large. From $u_k \to \varphi_1(m)$ we have $M(u_k^+) \to 1$, so that $E_V(u_k^+) \geq \lambda_1(m) M(u_k^+)$ for k sufficiently large. Thus

$$\lambda_1(m) \geq E_V(u_k) = E_V(u_k^+) + E_V(u_k^-) \geq \lambda_1(m) M(u_k^+) + E_V(u_k^-),$$

and consequently

$$E_V(u_k^-) \leq \lambda_1(m)(1 - M(u_k^+)) = \lambda_1(m) N(u_k^-). \tag{3.2}$$

We consider now three cases:

(1) If $N(u_k^-) = 0$ then $v_k = \frac{u_k^-}{||u_k^-||_p}$ is admissible in the definition of $\alpha(n)$. From (3.2) we get $E_V(v_k) \leq 0$, so that $\alpha(n) \leq 0$, a contradiction.

(2) If $N(u_k^-) > 0$, we set $v_k = \frac{u_k^-}{N(u_k^-)^{1/p}}$. Then

$$N(v_k) = 1 \quad \text{and} \quad \lambda_1(n) \leq E_V(v_k) \leq \lambda_1(m),$$

where the last inequality follows from (3.2). Hence $E_V(v_k)$ is bounded. From Lemma 3.2, we infer that v_k is bounded. On the other hand, by Lemma 3.1,

$$\frac{1}{||v_k||^p} = \frac{N(u_k^-)}{||u_k^-||^p} \longrightarrow 0,$$

and we get a contradiction.

(3) If $N(u_k^-) < 0$, we set $v_k = \frac{u_k^-}{[-N(u_k^-)]^{1/p}}$, so that $N(v_k) = -1$. In this case n is sign-changing, so

$$-\lambda_{-1}(n) \leq E_V(v_k) \leq -\lambda_1(m).$$

Thus v_k is bounded and we get a contradiction as before.

So we obtain a contradiction, proving that $\varphi_1(m)$ is a strict minimizer of $\tilde{E}_V$. A similar argument holds for $-\varphi_1(n)$. □

Some Palais-Smale and Palais-Smale-Cerami conditions are next established for $\tilde{E}_V$. We recall that u_k is a $(PS)_d$ (resp. $(PSC)_d$) sequence for $\tilde{E}_V$ if

$$\tilde{E}_V(u_k) \to d \quad \text{and} \quad \|\tilde{E}_V'(u_k)\|_* \to 0 \text{ (resp. } (1 + \|u_k\|)\|\tilde{E}_V'(u_k)\|_* \to 0),$$

where $\|\tilde{E}_V'(u_k)\|_*$ is the norm of $E_V'(u_k)$ restricted to $T_{u_k}(\mathcal{S}_{m,n})$, the tangent space to the manifold $\mathcal{S}_{m,n}$ at the point u_k. $\tilde{E}_V$ is said to satisfy the $(PS)_d$ (resp. $(PSC)_d$) condition if every $(PS)_d$ (resp. $(PSC)_d$) sequence has a convergent subsequence. Finally, $\tilde{E}_V$ satisfies the (PS) (resp. (PSC)) condition if $\tilde{E}_V$ satisfies the $(PS)_d$ (resp. $(PSC)_d$) condition for any d.

Proposition 3.5. *If $\alpha(m), \alpha(n) > 0$ then:*

(1) *$\tilde{E}_V$ satisfies the (PS) condition along bounded sequences.*
(2) *$\tilde{E}_V$ satisfies the $(PSC)_d$ condition for any $d > \max\{\lambda_1(m), \lambda_1(n)\}$.*

Proof.

(1) Let (u_k) be a (PS) sequence for $\tilde{E}_V$, i.e., there are $d \in \mathbb{R}$ and a sequence $\varepsilon_k \to 0$ such that

$$(PS1) \quad E_V(u_k) \to d$$
$$(PS2) \quad | < E_V'(u_k), \xi > | \leq \varepsilon_k ||\xi|| \quad \text{for all} \quad \xi \in T_{u_k}(\mathcal{S}_{m,n}).$$

Let us write, for $w \in W_0^{1,p}(\Omega)$,

$$a_k(w) := w - \frac{1}{p} < I_{m,n}'(u_k), w > u_k.$$

It is clear that $a_k(w) \in T_{u_k}(\mathcal{S}_{m,n})$. If (u_k) is bounded, we can assume that $u_k \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$ and $u_k \to u_0$ in $L^{pr'}(\Omega)$. If we choose $w = u_k - u_0$ and $\xi = a_k(w)$ in $(PS2)$ then $< I_{m,n}'(u_k), w > \to 0$, so that $\langle E_V'(u_k), u_k - u_0 \rangle \to 0$. From the (S^+) property of $-\Delta_p$, we get $u_k \to u_0$.

(2) Let (u_k) be a (PSC) sequence for $\tilde{E}_V$, i.e., for some sequence $\varepsilon_k \to 0$ and some $d \geq \min\{\lambda_1(m), \lambda_1(n)\}$, there holds

$$(PSC1) \quad E_V(u_k) \to d$$

$$(PSC2) \quad | < E'_V(u_k), \xi > | \leq \frac{\varepsilon_k}{1+||u_k||}||\xi|| \quad \text{for all} \quad \xi \in T_{u_k}(\mathcal{S}_{m,n}).$$

Assume that (u_k) is unbounded and set $v_k = \frac{u_k}{||u_k||}$. Up to a subsequence, there is some v_0 such that $v_k \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $v_n \to v_0$ in $L^{pr'}(\Omega)$. We choose $\xi = a_k(v_k - v_0)$ in $(PSC2)$ and divide it by $||u_k||^{p-1}$ to obtain

$$\begin{aligned} &\left| < E'_V(v_k), v_k - v_0 > -\frac{1}{p} < I'_{m,n}(v_k), v_k - v_0 > E_V(u_k) \right| \\ &\leq \varepsilon_k \frac{||u_k||}{1+||u_k||} \left\| \frac{v_k - v_0}{||u_k||^p} - \frac{1}{p} < I'_{m,n}(v_k), v_k - v_0 > v_k \right\|. \end{aligned} \tag{3.3}$$

Since $< I'_{m,n}(v_k), v_k - v_0 > \to 0$, it follows that $\langle E'_V(v_k), v_k - v_0 \rangle \to 0$. By the (S^+) property of $-\Delta_p$, we get $v_k \to v_0$. Now, from

$$||v_k|| = 1, \quad E_V(v_k) = \frac{E_V(u_k)}{||u_k||^p}, \quad \text{and} \quad I_{m,n}(v_k) = \frac{I_{m,n}(u_k)}{||u_k||^p},$$

it follows that

$$v_0 \not\equiv 0, \quad E_V(v_0) = 0 \quad \text{and} \quad I_{m,n}(v_0) = 0. \tag{3.4}$$

On the other hand, taking $\xi = a_k(w)$ in $(PSC2)$, for an arbitrary $w \in W_0^{1,p}(\Omega)$, and arguing as above, we find that v_0 is a solution of

$$-\Delta_p v_0 + V|v_0|^{p-2}v_0 = d[m(v_0^+)^{p-1} - n(v_0^-)^{p-1}].$$

We distinguish now two cases:

(a) If v_0 is sign-constant then $M(v_0) = 0$ or $N(v_0) = 0$, and d is a principal eigenvalue, which is impossible since $\alpha(m), \alpha(n) > 0$ (see Th. 2.1).

(b) If v_0 changes sign then, as $I_{m,n}(v_0) = 0$, one has either $M(v_0^+) \leq 0$ or $N(v_0^-) \leq 0$. We consider two subcases:

  (i) If $M(v_0^+) < 0$ then $-\lambda_{-1}(m) \leq \frac{E_V(v_0^+)}{-M(v_0^+)} = -d$, which is a contradiction since $d > \lambda_1(m) > \lambda_{-1}(m)$.

  (ii) If $M(v_0^+) = 0$ then $N(v_0^-) = 0$, so that, after L^p normalization, v_0^+ (resp. v_0^-) is admissible for $\alpha(m)$ (resp. $\alpha(n)$). Hence $E_V(v_0^+) > 0$ and $E_V(v_0^-) > 0$, so that

$$E_V(v_0) = E_V(v_0^+) + E_V(v_0^-) > 0,$$

  in contradiction with (3.4).

Therefore we conclude that $M(v_0^+) > 0$. In a similar way, we prove that $N(v_0^-) > 0$.

Thus $I_{m,n}(v_0) = 0$ is excluded, and it follows that (u_k) is bounded. We repeat the argument used for v_k to conclude that u_k converges, up to a subsequence.

□

Now we can deduce that $\beta(m,n)$ is an eigenvalue of (AP), which is strictly larger than $\lambda_1(m)$ and $\lambda_1(n)$.

Theorem 3.6. *If* $\alpha(m), \alpha(n) > 0$ *then* $\beta(m,n) > \max\{\lambda_1(m), \lambda_1(n)\}$ *and* $\beta(m,n)$ *is a critical value of* $\tilde{E}_V$. *In particular,* $c(m,n) \leq \beta(m,n)$.

Proof. The first inequality follows directly from Lemma 3.4 and [**2**, Lemma 6]. Let us observe that the latter one holds also when $\tilde{E}_V$ satisfies the (PS) condition along bounded sequences, which was shown in Prop. 3.5. We apply the mountain-pass theorem for a C^1 functional which satisfies the (PSC) condition on a C^1 manifold (cf. [**3**, Theorem 4.1]), to infer that $\beta(m,n)$ is a critical value of $\tilde{E}_V$. ☐

In the sequel, we show that $\beta(m,n) = c(m,n)$ when either $\lambda_1(m), \lambda_1(n) > 0$ (no matter they are ordered or not) or $\beta(m,n) < 0$. Let us recall that the ordered notion is defined in (2.5).

Lemma 3.7. *Assume that* $\alpha(m) > 0$.

1. *If* $0 < \lambda_1(m) < d$ *then there exists* $\hat{n}$ *such that* $\hat{n} \leq m$, $\lambda_1(\hat{n}) > d$, *and* $\lambda_1(m), \lambda_1(\hat{n})$ *are ordered.*
2. *If* $\lambda_1(m) < d < 0$ *then there exists* $\hat{n}$ *such that* $\hat{n} \geq m$, $\lambda_1(\hat{n}) > d$, *and* $\lambda_1(m), \lambda_1(\hat{n})$ *are ordered.*

Proof. We will use the fact that for $\lambda > 0$ (resp. $\lambda < 0$), $\mu_1(\lambda, m)$, defined in (2.4), is decreasing (resp. increasing) with respect to m. Recall that the zeros of $\lambda \mapsto \mu_1(\lambda, m)$ are the principal eigenvalues of (P_m).

1. We extract $\hat{n}$ from the sequence $m_k := \frac{m^+}{k} - m^-$, which satisfies $m_k \leq m$ for every k and, by Cor.3.3, $\lambda_1(m_k) \to \infty$. Moreover, from $\hat{n} \leq m$ it follows that $\mu_1(\lambda, \hat{n}) \geq \mu_1(\lambda, m)$ for every $\lambda > 0$. Thus $\lambda_{-1}(\hat{n}) < \lambda_{-1}(m) < \lambda_1(m) < \lambda_1(\hat{n})$, so that $\lambda_1(m), \lambda_1(\hat{n})$ are *ordered.*
2. We set $\hat{n} = m + k$, where $k > 0$ is a constant. Thus $\mu_1(\lambda, \hat{n}) \geq \mu_1(\lambda, m)$ for every $\lambda \leq 0$. Consequently, $\alpha(\hat{n}) \geq \alpha(m) > 0$. Finally, we choose $k > \frac{\mu_1(d,m)}{d}$, so that $\mu_1(d, \hat{n}) > 0$. Indeed, note that if $\|u\|_p = 1$ then
$$E_V(u) - d\int_\Omega \hat{n}|u|^p\,dx = E_V(u) - dM(u) - kd,$$
so that $\mu_1(d,\hat{n}) = \mu_1(d,m) - kd$. Since $\lambda_1(m) < 0$, it follows that $\mu_1(0,\hat{n}) = \lambda_1(1) < 0$, so that $\lambda_1(\hat{n}) > d$. Once again, $\hat{n} \geq m$ implies that $\lambda_1(m), \lambda_1(\hat{n})$ are ordered.

☐

Lemma 3.8. *Assume that* $\alpha(m) > 0$. *If either* $\lambda_1(m) > 0$ *or* $d < 0$ *then the set* $\mathcal{O} := \{u \in \mathcal{S}_{m,n};\ u \geq 0,\ E_V(u) < d\}$ *is arcwise connected. The same conclusion holds if either* $\lambda_1(n) > 0$ *or* $d < 0$ *and the condition* $u \geq 0$ *is replaced by* $u \leq 0$.

Proof. Since $\mathcal{O}$ is empty when $d \leq \lambda_1(m)$, we can assume either
$$0 < \lambda_1(m) < d \quad \text{or} \quad \lambda_1(m) < d < 0.$$
By Lemma 3.7, there is $\hat{n} \in L^r(\Omega)$ such that
$$\lambda_1(\hat{n}) > d, \quad \lambda_1(m), \lambda_1(\hat{n}) \text{ are ordered}, \quad \text{and} \quad \hat{n} \gtreqless m \text{ if } d \gtreqless 0.$$

Consider now the open set

$$\hat{\mathcal{O}} := \{u \in \mathcal{S}_{m,\hat{n}} : E_V(u) < d\}.$$

The restriction $\hat{E}_V$ of E_V to $\mathcal{S}_{m,\hat{n}}$ satisfies the (PSC) condition and one can show that, under this condition, Lemma 14 from [**2**] still holds (using the strong version of the Ekeland variational principle). Thus any nonempty component of $\hat{\mathcal{O}}$ contains a critical point of $\hat{E}_V$. Since $\lambda_1(m), \lambda_1(\hat{n})$ are *ordered*, these are the first two critical values of $\hat{E}_V$. From $\lambda_1(m) < d < \lambda_1(\hat{n})$, we infer that $\varphi_1(m)$ is the only critical point of $\hat{E}_V$ in $\hat{\mathcal{O}}$ and we conclude that $\hat{\mathcal{O}}$ is arcwise connected. Now, let $u_1, u_2 \in \mathcal{O}$. Since $\mathcal{O} \subset \hat{\mathcal{O}}$, there exists a path $\gamma \in \hat{\mathcal{O}}$ going from u_1 to u_2. By the choice of n̂, there holds

$$M(\gamma(t)) \gtreqless I_{m,\hat{n}}(\gamma(t)) = 1, \quad \text{if} \quad d \gtreqless 0,$$

so that $\gamma_1(t) := \frac{|\gamma(t)|}{M(\gamma(t))^{1/p}}$ is a path in $\mathcal{S}_{m,n}$, going from u_1 to u_2 and made of nonnegative functions. Let us remark that γ_1 is also well defined when $d < 0$. Indeed, since the function $t \mapsto M(\gamma(t))$ is continuous and

$$M(\gamma(0)) = M(\gamma(1)) = 1,$$

if $M(\gamma(t_0)) \leq 0$ for some $t_0 \in [0,1]$, then there exists $\tilde{t} \in [0,1]$ such that $M(\gamma(\tilde{t})) = 0$. Thus $\gamma(\tilde{t})$ is admissible in the definition of $\alpha(m)$, so that

$$\alpha(m) \leq E_V(\gamma(\tilde{t})) < d < 0,$$

a contradiction. Finally, one has

$$E_V(\gamma_1(t)) = \frac{E_V(\gamma(t))}{M(\gamma(t))} \leq E_V(\gamma(t)) < d$$

in either case. ☐

Theorem 3.9. *Assume that* $\alpha(m), \alpha(n) > 0$. *If either* $\lambda_1(m), \lambda_1(n) > 0$ *or* $\beta(m,n) < 0$ *then* $\beta(m,n) = c(m,n)$, *i.e.,* $\beta(m,n)$ *is the first non-principal eigenvalue of* (AP).

Proof. Assume by contradiction that there exists an eigenpair (λ, u) for (AP), with

$$\max\{\lambda_1(m), \lambda_1(n)\} < \lambda < \beta(m,n).$$

We know that u changes sign and

$$E_V(u^+) = \lambda M(u^+), \quad E_V(u^-) = \lambda N(u^-).$$

By Lemma 2.4, there holds $M(u^+), N(u^-) > 0$. It follows that

$$I_{m,n}(u^+ - tu^-) = M(u^+) + t^p N(u^-) > 0, \quad \forall t \in [0,1].$$

Now, arguing as in [**2**], the paths

$$\gamma_1(t) = \frac{u^+ - tu^-}{I_{m,n}(u^+ - tu^-)^{1/p}}, \quad \gamma_2(t) = \frac{(1-t)u^+ - u^-}{I_{m,n}((1-t)u^+ - u^-)^{1/p}}, \quad t \in [0,1],$$

go respectively from $\frac{u^+}{(M(u^+))^{1/p}}$ to $\frac{u}{I_{m,n}(u)^{1/p}}$ and from $\frac{u}{I_{m,n}(u)^{1/p}}$ to $\frac{-u^-}{N(u^-)^{1/p}}$. One can also see that $\gamma_1(t), \gamma_2(t) \in \mathcal{S}_{m,n}$ and

$$\tilde{E}_V(\gamma_1(t)) = \tilde{E}_V(\gamma_2(t)) = \lambda,$$

for every $t \in [0,1]$. Moreover, since either $\lambda_1(m) > 0$ (resp. $\lambda_1(n) > 0$) or $\lambda < 0$, using Lemma 3.8 with $d = \lambda$, one can construct a path in $\mathcal{S}_{m,n}$ made up of nonnegative functions (resp. nonpositive functions), going from $\frac{u^+}{(M(u^+))^{1/p}}$ to $\varphi_1(m)$ (resp. from $-\frac{u^-}{(N(u^-))^{1/p}}$ to $-\varphi_1(n)$) and staying at levels below $\beta(m,n)$. Patching everything together, one gets a path on $\mathcal{S}_{m,n}$ that goes from $\varphi_1(m)$ to $-\varphi_1(n)$ and stays at levels below $\beta(m,n)$, providing a contradiction with the definition of the latter one. □

Whenever $\beta(m,n)$ is the first non-principal eigenvalue of (AP), it is continuous with respect to m and n:

Proposition 3.10. *If* $\alpha(m), \alpha(n) > 0$ *and* $(m_k, n_k) \to (m,n)$ *in* $L^r(\Omega) \times L^r(\Omega)$ *then* $\limsup \beta(m_k,n_k) \le \beta(m,n)$. *If, in addition,* $\lambda_1(m), \lambda_1(n) > 0$ *or* $\beta(m,n) < 0$ *then* $\lim \beta(m_k,n_k) = \beta(m,n)$.

Proof. Let us first observe that by [**7**, Prop. 25], $\alpha(m)$ is continuous (resp. lower semi-continuous) if m change sign (resp. $m \ge 0$), so that $\min\{\alpha(m_k), \alpha(n_k)\} > 0$ for k sufficiently large. Let $\varepsilon > 0$ and take $\gamma \in \Gamma_0$ such that

$$\max_{t\in[0,1]} E_V(\gamma(t))) < \beta(m,n) + \varepsilon.$$

Since $I_{m,n}(\gamma(t))$ is continuous with respect to m, n, t, we deduce that, for k sufficiently large,

$$\max_{t\in[0,1]} \frac{E_V(\gamma(t))}{I_{m_k,n_k}(\gamma(t))} < \beta(m,n) + \varepsilon.$$

Hence $\beta(m_k,n_k) \le \beta(m,n) + \varepsilon$ and consequently

$$\limsup c(m_k,n_k) \le \beta(m,n) + \varepsilon.$$

Since ε is arbitrary, the upper semi-continuity follows.

We prove now the lower semi-continuity when either $\lambda_1(m), \lambda_1(n) > 0$ or $\beta(m,n) < 0$. Assume by contradiction that, for a subsequence, $\beta(m_k,n_k) \to c$, with $c < \beta(m,n)$. Let $u_k \in \mathcal{S}_{m_k,n_k}$ be a solution associated to $\lambda_k = \beta(m_k,n_k)$, i.e

$$-\Delta_p u_k + V|u_k|^{p-2}u_k = \lambda_k[m_k(u_k^+)^{p-1} - n_k(u_k^-)^{p-1}]. \tag{3.5}$$

We know that u_k changes sign. Let us prove that (u_k) is bounded. If not, set $v_k = u_k/||u_k||$. Then we can assume that $v_k \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $v_k \to v_0$ in $L^{pr'}(\Omega)$. We claim that v_0 must changes sign. Indeed, if $v_0 \ge 0$ then $u_k^- \to 0$ in measure, i.e., $|\Omega_k^-| \to 0$, where

$$\Omega_k^- := \{x \in \Omega;\ u_k(x) \le 0\}.$$

Multiplying (3.5) by u_k^-, one gets

$$\|u_k^-\|^p = \int_{\Omega_k^-} (\lambda_k n_k - V)(u_k^-)^p\,dx.$$

Applying Hölder inequality, we get

$$\|u_k^-\|^p \le \|\lambda_k n_k - V\|_r \|u_k^-\|_{pr'}^p |\Omega_k^-|^\sigma,$$

where $\sigma > 0$ is a constant depending on N, p, r. Moreover, as (λ_k) and (n_k) are bounded, there holds

$$\|u_k^-\|^p \le C\|u_k^-\|_{pr'}^p |\Omega_k^-|^\sigma,$$

for some $C > 0$. Finally, by Sobolev inequality, one deduces that $|\Omega_k^-| \geq C^{-\sigma} > 0$, for some constant $C > 0$. But this contradicts the fact that $|\Omega_k^-| \to 0$. A similar argument holds when $v_0 \leq 0$ in Ω. Hence v_0 changes sign in Ω. Furthermore, dividing (3.5) by $||u_k||^{p-1}$, one gets

$$-\Delta_p v_k + V|v_k|^{p-2}v_k = \lambda_k[m_k(v_k^+)^{p-1} - n_k(v_k^-)^{p-1}]. \tag{3.6}$$

Taking $(v_k - v_0)$ as test function in (3.6) and using the (S_+) property of the p-Laplacian, one gets $v_k \to v_0$ in $W_0^{1,p}(\Omega)$. On the other hand, taking $w \in W_0^{1,p}(\Omega)$ as test function in (3.6) and passing to the limit, one finds that v_0 is a solution of

$$-\Delta_p v_0 + V|v_0|^{p-2}v_0 = c[m(v_0^+)^{p-1} - n(v_0^-)^{p-1}].$$

Since $c < \beta(m, n) = c(m, n)$, one deduces that $v_0 = \varphi_1(m)$ or $v_0 = -\varphi_1(n)$, which contradicts the fact that v_0 changes sign. So u_k is bounded in $W_0^{1,p}(\Omega)$. Arguing as above, one gets that $u_k \to u_0$ in $W_0^{1,p}(\Omega)$ with u_0 a sign-changing solution of

$$-\Delta_p u_0 + V|u_0|^{p-2}u_0 = c[m(u_0^+)^{p-1} - n(u_0^-)^{p-1}],$$

contradicting again the assumption $c < \beta(m, n)$. Therefore, the lower semi-continuity is proved. □

4. The first non-principal asymmetric eigencurve

Our purpose now is to find the first non-principal eigenvalue when the mountain-pass approach seems to fail (in particular, when $\min\{\alpha(m), \alpha(n)\} < 0$). The method followed now is comparable to the one used in [**7**].

Let us consider the 'V-asymmetric' eigenvalue problem

$$-\Delta_p u + V_1(u^+)^{p-1} - V_2(u^-)^{p-1} = \lambda|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega) \tag{4.1}$$

with $V_1, V_2 \in L^r(\Omega)$ and r satisfying $(H1)$. This problem has clearly two principal eigenvalues, which are its two first eigenvalues. Proceeding as in [**7**], where the case $V = V_1 = V_2$ has been treated, one can prove that the first non-principal eigenvalue of (4.1) is given by

$$d(V_1, V_2) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_{V_1,V_2}(\gamma(t)), \tag{4.2}$$

where

$$E_{V_1,V_2}(u) := \int_\Omega [|\nabla u|^p + V_1(u^+)^p + V_2(u^-)^p]\, dx = E_{V_1}(u^+) + E_{V_2}(u^-)$$

and

$$\Gamma := \{\gamma \in \mathcal{C}([0, 1]; S) : \gamma(0) \geq 0 \geq \gamma(1)\}.$$

Here S is the L^p unit sphere in $W_0^{1,p}(\Omega)$. Furthermore, one can easily see that if $V_1 = V_2 = V$ then $d(V_1, V_2) = \lambda_2(1)$.

We introduce now the *first non-principal asymmetric eigencurve* of $-\Delta_p + V$, defined by

$$\sigma(\lambda) = \sigma(\lambda, m, n) := d(V - \lambda m, V - \lambda n), \quad \lambda \in \mathbb{R},$$

viz, the first non-principal eigenvalue of

$$-\Delta_p u + (V - \lambda m)(u^+)^{p-1} - (V - \lambda n)(u^-)^{p-1} = \sigma|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega). \tag{4.3}$$

The zeros of $\sigma(\lambda)$ are thus non-principal eigenvalues of (AP). We look for conditions on V, m and n that allow us to infer the existence of these zeros. For this purpose, we establish some useful properties of $d(V_1, V_2)$ (and consequently of $\sigma(\lambda)$). The first one concerns the continuity with respect to V_1 and V_2, while the second one gives another characterization for $d(V_1, V_2)$. This characterization is inspired by a formulation for the first non-trivial eigenvalue of $-\Delta_p$ on a riemannian manifold without boundary, from [**11**].

PROPOSITION 4.1. *If* $V_{1,k} \rightharpoonup V_1, V_{2,k} \rightharpoonup V_2$ *in* $L^r(\Omega)$ *then* $d(V_{1,k}, V_{2,k}) \to d(V_1, V_2)$.

PROOF. $E_{V_1,V_2}(u)$ is weakly continuous with respect to V_1 and V_2, so one can repeat the proof of [**7**, Prop. 26]. □

PROPOSITION 4.2. *There holds*

$$d(V_1, V_2) = \min_{(u,v)\in\mathcal{J}} \max\{E_{V_1}(u), E_{V_2}(v)\}, \tag{4.4}$$

where

$$\mathcal{J} = \{(u, v) \in (W_0^{1,p}(\Omega))^2; uv \equiv 0, ||u||_p = ||v||_p = 1\}.$$

Moreover, this minimum is achieved by (u, v) *if, and only if,* $(u, v) = \left(\frac{\varphi^+}{\|\varphi^+\|_p}, \frac{\varphi^-}{\|\varphi^-\|_p}\right)$, *where* φ *is an eigenfunction associated to* $d(V_1, V_2)$.

PROOF. Let

$$\tilde{d}(V_1, V_2) := \inf_{(u,v)\in\mathcal{J}} \max\{E_{V_1}(u), E_{V_2}(v)\}$$

and φ be an eigenfunction associated to $d(V_1, V_2)$. We know that φ changes sign and

$$E_{V_1}(\varphi^+) = d(V_1, V_2)||\varphi^+||_p^p, \quad E_{V_2}(\varphi^-) = d(V_1, V_2)||\varphi^-||_p^p.$$

L^p-normalizing φ^+ and φ^-, one gets a couple of admissible functions in the definition of $\tilde{d}(V_1, V_2)$, so that

$$\tilde{d}(V_1, V_2) \le \max\left\{\frac{E_{V_1}(\varphi^+)}{||\varphi^+||_p^p}, \frac{E_{V_2}(\varphi^-)}{||\varphi^-||_p^p}\right\} = d(V_1, V_2).$$

Now let $(u, v) \in \mathcal{J}$ and define the following path: $\gamma(t) = \frac{(1-t)|u|-t|v|}{[(1-t)^p+t^p]^{1/p}}$, $t \in [0, 1]$. Then $\gamma \in \mathit{\Gamma}$ and consequently $d(V_1, V_2) \le \max\limits_{t\in[0,1]} E_{V_1,V_2}(\gamma(t))$. But

$$E_{V_1,V_2}(\gamma(t)) = \frac{(1-t)^p E_{V_1}(u) + t^p E_{V_2}(v)}{(1-t)^p + t^p} \le \max\{E_{V_1}(u), E_{V_2}(v)\}, \quad \forall t \in [0, 1].$$

Thus $d(V_1, V_2) \le \max\{E_{V_1}(u), E_{V_2}(v)\}$ and it follows that $d(V_1, V_2) \le \tilde{d}(V_1, V_2)$, so that $d(V_1, V_2) = \tilde{d}(V_1, V_2)$ and $\tilde{d}(V_1, V_2)$ is achieved by (u, v). Finally, if (u, v) realizes $\tilde{d}(V_1, V_2)$, then γ defined above realizes the mountain-pass level, so that, for some $t_0 \in [0, 1]$, $\gamma(t_0)$ is an eigenfunction associated to $d(V_1, V_2)$. Thus, after L^p normalization we find $\gamma(t_0)^+ = u$ and $\gamma(t_0)^- = v$. □

From now on, we will take advantage of the expression

$$\sigma(\lambda) = \min_{(u,v)\in\mathcal{J}} \max\{E_{V-\lambda m}(u), E_{V-\lambda n}(v)\} \tag{4.5}$$

to deduce the existence of zeros for σ. Note that, by Prop.4.1, σ is a continuous map. Observe also that $\sigma(0) = \lambda_2(1)$ and that, for every $\lambda \in \mathbb{R}$,

$$\sigma(\lambda) > \max\{\mu_1(\lambda, m), \mu_1(\lambda, n)\},$$

so that

$$\sup \sigma(\lambda) \geq \max\{\alpha(m), \alpha(n)\}.$$

Let us now analyze the behavior of $\sigma(\lambda)$ when $\lambda \to \infty$. We include the case $n \leq 0$, where we need the following assumption:

$$(H_{m,n}) \qquad \begin{cases} |\text{supp } m^+ \cap \text{ supp } n| > 0 & \text{or} \\ \overline{(\text{supp } m^+)} \cap \partial\Omega_n \neq \emptyset & \text{or} \\ \overline{(\text{supp } m^+)} \cap \partial\Omega_n = \emptyset \text{ and } N \geq p, \end{cases}$$

where $\Omega_n := \Omega \setminus \mathit{supp}(n)$.

Proposition 4.3. *Assume that* $m^+ \not\equiv 0$.

(1) *If* $n^+ \not\equiv 0$ *then* $\lim_{\lambda\to\infty} \sigma(\lambda) = -\infty$.
(2) *If* $(H_{m,n})$ *holds,* $n \leq 0$ *and* $\alpha(n) < 0$ *then* $\lim_{\lambda\to\infty} \sigma(\lambda) < 0$.

Proof.

(1) Let u, v be such that $uv \equiv 0$ and $M(u) > 0$, $N(v) > 0$. Note that this choice of u and v is always possible: set

$$X := \text{supp } m^+ \cap \text{supp } n^+.$$

   (a) If $|X| = 0$, we obtain u and v by regularizing the characteristic function of supp m^+ and supp n^+, respectively.
   (b) If $|X| > 0$, we choose $\tilde{X} \subset X$ such that $|\tilde{X}| > 0$ and $|X \setminus \tilde{X}| > 0$. We regularize the characteristic functions of $\tilde{X}$ and $X \setminus \tilde{X}$ to get u and v.

   For such u and v, we have $\max\{E_{V-\lambda m}(u), E_{V-\lambda n}(v)\} \to -\infty$ as $\lambda \to \infty$, so that $\sigma(\lambda) \to -\infty$ as $\lambda \to \infty$.

(2) Let us initially consider $n \equiv 0$ (in this case $\Omega_n = \Omega$ and $\alpha(n) = \lambda_1(1)$). We will find a couple $(u, v) \in \mathcal{J}$ such that $M(u) > 0 \geq E_V(v)$. Consequently $\max\{E_{V-\lambda m}(u), E_V(v)\} \leq 0$ for λ sufficiently large. We follow an argument used in [**2**], where the authors constructed a sequence u_k that converges to $\varphi_1(m)$ and vanishes somewhere in Ω. To this end, they need to assume either $\overline{(\text{supp } m^+)} \cap \partial\Omega \neq \emptyset$ or $\overline{(\text{supp } m^+)} \cap \partial\Omega = \emptyset$ and $N \geq p$.

   (a) If $\overline{(\text{supp } m^+)} \cap \partial\Omega \neq \emptyset$, we make use of the continuous and increasing dependence of $\lambda_1(1, \Omega)$ with respect to Ω. In other words, let

   $$\Omega_\delta := \{x \in \Omega;\ \mathit{dist}(x, \partial\Omega) > \delta\}.$$

   Then $\lambda_1(1, \Omega_\delta) \searrow \lambda_1(1, \Omega)$ as $\delta \to 0$. We choose $\delta > 0$ sufficiently small in order to have $\lambda_1(1, \Omega_\delta) < 0$. We pick then $v = \varphi_1(\Omega_\delta)$ and $u \geq 0$, $u \not\equiv 0$ with supp $u \subset \overline{(\text{supp } m^+)} \cap (\Omega \setminus \Omega_\delta)$.

   (b) If $\overline{(\text{supp } m^+)} \cap \partial\Omega = \emptyset$ and $N \geq p$, we argue as in [**2**, Prop.35] to obtain a sequence v_k that converges to φ_1 and vanishes in a neighborhood of a point $x_0 \in \overline{(\text{supp } m^+)}$ (we choose this point x_0 non-isolated in $\overline{(\text{supp } m^+)}$). Therefore, we may find some $u \geq 0$, $u \not\equiv 0$ with supp $u \subset \overline{(\text{supp } m^+)}$ and $uv_k \equiv 0$, for k large enough. So we choose $v = v_k$

to have $E_V(v) < 0$.

If $n \not\equiv 0$, we find a couple $(u,v) \in \mathcal{J}$ such that $M(u) > 0 = N(v) > E_V(v)$ as follows: if $|\text{supp } m^+ \cap \text{supp } n| > 0$, we can find some $u \geq 0$, $u \not\equiv 0$ such that $M(u) > 0$ and $\text{supp } u \subset \text{supp } n$. Hence, if v is a minimizer for $\alpha(n)$, then $N(v) = 0 > E_V(v)$ and $uv \equiv 0$. In the two other cases, we proceed as above to find a sequence v_k that converges now to a minimizer for $\alpha(n)$ and vanishes in a positive measure subset of $\overline{(\text{supp } m^+)}$.

□

When $m, n \geq 0$, we are able to give a minimax formula for $\sup \sigma(\lambda)$. If, in addition, m and n vanish precisely on the same set X, this minimax gives also the second eigenvalue of $-\Delta_p + V$ on X, provided $\Omega \setminus X$ satisfies a regularity condition.

Let us recall that a property is said to hold *quasi-everywhere* (*q.e.*) on $\Omega' \subset \Omega$ if it holds on $\Omega' \setminus U$, where $U \subset \Omega'$ is such that $Cap_p(U, \Omega) = 0$. Here $Cap_p(U, \Omega)$ is the $W^{1,p}$-capacity of U. A function $u : \Omega \to \mathbb{R}$ is *quasi-continuous* if given $\varepsilon > 0$, there is an open set $U_\varepsilon \subset \mathbb{R}^N$ such that $Cap_p(U_\varepsilon, \Omega) < \varepsilon$ and u is continuous on $\Omega \setminus U_\varepsilon$. It is well-known that every function $u \in W^{1,p}(\mathbb{R}^N)$ has a *quasi-continuous* representative given *q.e.* by

$$\tilde{u}(x) := \lim_{R \to 0} \frac{1}{|B(x,R)|} \int_{B(x,R)} u.$$

An open set $D \subset \mathbb{R}^N$ is said to be *p-stable* if for every $u \in W^{1,p}(\mathbb{R}^N)$,

$$u = 0 \ q.e. \quad \text{in } \mathbb{R}^N \setminus \overline{D} \implies u = 0 \ q.e. \quad \text{in } \mathbb{R}^N \setminus D. \tag{4.6}$$

When $m \geq 0$, the p-stability of $\Omega \setminus \text{supp } m$ allows one to conclude that $u \in W_0^{1,p}(\Omega \setminus \text{supp } m)$ if $M(u) = 0$. Indeed, let $D \subset \Omega \subset \mathbb{R}^N$ be open sets. Then

$$u \in W_0^{1,p}(D) \iff u \in W_0^{1,p}(\Omega) \text{ and } \tilde{u} = 0 \ q.e. \ on \ \Omega \setminus D.$$

Simple examples of *p-stable* sets are given by Lipschitzian domains or, more generally, by domains satisfying the uniform exterior cone property. We refer to [**6, 9**] for more details on these issues.

The next proposition is somewhat similar to Prop.2.2. Recall that we are assuming that $\Omega \setminus \text{supp } m$ and $\Omega \setminus \text{supp } n$ are p-stable if $m, n \geq 0$.

Proposition 4.4. *Let* $\alpha_2(m,n) := \inf_{(u,v) \in \mathcal{K}} \max\{E_V(u), E_V(v)\}$, *where*

$$\mathcal{K} := \{(u,v) \in \mathcal{J}; M(u) = N(v) = 0\}.$$

If $m, n \geq 0$ *then:*

1. $\sigma(\lambda)$ *is a decreasing map.*
2. $\alpha_2(m,n)$ *is achieved and* $\alpha_2(m,n) = \sup_{\lambda \in \mathbb{R}} \sigma(\lambda)$. *In particular,* $\alpha_2(m,n) \geq \max\{\alpha(m), \alpha(n)\}$.
3. *If supp*$(m) =$ *supp*$(n) = X$ *and* $\Omega \setminus X$ *is a p-stable set, then* $\alpha_2(m,n) = \lambda_2(1, \Omega \setminus X)$.

Proof.

1. The decreasing of $\sigma(\lambda)$ follows directly from (4.5) and its dependence on λ.

(2) For any pair $(u,v) \in \mathcal{K}$ and any $\lambda \in \mathbb{R}$, one has

$$E_{V-\lambda m}(u) = E_V(u) \quad \text{and} \quad E_{V-\lambda m}(v) = E_V(v),$$

so that $\alpha_2(m,n) \geq \sigma(\lambda)$ for every λ. Since $\sigma(\lambda)$ is decreasing, we get $\sup\limits_{\lambda\in\mathbb{R}} \sigma(\lambda) = \lim\limits_{\lambda\to-\infty} \sigma(\lambda)$. Let $\lambda_k \to -\infty$ and $(u_k,v_k) \in \mathcal{J}$ realize $\sigma(\lambda_k)$. Then

$$\sigma(\lambda_k) = E_V(u_k) - \lambda_k M(u_k) \geq E_V(u_k) \geq C_1^{-1}(||u_k||^p - C_2),$$

for every $\lambda_k \leq 0$, where we used Lemma 3.2. A similar argument applies for v_k. Hence u_k and v_k are bounded. Going to a subsequence if necessary, there are u_0, v_0 such that

$$u_k \rightharpoonup u_0, v_k \rightharpoonup v_0, \quad u_k \to u_0, v_k \to v_0 \text{ a.e.}, \quad \|u_0\|_p = \|v_0\|_p = 1$$

and

$$M(u_0) = \lim_{k\to\infty} M(u_k), \quad N(v_0) = \lim_{k\to\infty} N(v_k).$$

Thus $(u_0,v_0) \in \mathcal{J}$ and

$$\alpha_2(m,n) \geq \lim_{k\to\infty} \sigma(\lambda_k) \geq E_V(u_0) - \lim_{k\to\infty} \lambda_k M(u_k),$$

so that $M(u_0) = 0$. In a similar way, we get $N(v_0) = 0$, so that $(u_0,v_0) \in \mathcal{K}$. Moreover

$$\alpha_2(m,n) \geq \lim_{k\to\infty} \sigma(\lambda_k) \geq \max(E_V(u_0), E_V(v_0)) \geq \alpha_2(m,n).$$

Therefore $\alpha_2(m,n) = \sup \sigma(\lambda)$ and (u_0,v_0) is a minimizing pair for $\alpha_2(m,n)$.

(3) First, note that, after L^p normalization and extension by zero to Ω, $(\varphi^+,\varphi^-) \in \mathcal{K}$, where φ is an eigenfunction associated to $\lambda_2(1,\Omega\setminus X)$. Hence

$$\alpha_2(m,n) \leq \max\{E_V(\varphi^+), E_V(\varphi^-)\} = \lambda_2(1,\Omega\setminus X).$$

On the other hand, as $\Omega\setminus X$ is p-stable, one can proceed as in [**7**, Prop.11] to show that if (u_0,v_0) achieves $\alpha_2(m,n)$ then $u_0, v_0 \in W_0^{1,p}(\Omega\setminus X)$, i.e., (u_0,v_0) is an admissible pair for $\lambda_2(1,\Omega\setminus X)$. Therefore we conclude that $\alpha_2(m,n) = \lambda_2(1,\Omega\setminus X)$.

□

Note that when $m^- \not\equiv 0$ or $n^- \not\equiv 0$, the decreasing of $\sigma(\lambda)$ is no longer known. As a consequence, we don't know whether $\sup_\lambda \sigma(\lambda) = \alpha_2(m,n)$. In order to overcome this lack of information, if there is no principal eigenvalue of (AP) we will assume that $\lambda_2(1) > 0$ (i.e., $\sigma(0) > 0$). This condition will provide the first positive eigenvalue of (AP).

In view of Prop. 4.3, a characterization of $c(m,n)$ can now be easily proved when $\alpha(m), \alpha(n) > 0$.

THEOREM 4.5. *Let m,n be such that $m^+,n^+ \not\equiv 0$ and $\min\{\alpha(m),\alpha(n)\} > 0$. The first non-trivial eigenvalue of (AP) is given by*

$$c(m,n) = \min_{(u,v)\in\mathcal{J}'} \max\{E_V(u), E_V(v)\}, \tag{4.7}$$

where

$$\mathcal{J}' = \{(u,v) \in (W_0^{1,p}(\Omega))^2;\ uv \equiv 0,\ M(u) = N(v) = 1\}. \tag{4.8}$$

Proof. Since $\sigma(\lambda) > \mu_1(\lambda, n)$ and $\mu_1(\lambda_1(n), n) = \mu_1(\lambda_1(m), m) = 0$, we have $\sigma(\lambda) > 0$ for $\lambda = \max\{\lambda_1(m), \lambda_1(n)\}$. Thus, as σ is continuous and goes to $-\infty$ when $\lambda \to \infty$, we find some $\overline{\lambda} > \max\{\lambda_1(m), \lambda_1(n)\}$ such that $\sigma(\overline{\lambda}) = 0$. We prove now that $\overline{\lambda}$ is equal to the minimax in (4.7). Let $(u_0, v_0) \in \mathcal{J}$ achieve $\sigma(\overline{\lambda}) = 0$, i.e.,

$$E_V(u_0) - \overline{\lambda} M(u_0) = E_V(v_0) - \overline{\lambda} N(v_0) = 0.$$

By Lemma 2.4, we have $M(u_0), N(v_0) > 0$. Moreover, given any $(u,v) \in \mathcal{J}'$, one has

$$E_V(u) - \overline{\lambda} M(u) \geq 0 \quad \text{or} \quad E_V(v) - \overline{\lambda} N(v) \geq 0,$$

so that

$$\overline{\lambda} \leq \max\{E_V(u), E_V(v)\}.$$

In fact, we proved that the minimax in (4.7) is the only zero of σ, which implies that it is the first non-trivial eigenvalue of (AP). □

We show now that in the ordered case, (AP) does not admit any eigenvalue between $\min\{\lambda_{-1}(m), \lambda_{-1}(n)\}$ and $\max\{\lambda_1(m), \lambda_1(n)\}$. This is no longer true if $\lambda_1(m), \lambda_1(n)$ are finite and non ordered.

Proposition 4.6. *If m, n change sign and $\lambda_1(m), \lambda_1(n)$ are ordered then (AP) has no non-principal eigenvalue λ such that*

$$\min\{\lambda_{-1}(m), \lambda_{-1}(n)\} < \lambda < \max\{\lambda_1(m), \lambda_1(n)\}.$$

Proof. Assume that λ is a non-principal eigenvalue of (AP) with

$$\min\{\lambda_{-1}(m), \lambda_{-1}(n)\} < \lambda < \max\{\lambda_1(m), \lambda_1(n)\}.$$

Then $\sigma = 0$ is a non-principal eigenvalue of (4.3). On the other hand, the first non-principal eigenvalue of (4.3) satisfies

$$\sigma(\lambda) > \max\{\mu_1(\lambda, m), \mu_1(\lambda, n)\} > 0,$$

and we get a contradiction. □

Remark 4.7. One can find V, m, n such that $\lambda_1(m)$, $\lambda_1(n)$ are non ordered and (AP) has a non-principal eigenvalue λ with

$$\min\{\lambda_1(m), \lambda_1(n)\} < \lambda < \max\{\lambda_{-1}(m), \lambda_{-1}(n)\}.$$

Indeed, let V be such that $\lambda_2(1) < 0$ (for instance, $V \equiv -V_0$, with $V_0 > \lambda_2$, the second eigenvalue of $-\Delta_p$). Now we take n_0 sign-changing and such that $\alpha(n_0^-) > 0$ (for instance, $n_0 = c_1 \mathbf{1}_{B_1} - c_2 \mathbf{1}_{B_2}$, where $c_1, c_2 > 0$ and B_1, B_2 are two disjoint balls with B_2 such that $\lambda_1(1, B_2) > 0$). We set $n_\varepsilon = \varepsilon n_0^+ - n_0^-$, so that, by lower semi-continuity, $\alpha(n_\varepsilon) > 0$ for $\varepsilon > 0$ sufficiently small. Thus we choose $n = n_\varepsilon$, and we can assume that $N(\varphi_1) < 0$, so that $0 < \lambda_{-1}(n) < \lambda_1(n)$. Moreover, it is clear that the above procedure also provides m such that $\lambda_{-1}(m) < \lambda_1(m) < 0$. Now, since $\sigma(\lambda_1(m)) > 0 > \lambda_2(1) = \sigma(0)$, by continuity, we get $\lambda \in]\lambda_1(m), 0[$ such that $\sigma(\lambda) = 0$.

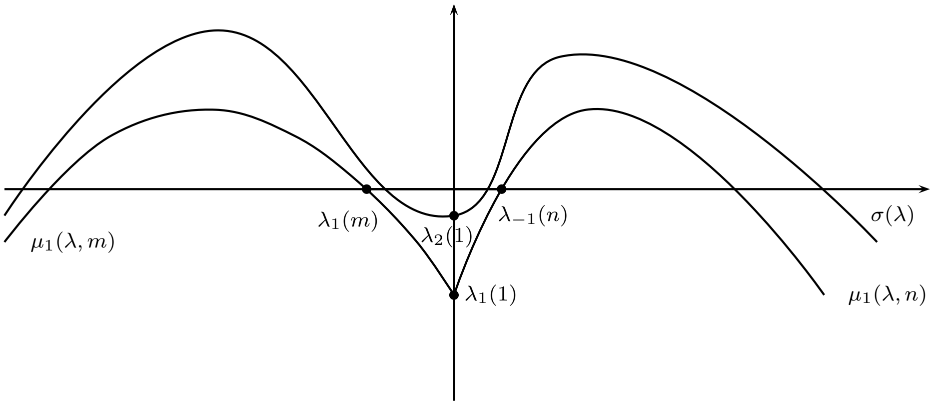

Figure 5: Existence of eigenvalues of (AP) between $\min\{\lambda_1(m), \lambda_1(n)\}$ and $\max\{\lambda_{-1}(m), \lambda_{-1}(n)\}$.

We shall now look for the first non-principal eigenvalue when either $\lambda_1(m) = -\infty$ or $\lambda_1(n) = -\infty$.

THEOREM 4.8. *Let m be such that $m^+ \not\equiv 0$.*

(1) *Let $n \geq 0$, $n \not\equiv 0$. If either $m \geq 0$ and $\alpha_2(m, n) > 0$ or m changes sign and $\alpha(m) \geq 0 \geq \alpha(n)$ then $\sigma(\overline{\lambda}) = 0$ for exactly one $\overline{\lambda} > \max\{\lambda_1(m), \lambda_1(n)\}$. Thus $c(m, n) = \overline{\lambda}$.*

(2) *Let n be such that $n^- \not\equiv 0$. If $n \leq 0$ assume $(H_{m,n})$ and $\alpha(n) \leq 0$.*
   (a) *If $\alpha(m) > 0$ (resp. $\alpha(m) \geq 0$) and $m \geq 0$ (resp. m changes sign) then $\sigma(\overline{\lambda}) = 0$ for at least one $\overline{\lambda} > \lambda_1(m)$. In this case,*

$$c(m, n) = \min\{\lambda > \lambda_1(m); \sigma(\lambda) = 0\}.$$

   (b) *If $\alpha(m) \leq 0$ (resp. $\alpha(m) < 0$), $m \geq 0$ (resp. m changes sign) and $\lambda_2(1) > 0$ then $\sigma(\overline{\lambda}) = 0$ for at least one $\overline{\lambda} > 0$, so that*

$$c(m, n) = \min\{\lambda > 0; \sigma(\lambda) = 0\}.$$

PROOF.

(1) We may assume, with no loss of generality, that $\max\{\lambda_1(m), \lambda_1(n)\} = \lambda_1(m)$ (which is the case if $\alpha(m) \geq 0 > \alpha(n)$). If $\lambda_1(m) > -\infty$ then $\sigma(\lambda_1(m)) > 0$. Hence, in any case, Prop. 4.4 and 4.3 yields that $\sigma(\lambda)$ vanishes at least once in $]\lambda_1(m), \infty[$. Assume that $\lambda_1(m) < \lambda' < \lambda''$ are such that $\sigma(\lambda') = \sigma(\lambda'') = 0$. Let $(u, v) \in \mathcal{J}$ realize $\sigma(\lambda')$, i.e., $E_{V-\lambda' m}(u) = E_{V-\lambda' n}(v) = 0$. We claim that $M(u) = N(v) = 0$. Indeed, let us first prove that $M(u) \geq 0$. If $m \geq 0$, there is nothing to prove. If m changes sign and $M(u) < 0$, then

$$-\lambda' = -\frac{E_V(u)}{M(u)} \geq -\lambda_{-1}(m) \geq -\lambda_1(m),$$

so that $\lambda' \leq \lambda_1(m)$, a contradiction. Therefore $M(u), N(v) \geq 0$ and so

$$\max\{E_{V-\lambda'' m}(u), E_{V-\lambda'' n}(v)\} \leq 0 = \sigma(\lambda'').$$

Thus (u, v) is a minimizing pair for $\sigma(\lambda'')$, so that

$$E_{V-\lambda'' m}(u) = E_{V-\lambda'' n}(v) = 0.$$

Consequently $M(u) = N(v) = 0$, i.e., $(u, v) \in \mathcal{K}$. Hence $E_V(u) > 0$ or $E_V(v) > 0$, which yields $E_{V-\lambda'' m}(u) > 0$ or $E_{V-\lambda'' n}(v) > 0$, a contradiction. Thus $\overline{\lambda} = c(m, n)$ is unique.

Let us now assume by contradiction that $\lambda < c(m, n)$ is a non-principal eigenvalue of (AP). Then $\sigma = 0$ is a non-principal eigenvalue of (4.3). On the other hand,

$$\max\{\mu_1(\lambda, m), \mu_1(\lambda, n)\} < 0 < \sigma(\lambda),$$

which is contrary to the fact that $\sigma(\lambda)$ is the first non-principal eigenvalue of (4.3).

(2)

(a) Since $\alpha_2(m, n) > \alpha(m) \geq 0$, one can argue as above to show that σ vanishes for at least one $\overline{\lambda} > \lambda_1(m)$. However, as $n^- \not\equiv 0$, we cannot prove that such $\overline{\lambda}$ is unique. If $\lambda \in\,]\lambda_1(m), c(m, n)[$ is an eigenvalue of (AP) and $\sigma(\lambda) \leq 0$, then σ vanishes for some value strictly less than $c(m, n)$, contrary to the definition of this one. Hence $\sigma(\lambda) > 0$ and we get a contradiction as before.

(b) Now there is no principal eigenvalue for (AP). The condition $\lambda_2(1) > 0$ implies $\sigma(0) > 0$. Since $\lim_{\lambda\to\infty} \sigma(\lambda) < 0$, $\sigma(\lambda)$ vanishes at least once in $]0, \infty[$ and we can prove as before that there is no positive eigenvalue less than $\min\{\lambda > \lambda_1(m); \sigma(\lambda) = 0\}$.

□

As a particular case of Prop.4.8, we obtain a condition on V and m for (P_m) to have a first non-principal eigenvalue in the absence of a principal one.

Corollary 4.9. *Let $m \geq 0$ be such that $\alpha_2(m) > 0 \geq \alpha(m)$, where $\alpha_2(m) := \alpha_2(m, m)$. Then (P_m) has a first (non-principal) eigenvalue, which is the unique zero of $\lambda \mapsto \sigma(\lambda, m, m)$.*

5. A first non-trivial Fučik curve

This section is devoted to the study of the non-trivial Fučik spectrum of (FP). We will restrict ourselves to Σ^+, the 'positive' part of Σ (as defined in (1.3)). Let us recall that $\lambda_1(m)$ and $\lambda_1(n)$ are not necessarily positive, not even necessarily finite, so that Σ^+ is not necessarily a subset of $\mathbb{R}^+ \times \mathbb{R}^+$.

5.1. The case $\min\{\alpha(m), \alpha(n)\} > 0$.

In the same way as [**2**], we will build a curve $\mathcal{C}$ made up of the points which are the first intersection of Σ^+ with a line of given slope. Instead of taking lines passing through the origin, we make a translation upon the point $(\lambda_1(m), \lambda_1(n))$: given $s > 0$, let l_s be the line defined by

$$B = B(A, s) := s(A - \lambda_1(m)) + \lambda_1(n), \quad \text{for } A \geq \lambda_1(m).$$

We look for the first point in l_s that belongs to Σ^+. This is equivalent to have

$$d(V - Am, V - B(A, s)n) = 0,$$

where d is defined in (4.2). So we are seeking the first zero of

$$\rho(A)=\rho(A,s):=d(V-Am,V-B(A,s)n),$$

defined for $A\geq\lambda_1(m)$.

Proposition 5.1. *Let m,n be such that $m^+,n^+\not\equiv 0$ and $\min\{\alpha(m),\alpha(n)\}>0$. For any $s>0$, l_s has a first intersection with Σ^+ given by $(A(s),B(s))$, where $B(s):=B(A(s),s)$,*

$$A(s)=A(s,m,n)=\min_{(u,v)\in\mathcal{J}'}\max\left\{E_V(u),\frac{1}{s}(E_V(v)-\lambda_1(n))+\lambda_1(m)\right\},\tag{5.1}$$

and $\mathcal{J}'$ is defined in (4.8).

Proof. As mentioned above, $(A(s),B(s))$ is the first cutting of l_s into Σ^+ provided $A(s)$ is the first zero of $\rho(A)=d(V-Am,V-B(A,s)n)$, defined for $A\geq\lambda_1(m)$. We have

$$\rho(\lambda_1(m))=d(V-\lambda_1(m)m,V-\lambda_1(n)n)>\lambda_1(V-\lambda_1(m)m,1).$$

One can easily see that $\lambda_1(V-\lambda_1(m)m,1)=0$, so that $\rho(\lambda_1(m))>0$. As $m^+,n^+\not\equiv 0$, $\rho(A)\to-\infty$ as $A\to\infty$. Consequently ρ vanishes at least once. Moreover, since $A\geq\lambda_1(m)$ and $B(A,s)\geq\lambda_1(n)$, if $\rho(A)=0$ and $(u,v)\in\mathcal{J}$ achieves $\rho(A)$, we can prove (as in Prop.4.8) that $M(u),N(v)>0$ and, as a consequence, that ρ vanishes exactly once. We set $A(s)$ as the unique zero of ρ and claim that

$$A(s)=\min_{(u,v)\in\mathcal{J}'}\max\left\{E_V(u),\frac{1}{s}(E_V(v)-\lambda_1(n))+\lambda_1(m)\right\}.\tag{5.2}$$

Indeed, as already observed, if (u,v) achieves $\rho(A(s))$ then $M(u),N(v)>0$, so that

$$A(s)=\frac{E_V(u)}{(M(u))^{1/p}}=\frac{1}{s}\left(\frac{E_V(v)}{(N(v))^{1/p}}-\lambda_1(n)\right)+\lambda_1(m),$$

and we deduce that $A(s)\geq c$, where c is the right-hand side in (5.2). Conversely, let (u,v) be such that $M(u)=N(v)=1$ with $uv\equiv 0$. From $\rho(A(s))=0$, we get

$$\max\{E_V(u)-A(s)M(u),E_V(v)-B(A(s),s)N(v)\}\geq 0.$$

Thus

$$E_V(u)-A(s)M(u)\geq 0 \text{ or } E_V(v)-B(A(s),s)N(v)\geq 0,$$

so that

$$A(s)\leq\max\left\{E_V(u),\frac{1}{s}(E_V(v)-\lambda_1(n))+\lambda_1(m)\right\},$$

which yields (5.1). □

Proposition 5.2. *Under the assumptions of Prop. 5.1, the maps $s\mapsto A(s),B(s)$ are continuous and respectively strictly decreasing and strictly increasing.*

Proof. The continuity and the strict decreasing of $A(s)$ follow easily from (5.1). It remains to prove that $B(s)$ is strictly increasing. Let us first show that

$$sA(s,m,n)=A(1,m/s,n)\quad\text{if}\quad s>0.$$

Indeed, if u is a solution of (AP) for $A=A(1,m/s,n)$ and $B=(A-\lambda_1(m/s))+\lambda_1(n)$, then the right-hand side of (AP) reads

$$A(1,m/s,n)m/s(u^+)^{p-1}-[A(1,m/s,n)-\lambda_1(m/s)+\lambda_1(n)]\,n(u^-)^{p-1}=$$

$$\frac{1}{s}A(1,m/s,n)m(u^+)^{p-1}-\left[s\left(\frac{1}{s}A(1,m/s,n)-\lambda_1(m)\right)+\lambda_1(n)\right]n(u^-)^{p-1},$$

where we used the well-known property $\lambda_1(m/s)=s\lambda_1(m)$.
We infer that

$$s^{-1}A(1,m/s,n)\geq A(s,m,n)$$

and likewise we show the reverse inequality. Hence we can write

$$B(s)=A(1,m/s,n)-\lambda_1(m/s)+\lambda_1(n).$$

Now let $0<s<s'$ and (u',v') achieve $A(1,m/s',n)$. Then $M(u')=s'$, $N(v')=1$ and

$$A(1,m/s',n)=E_V(u')=E_V(v')-\lambda_1(n)+\lambda_1(m/s').$$

We set $u=(\frac{s}{s'})^{\frac{1}{p}}u'$, so that $M(u)=s$ and $E_V(u)=\frac{s}{s'}A(1,m/s',n)$. Thus

$$A(1,m/s,n)\leq\max\left\{\frac{s}{s'}A(1,m/s',n),E_V(v')-\lambda_1(n)+\lambda_1(m/s)\right\},$$

so that

$$A(1,m/s,n)-\lambda_1(m/s)\leq\max\left\{\frac{s}{s'}A(1,m/s',n)-\lambda_1(m/s),E_V(v')-\lambda_1(n)\right\}.$$

Now, $E_V(v')-\lambda_1(n)=A(1,m/s',n)-\lambda_1(m/s')$ and

$$\begin{aligned}\frac{s}{s'}A(1,m/s',n)-\lambda_1(m/s) &= s\left(\frac{1}{s'}A(1,m/s',n)-\lambda_1(m)\right)\\ &< s'\left(\frac{1}{s'}A(1,m/s',n)-\lambda_1(m)\right)\\ &= A(1,m/s',n)-\lambda_1(m/s'),\end{aligned}$$

where we used that $A(1,m/s',n)>\lambda_1(m/s')$. Hence we conclude that

$$A(1,m/s,n)-\lambda_1(m/s)<A(1,m/s',n)-\lambda_1(m/s'),$$

so that $B(s)<B(s')$. □

Therefore, letting s vary in $]0,\infty[$, we get a curve

$$\mathcal{C}:=\{(A(s),B(s));s>0\}$$

in Σ^+, which is the first curve in Σ^+ in the sense that for any $(A,B)\in\Sigma^+$ either $A\geq A(s)$ or $B\geq B(s)$.

Next we study the asymptotic behavior of $\mathcal{C}$. Following the notation of [**2**], we set

$$A_0=\lim_{s\to0}A(s),\quad B_0=\lim_{s\to\infty}B(s),\quad A_\infty=\lim_{s\to\infty}A(s),\quad B_\infty=\lim_{s\to0}B(s),$$

$$\overline{A}:=\inf\{E_V(u^+):M(u^+)=1,N(u^-)>0\},$$

and

$$\overline{B}:=\inf\{E_V(u^-):N(u^-)=1,M(u^+)>0\}.$$

The following proposition extends the results proved in [**2**].

Proposition 5.3. *Under the assumptions of Prop. 5.1, there holds:*

(1) $A_0=B_0=\infty$.
(2) $A_\infty=\overline{A}$ *and* $B_\infty=\overline{B}$.

(3) *$\overline{A} = \lambda_1(m)$ if either $N \geq p$ or $N < p$ and $\overline{(supp\ n^+)} \cap \partial\Omega \neq \emptyset$. Otherwise $\overline{A} > \lambda_1(m)$. A similar statement holds for $\overline{B}$.*

PROOF.

(1) Let $s_k \to 0$ and $(u_k, v_k) \in \mathcal{J}'$ realize $A_k = A(s_k)$. We have

$$E_V(u_k) = A_k \quad \text{and} \quad E_V(v_k) = s_k(A_k - \lambda_1(m)) + \lambda_1(n).$$

If A_k is bounded then $E_V(u_k)$ is bounded and $E_V(v_k) \to \lambda_1(n)$, so that, by Lemma 3.2, u_k and v_k are bounded. Thus, up to a subsequence, there are u_0, v_0 such that

$$u_k \rightharpoonup u_0, v_k \rightharpoonup v_0, \quad M(u_0) = N(v_0) = 1, \quad \text{and} \quad u_0 v_0 \equiv 0.$$

Then $E_V(v_0) \leq \lambda_1(n)$ and we conclude that $v_0 = \pm\varphi_1(n)$. Hence $u_0 \equiv 0$, contradicting $M(u_0) = 1$. Therefore, we must have $A_k \to \infty$. A similar proof holds for B_0.

(2) Let u be an eigenfunction of (AP) associated to $(A(s), B(s))$, $s > 0$. Then

$$E_V(u^+) = A(s)M(u^+) \quad \text{and} \quad E_V(u^-) = B(s)N(u^-).$$

From $A(s) > \lambda_1(m)$ and $B(s) > \lambda_1(n)$, it follows once again that

$$M(u^+), N(u^-) > 0.$$

Set $\tilde{u} := \frac{u}{M(u^+)^{\frac{1}{p}}}$. Then $M((\tilde{u})^+) = 1$ and $N((\tilde{u})^-) > 0$, so that

$$A(s) = E_V(\tilde{u}^+) \geq \overline{A}.$$

Thus we proved that $A_\infty \geq \overline{A}$.
Assume now that $A_\infty > \overline{A}$. Then we can find w such that

$$M(w^+) = 1, \quad N(w^-) > 0 \quad \text{and} \quad \overline{A} < E_V(w^+) < A_\infty.$$

Moreover, by the decreasing of $A(s)$, one has $E_V(w^+) < A_\infty \leq A(s)$ for every $s > 0$. Since $\rho(A(s)) = 0$, given $(u, v) \in \mathcal{J}$, one has either

$$E_V(u) - A(s)M(u) \geq 0 \quad \text{or} \quad E_V(v) - B(s)N(v) \geq 0.$$

Let us choose, after L^p normalization, $u = w^+$ and $v = w^-$. We have

$$E_V(u) - A(s)M(u) = E_V(w^+) - A(s) < 0,$$

and consequently $E_V(w^-) \geq B(s)N(w^-)$. On the other hand, one has $E_V(w^+) > \lambda_1(m)$, since $M(w^+) = 1$ and $w^- \not\equiv 0$. Thus

$$\begin{aligned} B(s) &= s(A(s) - \lambda_1(m)) + \lambda_1(n) \\ &> s(E_V(w^+) - \lambda_1(m)) + \lambda_1(n) \to \infty \end{aligned}$$

as $s \to \infty$. So we get a contradiction, which shows that $A_\infty \leq \overline{A}$, and, as a consequence, $A_\infty = \overline{A}$. A similar argument shows that $B_\infty = \overline{B}$.

(3) The proof is similar to the second part of [**2**, Prop. 35], so we omit it.

□

5.2. The case $\alpha(m) > 0 \geq \alpha(n)$.

Now, since Σ^+ may be unbounded from below in the B direction, we allow the slope of the line to be negative. For each $s \in \mathbb{R}$, we denote by l_s the line defined by

$$B(A,s) := s(A - \lambda_1(m)), \quad A \geq \lambda_1(m).$$

Proposition 5.4. *Let m be such that $m^+ \not\equiv 0$.*

1. *Let $n \geq 0$, $n \not\equiv 0$ with $\alpha(n) \leq 0$. If either $\alpha(m) > 0$ and $m \geq 0$ or $\alpha(m) \geq 0$ and m changes sign then, for any $s > 0$ (resp. for any $s \leq 0$ if $(H_{m,n})$ holds), l_s has a first intersection with Σ^+ given by $(A(s), B(s))$, where $A(s)$ is the unique zero (resp. the least zero) of*

$$\rho(A) := d(V - Am, V - B(A,s)n)$$

   and $B(s) = B(A(s), s)$. Furthermore, $s \mapsto A(s)$ is decreasing, continuous on $]0, \infty[$ and lower semi-continuous on $]-\infty, 0[$.
2. *Let n be sign-changing with $\alpha(n) < 0$. If $\alpha(m) > 0$ (resp. $\alpha(m) \geq 0$) and $m \geq 0$ (resp. m changes sign) then, for any $s \in \mathbb{R}$, l_s has a first intersection with Σ^+ given by $(A(s), B(s))$, where $A(s)$ is the least zero of $\rho(A) := d(V - Am, V - B(A,s)n)$ and $B(s) = B(A(s), s)$.*

Proof.

1. We can proceed as in Prop.5.1 to show that for $s > 0$, ρ vanishes at least once. Now, as $A > \lambda_1(m)$ and $n \geq 0$, one can repeat the argument of Prop.4.8 to show that $M(u) > 0$ if (u, v) achieves $\rho(A) = 0$, so that ρ cannot vanish twice. If $s < 0$ then $B(A, s) < 0$ and we can argue as in Prop.4.8 to show that ρ has at least one zero. However, we cannot guarantee the uniqueness of such zero, since $s(A - \lambda_1(m))n \geq 0$.
   Let $s_k \to s_0$ and $A_k = A(s_k)$. We claim that A_k is bounded. Indeed, we may choose $A' > A(s_0)$ sufficiently large in order to have $\rho(A', s_0) < 0$. Thus, for k sufficiently large, $\rho(A', s_k) < 0$, so that $A_k \in]\lambda_1(m), A'[$. Now, up to a subsequence, there exists A such that $A_k \to A$, so that $\rho(a, s_0) = \lim \rho(A_k, s_k) = 0$. If $s_0 > 0$, by uniqueness we conclude that $A = A(s_0)$. If $s_0 < 0$ we may only infer that $A(s_0) \leq A$, i.e., $A(s_0) \leq \liminf A(s_k)$.
   In order to prove that $s \mapsto A(s)$ is decreasing, let $s' < s''$, $A' = A(s')$, $A'' = A(s'')$ and let (u, v) achieve $\rho(A', s') = 0$. As $n \geq 0$, it is clear that $s(A' - \lambda_1(m))N(v)$ is increasing with respect to s, so that

$$E_V(v) - s''(A' - \lambda_1(m))N(v) \leq 0.$$

   As a consequence, we get $\rho(A', s'') \leq 0$, which implies that $\rho(A, s'') = 0$ for some $A \leq A'$. Hence $A'' \leq A'$.
2. One can repeat the argument above to show that ρ vanishes at least once. Again, since n changes sign, we don't know whether uniqueness of zero holds for ρ.

□

We shall now consider

$$A_{-\infty} := \lim_{s \to -\infty} A(s), \quad B_{-\infty} := \lim_{s \to -\infty} B(s).$$

Let us also introduce

$$\underline{A} := \inf\{E_V(u^+) : M(u^+) = 1, N(u^-) \le 0, E_V(u^-) < 0\}.$$

PROPOSITION 5.5. *In addition to the assumptions of Prop. 5.4, if* n *is sign-changing assume either* $\overline{(\text{supp } m^+)} \cap \partial\Omega \neq \emptyset$ *or* $N \ge p$. *Then:*

1. $A_\infty = \overline{A} = \lambda_1(m)$ *if either* $N \ge p$ *or* $N < p$ *and* $\overline{(\text{supp } n^+)} \cap \partial\Omega \neq \emptyset$. *Otherwise,* $A_\infty \le \overline{A}$.
2. $A_{-\infty} \le \underline{A} < \infty$ *and* $B_{-\infty} = -\infty$.

PROOF.

1. One can check that, except the inequality $\overline{A} \le A_\infty$, the proof of (2),(3) in Prop.5.3 carries over to the present situation. Moreover, as $A_\infty \ge \lambda_1(m)$, the conclusion follows easily.
2. Let u be such that $M(u^+) = 1$, $N(u^-) \le 0$ and $E_V(u^-) < 0$. From $\rho(A(s)) = 0$, we find either

$$E_V(u^+) \ge A(s)M(u^+) \quad \text{or} \quad E_V(u^-) \ge B(s)N(u^-).$$

The latter inequality being excluded, we deduce that $A(s) \le E_V(u^+)$ for every $s < 0$, so that $A_{-\infty} \le \underline{A}$. Let us now prove that there exists u as above. For this purpose, we find a sequence v_k that vanishes is a positive measure subset of supp m^+ and is such that $N(v_k) \le 0$, $E_V(v_k) < 0$ for k sufficiently large.
If $n \ge 0$, we choose $v_k \ge 0$ such that v_k vanishes in a positive measure subset of supp m^+, $N(v_k) = 0$ and $v_k \to v$, where v is a minimizer for $\alpha(n)$. Such sequence exists by $(H_{m,n})$.
If n is sign-changing then any minimizer for $\alpha(n)$ is a solution of

$$-\Delta_p u + (V - \lambda n)|u|^{p-2}u = \alpha(n)|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega),$$

for some λ. Hence, it is positive on Ω. Now, since $\alpha(n) < 0$, we know that E_V is unbounded from below on $\mathcal{M}^-$ (see [**7**, Prop. 12]). To be more precise, if v is a minimizer for $\alpha(n)$ and $\varphi \ge 0$ is such that supp $\varphi \subset$ supp n^-, then

$$w_k := \frac{v + \frac{\varphi}{k}}{\left(-N(v + \frac{\varphi}{k})\right)^{\frac{1}{p}}}$$

satisfies $N(w_k) = -1$ and $E_V(w_k) \to -\infty$. We choose w such that $N(w) = -1$ and $E_V(w) < 0$, and we construct a sequence v_k such that $v_k \to w$ and v_k vanishes in a positive measure subset of supp m^+ (this is possible since we are assuming that $\overline{(\text{supp } m^+)} \cap \partial\Omega \neq \emptyset$ or $N \ge p$).
Once we obtained v_k, we may find $w \ge 0$ such that $M(w) = 1$ and $wv_k \equiv 0$. We choose then $u = w - v_k$, with k large enough. Therefore $\underline{A} < \infty$. Finally, for any $s < 0$, we have $A(s) > \lambda_1(m)$, so that $A_{-\infty} > \lambda_1(m)$ and $B(s) = s(A(s) - \lambda_1(m)) \to -\infty$ as $s \to -\infty$.

□

5.3. The case $\alpha(m), \alpha(n) \leq 0$.

To conclude this section, we briefly describe now the construction of $\mathcal{C}$ when neither (P_m) nor (P_n) admit a principal eigenvalue. We deal with non-negative weights m, n such that $\alpha(m), \alpha(n) \leq 0$. In this case, $\lambda_1(m) = \lambda_1(n) = -\infty$, so that $\Sigma = \Sigma^+$.

Let us denote by d_t the diagonal translated upon the point $(0, t)$, i.e., the line of equation

$$B = B(A, t) := A + t, \quad A \in \mathbb{R}.$$

Proposition 5.6.

1. *If* $\alpha_2(m, n) > 0$ *then, for every* $t \in \mathbb{R}$, d_t *has a first intersection with* Σ, *given by* $(A(t), B(t))$, *where* $A(t)$ *is the least zero of*

   $$\rho(A) = \rho(A, t) := d(V - Am, V - (A + t)n),$$

   defined for $A \in \mathbb{R}$, *and* $B(t) = B(A(t), t)$. *Moreover,* $t \mapsto A(t)$ *is decreasing and lower semi-continuous.*
2. $A(t) \leq \underline{A}$ *for every* $t \in \mathbb{R}$ *and* $\underline{A} < \infty$ *if* $(H_{m,n})$ *holds. A similar conclusion holds for* $B(t)$ *with respect to*

   $$\underline{B} := \inf\{E_V(u^-) : N(u^-) = 1, M(u^+) = 0, E_V(u^+) < 0\}.$$

   Finally, $\lim_{t \to \infty} A(t) = \lim_{t \to -\infty} B(t) = -\infty$.

Proof.

1. Since $m, n \geq 0$, it is clear that ρ is a decreasing map. Moreover, one can show that $\lim_{A \to \infty} \rho(A) = -\infty$ and $\sup_{A \in \mathbb{R}} \rho(A, t) = \alpha_2(m, n)$, for every $t > 0$. By continuity, ρ vanishes at least once. We set $A(t) := \min\{A; \rho(A, t) = 0\}$. If $(A, A + t) \in \Sigma$ with $A < A(t)$ then $\sigma = 0$ is a non-principal eigenvalue of

   $$-\Delta_p u + (V - Am)(u^+)^{p-1} - (V - (A + t)n)(u^-)^{p-1} = \sigma |u|^{p-2} u,$$

   so that $c(V - Am, V - (A + t)n) \leq 0$. By the definition of $A(t)$, we have $\rho(A, t) < 0$, so that ρ vanishes at some $A_0 < A(t)$, a contradiction. The decreasing of $t \mapsto A(t)$ follows from the dependence of $\rho(A, t)$ with respect to t, and the lower semi-continuity can be proved as in Prop.5.4-(a).
2. The proof is similar to Prop.5.4-(b), so we omit it.

□

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