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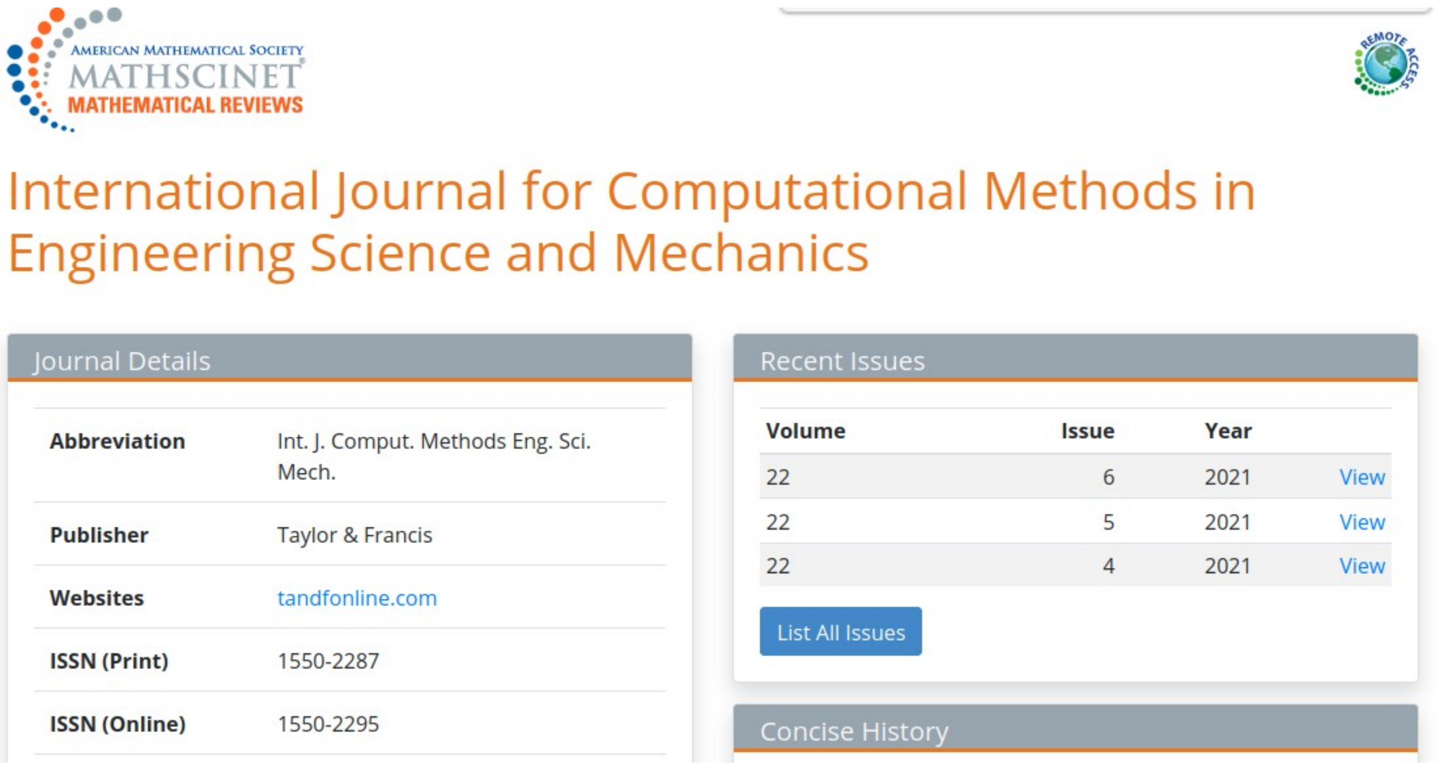

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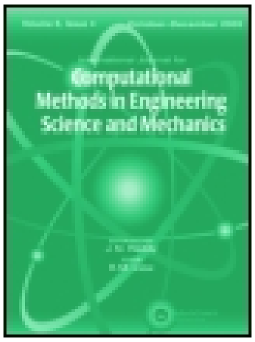

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Parameter-uniform finite difference method for singularly perturbed parabolic problem with two small parameters

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ABSTRACT
A parameter-uniform finite difference scheme is constructed and analyzed for solving singularly perturbed parabolic problems with two parameters. The solution involves boundary layers at both the left and right ends of the solution domain. A numerical algorithm is formulated based on uniform mesh finite difference approximation for time variable and appropriate piecewise uniform mesh for the spatial variable. The developed method is second-order convergent. Furthermore, the present method produces a more accurate solution than some methods.

KEYWORDS
Parameter-uniform; singularly perturbed; parabolic problems; two-parameters; and accurate solution

1. Introduction

Singular perturbation problems emerged as a result of modeling real-life applications and their solutions exhibit boundary layer phenomena. The best example to mention is the Navier–Stokes equations with large Reynolds number in fluid dynamics, the convective heat transport problems with large Péclet number [1–3]. Based on the number of perturbation parameters, continuity or discontinuity of the coefficient; and source function or initial and/or boundary conditions throughout the considered domain, singularly perturbed parabolic problems can be categorized into various types [4–18]. Generally, singularly perturbed one-dimensional parabolic problems have boundary or interior or both boundary and interior layers depending on the defined data. Hence, in this work, we consider a class of singularly perturbed parabolic problems with two parameters whose solutions exhibit boundary layers. These types of problems arise in various areas of applications such as fluid dynamics (linear Navier–Stokes equation), chemical reactor theory, heat, and mass transfer process in composite materials with small heat conduction. Classes of singularly perturbed parabolic problems involving single perturbation parameters and sub-divided into convection-diffusion and reaction-diffusion problems are recently studied [1–7].

Besides the books were given by Miller et al. [8] and Roos et al. [9], few researchers tried to develop different numerical schemes to solve singularly perturbed parabolic problems with two-parameter. This literature served us as a landmark to get a priori knowledge about the nature of the solution of these problems and helped us to get insight on how to develop the present numerical method. To mention some, spline difference scheme [10], a robust finite difference method [11], a robust layer adapted difference method [12], a parameter-uniform higher-order finite difference scheme [13–16] have been developed for solving the singularly perturbed parabolic problem with two-parameters. In most of these works, fitted mesh finite difference methods have been adopted, but they gave numerical solutions with less accuracy and low rate convergence. Further, some numerical methods have been developed recently for solving different types of singularly perturbed differential-difference and differential equations aroused from modeling of real-life applications [5, 18–24]. Recently, the most productive, innovative, and parameter-uniformly convergent numerical methods for solving any family of singularly perturbed problems have been provided in [25–32]. Furthermore, there are more efficient adaptive algorithm and complicated problems that attracts the attention of most researchers as long as in [33,34].

Singular perturbed parabolic problems are a recent and active research area in engineering and applied science. As a result, most numerical methods such as

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finite difference methods, finite element methods, and finite volume methods have been developed so far, but these methods produce satisfactory results only when the mesh length of the solution domain is less than the value of the perturbation parameter [13]. Moreover, these methods are not uniformly convergent. This difficulty is occurred due to the existence of the perturbation parameter(s) that induces the boundary layer where the solutions vary rapidly and behave smoothly away from the layer.

Hence, it is necessary to develop stable, convergent, and methods that produce more accurate numerical solutions for reasonable mesh length compared to perturbation parameters with a higher-order rate of convergence. Thus, in this work, we presented a more accurate numerical method that fulfills the criteria mentioned above for solving singularly perturbed parabolic problems with two parameters.

2. Statement of the problem

Consider the following singularly perturbed two-parameter parabolic initial-boundary value problem on the solution domain $(x,t) \in D := \Omega \times (0,T]$, $\Omega = (0,1)$

$$L_{x,t}u \equiv \varepsilon \frac{\partial^2 u}{\partial x^2} + \mu a(x,t)\frac{\partial u}{\partial x} - b(x,t)u(x,t) - \frac{\partial u}{\partial t} = f(x,t) \tag{1}$$

subject to the initial and boundary conditions

$$u(x,0) = s(x), x \in \bar{\Omega}, u(0,t) = q_0(t), u(1,t) = q_1(t), t \in [0,T] \tag{2}$$

The two perturbation parameters ε and μ satisfy $0 < \varepsilon, \mu \ll 1$. Coefficient functions $a(x,t), b(x,t)$ and source function $f(x,t)$ are sufficiently regular on $\bar{D}$ and content $a(x,t) \ge \alpha > 0, b(x,t) \ge \beta > 0$; α and β are real numbers. Also, we assume that sufficient regularity and compatibility conditions are imposed on the functions $s(x), q_0(t), q_1(t)$, and $f(x,t)$ so that a unique solution exists.

The problem in Eqs. (1) and (2) exhibits two boundary layers with different widths depending on the relation between the two parameters ε and μ. For chosen $\gamma \approx \min_{(x,t)\in\bar{D}} \frac{b(x,t)}{a(x,t)}$ if $\alpha\mu^2 \le \gamma\varepsilon$, then the reduced problem of Eq. (1) is

$$-b(x,t)u_0(x,t) - \frac{\partial u_0}{\partial t} = f(x,t), u_0(x,0) = s(x) \tag{3}$$

Thus, boundary layers of width $O\left(\sqrt{\varepsilon}\right)$ are expected in both neighborhoods of $x = 0$ and $x = 1$ if $u_0(0,t) \ne q_0(t)$ and $u_0(1,t) \ne q_1(t)$.

If $\alpha\mu^2 \ge \gamma\varepsilon$, then reduced problem

$$\begin{cases} \mu a(x,t)\dfrac{\partial u_\mu}{\partial x} - b(x,t)u_\mu(x,t) - \dfrac{\partial u_\mu}{\partial t} = f(x,t) \\ u_\mu(x,0) = s(x) \text{and} u_\mu(0,t) = q_0(t) \end{cases} \tag{4}$$

is again a singularly perturbed problem with perturbation parameter, μ. This is a first-order hyperbolic equation with initial data specified along two sides $t = 0$ and $x = 0$ of the domain $\bar{D}$. A boundary layer of width $O(\varepsilon/\mu)$ is predictable in the right neighborhood of $x = 0$ if $u_\mu(0,t) \ne q_0(t)$, and a boundary layer of width $O(\mu)$ is expected in a left neighborhood of $x = 1$ if $u_\mu(1,t) \ne q_1(t)$ [8,9, 12, 15]. When the parameter $\mu = 1$, the problem is the well-studied parabolic convection-diffusion problem in [1, 4] with the boundary layer of width $O(\varepsilon)$ appears in the neighborhood of $x = 0$ or $x = 1$. While the parameter $\mu = 0$, the problem is parabolic reaction-diffusion problem [6, 16] which have two boundary layers of width $O\left(\sqrt{\varepsilon}\right)$ near $x = 0$ and $x = 1$. Here, in this paper, we consider the problem in Eqs. (1) and (2), when the two perturbation parameters satisfy $0 < \varepsilon \ll 1$ and $0 < \mu \ll 1$, for which the problem has different layer widths on the opposite side of the space domain depending on the value of two perturbation parameters, ε and μ.

3. Properties of continuous solution

In this section, a priori estimate for the solution $u(x,t)$ of Eqs. (1) and (2) on the solution domain $\bar{D}$ are established. These estimates contain continuous minimum principle bounds of the solution and its derivatives, and then parameter uniform bounds on the regular and singular components to analyze the proposed scheme. The detailed proofs of both Lemmas 3.1 and 3.2 are provided in [11–17, 25].

Lemma 3.1 (Minimum principle). *Let $\varphi \in C^{2,1}(\bar{D})$. If $\varphi(x,t) \ge 0, \forall (x,t) \in \partial D$ and $L_{x,t}\varphi(x,t) \le 0, \forall (x,t) \in D$, then $\varphi(x,t) \ge 0, \forall (x,t) \in \bar{D}$.*

The direct importance of this minimum principle is the following stability estimate.

Lemma 3.2 (Uniform stability estimate). *Let $u(x,t)$be the solution of Eqs. (1) and (2). Then, we have*

$$\|u\|_{\bar{D}} \le C\left(\beta^{-1}\|f(x,t)\| + \max\left(|q_0(t)| + |q_1(t)|\right)\right), \forall (x,t) \in \bar{D}$$

where $\|.\|$ denotes the maximum norm on the domain $\bar{D}$ and β is a positive constant specified under Section 2.

The solution $u(x,t)$ and its derivatives satisfy the following bounds.

Lemma 3.3. *For any non-negative integers i, j such that $0 \leq i + 3j \leq 4$, the solution $u(x,t)$* satisfies

$$\left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\|_{\bar{D}} \leq C \begin{cases} \dfrac{1}{\left(\sqrt{\varepsilon}\right)^i}, & \text{if} \quad \alpha\mu^2 \leq \gamma\varepsilon \\ \left(\frac{\mu}{\varepsilon}\right)^i, & \text{if} \quad \alpha\mu^2 \geq \gamma\varepsilon \end{cases}$$

where the constant C is independent of ε, μ and is dependent only on

$$p\left|\left|\frac{\partial^{(p)} a}{\partial x^p}\right|\right|_{\bar{D}}, \left|\left|\frac{\partial^{(p)} b}{\partial x^p}\right|\right|_{\bar{D}}, \text{ and } \left\|\frac{\partial^{(p)} f}{\partial x^p}\right\|_{\bar{D}}, \text{ for } p = 0,1,2.$$

Proof. See [13]. □

Further to fix more firm error analysis for the proposed finite difference scheme; the solution $u(x,t)$ can be splitted as

$$u(x,t) = v(x,t) + w_L(x,t) + w_R(x,t), \forall (x,t) \in \bar{D} \quad (5)$$

where $v(x,t)$ is the smooth or regular component, $w_L(x,t)$ and $w_R(x,t)$ are the left and right singular components of the solution respectively.

Theorem 3.1. *For all non-negative integers i, j such that $0 \leq i + 3j \leq 4$, the regular component $v(x,t)$ satisfies the bounds*

$$\left\| \frac{\partial^{i+j} v}{\partial x^i \partial t^j} \right\|_{\bar{D}} \leq C \begin{cases} 1 + \dfrac{1}{\left(\sqrt{\varepsilon}\right)^{i-3}}, & \text{for} \quad \alpha\mu^2 \leq \gamma\varepsilon \\ 1 + \left(\frac{\varepsilon}{\mu}\right)^{3-i}, & \text{for} \quad \alpha\mu^2 \geq \gamma\varepsilon \end{cases}$$

where constant C is independent of both the perturbation parameters ε and μ. Detailed proof of this theorem is given by Gupta et al. [13].

Lemma 3.4. *When the solution to the problem in Eqs. (1) and (2) is decomposed as Eq. (5), the layer component $w_L(x,t)$ and $w_R(x,t)$ satisfies the following bounds and its proof is given by O'Riordan et al. [14].*

$$|w_L(x,t)| \leq C \exp(-\theta_1 x), |w_R(x,t)| \leq C \exp(-\theta_2 (1-x))$$

where $\theta_1 = \begin{cases} \sqrt{\dfrac{\gamma\alpha}{\varepsilon}}, \alpha\mu^2 \leq \gamma\varepsilon, \\ \dfrac{\alpha\mu}{\varepsilon}, \alpha\mu^2 \geq \gamma\varepsilon, \end{cases}$ and $\theta_2 = \begin{cases} \dfrac{1}{2}\sqrt{\dfrac{\gamma\alpha}{\varepsilon}}, \alpha\mu^2 \leq \gamma\varepsilon, \\ \dfrac{\gamma}{2\mu}, \alpha\mu^2 \geq \gamma\varepsilon. \end{cases}$

4. Description of the scheme

To describe the scheme, the argument splits into two steps; the time variable is discretized on uniform mesh and then the discretization of space variable on the piecewise uniform mesh will obtain the required scheme as follow.

4.1. Temporal discretization

To discretize the time variable with uniform mesh size k, the interval $[0, T]$ is partitioned into N equal sub-intervals and each nodal point satisfies $0 = t_0 < t_1 < \ldots < t_N = T$, for $t_n = nk$, $k = \frac{T}{N}$, $n = 0, 1, 2, \ldots, N$. Now, at the point $\left(x, t_{n+\frac{1}{2}}\right)$, Eq. (1) can be written as

$$\left(\varepsilon \frac{\partial^2 u}{\partial x^2} + \mu a \frac{\partial u}{\partial x} - bu - \frac{\partial u}{\partial t}\right)\left(x, t_{n+\frac{1}{2}}\right) = f\left(x, t_{n+\frac{1}{2}}\right) \quad (6)$$

By Taylor's series expansion about the point $\left(x, t_{n+\frac{1}{2}}\right)$, we have

$$\begin{cases} u(x, t_{n+1}) = u\left(x, t_{n+\frac{1}{2}}\right) + \dfrac{k}{2}\dfrac{\partial u}{\partial t} u\left(x, t_{n+\frac{1}{2}}\right) + \dfrac{k^2}{8}\dfrac{\partial^2 u}{\partial t^2} u\left(x, t_{n+\frac{1}{2}}\right) + \dfrac{k^3}{48}\dfrac{\partial^3 u}{\partial t^3} u\left(x, t_{n+\frac{1}{2}}\right) + \ldots \\ u(x, t_n) = u\left(x, t_{n+\frac{1}{2}}\right) - \dfrac{k}{2}\dfrac{\partial u}{\partial t} u\left(x, t_{n+\frac{1}{2}}\right) + \dfrac{k^2}{8}\dfrac{\partial^2 u}{\partial t^2} u\left(x, t_{n+\frac{1}{2}}\right) - \dfrac{k^3}{48}\dfrac{\partial^3 u}{\partial t^3} u\left(x, t_{n+\frac{1}{2}}\right) + \ldots \end{cases}$$

which gives

$$\frac{\partial u}{\partial t}\left(x, t_{n+\frac{1}{2}}\right) = \frac{u(x, t_{n+1}) - u(x, t_n)}{k} + \tau_T \quad (7)$$

where $\tau_T = -\frac{k^2}{24}\frac{\partial^3 u}{\partial t^3} u\left(x, t_{n+\frac{1}{2}}\right) + \ldots \equiv O(k^2)$.

From Eq. (6), let us take the average of all terms except the term involve time derivative written as

$$\varepsilon u_{xx}\left(x, t_{n+\frac{1}{2}}\right) + \mu a\left(x, t_{n+\frac{1}{2}}\right) u_x\left(x, t_{n+\frac{1}{2}}\right) - b\left(x, t_{n+\frac{1}{2}}\right) u\left(x, t_{n+\frac{1}{2}}\right) - f\left(x, t_{n+\frac{1}{2}}\right)$$
$$= \frac{1}{2}\left(L^*_{x,f} u(x, t_{n+1}) + L^*_{x,f} u(x, t_n\right) \quad (8)$$

where $L^*_{x,f} u(x,t_n) = L^*_x u(x,t_n) - f(x,t_n)$, $L^*_x u(x,t_n) = \varepsilon\ u_{xx}(x,t_n) + \mu a(x,t_n) u_x(x,t_n) - b(x,t_n) u(x,t_n)$.

Substituting both Eqs. (7) and (8) into Eq. (6) yields

$$\left(L_x^* - \frac{2}{k}I\right)u(x, t_{n+1}) = f(x, t_{n+1}) + f(x, t_n) - \left(L_x^* + \frac{2}{k}I\right)u(x, t_n) \quad (9)$$

subject to the initial and boundary conditions

$$\begin{aligned} u(0, t_{n+1}) &= q_0(t_{n+1}), u(1, t_{n+1}) = q_1(t_{n+1}), \\ u(x, 0) &= s(x), \forall x \in (0, 1) \end{aligned} \quad (10)$$

This semi-discrete approximation $u(x, t_{n+1})$ of Eqs. (9) and (10) to the exact solution $u(x, t)$ of Eqs. (1) and (2) at the time levels $t_{n+1} = (n+1)k$.

4.2. Space mesh generation and numerical discretization

Assume the solution has large gradients in both a narrow region near $x = 0$ and $x = 1$, then the mesh in this region will be fine and coarse everywhere else. Let M be a positive integer such that $M = 2^r$ with $r \geq 3$. The piecewise uniform Shishkin mesh type on the domain $\bar{\Omega}^M$ is defined by partitioning the domain $\bar{\Omega} = [0, 1]$ into three sub-intervals $\bar{\Omega}_1 = [0, \tau_1]$, $\bar{\Omega}_2 = [\tau_1, 1 - \tau_2]$, and $\bar{\Omega}_3 = [1 - \tau_2, 1]$ which are subdivided uniformly to contain $\frac{M}{4}, \frac{M}{2}$, and $\frac{M}{4}$ mesh elements respectively, such that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \bar{\Omega}_3$. Transition parameters τ_1 and τ_2 are chosen to be

$$\tau_1 = \begin{cases} \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\gamma\alpha}}\ln(M)\right\}, & \alpha\mu^2 \leq \gamma\varepsilon, \\ \min\left\{\frac{1}{4}, 2\frac{\varepsilon}{\alpha\mu}\ln(M)\right\}, & \alpha\mu^2 \geq \gamma\varepsilon, \end{cases}$$

$$\tau_2 = \begin{cases} \min\left\{\frac{1}{4}, 4\sqrt{\frac{\varepsilon}{\gamma\alpha}}\ln(M)\right\}, & \alpha\mu^2 \leq \gamma\varepsilon \\ \min\left\{\frac{1}{4}, 4\frac{\mu}{\gamma}\ln(M)\right\}, & \alpha\mu^2 \geq \gamma\varepsilon \end{cases}$$

Moreover, the set of mesh points determined by

$$x_m = mh_m, m = 1, 2, \ldots, M - 1, \text{for } x_0 = 0 \text{ and } x_M = 1$$

The mesh spacing $h_m = x_m - x_{m-1}$ is given by

$$h_m = \begin{cases} \frac{4\tau_1}{M}, & \text{if} \quad m = 1, 2, \ldots, \frac{M}{4} \\ \frac{2(1 - \tau_2 - \tau_1)}{M}, & \text{if}, m = \frac{M}{4} + 1, \frac{M}{4} + 2, \ldots, \frac{3M}{4} \\ \frac{4\tau_2}{M}, & \text{if} \quad m = \frac{3M}{4} + 1, \ldots, M \end{cases} \quad (11)$$

We represent the full discretization mesh by D_M^N and for the rest of the paper, any function $F(x, t)$ adopts the notation $F(x_m, t_n) = F_m^n$. We discretize the problem in Eq. (10) on D_M^N as:

$$\begin{aligned} L_M^N U_m^n &:\equiv L_{M,N}^* U_m^{n+1} - \frac{2}{k} U_m^{n+1} \\ &= f_m^{n+1} + f_m^n - L_{M,N}^* U_m^n - \frac{2}{k} U_m^n \end{aligned} \quad (12)$$

for $m = 1, 2, \ldots, M - 1 \mathrm{and} n = 0, 1, 2, \ldots, N$ subject to the boundary and initial conditions

$$U_0^{n+1} = q_0(t_{n+1}), U_M^{n+1} = q_1(t_{n+1}) \text{ and } U_m^0 = u_0(x_m) \quad (13)$$

where

$$L_{M,N}^* U_m^{n+1} = \varepsilon\delta_x^2 U_m^{n+1} + \mu a_m^{n+1} \delta_x^0 U_m^{n+1} - b_m^{n+1} U_m^{n+1}$$

$$L_{M,N}^* U_m^n = \varepsilon\delta_x^2 U_m^n + \mu a_m^n \delta_x^0 U_m^n - b_m^n U_m^n$$

$$\delta_x^2 U_m^n = \frac{2}{h_m + h_{m+1}}\left(\delta_x^+ U_m^n - \delta_x^- U_m^n\right),$$

$$\delta^0 U_m^n = \frac{U_{m+1}^n - U_{m-1}^n}{h_m + h_{m+1}}, \delta_x^+ U_m^n = \frac{U_{m+1}^n - U_m^n}{h_{m+1}},$$

$$\delta_x^- U_m^n = \frac{U_m^n - U_{m-1}^n}{h_m}$$

The obtained scheme in Eq. (12) can be written in the form:

$$E_m U_{m-1}^{n+1} - F_m U_m^{n+1} + G_m U_{m+1}^{n+1} = H_m^{n+1} \quad (14)$$

where $E_m = \frac{2\varepsilon}{h_m(h_m + h_{m+1})} - \frac{\mu a_m^{n+1}}{h_m + h_{m+1}}$, $F_m = \frac{2\varepsilon}{h_m h_{m+1}} + b_m^{n+1} + \frac{2}{k}$,

$$G_m = \frac{2\varepsilon}{h_m(h_m + h_{m+1})} + \frac{\mu a_m^{n+1}}{h_m + h_{m+1}},$$

$$\begin{aligned} H_m^{n+1} = &\left(\frac{-2\varepsilon}{h_m(h_m + h_{m+1})} + \frac{\mu a_m^n}{h_m + h_{m+1}}\right)U_{m-1}^n \\ &+ \left(\frac{2\varepsilon}{h_m h_{m+1}} + b_m^n - \frac{2}{k}\right)U_m^n - \left(\frac{2\varepsilon}{h_{m+1}(h_m + h_{m+1})}\right. \\ &\left. + \frac{\mu a_m^n}{h_m + h_{m+1}}\right)U_{m+1}^n + f_m^{n+1} + f_m^n. \end{aligned}$$

The matrix formed from the tri-diagonal system of Eq. (14) concerning the space direction with the coefficients $E_m^{n+1}, F_m^{n+1}, G_m^{n+1}$ and the right-hand side H_m^{n+1} are given that satisfy the conditions $\left|E_m^{n+1}\right| > 0, \left|F_m^{n+1}\right| > 0, \left|G_m^{n+1}\right| > 0$ and $\left|F_m^{n+1}\right| > \left|E_m^{n+1}\right| + \left|G_m^{n+1}\right|$ at each $(n+1)$ th level. These situations guarantee that the system is *diagonally dominant* and can be solved by a tri-diagonal solver.

5. Stability of the method

To investigate the stability and convergence analysis estimate for the formulated method, there are several procedures provided recently like in [20, 25–31, 35–39]. Researchers are advising to follow the more recent novel analysis procedures presented by Das and others [20, 25–31, 35–37]. In this paper, the Von Neumann stability technique is applied to investigate

Table 1. Comparison of maximum absolute errors for Example 1 at $M = 64 \text{and} k = \frac{0.125}{4}$.

$\mu \downarrow$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-10}$	$\varepsilon = 10^{-12}$
			Present method			
10^{-2}	1.5728e-05	4.2691e-03	3.6774e-05	1.2473e-05	1.2477e-05	1.2477e-05
10^{-4}	1.5531e-05	6.1635e-03	1.0582e-03	1.5111e-04	1.5903e-04	1.5912e-04
10^{-6}	1.5529e-05	6.1633e-03	1.0582e-03	1.0086e-04	8.5427e-06	6.7906e-06
10^{-8}	1.5529e-05	6.1633e-03	1.0582e-03	1.0086e-04	8.5430e-06	6.7341e-06
10^{-10}	1.5529e-05	6.1633e-03	1.0582e-03	1.0086e-04	8.5430e-06	6.7341e-06
			A parameter-uniform higher-order finite difference scheme [13]			
10^{-2}	5.7836e-03	5.8089e-03	1.5931e-02	9.1885e-03	9.1894e-03	9.1894e-03
10^{-4}	1.1314e-02	1.1271e-02	1.1273e-02	1.1290e-02	1.1291e-02	1.1291e-02
10^{-6}	1.1313e-02	1.1270e-02	1.1272e-02	1.1264e-02	1.1264e-02	1.1264e-02
10^{-8}	1.1313e-02	1.1270e-02	1.1272e-02	1.1264e-02	1.1263e-02	1.1263e-02
10^{-10}	1.1313e-02	1.1270e-02	1.1272e-02	1.1264e-02	1.1263e-02	1.1263e-02

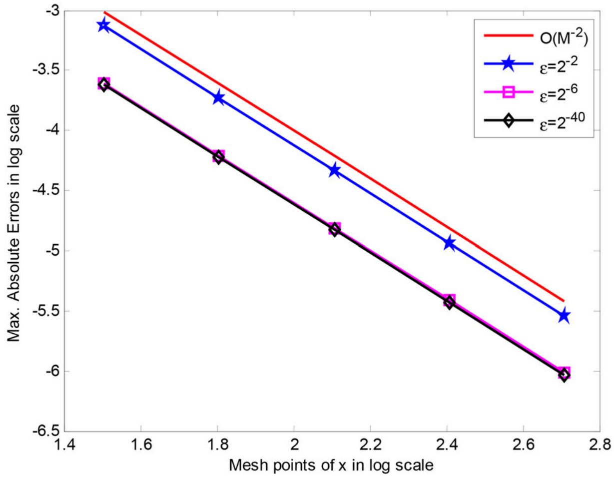

Figure 1. Log-log plot of maximum absolute errors correspond to the values of $\varepsilon = 2^{-5}$, $M = 2N$, $N = \{32, 64, 128, 256, 512\}$, and $T = 1$ for Example 1.

Table 2. Comparison of maximum absolute errors at $\varepsilon = 2^{-5}$, $M = 2N$, and $T = 1$ for Example 1.

$\mu \downarrow N \rightarrow$	16	32	64	128	256
			Present method		
2^{-2}	7.4036e-04	1.8444e-04	4.6115e-05	1.1526e-05	2.8814e-06
2^{-4}	3.1887e-04	7.9649e-05	1.9918e-05	4.9791e-06	1.2448e-06
2^{-6}	2.4622e-04	6.1550e-05	1.5392e-05	3.8477e-06	9.6196e-07
2^{-8}	2.3900e-04	5.9701e-05	1.4927e-05	3.7318e-06	9.3297e-07
2^{-10}	2.3799e-04	5.9449e-05	1.4859e-05	3.7146e-06	9.2865e-07
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-40}	2.3770e-04	5.9375e-05	1.4841e-05	3.7100e-06	9.2748e-07
			A robust finite difference method [11]		
2^{-2}	8.63e-3	3.95e-3	1.88e-3	9.20e-4	4.54e-4
2^{-4}	7.52e-3	3.29e-3	1.53e-3	7.35e-4	3.61e-4
2^{-6}	7.44e-3	3.23e-3	1.49e-3	7.18e-4	3.51e-4
2^{-8}	7.43e-3	3.23e-3	1.49e-3	7.16e-4	3.51e-4
2^{-10}	7.43e-3	3.22e-3	1.49e-3	7.16e-4	3.50e-4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-40}	7.43e-3	3.22e-3	1.49e-3	7.16e-4	3.50e-4

the stability of the developed scheme in Eq. (14), by assuming that the solution of Eq. (14) at the grid point (x_m, t_n) is given by

$$U_m^n = \xi^n e^{i\theta m} \tag{15}$$

where $i = \sqrt{-1}, \theta$ is the real number and ξ denotes the amplitude factor. Then, substituting Eq. (15) into the homogeneous part of Eq. (12) leads to:

$$\begin{aligned} &\left\{\frac{2\varepsilon}{h_m(h_m+h_{m+1})} - \frac{\mu a_m^{n+1}}{h_m+h_{m+1}}\right\}\xi^{n+1}e^{i\theta(m-1)} - \left\{\frac{2\varepsilon}{h_m h_{m+1}} + b_m^{n+1} + \frac{2}{k}\right\}\xi^{n+1}e^{i\theta m} \\ &+\left\{\frac{2\varepsilon}{h_m(h_m+h_{m+1})} + \frac{\mu a_m^{n+1}}{h_m+h_{m+1}}\right\}\xi^{n+1}e^{i\theta(m+1)} = \left(\frac{-2\varepsilon}{h_m(h_m+h_{m+1})} + \frac{\mu a_m^n}{h_m+h_{m+1}}\right)\xi^n e^{i\theta(m-1)} \\ &+\left(\frac{2\varepsilon}{h_m h_{m+1}} + b_m^n - \frac{2}{k}\right)\xi^n e^{i\theta m} - \left(\frac{2\varepsilon}{h_{m+1}(h_m+h_{m+1})} + \frac{\mu a_m^n}{h_m+h_{m+1}}\right)\xi^n e^{i\theta(m+1)} \end{aligned}$$

This can be simplified in the form of

$$\xi = \frac{\left(\frac{-2\varepsilon}{h_m(h_m+h_{m+1})} + \frac{\mu a_m^n}{h_m+h_{m+1}}\right)e^{-i\theta} + \frac{2\varepsilon}{h_m h_{m+1}} + b_m^n - \frac{2}{k} - \left(\frac{2\varepsilon}{h_{m+1}(h_m+h_{m+1})} + \frac{\mu a_m^n}{h_m+h_{m+1}}\right)e^{i\theta}}{\left(\frac{2\varepsilon}{h_m(h_m+h_{m+1})} - \frac{\mu a_m^{n+1}}{h_m+h_{m+1}}\right)e^{-i\theta} - \frac{2\varepsilon}{h_m h_{m+1}} - b_m^{n+1} - \frac{2}{k} + \left(\frac{2\varepsilon}{h_m(h_m+h_{m+1})} + \frac{\mu a_m^{n+1}}{h_m+h_{m+1}}\right)e^{i\theta}}$$

The stability condition $|\xi| \le 1$ satisfied, since

$$|\xi| = \left|\frac{\left(\frac{-2\varepsilon}{h_m(h_m+h_{m+1})} + \frac{\mu a_m^n}{h_m+h_{m+1}}\right)e^{-i\theta} + \frac{2\varepsilon}{h_m h_{m+1}} + b_m^n - \frac{2}{k} - \left(\frac{2\varepsilon}{h_{m+1}(h_m+h_{m+1})} + \frac{\mu a_m^n}{h_m+h_{m+1}}\right)e^{i\theta}}{\left(\frac{2\varepsilon}{h_m(h_m+h_{m+1})} - \frac{\mu a_m^{n+1}}{h_m+h_{m+1}}\right)e^{-i\theta} - \frac{2\varepsilon}{h_m h_{m+1}} - b_m^{n+1} - \frac{2}{k} + \left(\frac{2\varepsilon}{h_m(h_m+h_{m+1})} + \frac{\mu a_m^{n+1}}{h_m+h_{m+1}}\right)e^{i\theta}}\right| \le 1$$

Table 3. Rate of convergence when $\varepsilon = 2^{-5}$, $M = 2N$, and $T = 1$ for Example 1.

$\mu \downarrow$	$M = 32$	$M = 64$	$M = 128$	$M = 256$
2^{-2}	2.0051	1.9998	2.0003	2.0001
2^{-4}	2.0012	1.9996	2.0001	2.0000
2^{-6}	2.0001	1.9996	2.0001	1.9999
2^{-8}	2.0012	1.9998	2.0000	2.0000
2^{-10}	2.0012	2.0003	2.0001	2.0000
⋮	⋮	⋮	⋮	⋮
2^{-40}	2.0012	2.0003	2.0001	2.0000

Table 4. Maximum absolute errors and rate of convergence, $\varepsilon = 2^{-10}$ and $M = 2N$ for Example 2.

$\mu \downarrow N \rightarrow$	16	32	64	128	256
2^{-6}	4.0333e-05	1.9579e-05	5.5635e-06	1.4468e-06	3.6843e-07
	1.0427	1.8152	1.9431	1.9734	
2^{-10}	2.6269e-05	6.5702e-06	1.6433e-06	4.1080e-07	1.0270e-07
	1.9994	1.9993	2.0001	2.0000	
2^{-14}	2.6256e-05	6.5664e-06	1.6423e-06	4.1055e-07	1.0264e-07
	1.9995	1.9994	2.0001	2.0000	
2^{-18}	2.6255e-05	6.5661e-06	1.6423e-06	4.1054e-07	1.0264e-07
	1.9995	1.9993	2.0001	1.9999	
2^{-22}	2.6255e-05	6.5661e-06	1.6423e-06	4.1054e-07	1.0264e-07
	1.9995	1.9993	2.0001	1.9999	
2^{-26}	2.6255e-05	6.5661e-06	1.6423e-06	4.1054e-07	1.0264e-07
	1.9995	1.9993	2.0001	1.9999	
⋮	⋮	⋮	⋮	⋮	⋮
2^{-40}	2.6255e-05	6.5661e-06	1.6423e-06	4.1054e-07	1.0264e-07
	1.9995	1.9993	2.0001	1.9999	

From the statement of the problem, we have $a(x,t) \geq \alpha > 0$. Thus, assume that for the discrete $a_m^{n+1} \geq a_m^n \geq \alpha > 0$, and $b_m^{n+1} \geq b_m^n \geq \beta > 0$.

Hence, $|\xi| \leq 1$, which implies the developed finite difference method in Eq. (14) is unconditionally stable. Further procedures to investigate the parameter-uniformly convergent analysis, one can read and follow the works in [20, 25–31, 35–39].

6. Numerical illustrations

In this section, two test examples are considered and numerical results are computed to demonstrate the effectiveness of the present scheme. Since the exact solution for such type of problems is not available, the maximum absolute errors at all the mesh points are evaluated using the formula

$$E_{\varepsilon,\mu}^{M,N} = \max_{0 \leq m \leq M; 0 \leq n \leq N} \left| U_m^n - U_{2m}^{2n} \right|$$

where U_m^n is an approximate solution obtained using a constant space mash size h_m and time step k and U_{2m}^{2n} is also an approximate solution produced using space step $\frac{h_m}{2}$ with time step, $\frac{k}{2}$. Likewise, we compute the numerical rates of convergence by $R = \frac{\log E_{\varepsilon,\mu}^{M,N} - \log E_{\varepsilon,\mu}^{2M,2N}}{\log 2}$.

Example 1. Consider the parabolic initial-boundary value problem

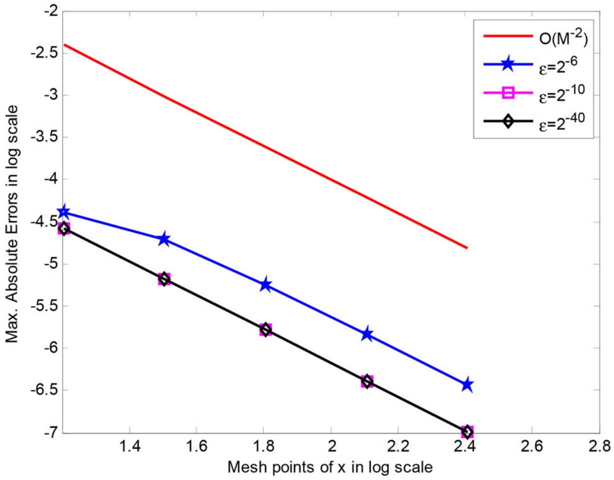

Figure 2. Log-log plot of maximum absolute errors correspond to the values of $\varepsilon = 2^{-10}$, $M = 2N$, $N = \{16, 32, 64, 128, 256\}$ for Example 2.

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \mu(1+x)\frac{\partial u}{\partial x} - u(x,t) - \frac{\partial u}{\partial t}$$
$$= 16x^2(1-x)^2, (x,t) \in (0,1) \times (0,T]$$

subject to the conditions

$$u(x,0) = 0, x \in [0,1] \text{ and } u(0,t) = 0 = u(1,t), t \in [0,T]$$

Numerical results for this example are presented in Tables 1–3 and Figure 1.

Table 5. Maximum absolute errors for Example 2 when $M = 64 = N$.

$\mu \downarrow$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\varepsilon = 10^{-10}$	$\varepsilon = 10^{-12}$
10^{-2}	2.8972e-06	5.0287e-03	1.1803e-04	9.7890e-05	9.7887e-05	9.7886e-05
10^{-4}	5.6589e-06	6.0059e-03	1.0114e-03	1.3414e-04	1.3414e-04	1.3414e-04
10^{-6}	5.6880e-06	6.0070e-03	1.0110e-03	1.0593e-04	1.0647e-05	1.7813e-06
10^{-8}	5.6883e-06	6.0070e-03	1.0110e-03	1.0592e-04	1.0641e-05	1.7170e-06
10^{-10}	5.6883e-06	6.0070e-03	1.0110e-03	1.0592e-04	1.0641e-05	1.7170e-06

Example 2. This example corresponds to the following initial boundary value problem

$$\varepsilon \frac{\partial^2 u}{\partial x^2} + \mu(1 + x - x^2 + t^2)\frac{\partial u}{\partial x} - (1 + 5xt)u(x,t) - \frac{\partial u}{\partial t} = x(1-x)(e^t - 1), (x,t) \in (0,1) \times (0,T]$$

subject to the conditions: $u(x,0) = 0, x \in [0,1]$ and $u(0,t) = 0 = u(1,t),\ t \in [0,1]$.

Maximum absolute errors and rate of convergence for Example 2 are presented in Table 4 and log-log plot as Figure 2.

7. Discussions and conclusion

Computed maximum absolute errors and the corresponding order of convergence for Examples 1 and 2 are tabulated in Tables 1–5, for various values of perturbation and mesh parameters. We considered different values of the perturbation parameters for the test examples for the sake of comparison to the existing results in the literature. Specifically, Tables 1 and 2 show the comparison of maximum absolute errors that demonstrates the advancement of the present method. Results in Tables 2 and 4 demonstrate that the maximum absolute error has monotonically decreasing behavior with an increasing number of mesh points and this confirms the convergence of the present method. Further, Tables 3 and 4 validate the parametric uniform convergence of the present method and show that the order of convergence is in agreement with Theorem 5.1. Figures 1 and 2 are the log-log plot for numerical results given in Tables 2 and 4, and show that the order of convergence for the discrete scheme is in support of our theoretical error estimates. Hence, we can conclude that the present method is parametric uniform, second-order convergent, and gives a more accurate solution than some existing numerical methods.

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Authors' contributions

All the authors have made substantive contributions to the article and assume full responsibility for its content. These three authors read and approved the final manuscript.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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