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Beta functions of $U(1)^d$ gauge invariant just renormalizable tensor models

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This manuscript reports the first order β-functions of recently proved just renormalizable random tensor models endowed with a $U(1)^d$ gauge invariance. The models that we consider are polynomial Abelian φ_6^4 and φ_5^6 models. We show in this work that both models are asymptotically free in the UV.

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I. INTRODUCTION

Many interesting physical systems can be represented mathematically as random matrix problems. In particular, matrix models, celebrated in the 1980's, provide a unique and well-defined framework for addressing quantum gravity in two dimensions and its cortege of consequences on integrable systems [1]. The generalization of such models to higher dimensions is called random tensor models [2]. Recently, these tensor models have acknowledged a strong revival thanks to the discovery by Gurau of the analogue of the t'Hooft $1/N$-expansion for the tensor situation [3–7] and of tensor renormalizable actions [8–12]. The tensor model framework begins to take a growing role in the problem of quantum gravity and rises as a true alternative to several known approaches [13–15].

Tensorial group field theory (TGFT) [14,15] is a recent proposal for the same problem. It aims at providing a content to a phase transition called geometrogenesis scenario by relating a discrete quantum pregeometric phase of our spacetime to the classical continuum limit consistent with Einstein general relativity. In short, within this approach, our spacetime and its geometry has to be reconstructed or must emerge from more fundamental and discrete degrees of freedom.

Matrix models expand in graphs via ordinary perturbations of the Feynman path integral. These graphs can be seen as dual to triangulations of two dimensional surfaces. Here, the discrete degrees of freedom refer to matrices, or more appropriately to their indices, or dually to triangles which glue to form a discrete version of a surface. In tensor models, this idea generalizes. Feynman graphs in such tensor models are dual to triangulations of a D dimensional object. The tensor field possesses discrete indices and it is dually related to a basic D dimensional simplex which should be glued to others in order to form a discretization of a D dimensional manifold.

As for any quantum field theory, the question of renormalizability of TGFT has been addressed and solved under specific prescriptions [8–12]. Those conditions identify as the introduction of a Laplacian dynamics for the action kinetic term [16] and the use of nonlocal interaction of the tensor invariant form [17,18]. Furthermore, as another important feature, the UV asymptotic freedom of some TGFTs has been proved in 3D [9] and 4D [19] (see also [20] for a shorter summary). This is very encouraging for the geometrogenesis scenario. Indeed, the asymptotic freedom means that, after some scales towards the IR direction, the renormalized coupling constant of the theory starts to blow up and, certainly, this entails a phase transition toward new degrees of freedom. This is analogue of the asymptotic freedom of non-Abelian Yang Mills theory leading to the better understanding of the quark confinement. However, the new degrees of freedom in TGFTs have not yet been investigated.

New TGFT models, of the form of φ_6^4 and φ_5^6 theories, equipped with tensor fields obeying a gauge invariance condition were recently shown just renormalizable at all orders of perturbation [12]. The gauge-invariant condition on tensor fields will help for the emergence of a well-defined metric on the space after phase transition [10,13]. The renormalization of the model followed from a multiscale analysis and a generalized locality principle leading to a power-counting theorem [21].

In the present work, we calculate the first order β-function of both models and prove that these models are asymptotically free in the UV regime. This paper also emphasizes that this asymptotic freedom could be a generic feature of all TGFTs for a model with and without gauge invariance [15]. Such a feature will strengthen the status of TGFTs as pertinent candidates for gravity emergent scenario.

The paper is organized as follows. We recall in Sec. II the main results concerning the renormalizability of φ_6^4 and φ_5^6-tensor models as proved in [12]. Section III is devoted to the study of the one-loop β-function of the φ_6^4-model and Sec. IV addresses the computation of the same quantity, this time at higher order loops, for the φ_5^6-model. Finally, an appendix gathers technical points useful for the proof of our statements.

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II. ABELIAN TGFT WITH GAUGE INVARIANCE

This section addresses a summary of the results obtained in [12]. We mainly present the model and its renormalization.

TGFTs over a group G are defined by a complex field φ over d copies of group G, i.e.,

$$\varphi\colon G^d \to \mathbb{C} \qquad (g_1, \ldots, g_d) \mapsto \varphi(g_1, \ldots, g_d). \tag{1}$$

The gauge invariance condition [13] is achieved by imposing that the fields obey the relation

$$\varphi(hg_1, \ldots, hg_d) = \varphi(g_1, \ldots, g_d), \quad \forall\ h \in G. \tag{2}$$

For Abelian TGFTs, one fixes the group $G = U(1)$. In the momentum representation, the field writes

$$\varphi(g_1, \ldots, g_d) = \sum_p \varphi_{[p]} e^{ip_1\theta_1} e^{ip_2\theta_2} \cdots e^{ip_d\theta_d}, \quad \theta_k \in [0, 2\pi),$$

where we denote $\varphi_{[p]} = \varphi_{12\cdots d} := \varphi(p_1, p_2, \ldots, p_d)$, with $p_k \in \mathbb{Z}$ and $g_k = e^{i\theta_k} \in U(1)$.

The generalized locality principle of the TGFTs considered in [12] requires us to define the interactions as the sum of tensor invariants [3]. From now, we will focus on $d = 6, 5$, and define two models described by

$$\mathcal{S}_4[\bar\varphi, \varphi] = \sum_{p_1,\ldots,p_6} \bar\varphi_{123456}\,\delta\Big(\sum_i^6 p_i\Big)(p^2 + m^2)\varphi_{123456} + \frac{1}{2}\lambda_{4,1}^{(4)} V_{4,1}^6, \tag{3}$$

$$\mathcal{S}_6[\bar\varphi, \varphi] = \sum_{p_1,\ldots,p_5} \bar\varphi_{12345}\,\delta\Big(\sum_i^5 p_i\Big)(p^2 + m^2)\varphi_{12345} + \frac{1}{2}\lambda_{4,1}^{(6)} V_{4,1}^5 + \frac{1}{2}\lambda_{4,2} V_{4,2} + \frac{1}{3}\lambda_{6,1} V_{6,1} + \lambda_{6,2} V_{6,2}, \tag{4}$$

where $\delta(\sum_i^d p_i)$ should be understood as a Kronecker symbol $\delta_{\sum_i^d p_i, 0}$ and $p^2 = \sum_i^d p_i^2$, $d = 6, 5$, respectively, and where the interactions are of the form given by

$$V_{4,1}^6 = \sum_{\mathbb{Z}^{12}} \bar\varphi_{123456}\varphi_{12'3'4'5'6'}\bar\varphi_{1'2'3'4'5'6'}\varphi_{1'23456} + \text{permutations}, \tag{5}$$

$$V_{4,1}^5 = \sum_{\mathbb{Z}^{12}} \bar\varphi_{12345}\varphi_{12'3'4'5'}\bar\varphi_{1'2'3'4'5'}\varphi_{1'2345} + \text{permutations}, \tag{6}$$

$$V_{4,2} = \Big(\sum_{\mathbb{Z}^5} \bar\varphi_{12345}\varphi_{12345}\Big)^2, \tag{7}$$

$$V_{6,1} = \sum_{\mathbb{Z}^{15}} \bar\varphi_{12345}\varphi_{1'2345}\bar\varphi_{1'2'3'4'5'}\varphi_{1''2'3'4'5'}\bar\varphi_{1''2''3''4''5''}\varphi_{12''3''4''5''} + \text{permutations}, \tag{8}$$

$$V_{6,2} = \sum_{\mathbb{Z}^{15}} \bar\varphi_{54321}\varphi_{1'2345}\bar\varphi_{5'4'3'2'1'}\varphi_{1''2''3''4''5'}\bar\varphi_{1''2''3''4''5''}\varphi_{12'3'4'5''} + \text{permutations}. \tag{9}$$

The "permutations" are performed on the color indices. The vertices are graphically represented in Figs. 1 and 2. As one notices, there are two kinds of lines in the vertices. The first type are parametrized by $1, 2, \ldots, d$ and one external half-line without any number. Call by 0 the color of this half-line.

The propagator of each model reads:

$$C([p]) = \frac{1}{\sum_{i=1}^d p_i^2 + m^2}\,\delta\Big(\sum_{i=1}^d p_i\Big), \quad d = 6, 5, \tag{10}$$

and it is represented graphically as a line with d strands, see Fig. 3.

A Feynman graph is a graph composed with lines of color 0 (propagators) and vertices. Hence, whenever we refer to a line in the following it will be always a 0-color line and $\mathcal{G}$ is an uncolored tensor graph in the sense of [3,18] which have d-strand lines of color 0.

Let $\mathcal{L}$ and $\mathcal{F}$ be the sets of internal lines and faces of the graph $\mathcal{G}$. The multiscale analysis shows that the divergence

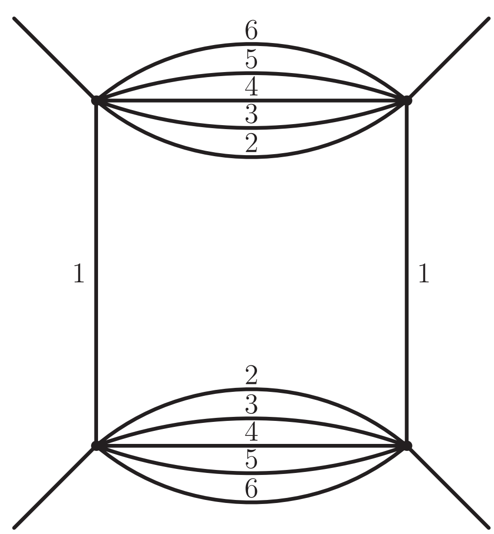

FIG. 1. Vertex representation of φ_6^4-model.

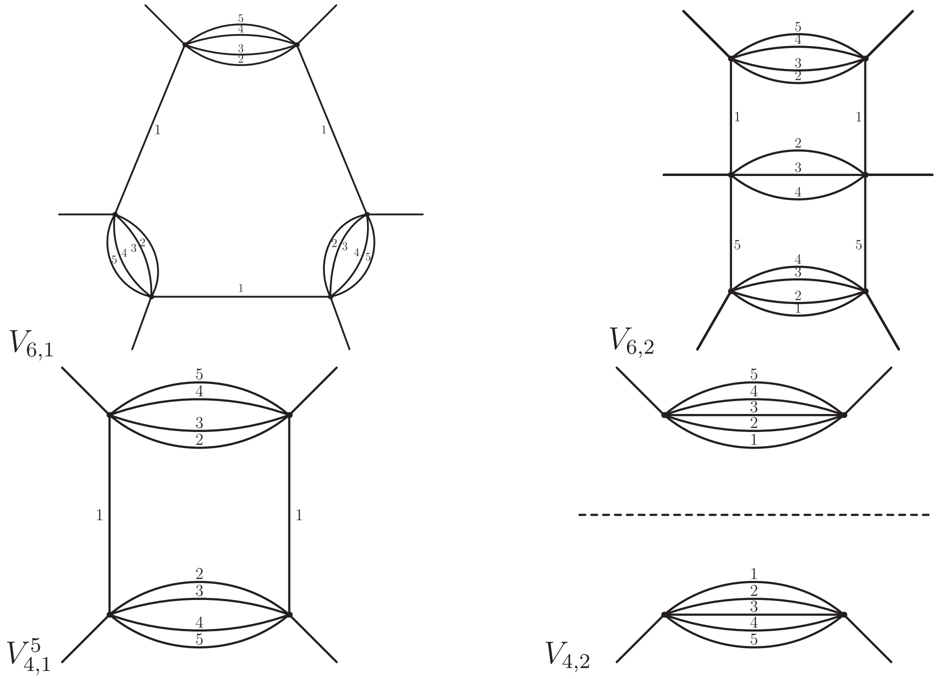

FIG. 2. Vertex representation of φ^6_5-model.

degree of the amplitude of a graph associated with both models can be written

$$\omega_d(\mathcal{G}) = 2L - F + R \tag{11}$$

where $L = |\mathcal{L}|$, $F = |\mathcal{F}|$, and R is the rank of matrix $(\epsilon_{lf}, l \in \mathcal{L}, f \in \mathcal{F})$, defined by

$$\epsilon_{lf}(\mathcal{G}) = \begin{cases} 1 & \text{if } l \in f \text{ and their orientation match,} \\ -1 & \text{if } l \in f \text{ and their orientation do not match,} \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

The following statement holds [12]:

Theorem 1. The models φ^4_6 defined by $\mathcal{S}_4$ and φ^6_5 defined by $\mathcal{S}_6$ are perturbatively renormalizable at all orders.

The proof of this statement rests on a power counting theorem which can be summarized by the following table giving the list of primitively divergent graphs (for precisions and notations, see [12]):

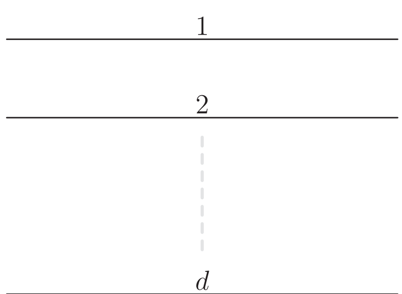

FIG. 3. Propagator of d-dimensional tensor model.

	N	$\omega(\mathcal{G})$	$\omega(\partial\mathcal{G})$	$C_{\partial\mathcal{G}} - 1$	$\omega_d(\mathcal{G})$
φ^4_6	4	0	0	0	0
	2	0	0	0	2
φ^6_5	6	0	0	0	0
	4	0	0	0	1
	4	0	0	1	0
	2	0	0	0	2
	2	0	0	0	1

(13)

Table: Divergent graphs of models φ^4_6 and φ^6_5

with N the number of external fields, $\omega(\mathcal{G})$ the degree of the colored extension of the graph $\mathcal{G}$, $\omega(\partial\mathcal{G})$ the degree of the colored extension of the boundary $\partial\mathcal{G}$ of the graph $\mathcal{G}$, $C_{\partial\mathcal{G}}$ the number of connected component of the boundary graph $\partial\mathcal{G}$.

Using this table, we are now in position to compute renormalized coupling equations.

III. ONE-LOOP β-FUNCTION OF φ^4_6-MODEL

This section is devoted to the one-loop evaluation of the β-function of φ^4_6. To proceed, we enlarge the space of coupling constants so that (3) becomes

$$\mathcal{S}_4[\bar\varphi, \varphi] = \sum_{p_1,\dots,p_6} \bar\varphi_{123456}\,\delta\Big(\sum_i^6 p_i\Big)(p^2 + m^2)\varphi_{123456} + \frac{1}{2}\sum_{\rho=1}^{6} \lambda_{4,1;\rho} V^6_{4,1;\rho}. \tag{14}$$

Only at the end will we perform a merging of all coupling at the same value $\lambda_{4,1;\rho} = \lambda_{4,1}$. Thus by introducing a distinction between the colors, $\rho = 1, 2, \ldots, 6$, the combinatorics becomes less involved.

We have the following theorem:

Theorem 2. At one-loop, the renormalized coupling constant associated with λ_4 is given by

$$\lambda_4^{\rm ren} = \lambda_4 + \frac{19\pi^2}{5\sqrt{5}}\lambda_4^2 I + O(\lambda_4^2), \quad \text{with} \qquad I = \int_0^\infty \frac{e^{-\alpha m^2}}{\alpha} d\alpha \tag{15}$$

such that the β-function of the model with single wave function renormalization and single coupling constant is given by $\beta = -\frac{19\pi^2}{5\sqrt{5}}$.

We now prove Theorem 2. Let Z be the wave function renormalization which writes:

$$Z = 1 - \frac{\partial^2}{\partial b_\rho^2}\Sigma|_{[b]=0}, \quad \rho = 1, 2, \ldots, 6, \tag{16}$$

where Σ is called the self-energy or the sum of all amputated one-particle irreducible (1PI) two-point functions which must be evaluated at one-loop. The derivative on Σ is with respect to an external argument. The β-function of the model φ_6^4 is encoded by the following quotient

$$\lambda_4^{\rm ren} = -\frac{\Gamma_4(0)}{Z^2}, \tag{17}$$

where Γ_4 is the sum of all amputated 1PI four-point functions computated at one-loop and at low external momenta that we symbolize by a unique argument (0).

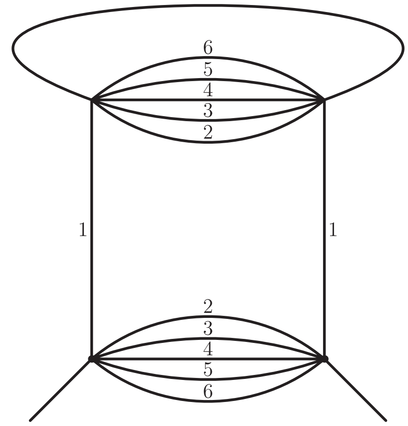

FIG. 4. Tadpole graphs T_1.

A. Self-energy and wave function renormalization

Having a look on (16) only is relevant the dependance in some color ρ of Σ. We will evaluate only this part in the self-energy at one-loop. For two sets of external arguments $[b]$ and $[b']$, one has

$$\Sigma([b],[b']) = \langle \bar\varphi_{[b]}\varphi_{[b']}\rangle^t_{\rm 1PI} = \sum_{\mathcal{G}} K_{\mathcal{G}} A_{\mathcal{G}}([b],[b']), \tag{18}$$

where $K_{\mathcal{G}}$ is a combinatorial factor and $A_{\mathcal{G}}$ is the amplitude of the graph $\mathcal{G}$. At one-loop, there exist six tadpole graphs T_ρ, $\rho = 1, \ldots, 6$, that contribute to the relation (18). For instance T_1 is represented in Fig. 4. We start by applying the Feynman rules to obtain the tadpole amplitude

$$\begin{aligned} A_{T_\rho} &= -\frac{\lambda_{4,\rho}}{2}\sum_{p_1,\ldots,p_5}\delta\Big(\sum_{i=1}^5 p_i + b_\rho\Big)\frac{1}{\sum_{i=1}^5 p_i^2 + b_\rho^2 + m^2} \\ &= -\frac{\lambda_{4,\rho}}{2}\sum_{p_1,\ldots,p_4}\frac{1}{\sum_{i=1}^4 p_i^2 + (\sum_{i=1}^4 p_i + b_\rho)^2 + b_\rho^2 + m^2} \\ &= -\frac{\lambda_{4,\rho}}{2}\sum_{p_1,\ldots,p_4}\frac{1}{\sum_{i=1}^4 + (\sum_{i=1}^4 p_i)^2 + 2b_\rho\sum_{i=1}^4 p_i + 2b_\rho^2 + m^2}. \end{aligned} \tag{19}$$

We denote this sum by

$$S^0(b) = \sum_{p_1,\ldots,p_4}\Bigg[\Big(\sum_{i=1}^4 p_i^2\Big) + \Big(\sum_{i=1}^4 p_i\Big)^2 + 2b\sum_{i=1}^4 p_i + 2b^2 + m^2\Bigg]^{-1} \tag{20}$$

and the amplitude of the tadpole T_ρ is given by

$$A_{T_\rho} = -\frac{\lambda_{4,\rho}}{2}S^0(b_\rho). \tag{21}$$

The combinatorial weight of these graphs T_ρ is $K_{T_\rho} = 2$. Also after introducing $S^0(b)$, an equation

$$\frac{\partial S^0(b)}{\partial b} = -\sum_{p_1,\ldots,p_4}\frac{2\sum_{i=1}^4 p_i + 4b}{\sum_{i=1}^4 + (\sum_{i=1}^4 p_i)^2 + 2b\sum_{i=1}^4 p_i + 2b^2 + m^2} \tag{22}$$

implies that

$$\frac{\partial^2 S^0(b)}{\partial b^2} = -4S^{01} + 8S^{02}, \tag{23}$$

where

$$S^{01}(b) = \sum_{p_1,\ldots,p_4}\bigg[\bigg(\sum_{i=1}^4 p_i^2\bigg) + \bigg(\sum_{i=1}^4 p_i\bigg)^2 + 2b\sum_{i=1}^4 p_i + 2b^2 + m^2\bigg]^{-2} \tag{24}$$

and

$$S^{02}(b) = \sum_{p_1,\ldots,p_4}\frac{(\sum_{i=1}^4 p_i + 2b)^2}{[\sum_{i=1}^4 p_i^2 + (\sum_{i=1}^4 p_i)^2 + 2b\sum_{i=1}^4 p_i + 2b^2 + m^2]^3}. \tag{25}$$

Then (18) is reexpressed as

$$\Sigma([b]) = -\sum_{\rho=1}^6 \lambda_{4,\rho} S^0(b_\rho). \tag{26}$$

We have the following relation (see Appendix A for details):

$$S^{01}(0) = \frac{\pi^2}{\sqrt{5}}I, \qquad S^{02}(0) = \frac{\pi^2}{5\sqrt{5}}I, \qquad I = \int_0^\infty d\alpha \frac{e^{-\alpha m^2}}{\alpha}, \tag{27}$$

then

$$\frac{\partial^2\Sigma[b]}{\partial b_\rho^2}\bigg|_{[b]=0} = 4\lambda_{4,\rho}(S^{01}(b_\rho) - 2S^{02}(b_\rho))|_{[b]=0} = \frac{12\pi^2}{5\sqrt{5}}\lambda_{4,\rho}I. \tag{28}$$

Using the fact that the tadpole amplitudes are symmetric with respect to the external variables, we reduce all coupling constants to the same value, i.e., $\lambda_{4,\rho} = \lambda_4$, and get the wave function renormalization as

$$Z = 1 - \frac{12\pi^2}{5\sqrt{5}}\lambda_4 I + O(\lambda_4^2). \tag{29}$$

B. Four-point functions

The 1PI four-point function amplitudes $\Gamma_{4,\rho}$, $\rho = 1, 2, \ldots, 6$, are given by

$$\Gamma_{4,\rho}([b],[b']) = \langle\bar\varphi_{[b]_1}\varphi_{[b]_2}\bar\varphi_{[b']_1}\varphi_{[b']_2}\rangle^t_{1\mathrm{PI}} = \sum_{\mathcal{G}} K_{\mathcal{G}} A_{\mathcal{G}}([b],[b']), \tag{30}$$

where $[b]_j$, $[b']_j$, $j = 1, 2$, are the external strand indices. Using the cyclic permutation over the six indices ρ, the four-point functions are explicitly given by

$$\Gamma_{4,1}(b_1,\ldots,b_6,b'_1,\ldots,b'_6) = \langle\bar\varphi_{123456}\varphi_{1'2'3'4'5'6}\bar\varphi_{1'2'3'4'5'6'}\varphi_{123456'}\rangle^t_{1\mathrm{PI}}, \tag{31}$$

$$\Gamma_{4,2}(b_1,\ldots,b_6,b'_1,\ldots,b'_6) = \langle\bar\varphi_{123456}\varphi_{1'2'3'4'56'}\bar\varphi_{1'2'3'4'5'6'}\varphi_{12345'6}\rangle^t_{1\mathrm{PI}}, \tag{32}$$

$$\Gamma_{4,3}(b_1,\cdots,b_6,b'_1,\cdots,b'_6) = \langle\bar\varphi_{123456}\varphi_{1'2'3'45'6'}\bar\varphi_{1'2'3'4'5'6'}\varphi_{1234'56}\rangle^t_{1\mathrm{PI}}, \tag{33}$$

$$\Gamma_{4,4}(b_1,\ldots,b_6,b'_1,\ldots,b'_6) = \langle\bar\varphi_{123456}\varphi_{1'2'34'5'6'}\bar\varphi_{1'2'3'4'5'6'}\varphi_{123'456}\rangle^t_{1\mathrm{PI}}, \tag{34}$$

$$\Gamma_{4,5}(b_1,\ldots,b_6,b'_1,\ldots,b'_6) = \langle\bar\varphi_{123456}\varphi_{1'23'4'5'6'}\bar\varphi_{1'2'3'4'5'6'}\varphi_{12'3456}\rangle^t_{1\mathrm{PI}}, \tag{35}$$

$$\Gamma_{4,6}(b_1,\ldots,b_6,b'_1,\ldots,b'_6) = \langle\bar\varphi_{123456}\varphi_{12'3'4'5'6'}\bar\varphi_{1'2'3'4'5'6'}\varphi_{1'23456}\rangle^t_{1\mathrm{PI}}. \tag{36}$$

At one-loop, there is a unique graph contributing to $\Gamma_{4;\rho}$. It is of the form given by Fig. 5. The combinatorial factor of this graph is always $K_{G_\rho} = 2\cdot 2\cdot 2$. Using the Feynman rules the amplitude of a four point graph with external indices $b_{1\rho}$ and $b_{2\rho}$ on the faces transmitted from left to right in Fig. 5 is

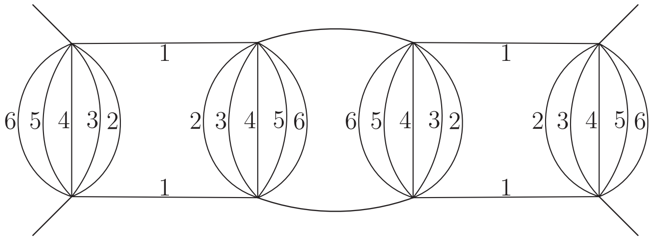

FIG. 5. The melonic one-loop four-point graph G_1.

$$A_{G_\rho}(b_{1\rho}, b_{2\rho}) = \frac{\lambda_{4,\rho}^2}{2^2.2} \sum_{p_1,\ldots,p_5} \delta\Big(\sum_{i=1}^5 p_i + b_{1\rho}\Big) \times \frac{1}{\sum_{i=1}^5 p_i^2 + b_{1\rho}^2 + m^2} \delta\Big(\sum_{i=1}^5 p_i + b_{2\rho}\Big) \times \frac{1}{\sum_{i=1}^5 p_i^2 + b_{2\rho}^2 + m^2} \tag{37}$$

which becomes, after setting $b_{1\rho} = b_{2\rho} = 0$,

$$A_{G_\rho}(0,0) = \frac{\lambda_{4,\rho}^2}{2^2.2} S^{01}(0). \tag{38}$$

We obtain

$$\Gamma_4(0) = -\lambda_4 + \lambda_4^2 S^{01}(0) + O(\lambda_4^2) = -\lambda_4 + \frac{\pi^2}{\sqrt{5}} \lambda_4^2 \mathcal{I} + O(\lambda_4^2). \tag{39}$$

The renormalizable coupling constant is finally given by

$$\lambda_4^{\rm ren} = -\frac{\Gamma_4(0,0)}{Z^2} = \lambda_4 + \frac{19\pi^2}{5\sqrt{5}} \lambda_4^2 \mathcal{I} + O(\lambda_4^2). \tag{40}$$

This result shows that the φ_6^4 model is asymptotically free in the UV regime. The β-function at one-loop of the model reads from (40):

$$\beta = -\frac{19\pi^2}{5\sqrt{5}}. \tag{41}$$

IV. TWO-LOOP β-FUNCTIONS OF THE φ_5^6-MODEL

In the φ_5^6-model, there are two types of coupling constants and so we must evaluate two renormalized coupling equations. In order to compute the β-functions of the φ_5^6 model it is important to note that the vertices of the type $V_{6,1}$ are parametrized by five indices $\rho = 1, 2, \ldots, 5$, and the vertices contributing to $V_{6,2}$ are parametrized by ten indices $\rho\rho' = 1.2, 1.3, 1.4, 1.\ 5, 2.3, 2.4, 2.5, 3.4, 3.5, 4.5$. The couple $\rho\rho'$ will be totally symmetric, i.e., $\rho\rho' = \rho'\rho$. For simplicity, the graphs of Fig. 6 represent henceforth the vertices of φ_5^6 model. For the same combinatorial reasons evoked above, we enlarge again the space of coupling and write (4) as

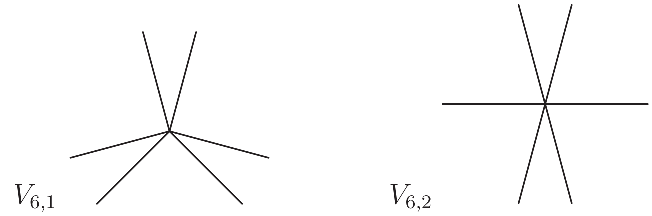

FIG. 6. New graphical representation of vertices $V_{6,1}$ and $V_{6,2}$.

$$\mathcal{S}_6[\bar\varphi, \varphi] = \sum_{p_1,\ldots,p_5} \bar\varphi_{12345} \delta\Big(\sum_i^5 p_i\Big)(p^2 + m^2)\varphi_{12345} + \frac{1}{3}\sum_\rho \lambda_{6,1;\rho} V_{6,1;\rho} + \sum_{\rho\rho'} \lambda_{6,2;\rho} V_{6,2;\rho\rho'} + \frac{1}{2}\sum_\rho \lambda_{4,1;\rho} V_{4,1;\rho} + \frac{1}{2}\sum_\rho \lambda_{4,2} V_{4,2}. \tag{42}$$

We have the following theorem:

Theorem 3. The renormalized coupling constants $\lambda_{6,1}^{\rm ren}$ and $\lambda_{6,2}^{\rm ren}$ satisfy the equations

$$\lambda_{6,1}^{\rm ren} = \lambda_{6,1} + \frac{9\pi^3}{4} \lambda_{6,1}^2 \mathcal{I}' + 12\Big(\frac{49}{31\sqrt{31}} + \frac{5}{8}\Big)\pi^3 \lambda_{6,1}\lambda_{6,2}\mathcal{I}' + O(\lambda_{6,1}^p \lambda_{6,2}^{3-p}), \tag{43}$$

and

$$\lambda_{6,2}^{\rm ren} = \lambda_{6,2} + 4\Big(\frac{178}{31\sqrt{31}} + \frac{11}{8}\Big)\pi^3 \lambda_{6,2}^2 \mathcal{I}' + \frac{11\pi^3}{4}\lambda_{6,1}\lambda_{6,2}\mathcal{I}' + O(\lambda_{6,1}^p \lambda_{6,2}^{3-p}), \tag{44}$$

$p = 0, 1, 2, 3$.

A. Self-energy and wave function renormalization

The following proposition holds:

Proposition 1. The wave function renormalization of the model is given by

$$Z = 1 - \frac{5\pi^3}{4}\lambda_{6,1}\mathcal{I}' - 4\Big(\frac{80}{31\sqrt{31}} + \frac{5}{8}\Big)\pi^3 \lambda_{6,2}\mathcal{I}' + O(\lambda_{6,1}^p \lambda_{6,2}^{2-p}), \tag{45}$$

$p = 0, 1, 2$, and where $\mathcal{I}'$ writes

$$\mathcal{I}' = \int_0^\infty \int_0^\infty d^2\alpha \frac{e^{-2\alpha m^2}}{\alpha^2}. \tag{46}$$

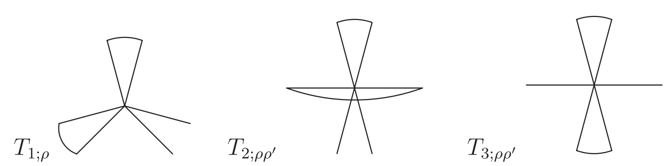

FIG. 7. Divergent tadpoles graphs of φ_5^6-model.

Proof: The graphs contributing to the self-energy are of the form listed in Fig. 7. The amplitude corresponding to the tadpoles graphs $T_{1;\rho}$ with the external strand index b_ρ is given by the following relation

$$A_{T_{1;\rho}}(b_\rho) = -\frac{\lambda_{6,1;\rho}}{3}\sum_{p_1,\ldots,p_4}\frac{\delta(\sum_{i=1}^4 p_i + b_\rho)}{\sum_{i=1}^4 p_i^2 + b_\rho^2 + m^2}\sum_{q_1,\ldots,q_4}\frac{\delta(\sum_{i=1}^4 q_i + b_\rho)}{\sum_{i=1}^4 q_i^2 + b_\rho^2 + m^2} = -\frac{\lambda_{6,1;\rho}}{3}S^1(b_\rho), \tag{47}$$

where

$$S^1(b) = \sum_{\substack{p_1,p_2,p_3\\q_1,q_2,q_3}}\left\{\left[\sum_{k=1}^3 p_k^2 + \left(\sum_{k=1}^3 p_k\right)^2 + 2b\sum_{k=1}^3 p_k + 2b^2 + m^2\right]^{-1}\left[\sum_{k=1}^3 q_k^2 + \left(\sum_{k=1}^3 q_k\right)^2 + 2b\sum_{k=1}^3 q_k + 2b^2 + m^2\right]^{-1}\right\}. \tag{48}$$

Using the combinatorial number associated to the tadpole graph $T_{1;\rho}$ given by $K_{T_{1;\rho}} = 3$, the sum of 1PI two-point functions are given by

$$\Omega_{6,1}(b_\rho) = 3A_{T_{1;\rho}}(b_\rho). \tag{49}$$

Similarly, the amplitude corresponding to the tadpole graphs $T_{2;1\rho}$ and $T_{3;1\rho}$ are respectively given by relations

$$\begin{aligned}A_{T_{2;1\rho}}(b_1) &= -\lambda_{6,2;1\rho}\sum_{p_1,\ldots,p_5}\sum_{q_1,\ldots,q_4}\left[\frac{\delta(\sum_{i=1}^5 p_i)}{\sum_{i=1}^5 p_i^2 + m^2}\delta(p_5 - q_4)\frac{\delta(\sum_{i=1}^4 q_i + b_1)}{\sum_{i=1}^4 q_i^2 + b_1^2 + m^2}\right]\\ &= -\lambda_{6,2;1\rho}\sum_{\substack{p_1,\ldots,p_4\\q_1,q_2,q_3}}\left\{\left(\frac{1}{\sum_{i=1}^4 p_i^2 + (\sum_{i=1}^4 p_i)^2 + m^2}\right)\left(\frac{\delta(\sum_{i=1}^3 q_i + \sum_{i=1}^4 p_i + b_1)}{\sum_{i=1}^3 q_i^2 + (\sum_{i=1}^4 p_i)^2 + b_1^2 + m^2}\right)\right\} = -\lambda_{6,2;1\rho}S^{12}(b_1),\end{aligned} \tag{50}$$

and

$$A_{T_{3;1\rho'}}(b_1, b_\rho) = -\lambda_{6,2;1\rho}\sum_{p_1,\ldots,p_4}\frac{\delta(\sum_{i=1}^4 p_i + b_1)}{\sum_{i=1}^4 p_i^2 + b_1^2 + m^2}\sum_{q_1,\ldots,q_4}\frac{\delta(\sum_{i=1}^4 q_i + b_\rho)}{\sum_{i=1}^4 q_i^2 + b_\rho^2 + m^2} = -\lambda_{6,2;1\rho}S^{13}(b_1, b_\rho), \tag{51}$$

where

$$\begin{aligned}S^{12}(b) = \sum_{\substack{p_1,p_2,p_3,p_4\\q_1,q_2}}&\left\{\left(\sum_{k=1}^4 p_k^2 + \left(\sum_{k=1}^4 p_k\right)^2 + m^2\right)^{-1}\left(\sum_{k=1}^2 q_k^2 + \left(\sum_{k=1}^2 q_k\right)^2 + 2b\sum_{k=1}^2 q_k + 2\left(\sum_{k=1}^4 p_k\right)^2 - 2b\sum_{k=1}^4 p_k\right.\right.\\ &\left.\left.- 2\sum_{k=1}^4 p_k\sum_{k=1}^2 q_k + 2b^2 + m^2\right)^{-1}\right\},\end{aligned} \tag{52}$$

and

$$S^{13}(b, b') = \sum_{\substack{p_1,p_2,p_3\\q_1,q_2,q_3}}\left\{\left[\sum_{k=1}^3 p_k^2 + \left(\sum_{k=1}^3 p_k\right)^2 + 2b\sum_{k=1}^3 p_k + 2b^2 + m^2\right]^{-1}\left[\sum_{k=1}^3 q_k^2 + \left(\sum_{k=1}^3 q_k\right)^2 + 2b'\sum_{k=1}^3 q_k + 2b'^2 + m^2\right]^{-1}\right\}. \tag{53}$$

The combinatorial factors are $K_{T_{2;1\rho}} = 1$ and $K_{T_{3;1\rho}} = 1$. Therefore the sum of these contributions yield

$$\Omega_{6,2}(b_1, b_\rho) = A_{T_{2;1\rho}}(b_1) + A_{T_{3,1\rho}}(b_1, b_\rho). \tag{54}$$

Combining the relations (49) and (54), we get a part of the self-energy involving the variable b_1

$$\begin{aligned}\Sigma_6(b_1, b_\rho) = 3A_{T_{1;1}}(b_1) + \sum_\rho[A_{T_{2;1\rho}}(b_1) + A_{T_{3;1\rho}}(b_1, b_\rho)]\\ + O(\lambda_{6,1}^p\lambda_{6,2}^{2-p}).\end{aligned} \tag{55}$$

The wave function renormalization of the model is given by

$$Z = 1 - \frac{\partial^2}{\partial b_1^2}\Sigma_6(b_1, b_\rho)|_{b_1=b_\rho=0}. \tag{56}$$

Using Appendix B, we have the following relations:

$$\begin{aligned}\frac{\partial^2}{\partial b_1^2}\Omega_{6,1}(b_1)|_{b_1=0} &= \frac{5\pi^3}{4}\lambda_{6,1;1}I',\\ I' &= \int_0^\infty\int_0^\infty d^2\alpha\frac{e^{-2\alpha m^2}}{\alpha^2}\end{aligned} \tag{57}$$

$$\frac{\partial^2}{\partial b_1^2}\Omega_{6,2}(b_1,b_\rho)|_{b_1=b_\rho=0}=\left(\frac{80}{31\sqrt{31}}+\frac{5}{8}\right)\pi^3\lambda_{6,2,1\rho}I'. \quad (58)$$

We restrict from now the coupling constants in each sector such that $\lambda_{6,1;\rho}=\lambda_{6,1}$ and $\lambda_{6,2;\rho\rho'}=\lambda_{6,2}$ so that the wave function renormalization is

$$Z=1-\frac{5\pi^3}{4}\lambda_{6,1}I'-4\left(\frac{80}{31\sqrt{31}}+\frac{5}{8}\right)\pi^3\lambda_{6,2}I' + O(\lambda_{6,1}^p\lambda_{6,2}^{2-p}), \quad (59)$$

$p=0,1,2.$ □

B. Six-point functions

The initial calculation of the six-point functions shows that they prolifer quickly [19]. However, in the present gauge invariant model, which is more constrained, several of these should be nonrenormalized because either are convergent (pay attention to the fact that gauge invariant models are less divergent than the ordinary one) or turn out to violate the face-connectedness condition [see discussion below and (9)] [10,12].

At the end, we will focus on the six-point functions which are face-connected graphs of type $V_{6,1}-V_{6,1}$ and $V_{6,1}-V_{6,2}$, see Fig. 8. This will be used for the calculation of the sum of 1PI six-point functions $\Gamma_{6,1;\rho}$ and $\Gamma_{6,2;\rho\rho'}$. The renormalized coupling constant equations for $\lambda^{\rm ren}_{6,1;1}$ and $\lambda^{\rm ren}_{6,2;1\rho'}$ are defined by

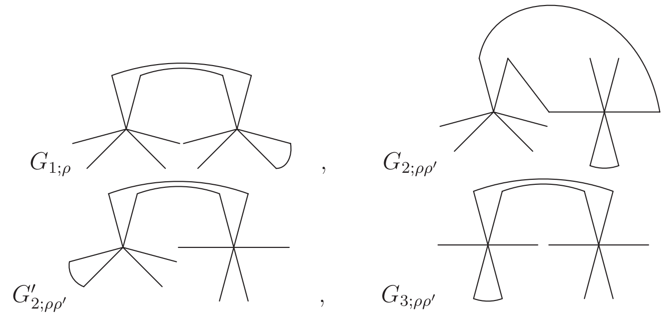

FIG. 8. Face-connected divergent six-point graphs of φ_5^6-model.

$$\lambda^{\rm ren}_{6,1;\rho}=-\frac{\Gamma_{6,1;\rho}(0,0)}{Z^3},\qquad \lambda^{\rm ren}_{6,2;\rho\rho'}=-\frac{\Gamma_{6,2;\rho\rho'}(0,0)}{Z^3}. \quad (60)$$

Proof of Theorem 3

The first part of this proof is about the evaluation of amplitudes of various graphs of Fig. 8. We introduce some formal sums:

$$S^3=\sum_{\substack{p_1,p_2,p_3\\q_1,q_2,q_3}}\left(\sum_{k=1}^3 p_k^2+\left(\sum_{k=1}^3 p_k\right)^2+m^2\right)^{-2}\left(\sum_{k=1}^3 q_k^2+\left(\sum_{k=1}^3 q_k\right)^2+m^2\right)^{-1},$$

$$S^{31}=\sum_{\substack{p_1,p_2,p_3\\q_1,q_2,q_3}}\left(\sum_{k=1}^3 p_k^2+\left(\sum_{k=1}^3 p_k\right)^2+m^2\right)^{-2}\left(\sum_{k=1}^3 q_k^2+\left(\sum_{k=1}^3(p_k-q_k)\right)^2+\left(\sum_{k=1}^3 p_k\right)^2+m^2\right)^{-1},$$

$$=\sum_{\substack{p_1,p_2\\q_1,q_2,q_3,q_4}}\left(\sum_{k=1}^2 p_k^2+\left(\sum_{k=1}^4 q_k-\sum_{k=1}^2 p_k\right)^2+\left(\sum_{k=1}^4 q_k\right)^2+m^2\right)^{-2}\left(\sum_{k=1}^4 q_k^2+\left(\sum_{k=1}^4 q_k\right)^2+m^2\right)^{-1}. \quad (61)$$

A calculation yields, at low external momenta,

$$A_{G_{1;\rho}}(0,\ldots,0)=\frac{\lambda_{6,1;\rho}^2}{3^2\times 2!}K_{G_{1;\rho}}S^3=3\cdot 2.\lambda^2_{6,1;\rho}S^3, \quad (62)$$

$$A_{G_{2;\rho\rho'}}(0,\ldots,0)=\frac{1}{3}\lambda_{6,1;\rho}\sum_{\rho'}\lambda_{6,2;\rho\rho'}K_{G_{2;\rho\rho'}}S^{31}=3\lambda_{6,1;\rho}\left[\sum_{\rho'\neq\rho}\lambda_{6,2;\rho\rho'}\right]S^{31}, \quad (63)$$

$$A_{G'_{2;\rho\rho'}}(0,\ldots,0)=\frac{1}{3}(\lambda_{6,1;\rho}+\lambda_{6,1;\rho'})\lambda_{6,2;\rho\rho'}K_{G'_{2;\rho\rho'}}S^3=2(\lambda_{6,1;\rho}+\lambda_{6,1;\rho'})\lambda_{6,2;\rho\rho'}S^3, \quad (64)$$

$$A_{G_{3;\rho\rho'}}(0,\ldots,0)=\lambda_{6,2;\rho\rho'}\left[\sum_{\tilde\rho\neq\rho}\lambda_{6,2;\rho\tilde\rho}+\sum_{\tilde\rho\neq\rho'}\lambda_{6,2;\rho'\tilde\rho}\right](S^3+S^{31}), \quad (65)$$

$$K_{G_{1;\rho}}=3^3\cdot 2^2,\qquad K_{G_{2;\rho\rho'}}=3\cdot 3,\qquad K_{G'_{2;\rho\rho'}}=3\cdot 2. \quad (66)$$

The contributions to $\Gamma_{6,1;\rho}$ are obtained from $G_{1;\rho}$ and $G_{2;\rho\rho'}$. Using these, we get

$$\Gamma_{6,1;\rho}(0,\ldots,0) = -\lambda_{6,1;\rho} + \lambda_{6,1;\rho}\Big[6\lambda_{6,1;\rho}S^3 + 3\Big[\sum_{\rho'\neq\rho}\lambda_{6,2;\rho\rho'}\Big]S^{31}\Big] + O(\lambda_{6,1}^p\lambda_{6,2}^{3-p}). \tag{67}$$

The contributions to $\Gamma_{6,2;\rho\rho'}$ are obtained from $G'_{2;\rho\rho'}$ and $G_{3;\rho\rho'}$. One finds

$$\Gamma_{6,2;\rho\rho'}(0,\ldots,0) = -\lambda_{6,2;\rho\rho'} + 2(\lambda_{6,1;\rho} + \lambda_{6,1;\rho'})\lambda_{6,2;\rho\rho'}S^3 + \lambda_{6,2;\rho\rho'}\Big[\sum_{\tilde\rho\neq\rho}\lambda_{6,2;\rho\tilde\rho} + \sum_{\tilde\rho\neq\rho'}\lambda_{6,2;\rho'\tilde\rho}\Big](S^3 + S^{31}) + O(\lambda_{6,1}^p\lambda_{6,2}^{3-p}) \tag{68}$$

Reducing to the smaller space of couplings $\lambda_{6,1;\rho} = \lambda_{6,1}$ and $\lambda_{6,2;\rho\rho'} = \lambda_{6,2}$, we get

$$\Gamma_{6,1}(0,\ldots,0) = -\lambda_{6,1} + 6\lambda_{6,1}^2S^3 + 12\lambda_{6,1}\lambda_{6,2}S^{31} + O(\lambda_{6,1}^p\lambda_{6,2}^{3-p}),$$
$$\Gamma_{6,2}(0,\ldots,0) = -\lambda_{6,2} + 8\lambda_{6;2}^2(S^3 + S^{31}) + 4\lambda_{6,2}\lambda_{6,1}S^3 + O(\lambda_{6,1}^p\lambda_{6,2}^{3-p}). \tag{69}$$

Asymptotically, we can obtain the relation

$$S^3 = \frac{\pi^3}{4}I', \qquad S^{31} = \frac{\pi^3}{\sqrt{31}}I' \tag{70}$$

(see Appendix B for more detail). At one-loop the renormalized coupling constant $\lambda_{6,1}^{\text{ren}}$ and $\lambda_{6,2}^{\text{ren}}$ are given by

$$\lambda_{6,1}^{\text{ren}} = \lambda_{6,1} + \frac{9\pi^3}{4}\lambda_{6,1}^2I' + 12\Big(\frac{49}{31\sqrt{31}} + \frac{5}{8}\Big)\pi^3\lambda_{6,1}\lambda_{6,2}I' + O(\lambda_{6,1}^p\lambda_{6,2}^{3-p}), \tag{71}$$

and

$$\lambda_{6,2}^{\text{ren}} = \lambda_{6,2} + 4\Big(\frac{178}{31\sqrt{31}} + \frac{11}{8}\Big)\pi^3\lambda_{6,2}^2I' + \frac{11\pi^3}{4}\lambda_{6,1}\lambda_{6,2}I' + O(\lambda_{6,1}^p\lambda_{6,2}^{3-p}). \tag{72}$$

□

C. Discussion

(i) Let us come back to the subtle issue about the notion of connectedness in this theory. The correct notion of connectedness should be the one of face connectedness. Several graphs which *a priori* are divergent should not renormalize any coupling constant. For instance, graphs of the form given in Fig. 9 are face-disconnected divergent six-point graphs. They do not contribute to the 1PI six-point functions. The amplitudes of the graphs are

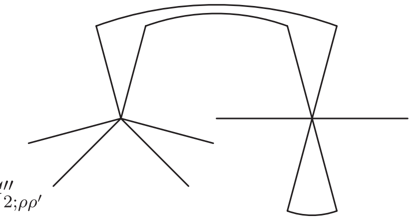

FIG. 9. Face-disconnected and divergent six-point graphs of φ_5^6-model.

$$A_{G''_{2;\rho\rho'}}(0,\ldots,0) = \frac{1}{3}\lambda_{6,1;\rho}\sum_{\rho'}\lambda_{6,2;\rho\rho'}K_{G''_{2;\rho\rho'}}S^3 = 3\lambda_{6,1;\rho}\Big[\sum_{\rho'\neq\rho}\lambda_{6,2;\rho\rho'}\Big]S^3, \tag{73}$$

$$K_{G''_{2;\rho\rho'}} = 3\cdot 3. \tag{74}$$

(ii) We now discuss the results of Theorem 3. Equation (71) can be reexpressed as

$$\lambda_{6,1}^{\text{ren}} = \lambda_{6,1} - \beta_1\lambda_{6,1}^2I' - \beta_{12}\lambda_{6,1}\lambda_{6,2}I' + O(\lambda_{6,1}^p\lambda_{6,2}^{3-p}), \tag{75}$$

where, at this order of perturbation, the β-function splits into coefficients β_1 and β_{12} given by

$$\beta_1 = -\frac{9\pi^3}{4}, \qquad \beta_{12} = -12\Big(\frac{49}{31\sqrt{31}} + \frac{5}{8}\Big)\pi^3. \tag{76}$$

This clearly shows that $\lambda_{6,1}^{\text{ren}} \geq \lambda_{6,1}$ proving that this sector is asymptotically free, provided all couplings are positive. In the same way, Eq. (72) can be reexpressed as

$$\lambda_{6,2}^{\text{ren}} = \lambda_{6,2} - \beta_2\lambda_{6,2}^2I' - \beta_{21}\lambda_{6,1}\lambda_{6,2}I' + O(\lambda_{6,1}^p\lambda_{6,2}^{3-p}) \tag{77}$$

where the β-functions β_2 and β_{21} are given by

$$\beta_2 = -4\Big(\frac{178}{31\sqrt{31}} + \frac{11}{8}\Big)\pi^3, \qquad \beta_{21} = -\frac{11\pi^3}{4}. \tag{78}$$

The same conclusion holds for the sector $\lambda_{6,2}$ which is asymptotically free. Both relations (76) and (78) show that the model with both interactions is asymptotically free in the UV regime. Hence, gauge invariant TGFT models of the form presented here make sense at arbitrary small scales yielding, far in the UV, a theory of noninteracting spheres. Indeed, according to [3], all interactions presented here (called melonic interactions) are dual with simplicial topological sphere. The present results also show that both models might experience a phase transition when the renormalized coupling constants become larger and larger in the IR. This feature deserves full investigation.

(iii) We will now discuss renormalized coupling constants $\lambda_{4,1}^{\rm ren}$ and $\lambda_{4,2}^{\rm ren}$. We have already shown that, at high scale, the bare values of coupling constants $\lambda_{6,1}$ and $\lambda_{6,2}$ vanish. Further the divergent four-point functions must not have more than one vertex type $V_{4,1}$, or $V_{4,2}$, the only divergent graphs are those couples with $V_{6,1}$ or $V_{6,2}$. Using relations (71) and (72) we come to the conclusion that

$$\lambda_{4,1}^{\rm ren} = \lambda_{4,1} + O(\lambda_{4,1}^{p}\lambda_{6,k}^{3-p}), \quad k = 1 \text{ or } k = 2, \tag{79}$$

$$\lambda_{4,2}^{\rm ren} = \lambda_{4,2} + O(\lambda_{4,2}^{p}\lambda_{6,k}^{3-p}), \quad k = 1 \text{ or } k = 2. \tag{80}$$

Then the φ^4 sector is safe at all loops and the β-functions are given by

$$\beta_{4,1} = \beta_{4,2} = 0. \tag{81}$$

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APPENDIX A: DIVERGENT SERIES FOR φ_6^4-MODEL

Propostion 2. Let $I = \int_0^\infty d\alpha \frac{e^{-\alpha m^2}}{\alpha}$ be a logarithmically divergent quantity in the UV regime. The series $S^{01}(0)$ and $S^{02}(0)$ asymptotically write as

$$S^{01}(0) = \frac{\pi^2}{\sqrt{5}} I, \qquad S^{02}(0) = \frac{\pi^2}{5\sqrt{5}} I. \tag{A1}$$

The rest of this section is devoted to the proof of this proposition. Let us recall the Schwinger formula: Let A be a positive define operator and n is an integer then we get

$$\frac{1}{A^{n+1}} = \frac{1}{n!}\int_0^\infty d\alpha \alpha^n e^{-\alpha A}. \tag{A2}$$

For $A = \sum_{k=1}^4 p_k^2 + (\sum_{k=1}^4 p_k)^2 + m^2$, we arrive at expression

$$\begin{aligned}\sum_{[p_{14}]\in\mathbb{Z}^4} \frac{1}{A^2} &= \lim_{\Lambda\to 0}\lim_{\Lambda'\to 0}\sum_{[p_{14}]}^{\Lambda}\int_{\Lambda'}^\infty d\alpha \alpha^n e^{-\alpha A}\\ &= \lim_{\Lambda'\to 0}\int_{\Lambda'}^\infty d\alpha\alpha^n \lim_{\Lambda\to 0}\sum_{[p_{14}]}^{\Lambda} e^{-\alpha A}\\ &= \int_0^\infty d\alpha \alpha e^{-\alpha m^2}\sum_{[p_{14}]\in\mathbb{Z}^4} e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_ip_j]},\end{aligned} \tag{A3}$$

$|p_{14}|^2 = \sum_{k=1}^4 p_k^2$, $[p_{ij}] = (p_i, p_{i+1}, \ldots, p_j)$, $i < j$. We have the following lemma

Lemma 1. Let $-\infty < p < \infty$. For $n \to \infty$, uniformly in any finite interval of positive β, we get

$$\sum_{p=-\infty}^{\infty} e^{-\frac{\beta}{n}p^2} = \sqrt{\frac{n\pi}{\beta}}. \tag{A4}$$

Proof. The proof of this lemma is given in [22]. □

Noting that in the previous lemma $\frac{\beta}{n} \to 0$ as $\alpha = M^{-2i} \to 0$. Then $\sum_{p=-\infty}^{\infty} e^{-\alpha p^2} = \sqrt{\frac{\pi}{\alpha}}$. Then

$$\begin{aligned}&\sum_{[p_{14}]\in\mathbb{Z}^4} e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_ip_j]}\\ &= \sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}\sqrt{\frac{3\pi}{4\alpha}}\sqrt{\frac{4\pi}{5\alpha}}\\ &= \frac{\pi^2}{\alpha^2\sqrt{5}}.\end{aligned} \tag{A5}$$

We arrive at the expression

$$\begin{aligned}&\int_0^\infty d\alpha\alpha e^{-\alpha m^2}\sum_{[p_{14}]\in\mathbb{Z}^4} e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_ip_j]}\\ &= \frac{\pi^2}{\sqrt{5}}\int_0^\infty d\alpha \frac{e^{-\alpha m^2}}{\alpha} = \frac{\pi^2}{\sqrt{5}} I.\end{aligned} \tag{A6}$$

Finally $S^{01}(0) = \frac{\pi^2}{\sqrt{5}} I$. Using the same argument, we get

$$\begin{aligned}
\frac{(\sum_{k=1}^4 p_k)^2}{A^3} &= \frac{1}{2}\int_0^\infty d\alpha\alpha^2 e^{-\alpha m^2}\sum_{[p_{14}]\in\mathbb{Z}^4}\Big(|p_{14}|^2 + 2\sum_{i=1,i<j}^4 p_i p_j\Big)e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_i p_j]} \\
&= \frac{1}{2}\int_0^\infty d\alpha\alpha^2 e^{-\alpha m^2}\sum_{[p_{14}]\in\mathbb{Z}^4}\Big(|p_{14}|^2 + \sum_{i=1,i<j}^4 p_i p_j\Big)e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_i p_j]} \\
&\quad + \frac{1}{2}\int_0^\infty d\alpha\alpha^2 \sum_{[p_{14}]\in\mathbb{Z}^4}\sum_{i=1,i<j}^4 p_i p_j e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_i p_j]} \\
&= -\frac{1}{4}\int_0^\infty d\alpha\alpha^2 e^{-\alpha m^2}\frac{\partial}{\partial\alpha}\sum_{[p_{14}]\in\mathbb{Z}^4} e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_i p_j]} \\
&\quad + \frac{1}{2}\int_0^\infty d\alpha\alpha^2 e^{-\alpha m^2}\sum_{[p_{14}]\in\mathbb{Z}^4}\sum_{i=1,i<j}^4 p_i p_j e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_i p_j]} \\
&= X_1 + X_2,
\end{aligned} \tag{A7}$$

with

$$X_1 = -\frac{1}{4}\int_0^\infty d\alpha\alpha^2 e^{-\alpha m^2}\frac{\partial}{\partial\alpha}\sum_{[p_{14}]\in\mathbb{Z}^4} e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_i p_j]}, \tag{A8}$$

$$X_2 = \frac{1}{2}\int_0^\infty d\alpha\alpha^2 e^{-\alpha m^2} \times \sum_{[p_{14}]\in\mathbb{Z}^4}\sum_{i=1,i<j}^4 p_i p_j e^{-2\alpha[|p_{14}|^2+\sum_{i=1,i<j}^4 p_i p_j]}, \tag{A9}$$

and we get

$$X_1 = -\frac{1}{4}\int_0^\infty d\alpha\alpha^2 e^{-\alpha m^2}\frac{\partial}{\partial\alpha}\Big(\frac{\pi^2}{\alpha^2\sqrt{5}}\Big) = \frac{\pi^2}{2\sqrt{5}}I. \tag{A10}$$

To compute X_2 let us give the following lemma

Lemma 2. Let $-\infty < p < \infty$. For $\alpha \to 0$ uniformly in any finite interval of constant c, we get

$$\begin{aligned}
\sum_{p=-\infty}^{\infty} p e^{-\alpha p^2+2cp} &= \frac{c}{\alpha}\sqrt{\frac{\pi}{\alpha}}e^{\frac{\alpha^2}{c}}, \\
\sum_{p=-\infty}^{\infty} p^n e^{-\alpha p^2+2cp} &= \frac{1}{2^{n-1}\alpha}\sqrt{\frac{\pi}{\alpha}}\frac{d}{dc}(ce^{\frac{\alpha^2}{c}}).
\end{aligned} \tag{A11}$$

Using this lemma we get easily

$$X_2 = \frac{1}{2}\int_0^\infty d\alpha\alpha^2 e^{-\alpha m^2}\Big(-\frac{3\pi^2}{5\alpha^3\sqrt{5}}\Big) = -\frac{3\pi^2}{10\sqrt{5}}I. \tag{A12}$$

Therefore, $S^{02}(0) = \frac{\pi^2}{5\sqrt{5}}I$.

APPENDIX B: DIVERGENT SERIES FOR φ_5^6-MODEL

In this section, we will focus on the divergent terms of the φ_5^6-model. Let us consider the functions $\Omega_{6,1}(b)$ and $\Omega_{6,2}(b,b')$. The partial derivative of $\Omega_{6,1}(b)$ and $\Omega_{6,2}(b,b')$ respect to external strand b participated to the expression of the wave function. The goal of this part is the proof of the following proposition.

Proposition 3. Let $I' = \int_0^\infty\int_0^\infty d^2\alpha\frac{e^{-2\alpha m^2}}{\alpha^2}$ be a logarithmically divergent quantity in the UV regime. The partial derivative of $\Omega_{6,1}(b)$ and $\Omega_{6,2}(b,b')$ are respectively given by

$$\frac{\partial^2}{\partial b^2}\Omega_{6,1}(b)|_{b=0} = \frac{5\pi^3}{4}\lambda_{6,1;1}I', \tag{B1}$$

$$\frac{\partial^2}{\partial b^2}\Omega_{6,2}(b,b')|_{b=b'=0} = \Big[\frac{80}{31\sqrt{31}} + \frac{5}{8}\Big]\pi^3\lambda_{6,2;1\rho'}I'. \tag{B2}$$

The rest of this section is devoted to the proof of the above proposition. We have

$$\frac{\partial^2}{\partial b_1^2}\Omega_{6,1}(b)|_{b=0} = 8\lambda_{6,1;1}\Big\{\sum_{\substack{p_1,p_2,p_3\\q_1,q_2,q_3}}\Big[\Big(\frac{1}{\chi^2_{(3)}(p)} - 2\frac{(\sum p_k)^2}{\chi^3_{(3)}(p)}\Big)\Big] \times\Big[\frac{1}{\chi_{(3)}(q)}\Big] - \frac{\sum p_k}{\chi^2_{(3)}(p)}\frac{\sum q_k}{\chi^2_{(3)}(q)}\Big\}, \tag{B3}$$

where $\chi_{(n)}(p) = \sum_{k=1}^n p_k^2 + (\sum_{k=1}^n p_k)^2 + m^2$. By using the Schwinger formula (A2), we find

$$\begin{aligned}\sum_{[p]\in\mathbb{Z}^3}\sum_{[q]\in\mathbb{Z}^3}\left(\frac{1}{\chi^2_{(3)}(p)}\frac{1}{\chi_{(3)}(q)}\right) &= \int_0^\infty\int_0^\infty \alpha d\alpha d\beta \sum_{[p]\in\mathbb{Z}^3} e^{-\alpha\chi_{(3)}(p)}\sum_{[q]\in\mathbb{Z}^3} e^{-\beta\chi_{(3)}(q)}\\ &= \int_0^\infty\int_0^\infty \alpha e^{-\alpha m^2}e^{-\beta m^2}d\alpha d\beta\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}\sqrt{\frac{3\pi}{4\alpha}}\sqrt{\frac{\pi}{2\beta}}\sqrt{\frac{2\pi}{3\beta}}\sqrt{\frac{3\pi}{4\beta}}\\ &= \frac{\pi^3}{4}\int_0^\infty\int_0^\infty \alpha\frac{e^{-\alpha m^2}}{\alpha^{\frac{3}{2}}}\frac{e^{-\beta m^2}}{\beta^{\frac{3}{2}}}d\alpha d\beta\\ &= \frac{\pi^3}{4}\int_0^\infty\int_0^\infty d^2\alpha\frac{e^{-2\alpha m^2}}{\alpha^2}\\ &= \frac{\pi^3}{4}\mathcal{I}'. \end{aligned}\tag{B4}$$

In the same manner, we get

$$\sum_{\substack{[p]\in\mathbb{Z}^3\\ [q]\in\mathbb{Z}^3}}\frac{(\sum p_k)^2}{\chi^3_{(3)}(p)}\frac{1}{\chi_{(3)}(q)} = \frac{1}{2}\int_0^\infty\int_0^\infty \alpha^2 e^{-\alpha m^2}e^{-\beta m^2}\mathcal{P}(\alpha,\beta)d\alpha d\beta, \tag{B5}$$

where

$$\mathcal{P}(\alpha,\beta) = \sum_{[p]\in\mathbb{Z}^3}\left(\sum p_k\right)^2 e^{-\alpha(\chi_{(3)}(p)-m^2)}\sum_{[q]\in\mathbb{Z}^3} e^{-\beta(\chi_{(3)}(q)-m^2)}$$

can be computed by using the following results:

$$\sum_{[p]\in\mathbb{Z}^3}\left(\sum p_k\right)^2 e^{-\alpha(\chi_{(3)}(p)-m^2)} = \frac{3}{8\alpha}\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}\sqrt{\frac{3\pi}{4\alpha}}, \tag{B6}$$

$$\sum_{[q]\in\mathbb{Z}^3} e^{-\beta(\chi_{(3)}(q)-m^2)} = \sqrt{\frac{\pi}{2\beta}}\sqrt{\frac{2\pi}{3\beta}}\sqrt{\frac{3\pi}{4\beta}}. \tag{B7}$$

We get $\sum_{\substack{[p]\in\mathbb{Z}^3\\ [q]\in\mathbb{Z}^3}}\frac{(\sum p_k)^2}{\chi^3_{(3)}(p)}\frac{1}{\chi_{(3)}(q)} = \frac{3\pi^3}{64}\mathcal{I}$. A simple routine checking shows that

$$\sum\frac{p_k}{\chi^2_{(3)}(p)}\sum\frac{q_k}{\chi^2_{(3)}(q)} = 0. \tag{B8}$$

Finally,

$$\frac{\partial^2}{\partial b^2}\Omega_{6,1}(b)|_{b=0} = \frac{5\pi^3}{4}\lambda_{6,1;1}\mathcal{I}.$$

The second-order partial derivative of $\Omega_{6,2}(b,b')$ with respect to the external strand b is written as

$$\begin{aligned}\frac{\partial^2}{\partial b^2}\Omega_{6,2}(b,b')|_{b=b'=0} = 4\lambda_{6,2;1\rho'}\Bigg\{&\sum_{\substack{p_1,p_2,p_3,p_4\\ q_1,q_2}}\frac{1}{\chi_{(4)}(p)}\left[\frac{1}{\chi^2_{(2,4)}(q,p)} - 2\frac{(\sum_{k=1}^2 q_k - \sum_{k=1}^4 p_k)^2}{\chi^3_{(2,4)}(q,p)}\right]\\ &+\sum_{\substack{p_1,p_2,p_3\\ q_1,q_2,q_3}}\left[\frac{1}{\chi^2_{(3)}(p)} - 2\frac{(\sum p_k)^2}{\chi^3_{(3)}(p)}\right]\frac{1}{\chi_{(3)}(q)}\Bigg\},\end{aligned} \tag{B9}$$

where

$$\chi_{(m,n)}(q,p) = \sum_{k=1}^m q_k^2 + \left(\sum_{k=1}^m q_k\right)^2 + 2\left(\sum_{k=1}^n p_k\right)^2 - 2\sum_{k=1}^n p_k\sum_{k=1}^m q_k + m^2.$$

Let us compute the series $\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\frac{1}{\chi^2_{(2,4)}(q,p)}\frac{1}{\chi_{(4)}(p)}$ and $\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\frac{1}{\chi_{(4)}(p)}\frac{(\sum_{k=1}^2 q_k - \sum_{k=1}^4 p_k)^2}{\chi^3_{(2,4)}(q,p)}$. Using the Schwinger formula we can write that

$$\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\frac{1}{\chi^2_{(2,4)}(q,p)}\frac{1}{\chi_{(4)}(p)}=\int_0^\infty\int_0^\infty \alpha e^{-\alpha m^2}e^{-\beta m^2}d\alpha d\beta\,\mathcal{Q}(\alpha,\beta), \tag{B10}$$

where

$$\mathcal{Q}(\alpha,\beta)=\sum_{[p]\in\mathbb{Z}^4}\sum_{[q]\in\mathbb{Z}^2}e^{-\alpha(\chi_{(2,4)}(q,p)-m^2)}e^{-\beta(\chi_{(4)}(p)-m^2)}.$$

Now by lemma 2, we reach

$$\sum_{[q]\in\mathbb{Z}^2}e^{-\alpha(\chi_{(2,4)}(q,p)-m^2)}=\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}e^{-\frac{4}{3}\alpha(\sum_k p_k)^2}. \tag{B11}$$

Moreover, $\mathcal{Q}(\alpha,\beta)$ is given by

$$\mathcal{Q}(\alpha,\beta)=\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}\sum_{[p]\in\mathbb{Z}^4}e^{-(\beta+\frac{4}{3}\alpha)(\sum_k p_k)^2-\beta|p_{14}|^2}=\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}\sqrt{\frac{\pi}{a}}\sqrt{\frac{\pi}{a'}}\sqrt{\frac{\pi}{a''}}\sqrt{\frac{\pi}{a'''}}, \tag{B12}$$

where

$$\begin{aligned} &a=2\beta+\frac{4}{3}\alpha, \qquad b=\beta+\frac{4}{3}\alpha, \qquad a'=a-\frac{b^2}{a}, \qquad b'=-b+\frac{b^2}{a},\\ &a''=a'-\frac{b'^2}{a'}, \qquad b''=b'+\frac{b'^2}{a'}, \qquad a'''=a''-\frac{b''^2}{a''}. \end{aligned} \tag{B13}$$

Then for $\alpha=\beta$ we get

$$a=\frac{10\alpha}{3}, \qquad a'=\frac{17\alpha}{10}, \qquad a''=\frac{24\alpha}{17}, \qquad a'''=\frac{31\alpha}{24}, \tag{B14}$$

and

$$\mathcal{Q}(\alpha,\alpha)=\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}\sqrt{\frac{3\pi}{10\alpha}}\sqrt{\frac{10\pi}{17\alpha}}\sqrt{\frac{17\pi}{24\alpha}}\sqrt{\frac{24\pi}{31\alpha}}.$$

Finally,

$$\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\frac{1}{\chi^2_{(2,4)}(q,p)}\frac{1}{\chi_{(4)}(p)}=\frac{\pi^3}{\sqrt{31}}\int_0^\infty\int_0^\infty d^2\alpha\frac{e^{-2\alpha m^2}}{\alpha^2}=\frac{\pi^3}{\sqrt{31}}\mathcal{I}'. \tag{B15}$$

$$\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\frac{1}{\chi_{(4)}(p)}\frac{(\sum_{k=1}^2 q_k-\sum_{k=1}^4 p_k)^2}{\chi^3_{(2,4)}(q,p)}=\frac{1}{2}\int_0^\infty\int_0^\infty d^2\alpha\,\alpha^2e^{-2\alpha m^2}\mathcal{R}(\alpha,\alpha), \tag{B16}$$

where $\mathcal{R}(\alpha,\alpha)=\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}(\sum_{k=1}^2 q_k-\sum_{k=1}^4 p_k)^2e^{-\alpha(\chi_{(2,4)}(q,p)-m^2)}e^{-\alpha(\chi_{(4)}(p)-m^2)}$ This quantity can be written as

$$\begin{aligned}\mathcal{R}(\alpha,\alpha)&=-\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\frac{\partial}{\partial\alpha}(e^{-\alpha(\chi_{(2,4)}(q,p)-m^2)})e^{-\alpha(\chi_{(4)}(p)-m^2)}\\ &\quad-\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\Big(|q_{12}|^2+\Big(\sum_k p_k\Big)^2\Big)e^{-\alpha(\chi_{(2,4)}(q,p)-m^2)}e^{-\alpha(\chi_{(4)}(p)-m^2)}\\ &=-\sum_{p\in\mathbb{Z}^4}\frac{\partial}{\partial\alpha}\Big(\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}e^{-\frac{4}{3}\alpha(\sum_k p_k)^2}\Big)e^{-\alpha(|p_{14}|^2+(\sum_k p_k)^2)}\\ &\quad-\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\Big(|q_{12}|^2+\Big(\sum_k p_k\Big)^2\Big)e^{-\alpha(\chi_{(2,4)}(q,p)-m^2)}e^{-\alpha(\chi_{(4)}(p)-m^2)}\\ &=\mathcal{R}_1+\mathcal{R}_2,\end{aligned} \tag{B17}$$

where

$$\mathcal{R}_1 = -\sum_{p\in\mathbb{Z}^4} \frac{\partial}{\partial\alpha}\left(\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}e^{-\frac{4}{3}\alpha(\sum_k p_k)^2}\right)e^{-\alpha(|p_{14}|^2+(\sum_k p_k)^2)}$$

and

$$\mathcal{R}_2 = -\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\left(|q_{12}|^2 + \left(\sum_k p_k\right)^2\right)e^{-\alpha(\chi_{(2,4)}(q,p)-m^2)}e^{-\alpha(\chi_{(4)}(p)-m^2)}.$$

The additional contribution $\mathcal{R}_1$ can be evaluated as

$$\begin{aligned}\mathcal{R}_1 &= -\sum_{p\in\mathbb{Z}^4} \frac{\partial}{\partial\alpha}\left(\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}e^{-\frac{4}{3}\alpha(\sum_k p_k)^2}\right)e^{-\alpha(|p_{14}|^2+(\sum_k p_k)^2)}\\ &= \sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}\left[\frac{1}{\alpha}\sum_{p\in\mathbb{Z}^4} e^{-\frac{7}{3}\alpha(\sum_k p_k)^2-\alpha|p_{14}|^2} + \frac{4}{3}\sum_{p\in\mathbb{Z}^4}\left(\sum_k p_k\right)^2 e^{-\frac{7}{3}\alpha(\sum_k p_k)^2-\alpha|p_{14}|^2}\right]\\ &= \mathcal{R}_{11} + \mathcal{R}_{12}.\end{aligned} \tag{B18}$$

In the above expression

$$\mathcal{R}_{11} = \frac{1}{\alpha}\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}\sqrt{\frac{3\pi}{10\alpha}}\sqrt{\frac{10\pi}{17\alpha}}\sqrt{\frac{17\pi}{24\alpha}}\sqrt{\frac{24\pi}{31\alpha}} = \frac{\pi^3}{\alpha^4\sqrt{31}}$$

and

$$\mathcal{R}_{12} = \frac{4}{3}\sqrt{\frac{\pi}{2\alpha}}\sqrt{\frac{2\pi}{3\alpha}}\sum_{p\in\mathbb{Z}^4}\left(\sum_k p_k\right)^2 e^{-\frac{7}{3}\alpha(\sum_k p_k)^2-\alpha|p_{14}|^2}.$$

We also have

$$U = \sum_{p\in\mathbb{Z}^4}\left(\sum_k p_k\right)^2 e^{-\frac{7}{3}\alpha(\sum_k p_k)^2-\alpha|p_{14}|^2} = \frac{18\pi^2}{31\alpha^3\sqrt{93}}. \tag{B19}$$

Therefore, $\mathcal{R}_{12} = \frac{8\pi^3}{31\alpha^4\sqrt{31}}$ and then $\mathcal{R}_1 = \frac{39\pi^3}{31\alpha^4\sqrt{31}}$. Using the same above argument, we can prove that $\mathcal{R}_2 = -\frac{28\pi^3}{31\alpha^4\sqrt{31}}$. Finally, it is straightforward to check following relation

$$\sum_{p\in\mathbb{Z}^4}\sum_{q\in\mathbb{Z}^2}\frac{1}{\chi_{(4)}(p)}\frac{(\sum_{k=1}^2 q_k - \sum_{k=1}^4 p_k)^2}{\chi^3_{(2,4)}(q,p)} = \frac{11\pi^3}{62\sqrt{31}}I'. \tag{B20}$$

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