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## ABSTRACT

We study the relativistic dynamics of a particle carrying a non-Abelian charge in the presence of a non-Abelian background electromagnetic field. To this end, we extract the non-Abelian Dirac Hamiltonian from a system describing the interaction between the Yang–Mills field and a spin-1/2 field. The dynamics of a particle with non-Abelian charge is quantized directly by analogy with its quantum theory. By choosing a suitable non-Abelian gauge field, we investigate the spectrum in two-dimensional space, paying particular attention to the role of the total angular momentum. Relativistic Landau levels are obtained explicitly by means of an analytical method. The wave functions of the system are obtained in terms of the generalized Laguerre polynomials. Interesting features of such models are discussed through the spectrum.

## I. INTRODUCTION

The symmetry properties of the elementary particles are formulated in terms of invariance, highlighted mathematically by the symmetry groups. Yang–Mills theory is formulated in terms of an internal symmetry of group, which is associated with the isospin invariance of strong interactions, a consequence of the fact that the strong nuclear forces are independent of the electric charges of the particles. Thus, Yang and Mills felt the need to introduce a new isospin potential field that has a local gauge group—the symmetry group $SU(2)$—by analogy with the phase group $U(1)$, which determines the symmetry of electromagnetism. In Ref. 1, a system describing the interaction between the Yang–Mills field and isotopic-spin-carrying particles in the classical limit is given. A set of classical equations for the motion of a non-Abelian charge particle is extracted.[2–5] One of the equations of motion, which play an essential role in theoretical physics, is the emblematic Dirac equation. It is an exactly solvable equation introduced in the context of many equations describing the dynamics of particles in relativistic quantum mechanics.[6] As a simplified model for complex interactions, the Dirac equation is mainly used in relativistic quantum mechanics. Moreover, since it can be exactly solved, it is, thus, particularly useful.[7–10] By means of a linear vector potential, the Dirac equation may be transformed into the Dirac oscillator equation that has attracted a lot of interest because of its numerous physical applications. One of the most interesting applications of the Dirac oscillator equation is that when its non-relativistic limit is taken, the associated Klein–Gordon equations describe a harmonic oscillator in the presence of a strong spin–orbit coupling.[11] A further key application is that the $(2+1)$-dimensional Dirac oscillator contains in itself another germane model in quantum physics: the Jaynes–Cummings model or, more rigorously, the anti-Jaynes–Cummings version.[12–16] The powerfulness of the Jaynes–Cummings model relies on its simplicity and its capacity to embrace very different physical situations such as light–matter interaction, dynamics of trapped ions, or Bose–Einstein condensates.[17] Furthermore, the Dirac oscillator in $(2+1)$-dimensional has been used as a framework to study some condensed matter physical phenomena such as the study of electrons in two dimensional materials, which can be applied to study some aspects of the physics of graphene.[18,19] Additionally, the Dirac oscillator plays an important role in the description of relativistic many-body problems and supersymmetric relativistic quantum mechanics.[20–23] Dirac oscillator representation is also proposed in quantum chromodynamics, particularly, in connection with quark confinement models in mesons and baryons.[24] It is now accepted in particle physics that all fundamental fermions, electrons, muons, neutrinos, and quarks follow a Dirac equation.

In this work, we extract the Dirac equation from a Lagrangian describing the interaction between the non-Abelian gauge field and a spin-1/2 field. The equation obtained describes the relativistic dynamics of a particle carrying a non-Abelian charge in a non-Abelian background electromagnetic field. Despite a large variety of works that have been published concerning the Dirac equation,[25–31] no one has reported the dynamics of a particle carrying a non-Abelian charge. Recently, the study of a particle carrying a non-Abelian charge has renewed interest. The creation of non-Abelian fields in a laboratory is still a challenge, but a very fast progress in this field is being done and there are several very promising proposals.[32–35] Reference 36 has indicated how non-Abelian external gauge fields can be generated with the help of external lasers. In Ref. 37, the Landau problem on the sphere and the charge and spin dynamics induced by the insertion of non-Abelian flux in an otherwise Abelian background are analyzed and solved. More recently, the spectrum of the $N = 1$ supersymmetric Wong equations and the non-Abelian Landau problem is obtained in the specific case of a spin-1/2 particle in a nontrivial static non-Abelian background magnetic field.[38] The authors of Ref. 39 have extracted the wave functions for a Dirac particle moving under the action of a non-Abelian field. These wave functions extracted are found to be the same as those obtained by direct resolution of the Dirac equation. All these results have revived interest in the study of the motion of particles carrying non-Abelian charges in a non-Abelian background field.

The aim of this work is to present a systematic investigation of the analytical spectrum for the non-Abelian Dirac Hamiltonian. We exploit the rotation symmetry in the system by means of the constant of motion, which is the total angular momentum. The spectrum of the model is obtained in terms of the generalized Laguerre polynomials. Then, we identify through the energy spectrum the Dirac point, which is one of the main features for the massless Dirac Hamiltonian.

This paper is organized as follows: In Sec. II, we define the Lagrangian density that describes the interaction between the Yang–Mills field and a spin-1/2 field. We obtain the Dirac equation for a particle carrying a non-Abelian charge in a non-Abelian background electromagnetic field. The Hamiltonian that describes the relativistic dynamics of the Dirac equation is identified. Then, the quantum dynamics of the system is given in Sec. III. In Sec. IV, we exactly solve the non-Abelian Dirac Hamiltonian in the framework of relativistic quantum mechanics. Concerning Sec. V, we present some graphics representing energies as functions of various values of non-Abelian field parameters. We present the results and discussions in Sec. VI. Section VII is devoted to the conclusion. Finally, some calculation details related to the spectra of the total angular momentum and non-Abelian Dirac Hamiltonian are reported in Appendixes A and B.

## II. LAGRANGIAN DENSITY IN THE CLASSICAL LIMIT

Yang–Mills theory is a non-Abelian formalism with the symmetry gauge group $U(2) = U(1) \times SU(2)$. The gauge field, which mediates the interaction between the charged spin-1/2 fields, is the non-Abelian electromagnetic field. The Lagrangian density describing the physical system is given by[1]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu A}F^{\mu\nu A} + i\overline{\psi}\gamma^{\mu}D_{\mu}\psi - \frac{mc}{\hbar}\overline{\psi}\psi,\ \ D_{\mu} = \partial_{\mu} + iA_{\mu}. \tag{1}$$

The notation $A = (0, a)$ is adopted for the generators of the gauge group $U(2)$, since we wish to keep the $U(1)$ part in the gauge group corresponding to the generator $T^0 = \mathbb{1}$, while the $T^a$ are the Hermitian generators of the $SU(2)$ group with the Lie algebra $[T^a, T^b] = if^{abc}T^c$, $a = 1, 2, 3$. The structure constants $f^{abc}$ will be chosen as real and totally antisymmetric. We have a non-Abelian gauge field associated with a gauge group $U(2)$ which we describe by the potential $A_\mu = A_\mu^A T^A$ and the non-Abelian field strength tensor,

$$\begin{aligned} F_{\mu\nu} &= -i[D_\mu, D_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \end{aligned} \tag{2}$$

In expression (1), a possible choice of the Clifford–Dirac algebra of $\gamma^\mu$ matrices ($\mu = 0, i$) is $\gamma^0 = \Sigma_3$ and $\gamma^0\gamma^i = \Sigma_i$, while the $\Sigma_i (i = 1, 2)$ stand for the Pauli matrices. $\psi, m \geq 0$ denote the Dirac spinor field and a fermionic mass term, respectively. $\overline{\psi} = \psi^\dagger\gamma^0$, called "psi-bar," is sometimes referred to as a Dirac adjoint. Remark that in expression (1) the non-Abelian charges are not visible explicitly. As will be clear hereafter—at a quantum level—these charges will be measured in units of $\hbar$. The non-Abelian electric and magnetic fields in the Lie-algebra valued form are given by

$$E_i = F_{0i} = \partial_0 A_i - \partial_i A_0 + i[A_0, A_i],\ F_{ij} = \partial_i A_j - \partial_j A_i + i[A_i, A_j]. \tag{3}$$

These background electric and magnetic fields are classic and given by some choice of configuration space function, as a function of time, within the internal space through the components labeled by $A$.

A careful consideration finds that the density Lagrangian (1) is invariant under the non-Abelian gauge transformations, leading to gauge invariant equations of motion. The Euler–Lagrange equations of motion are given by

$$\partial_\mu F_{\mu\nu} + i[F_{\mu\nu}, A_\mu] = \overline{\psi}\gamma^\nu\psi, \tag{4}$$

$$i\gamma^\mu(\partial_\mu + iA_\mu)\psi - \frac{mc}{\hbar}\psi = 0. \tag{5}$$

Let us concentrate on Eq. (5) that stands for the Dirac equation for a non-Abelian charged particle in a non-Abelian background electromagnetic field. It can be written in the form

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi, \tag{6}$$

where $H$ is the non-Abelian Dirac Hamiltonian, describing the relativistic dynamics of a particle moving on a plane in the presence of non-Abelian external electric and magnetic fields.

### III. QUANTUM DYNAMICS

A starting point of the quantization procedure, for a system, is to adopt its classical description as a framework. However, the dynamics of a particle with non-Abelian charge is quantized directly at the opposite of the Abelian case of an electrically charged particle. In the representation of $U(2) = U(1) \times SU(2)$, we have

$$T^0 = \mathbb{1}, \quad [T^a, T^b] = if^{abc}T^c, \quad a, b, c = 1, 2, 3, \tag{7}$$

leading to the non-Abelian charges $Q^a = \hbar\frac{T^a}{2}$ and the Abelian charge $Q^0 = \hbar\mathbb{1}$. We have the following commutation relation:

$$[Q^a; Q^b] = 2i\hbar f^{abc}Q^c. \tag{8}$$

Consequently, the non-Abelian Dirac Hamiltonian is given by

$$H = c\Sigma_i(p_i + \mathcal{A}_i) + \phi + mc^2\Sigma_3, \quad i = 1, 2, \tag{9}$$

where

$$\mathcal{A}_i = A_i^A Q^A = \hbar A_i, \quad \phi = \phi^A Q^A = c\hbar A_0, \quad A = 0, a. \tag{10}$$

Here, $\phi$ and $\mathcal{A}_i$ are the non-Abelian scalar and vector potentials, respectively. $\Sigma_i$ and $\Sigma_3$ are Dirac matrices, $p_i$ is the momentum operator, $m$ is the rest mass of the electron, $c$ is the speed of light, and $\hbar$ is the reduced Planck's constant.

We study the Dirac Hamiltonian describing the movement of the relativistic electron carrying a non-Abelian charge and undergoing non-Abelian electric and magnetic fields in the plane. The dynamics of the system is determined by the algebra, spanned by the following commutation relations:

$$[x_1, p_1] = i\hbar\mathbb{1}, \ [x_2, p_2] = i\hbar\mathbb{1}, \tag{11}$$

$$[\Sigma_1, \Sigma_2] = 2i\Sigma_3, \ [\Sigma_2, \Sigma_3] = 2i\Sigma_1, \ [\Sigma_3, \Sigma_1] = 2i\Sigma_2, \tag{12}$$

$$[Q_1, Q_2] = 2i\hbar Q_3, \ [Q_2, Q_3] = 2i\hbar Q_1, \ [Q_3, Q_1] = 2i\hbar Q_2. \tag{13}$$

We note with interest that the non-Abelian charge operators still commute with the Dirac matrices,

$$\begin{gathered}[\Sigma_1, Q_1] = 0, \ [\Sigma_2, Q_2] = 0, \ [\Sigma_3, Q_3] = 0, \\ [\Sigma_1, Q_2] = 0, \ [\Sigma_2, Q_3] = 0, \ [\Sigma_3, Q_1] = 0,\end{gathered} \tag{14}$$

likewise the fermionic operators still commute with the bosonic operators.

In the Heisenberg picture, by using the above commutation relations, we obtain from the Dirac Hamiltonian (9) the following equation of motion, which is Wong's equation, in the form

$$\begin{aligned} m\ddot{x}_i &= \dot{x}_j F_{ij}^A Q^A + E_i^A Q^A \\ &= \dot{x}_j \mathcal{F}_{ij} + \mathcal{E}_i, \end{aligned} \tag{15}$$

where the right-hand side represents a generalization of the Lorentz force. This equation describes the non-relativistic dynamics of a spinless particle, with color or internal symmetry degrees of freedom, moving in a background electromagnetic field.

At this stage of transformation of the model, we take note of the fact that the fields present the expected configurations, namely, the non-Abelian magnetic field $F_{ij}^A$ is perpendicular to the plane, while the non-Abelian electric field $E_i^A$ lies in the latter. Hence, we have the following general form for the non-Abelian gauge fields:

$$\mathcal{A}_i = -\frac{1}{2}B\varepsilon_{ij}x_j - \beta\sum_{a=1}^{3}\varepsilon_{ia}Q_a, \quad \phi = -x_iE_i + \phi_aQ_a. \tag{16}$$

This choice leads to the non-Abelian magnetic field $\mathcal{F}_{ij}$ and electric field $\mathcal{E}_i$, given by

$$\mathcal{F}_{12} = B\mathbb{1} - 2\beta^2Q_3, \tag{17}$$

$$\mathcal{E}_1 = \frac{E_1}{c}\mathbb{1} - 2\frac{\beta}{c}(\phi_1Q_3 - \phi_3Q_1), \ \mathcal{E}_2 = \frac{E_2}{c}\mathbb{1} - 2\frac{\beta}{c}(\phi_2Q_3 - \phi_3Q_2), \tag{18}$$

with the notation

$$E_i = \hbar E_i^0, \ B = \hbar B^0. \tag{19}$$

From here on, solving the problem follows a standard path. One way to relate the Hamiltonian $H$ to an appropriate Lie algebra is to construct its bosonic and fermionic representations. For this purpose, let us consider the Fock operators (see, for example, Ref. 40),

$$a_i = \sqrt{\frac{\omega}{2\hbar}}\left(x_i + \frac{i}{\omega}p_i\right), \quad a_i^\dagger = \sqrt{\frac{\omega}{2\hbar}}\left(x_i - \frac{i}{\omega}p_i\right), \tag{20}$$

with

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}\mathbb{1}. \tag{21}$$

Here, $\omega = \frac{B}{2}$, with $B$ being the Abelian magnetic field. We introduce the chiral oscillators $(a_\pm, a_\pm^\dagger)$ that verify the Fock algebra $\left[a_\pm, a_\pm^\dagger\right] = \mathbb{1}$, defined through the expressions

$$a_\pm = \frac{1}{\sqrt{2}}(a_1 \mp ia_2), \quad a_\pm^\dagger = \frac{1}{\sqrt{2}}\left(a_1^\dagger \pm ia_2^\dagger\right). \tag{22}$$

All these expressions may be inverted to express the original $x_i$ and $p_i$ in terms of $a_\pm$ and $a_\pm^\dagger$.

Concerning the fermionic representations, we set for the fermionic Fock operators

$$\Sigma_\pm = \frac{1}{2}(\Sigma_1 \pm i\Sigma_2). \tag{23}$$

These quantities suitably satisfy the following commutation relations:

$$[\Sigma_+, \Sigma_-] = \Sigma_3, \quad [\Sigma_3, \Sigma_\pm] = \pm 2\Sigma_\pm. \tag{24}$$

The sector concerning the ladder operators is given by

$$Q_\pm = \frac{1}{2}(Q_1 \pm iQ_2). \tag{25}$$

These operators satisfy the following commutation relations:

$$[Q_+, Q_-] = \hbar Q_3, \quad [Q_3, Q_\pm] = \pm 2\hbar Q_\pm. \tag{26}$$

We obtain the following commutation relations:

$$[\Sigma_\mp, Q_\pm] = 0, \quad [\Sigma_\mp, Q_\mp] = 0, \quad [\Sigma_\mp, Q_3] = 0, \quad [Q_\mp, \Sigma_3] = 0. \tag{27}$$

By using the non-Abelian gauge potentials (16) and the above representations of the bosonic and fermionic sectors, one gets the following expression for the non-Abelian Dirac Hamiltonian:

$$\begin{aligned} H = &-2ic\Sigma_+(\sqrt{\hbar\omega}a_+ + \beta Q_-) + 2ic\Sigma_-(\sqrt{\hbar\omega}a_+^\dagger + \beta Q_+) \\ &- \sqrt{\frac{\hbar}{4\omega}}\Big[(E_1 + iE_2)(a_+ + a_-^\dagger) + (E_1 - iE_2)(a_- + a_+^\dagger)\Big] \\ &+ (\phi_1 - i\phi_2)Q_+ + (\phi_1 + i\phi_2)Q_- \\ &+ mc^2\Sigma_3 + \phi_3 Q_3. \end{aligned} \tag{28}$$

## IV. SPECTRUM OF THE NON-ABELIAN DIRAC HAMILTONIAN

Now, we should try to diagonalize the Hamiltonian (28). In order to restrict the analysis toward a solvable situation, let us set the following only vanishing components:

$$E_i = 0, \;\; \phi_i = 0, \;\; i = 1, 2. \tag{29}$$

Indeed, the rotations in internal space around the $a = 3$ direction are still possible for this choice. We note that $E^0$ and $B^0$ are the Abelian electric and magnetic background fields, while their non-Abelian counterparts are given by $\phi_3 = \rho$ and $\beta$, respectively. The Hamiltonian (28) becomes

$$\begin{aligned} H = &-2ic\Sigma_+(\sqrt{\hbar\omega}a_+ + \beta Q_-) + 2ic\Sigma_-(\sqrt{\hbar\omega}a_+^\dagger + \beta Q_+) \\ &+ mc^2\Sigma_3 + \rho Q_3. \end{aligned} \tag{30}$$

The aim of this part is to solve (30) in the standard context where the commutation relations (11)–(14) hold, paying particular attention to the role of the total angular momentum.

It is easy to show that the total angular momentum of the Hamiltonian (30), defined as follows:

$$L = \hbar(a_+^\dagger a_+ - a_-^\dagger a_- + \frac{1}{2}\Sigma_3) + \frac{1}{2}Q_3, \tag{31}$$

is a conserved quantity of the system. Hence, the system possesses two conserved quantities, whose Poisson bracket vanishes on account of their gauge invariance under the non-Abelian gauge transformations. These may be diagonalized in a common basis of eigenstates. We will then consider the Hilbert space of physical fermionic states as being the $2^2 = 4$ dimensional tensor product of two copies of the Hilbert space spanned by $|+\rangle$ and $|-\rangle$. We therefore get

$$|+,+\rangle = |+\rangle \otimes |+\rangle, \;\; |+,-\rangle = |+\rangle \otimes |-\rangle, \;\; |-,+\rangle = |-\rangle \otimes |+\rangle, \;\; |-,-\rangle = |-\rangle \otimes |-\rangle. \tag{32}$$

The states space can be reported to an orthonormal basis defined by

$$\{|+,+\rangle, |+,-\rangle, |-,+\rangle, |-,-\rangle\}. \tag{33}$$

All the fermionic operators of the system may be represented in this basis.

We finally associate with the fermionic operators the following tensor products of the usual Pauli matrices $\sigma_\pm$ and $\sigma_3$:

$$\Sigma_\pm = \sigma_\pm \otimes \mathbb{1}, \quad \Sigma_3 = \sigma_3 \otimes \mathbb{1}, \tag{34}$$

$$Q_\pm = \hbar\mathbb{1} \otimes \sigma_\pm, \quad Q_3 = \hbar\mathbb{1} \otimes \sigma_3, \tag{35}$$

where

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{36}$$

Set $|+\rangle$ and $|-\rangle$ by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively. The following matrices are defined for the base vectors:

$$|+,+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \;\; |+,-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{37}$$

$$|-,+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \;\; |-,-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{38}$$

The particular representation of the $(Q_\pm, Q_3)$ algebra considered here is given by

$$\begin{aligned} Q_3|\pm,\pm\rangle &= \pm\hbar|\pm,\pm\rangle, \quad Q_3|\pm,\mp\rangle = \mp\hbar|\pm,\mp\rangle, \\ Q_\pm|\pm,\mp\rangle &= \hbar|\pm,\pm\rangle, \quad Q_\pm|\mp,\mp\rangle = \hbar|\mp,\pm\rangle. \end{aligned} \tag{39}$$

Concerning the $(\Sigma_\pm, \Sigma_3)$ algebra, its representation is given by

$$\begin{aligned} \Sigma_3|\pm,\pm\rangle &= \pm|\pm,\pm\rangle, \quad \Sigma_3|\pm,\mp\rangle = \pm|\pm,\mp\rangle, \\ \Sigma_\pm|\mp,\pm\rangle &= |\pm,\pm\rangle, \quad \Sigma_\pm|\mp,\mp\rangle = |\pm,\mp\rangle. \end{aligned} \tag{40}$$

The construction of the representation of the whole system is now straightforward if we consider the tensor product of the representations of the bosonic and fermionic sectors.

The total angular momentum is, therefore, given by

$$L = \hbar\begin{pmatrix} a_+^\dagger a_+ - a_-^\dagger a_- + 1 & 0 & 0 & 0 \\ 0 & a_+^\dagger a_+ - a_-^\dagger a_- & 0 & 0 \\ 0 & 0 & a_+^\dagger a_+ - a_-^\dagger a_- & 0 \\ 0 & 0 & 0 & a_+^\dagger a_+ - a_-^\dagger a_- - 1 \end{pmatrix}. \tag{41}$$

Concerning the non-Abelian Dirac Hamiltonian, it is expressed as follows:

$$H = \begin{pmatrix} mc^2 + \hbar\rho & 0 & -2ic\sqrt{\hbar\omega}a_+ & 0 \\ 0 & mc^2 - \hbar\rho & -2ic\hbar\beta & -2ic\sqrt{\hbar\omega}a_+ \\ 2ic\sqrt{\hbar\omega}a_+^\dagger & 2ic\hbar\beta & -mc^2 + \hbar\rho & 0 \\ 0 & 2ic\sqrt{\hbar\omega}a_+^\dagger & 0 & -mc^2 - \hbar\rho \end{pmatrix}. \tag{42}$$

Thus, one can write the eigenvalue equation of the total angular momentum[41] (the details are given in Appendix A),

$$L|\Psi\rangle = \hbar(\ell + 1)|\Psi\rangle, \quad \ell = n_+ - n_-, \quad (\ell = 0, 1, 2, \cdots), \tag{43}$$

with the eigenfunction given by

$$\begin{aligned} |\Psi\rangle = {} & [a_+^\dagger]^\ell \varphi_1(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \otimes |+,+\rangle + [a_+^\dagger]^{\ell+1}\varphi_2(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \otimes |+,-\rangle \\ & + [a_+^\dagger]^{\ell+1}\varphi_3(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \otimes |-,+\rangle + [a_+^\dagger]^{\ell+2}\varphi_4(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \otimes |-,-\rangle, \end{aligned} \tag{44}$$

where $|\Omega_0\rangle$ is the vacuum state for both bosons, i.e., $a_\pm|\Omega_0\rangle = 0$. $\varphi_1(a_+^\dagger a_-^\dagger)$, $\varphi_2(a_+^\dagger a_-^\dagger)$, $\varphi_3(a_+^\dagger a_-^\dagger)$, $\varphi_4(a_+^\dagger a_-^\dagger)$ are the power series in the product of the creation operators starting with power zero.

We note that the Hamiltonian $H$ and the total angular momentum $L$ commute, the eigenfunctions (44) are also the eigenfunctions of the Hamiltonian $H$. Therefore, we may write the eigenvalue equation

$$H|\Psi\rangle = E|\Psi\rangle. \tag{45}$$

Equation (45) can be solved exactly (the details are given in Appendix B). We obtain the energy spectrum,[42]

$$E^\tau_{(n_-,\pm)} = \pm\sqrt{4c^2\mathcal{P}^2_{(n_-,\tau)} + m^2c^4 + \hbar^2\rho^2}, \quad \tau = \pm, \quad n_- = 0, 1, 2, \ldots, \tag{46}$$

where

$$\begin{aligned} \mathcal{P}^2_{(n_-,\tau)} = {} & \hbar\omega(n_- + 1) + \frac{\hbar^2\beta^2}{2} \\ & + \tau\frac{1}{2}\sqrt{4\hbar^3\omega(\beta^2 + \rho^2)(n_- + 1) + \hbar^4\beta^4 + m\hbar^2\rho(m\rho - 2\hbar\beta^2)}. \end{aligned} \tag{47}$$

The corresponding abstract eigenstates are given by

$$
\begin{aligned}
\frac{1}{\mathcal{N}_{(n_-,\pm)}}|\Psi\rangle ={}& -i\frac{2c\sqrt{\hbar\omega}}{(E^{\tau}_{(n_-,\pm)} - mc^2 - \hbar\rho)}(n_- + 1)|n_+, n_-\rangle \otimes |+,+\rangle \\
&+ i\frac{(E^{\tau}_{(n_-,\pm)} + mc^2 - \hbar\rho)}{2c\hbar\beta} \times \left[\frac{4c^2\hbar\omega}{(E^{\tau}_{(n_-,\pm)} - \hbar\rho)^2 - m^2c^4}(n_- + 1) - 1\right]|n_+ + 1, n_-\rangle \otimes |+,-\rangle \\
&+ |n_+ + 1, n_-\rangle \otimes |-,+\rangle \\
&+ \frac{\sqrt{\hbar\omega}}{\hbar\beta}\frac{(E^{\tau}_{(n_-,\pm)} + mc^2 - \hbar\rho)}{(E^{\tau}_{(n_-,\pm)} + mc^2 + \hbar\rho)} \times \left[1 - \frac{4c^2\hbar\omega}{(E^{\tau}_{(n_-,\pm)} - \hbar\rho)^2 - m^2c^4}(n_- + 1)\right]|n_+ + 2, n_-\rangle \otimes |-,-\rangle,
\end{aligned} \tag{48}
$$

with $\mathcal{N}_{(n_-,\pm)}$ being a normalization factor still to be determined from some independent requirement. $|n_+, n_-\rangle = \frac{1}{\sqrt{(n_+)!(n_-)!}}(a_+^{\dagger})^{n_+}(a_-^{\dagger})^{n_-}|\Omega_0\rangle$ are normalized chiral Fock states. It may be shown that, in the configuration space representation, the wave functions of these chiral Fock states are given as[40]

$$
\langle x_1, x_2|n_+, n_-\rangle = \frac{(-1)^n}{\sqrt{2\pi\hbar}}\sqrt{\frac{n!}{(n+|\ell|)!}}\,u^{\frac{|\ell|}{2}}e^{i\ell\theta}e^{-\frac{r}{2}}L_n^{|\ell|}(r), \tag{49}
$$

where $\ell = n_+ - n_-$, $n = \min(n_-, n_+) = n_- + \frac{1}{2}(\ell - |\ell|)$ and $L_n^{|\ell|}(r)$ are the generalized Laguerre polynomials, while $r = \frac{\omega}{\hbar}(x_1^2 + x_2^2)$, $e^{i\theta} = \frac{x_1 + ix_2}{\sqrt{x_1^2 + x_2^2}}$.

## V. GRAPHICAL REPRESENTATIONS OF THE ENERGY LEVELS

In this section, we present some graphics representing energies as functions of various values of non-Abelian field parameters for the sake of more transparency and also compare them with existing results in the literature. Moreover, this will allow us to appreciate the effects of non-Abelian field as compared to the Abelian case. Let us adopt the following notations: $m = 1, c = 1, \hbar = 1$. Figures 1 and 2 show the graphic representations of the eigenvalues (46) for various values of the non-Abelian field parameters, while Fig. 3 presents the non-relativistic approximation of these eigenvalues.

For $\beta \to 0$ and $\rho \to 0$, we recover the standard Abelian magnetic field $B$ and the expected result, which is that the Landau levels are doubly degenerate. In addition, the non-Abelian parameters remove the degeneracy and the Landau levels are then shown.

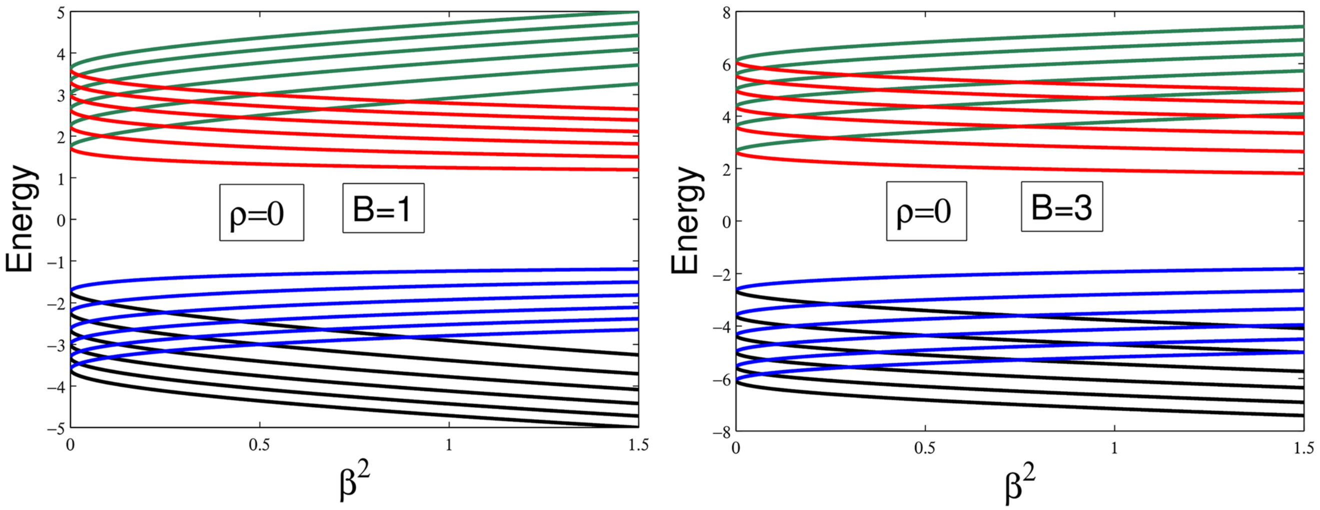

**FIG. 1.** $E^{\tau}_{(n_-,\pm)}$ as a function of $\beta^2$ for $n_- = 0, \ldots, 5$ and for a fixed value of the quantum number $n_+$. The black and blue lines represent the eigenvalues $E^{+}_{(n_-,-)}$ and $E^{-}_{(n_-,-)}$, respectively, while the red and green lines represent the eigenvalues $E^{-}_{(n_-,+)}$ and $E^{+}_{(n_-,+)}$, respectively. For $\beta \to 0$, the Landau levels are doubly degenerate.

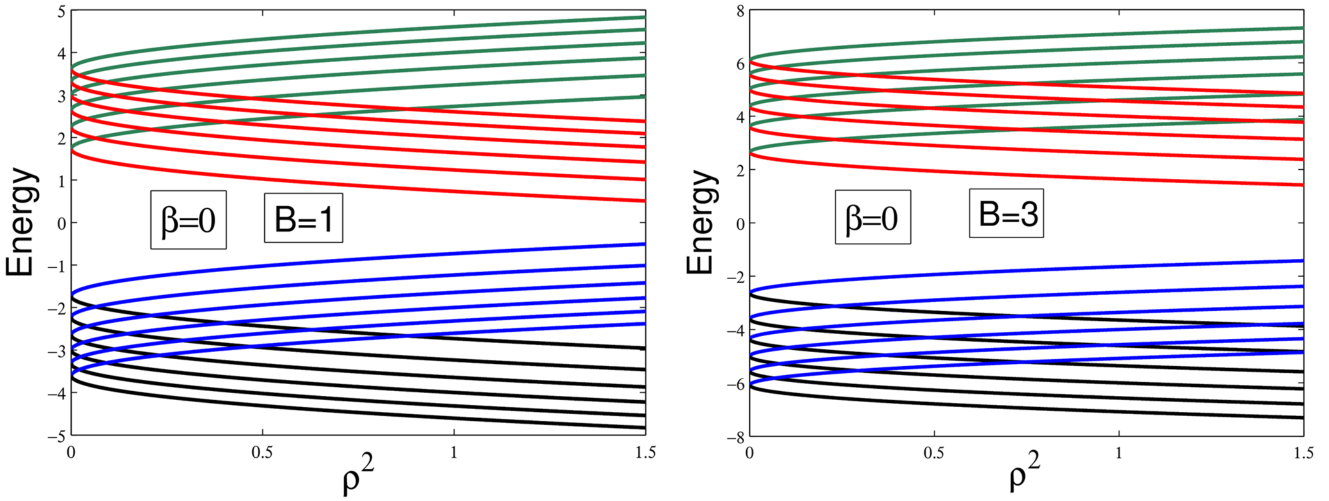

**FIG. 2.** Plots of $E^{\tau}_{(n_-,\pm)}$ as a function of $\rho^2$ for $n_- = 0, \ldots, 5$ and for a fixed value of the quantum number $n_+$. The black lines represent the eigenvalues $E^{+}_{(n_-,-)}$, the blue lines represent the eigenvalues $E^{-}_{(n_-,-)}$, the red lines represent the eigenvalues $E^{-}_{(n_-,+)}$, and the green lines represent the eigenvalues $E^{+}_{(n_-,+)}$. For $\rho \to 0$ the Landau levels are doubly degenerate.

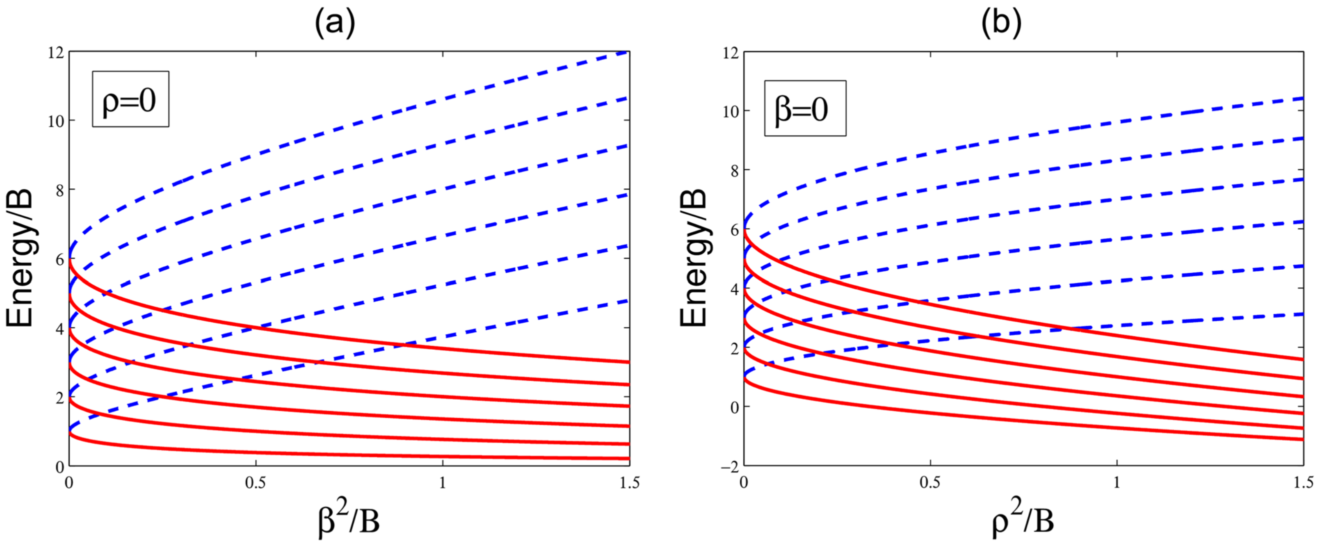

**FIG. 3.** $E^{NR}_{(n_-,\pm)}$ (the non-relativistic approximation of $E^{\tau}_{(n_-,\pm)}$), as a function of $\beta^2/B$ in (a) and as a function of $\rho^2/B$ in (b), for $n_- = 0, \ldots, 5$ and for a fixed value of the quantum number $n_+$. The red solid and blue dashed lines represent the eigenvalues $E^{NR}_{(n_-,-)}$ and $E^{NR}_{(n_-,+)}$, respectively. For $\beta \to 0$ and $\rho \to 0$, the Landau levels are doubly degenerate.

## VI. RESULTS AND DISCUSSIONS

We have exactly solved, in this paper, the Hamiltonian describing the relativistic dynamics of a particle carrying a non-Abelian charge in a non-Abelian background electromagnetic field. The results found are, to some large extent, original and are in well agreement with those obtained in the literature for particular cases. For example, when we remove the contribution of the non-Abelian terms, namely, by setting $\beta = 0$ and $\rho = 0$, we obtain the two-dimensional Dirac equation in the presence of an external Abelian magnetic field and the eigenvalues (46) become as expected

$$E_{(n_-,\pm)} = \pm\sqrt{4c^2\hbar\omega(n_- + 1) + m^2c^4}. \tag{50}$$

The corresponding eigenstates are given by

$$\begin{aligned}\frac{1}{N_{(n_-,\pm)}}|\varphi\rangle &= -i\frac{2c\sqrt{\hbar\omega}}{(E_{(n_-,\pm)} - mc^2)}(n_- + 1)|n_+, n_-\rangle \otimes |+,+\rangle \\ &\quad + |n_+ + 1, n_-\rangle \otimes |-,+\rangle,\end{aligned} \tag{51}$$

with $N_{(n_-,\pm)}$ being a normalization factor. We recover exactly the same energy spectra as in Refs. 8 and 9.

Furthermore, the Dirac points play a central role in many phenomena in condensed matter physics. One meet them, in particular, in the graphene, where they favor electron conduction in ultra-relativistic movement. Our two-dimensional non-Abelian Dirac model can describe such properties of electrons in graphene. Indeed, it is sufficient that the energy of the model is identified with that of an ultra-relativistic particle in the zero mass limit. Effectively, for massless charge carriers, the energy spectrum (46) reduces to

$$E^{\tau}_{(n_-,\pm)} = \pm\sqrt{4c^2\mathcal{P}^2_{(n_-,\tau)} + \hbar^2\rho^2}, \tag{52}$$

where

$$\mathcal{P}^2_{(n_-,\tau)} = \hbar\omega(n_- + 1) + \frac{\hbar^2\beta^2}{2} + \tau\frac{1}{2}\sqrt{4\hbar^3\omega(\beta^2 + \rho^2)(n_- + 1) + \hbar^4\beta^4}. \tag{53}$$

The above result is discussed in Refs. 43 and 44, where the relativistic particle has been identified to the neutrino in the Abelian magnetic field. In these works, the velocity of the light is replaced by the Fermi velocity, which is approximate to $v_F \approx \frac{1}{300}c$.

On the other hand, for $m \neq 0$ and $\hbar\rho \ll mc^2$, the non-relativistic limit of (46) leads to the following result:

$$\begin{aligned}E^{NR}_{(n_-,\pm)} &= \hbar\omega_c(n_- + 1) + \frac{\hbar^2\beta^2}{m} \\ &\quad \pm\hbar\sqrt{2\frac{\hbar\omega_c}{m}(\beta^2 + \rho^2)(n_- + 1) + \frac{\hbar^2\beta^4}{m^2} + \frac{\rho}{m}(m\rho - 2\hbar\beta^2)},\end{aligned} \tag{54}$$

where $E^{NR}_{(n_-,\pm)}$ is the non-relativistic limit of the energy spectrum and $\omega_c = \frac{B}{m}$ is the cyclotron frequency. For the case $\rho = 0$, i.e., for a vanishing non-Abelian electric field, the energy spectrum (54) is similar to that found in Ref. 38.

## VII. CONCLUSION

We extracted the Dirac equation from the Lagrangian describing the interaction between the non-Abelian vector field and a spin-1/2 field in the classical limit. The quantum dynamics of a particle with non-Abelian charge has been obtained by means of a direct quantization procedure, paying attention to the physical dimension of the obtained quantum variables. Wong's equation, which generalizes the Lorentz force in the non-Abelian case, is obtained. Exact solutions are then obtained for the Hamiltonian describing the relativistic dynamics of a particle carrying a non-Abelian charge in the presence of non-Abelian external electromagnetic field. The analytical expressions of the wave functions describing the relativistic energy levels are obtained in terms of the generalized Laguerre polynomials. We found that in the limit where $\beta$ and $\rho$ tend to zero, the standard Abelian magnetic field $B$ is recovered. Moreover, the expected result, which is that the Landau levels are doubly degenerate, is obtained. We have also found that, similarly to the graphene case, the so-called Dirac points are present in the energy spectrum. Around these points, the dispersion relation is linear and atoms behave like massless particles moving with a modified speed tending to that of the light. In a forthcoming paper, we will investigate various generalizations of the present model, namely, the Dirac equation with a non-Abelian charged with nonlinear couplings or the non-Abelian Dirac equation in noncommutative phase space.

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## APPENDIX A: ON THE DERIVATION OF THE SPECTRUM OF THE TOTAL ANGULAR MOMENTUM

Here, we give some details about the determination of the total angular momentum quantum states. In a Bargmann–Fock space setting, the angular momentum has the following differential representation:

$$L = \hbar\begin{pmatrix} x_+\partial_{x_+} - x_-\partial_{x_-} + 1 & 0 & 0 & 0 \\ 0 & x_+\partial_{x_+} - x_-\partial_{x_-} & 0 & 0 \\ 0 & 0 & x_+\partial_{x_+} - x_-\partial_{x_-} & 0 \\ 0 & 0 & 0 & x_+\partial_{x_+} - x_-\partial_{x_-} - 1 \end{pmatrix}. \tag{A1}$$

We look for eigenstates expressed in terms of eigenfunctions of separate variables. We then write

$$|\Psi\rangle = \begin{pmatrix} \phi_+^{(1)}(x_+)\phi_-^{(1)}(x_-) \\ \phi_+^{(2)}(x_+)\phi_-^{(2)}(x_-) \\ \phi_+^{(3)}(x_+)\phi_-^{(3)}(x_-) \\ \phi_+^{(4)}(x_+)\phi_-^{(4)}(x_-) \end{pmatrix}. \tag{A2}$$

The eigenvalue equation $L|\Psi\rangle = \hbar\kappa|\Psi\rangle$, where $\kappa$ stands for the quantum number, splits as follows:

$$\hbar(x_+\partial_{x_+} - x_-\partial_{x_-})\phi_+^{(1)}(x_+)\phi_-^{(1)}(x_-) = \hbar(\kappa - 1)\phi_+^{(1)}(x_+)\phi_-^{(1)}x_-\big), \tag{A3}$$

$$\hbar(x_+\partial_{x_+} - x_-\partial_{x_-})\phi_+^{(2)}(x_+)\phi_-^{(2)}(x_-) = \hbar\kappa\phi_+^{(2)}(x_+)\phi_-^{(2)}(x_-), \tag{A4}$$

$$\hbar(x_+\partial_{x_+} - x_-\partial_{x_-})\phi_+^{(3)}(x_+)\phi_-^{(3)}(x_-) = \hbar\kappa\phi_+^{(3)}(x_+)\phi_-^{(3)}(x_-), \tag{A5}$$

$$\hbar(x_+\partial_{x_+} - x_-\partial_{x_-})\phi_+^{(4)}(x_+)\phi_-^{(4)}(x_-) = \hbar(\kappa + 1)\phi_+^{(4)}(x_+)\phi_-^{(4)}(x_-). \tag{A6}$$

Equation (A3) may be rewritten as follows:

$$\frac{x_+\partial x_+\phi_+^{(1)}(x_+)}{\phi_+^{(1)}(x_+)} - \frac{x_-\partial x_-\phi_-^{(1)}(x_-)}{\phi_-^{(1)}(x_-)} = \kappa - 1. \tag{A7}$$

The solution of Eq. (A7) is given by

$$\phi_+^{(1)}(x_+) = c_+^{(1)}x_+^{n_+},\quad \phi_-^{(1)}(x_+) = c_-^{(1)}x_-^{n_-}, \tag{A8}$$

where $c_\pm^{(1)}$ are arbitrary coefficients. We require that the constants $n_\pm$ satisfy the relation $n_+ - n_- = \kappa - 1$. If the functions $\phi_\pm^{(1)}$ are analytical in the complex plane, the constants $n_\pm$ must take non-negative integer values. Therefore, we have $\kappa = \ell + 1$, $\ell$ running through non-negative integer values. Then, we obtain

$$\begin{aligned} \phi_+^{(1)}(x_+)\phi_-^{(1)}(x_-) &= c_+^{(1)}c_-^{(1)}x_+^{\ell}(x_+x_-)^{n_-} \\ &= x_+^{\ell}\varphi^{(1)}(x_+x_-), \end{aligned} \tag{A9}$$

where $n_-$ and $\ell$ are non-negative integers. $\varphi^{(1)}(x_+x_-)$ being the analytical functions in the complex plane. The solutions for Eqs. (A4)–(A6) may be constructed in the same manner. We have

$$\begin{aligned} \phi_+^{(2)}(x_+)\phi_-^{(2)}(x_-) &= c_+^{(2)}c_-^{(2)}x_+^{\ell+1}(x_+x_-)^{n_-} \\ &= x_+^{\ell+1}\varphi^{(2)}(x_+x_-), \end{aligned} \tag{A10}$$

$$\begin{aligned} \phi_+^{(3)}(x_+)\phi_-^{(3)}(x_-) &= c_+^{(3)}c_-^{(3)}x_+^{\ell+1}(x_+x_-)^{n_-} \\ &= x_+^{\ell+1}\varphi^{(3)}(x_+x_-), \end{aligned} \tag{A11}$$

$$\begin{aligned} \phi_+^{(4)}(x_+)\phi_-^{(4)}(x_-) &= c_+^{(4)}c_-^{(4)}x_+^{\ell+1}(x_+x_-)^{n_-} \\ &= x_+^{\ell+2}\varphi^{(4)}(x_+x_-). \end{aligned} \tag{A12}$$

In conclusion, the abstract spectrum for the total angular momentum presents as follows:

$$|\Psi\rangle = \begin{pmatrix} (a_+^\dagger)^{\ell}\varphi^{(1)}(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \\ (a_+^\dagger)^{\ell+1}\varphi^{(2)}(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \\ (a_+^\dagger)^{\ell+1}\varphi^{(3)}(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \\ (a_+^\dagger)^{\ell+2}\varphi^{(4)}(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \end{pmatrix}, \ \hbar\kappa = \hbar(\ell+1), \ \ell \geq 0. \tag{A13}$$

The same procedure also applies for Eqs. (A3)–(A6) with $\ell < 0$. The eigenstates and the eigenvalues of the angular momentum $L$ are given by

$$|\Psi\rangle = \begin{pmatrix} (a_+^\dagger)^{-(\ell)}\varphi^{(1)}(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \\ (a_+^\dagger)^{-(\ell+1)}\varphi^{(2)}(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \\ (a_+^\dagger)^{-(\ell+1)}\varphi^{(3)}(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \\ (a_+^\dagger)^{-(\ell+2)}\varphi^{(4)}(a_+^\dagger a_-^\dagger)|\Omega_0\rangle \end{pmatrix}, \ \hbar\kappa = \hbar(\ell+1), \ \ell < 0. \tag{A14}$$

## APPENDIX B: ON THE DERIVATION OF THE SPECTRUM OF THE HAMILTONIAN *H*

We provide in the following some details of the eigenvalue of the non-Abelian Dirac Hamiltonian. Substituting the state (A13) into the Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$, we get the following system of equations:

$$\begin{aligned} &(mc^2+\rho)\varphi^{(1)}(a_+^\dagger a_-^\dagger) - 2ic\sqrt{\hbar\omega}a_+a_+^\dagger\varphi^{(3)}(a_+^\dagger a_-^\dagger) = E\varphi^{(1)}(a_+^\dagger a_-^\dagger), \\ &(mc^2-\rho)\varphi^{(2)}(a_+^\dagger a_-^\dagger) - 2ic\hbar\beta\varphi^{(3)}(a_+^\dagger a_-^\dagger) - 2ic\sqrt{\hbar\omega}a_+a_+^\dagger\varphi^{(4)}(a_+^\dagger a_-^\dagger) = E\varphi^{(2)}(a_+^\dagger a_-^\dagger), \\ &2ic\sqrt{\hbar\omega}\varphi^{(1)}(a_+^\dagger a_-^\dagger) + 2ic\hbar\beta\varphi^{(2)}(a_+^\dagger a_-^\dagger) - (mc^2-\rho)\varphi^{(3)}(a_+^\dagger a_-^\dagger) = E\varphi^{(3)}(a_+^\dagger a_-^\dagger), \\ &2ic\sqrt{\hbar\omega}\varphi^{(2)}(a_+^\dagger a_-^\dagger) - (mc^2+\rho)\varphi^{(4)}(a_+^\dagger a_-^\dagger) = E\varphi^{(4)}(a_+^\dagger a_-^\dagger), \end{aligned} \tag{B1}$$

where the expression of the Hamiltonian (28) has been used.

Clearly, the system of Eq. (B1) reduces as follows:

$$\Big[\hbar^2\omega^2\varepsilon(a_+^\dagger a_+)^2 + \hbar\omega\varepsilon[3\hbar\omega - \varepsilon_+ + \varepsilon_-]a_+^\dagger a_+ + \varepsilon\hbar\omega(\hbar\omega - \varepsilon_+ - \varepsilon_-) - \varepsilon_+\varepsilon_-(\varepsilon - \hbar^2\beta^2)\Big]\varphi^{(3)}(a_+^\dagger a_-^\dagger) = 0, \tag{B2}$$

where

$$\varepsilon_\pm = \frac{(E+mc^2)(E-mc^2) \pm 2\hbar\rho E}{4c^2}, \quad \varepsilon = \frac{(E+mc^2)(E-mc^2) + 2\hbar\rho mc^2}{4c^2}. \tag{B3}$$

In a Bargmann–Fock space by using the expression of the analytical function $\varphi^{(3)}(x_+x_-)$, Eq. (B2) becomes

$$\Big[\hbar^2\omega^2\varepsilon n_-(n_- - 1) + \hbar\omega\varepsilon[3\hbar\omega - \varepsilon_+ + \varepsilon_-]n_- + \varepsilon\hbar\omega(\hbar\omega - \varepsilon_+ - \varepsilon_-) - \varepsilon_+\varepsilon_-(\varepsilon - \hbar^2\beta^2)\Big] = 0. \tag{B4}$$

A straightforward computation of Eq. (B4) gives the following expressions for the energy spectrum:

$$E^{\tau}_{(n_-,\pm)} = \pm\sqrt{4c^2\mathcal{P}^2_{(n_-,\tau)} + m^2c^4 + \hbar^2\rho^2}, \quad \tau = \pm, \ n_- = 0, 1, 2, \ldots, \tag{B5}$$

where

$$\mathcal{P}^2_{(n_-,\tau)} = \hbar\omega(n_- + 1) + \frac{\hbar^2\beta^2}{2} + \tau\frac{1}{2}\sqrt{4\hbar^3\omega(\beta^2+\rho^2)(n_- + 1) + \hbar^4\beta^4 + m\hbar^2\rho(m\rho - 2\hbar\beta^2)}. \tag{B6}$$

The corresponding eigenfunctions are given by

$$|\Psi\rangle = \begin{pmatrix} (x_+)^{\ell}\varphi^{(1)}(x_+x_-) \\ (x_+)^{\ell+1}\varphi^{(2)}(x_+x_-) \\ (x_+)^{\ell+1}\varphi^{(3)}(x_+x_-) \\ (x_+)^{\ell+2}\varphi^{(4)}(x_+x_-) \end{pmatrix}, \tag{B7}$$

where

$$\varphi^{(1)}(x_+x_-) = -i\frac{2c\sqrt{\hbar\omega}}{(E^{\tau}_{(n_-,\pm)} - mc^2 - \hbar\rho)}(n_- + 1)\varphi^{(3)}(x_+x_-), \tag{B8}$$

$$\varphi^{(2)}(x_+x_-) = i\frac{(E^{\tau}_{(n_-,\pm)} + mc^2 - \hbar\rho)}{2c\hbar\beta}\left[\frac{4c^2\hbar\omega}{(E^{\tau}_{(n_-,\pm)} - \hbar\rho)^2 - m^2c^4}(n_- + 1) - 1\right]\varphi^{(3)}(x_+x_-), \tag{B9}$$

$$\varphi^{(4)}(x_+x_-) = \frac{\sqrt{\hbar\omega}}{\hbar\beta}\frac{(E^{\tau}_{(n_-,\pm)} + mc^2 - \hbar\rho)}{(E^{\tau}_{(n_-,\pm)} + mc^2 + \hbar\rho)}\left[1 - \frac{4c^2\hbar\omega}{(E^{\tau}_{(n_-,\pm)} - \hbar\rho)^2 - m^2c^4}(n_- + 1)\right]\varphi^{(3)}(x_+x_-), \tag{B10}$$

$$\varphi^{(3)}(x_+x_-) = c(3)(x_+x_-)^{n_-}, \tag{B11}$$

where $c(3)$ is an appropriate normalization.

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