

Research Article

Error Estimation of Euler Method for the Instationary Stokes–Biot Coupled Problem

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In this paper, we study a finite element computational model for solving the interaction between a fluid and a poroelastic structure that couples the Stokes equations with the Biot system. Equilibrium and kinematic conditions are imposed on the interface. A mixed Darcy formulation is employed, resulting in continuity of flux condition of essential type. A Lagrange multiplier method is used to impose weakly this condition. With the obtained finite element solutions, the error estimators are performed for the fully discrete formulations.

1. Introduction

The paper presents a computation friendly finite element formulation of coupled fluid motions at a free-poroelastic interface, where Navier–Stokes equations are employed for free fluid and Darcy's law is used for the permeable material. A reliable and efficient a posteriori error estimator is analysed. This is a challenging multiphysics problem that has a wide range of applications, including processes arising in gas and oil extraction from naturally or hydraulically fractured reservoirs, designing industrial filters, and blood-vessel interaction. In these applications, it is important to model properly the interaction between the free fluid with the fluid within the porous medium and to take into account the effect of the deformation of the medium. For example, geomechanical effects play an important role in hydraulic fracturing, as well as in modeling phenomena such as subsidence and compaction.

The free fluid region can be modeled by the Stokes or Navier–Stokes equations, while the flow through the deformable porous medium is modeled by the quasi-static Biot systems of poroelasticity [1]. The two regions are coupled via dynamic and kinematic interface conditions, including

balance of forces, continuity of normal velocity, and a no slip or slip with friction tangential velocity condition [2–10]. These multiphysics models exhibit features of coupled Stokes–Darcy flows and fluid-structure interaction (FSI) [11–15].

The well-posedness of the mathematical model based on the Stokes–Biot system for the coupling between a fluid and a poroelastic structure is studied in [16]. A numerical study of the problem, using a Navier–Stokes equations for the fluid, is presented in [11, 17], utilizing a variational multi-scale approach to stabilize the finite element spaces. The problem is solved using both a monolithic and a partitioned approach, with the latter requiring subiterations between the two problems.

Finite element analysis of an arbitrary Lagrangian–Eulerian method for Stokes/parabolic moving interface problem with jump coefficients has been studied in [18]. The authors in [19] studied a numerical solution of the coupled system of the time-dependent Stokes and fully dynamic Biot equations. They established stability of the scheme and derived error estimates for the fully discrete coupled scheme. Numerical errors and convergence rates for smooth problems as well as tests on realistic material parameters have

been presented. In [20], Jing Wen and Yinnian He considered a strongly conservative discretization for the rearranged Stokes–Biot model based on the interior penalty discontinuous Galerkin method and mixed finite element method. The existence and uniqueness of solution of the numerical scheme have been presented. Then, the analysis of stability and priori error estimates has been derived. The numerical examples under uniform meshes which well validate the analysis of convergence and the strong mass conservation are presented. A staggered finite element procedure for the coupled Stokes–Biot system with fluid entry resistance has been studied by Bergkamp et al. in [21] while Ambartsumyan et al. in [22] studied flow and transport in fractured poroelastic media using Stokes flow in the fractures and the Biot model in the porous media. In Section 6 in [23], fully discrete continuous approximation has been proposed for the weak coupled mixed formulation. For the discretization of the fluid velocity and pressure, the authors have used the finite elements which include the MINI-elements, the Taylor–Hood elements, and the conforming Crouzeix–Raviart elements. For the discretization of the porous medium problem, they choose the spaces that include Raviart–Thomas and Brezzi–Douglas–Marini elements. An a priori error analysis is performed with some numerical tests confirming the convergence rates.

A posteriori error estimators are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. They are essential to design adaptive mesh refinement algorithms which aquidistribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuška and Rheinboldt [24–27], adaptive finite element methods based on a posteriori error estimates have been extensively investigated.

In [28], we have proposed a family error indicator for semidiscrete approximation for the Stokes–Biot system. To the best of our knowledge, there is no a posteriori error estimation for the Stokes–Biot fluid–poroelastic structure interaction model for fully discrete finite element methods. Here, we develop such a posteriori error analysis for the fully discrete conforming finite element methods. We have got a new family of a local indicator error Θ_K (see equation (59) in Definition 3) and global Θ (equation (60)). The difference between this paper and our previous work [28] is that our discretization is fully discrete formulation. As an advantage, the error indicators in this work are more accessible to computation.

The schedule of the paper is as follows. Section 2 is devoted to notations and basic results that are used throughout the document. Our main results regarding a posteriori error analysis are stated in Section 3. We prove that our indicator errors are efficient, reliable, and then optimal. The global inf-sup condition is the main tool yielding the reliability. In turn, the local efficiency result is derived using the technique of bubble function introduced by R. Verfürth [29] and used in similar context by C. Carstensen [30]. Finally, this paper is summarized with further works in Section 4.

2. Preliminaries and Notations

2.1. Stokes–Biot Model Problem. We consider a multiphysics model problem for free fluid’s interaction with a flow in a deformable porous media, where the simulation domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a union of nonoverlapping regions Ω_f and Ω_p . Here, Ω_f is a free fluid region with flow governed by the Stokes equations and Ω_p is a poroelastic material governed by the Biot system. For simplicity of notation, we assume that each region is connected. The extension to nonconnected regions is straightforward. Let $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$ (see Figure 1).

Let (\mathbf{u}_*, p_*) be the velocity–pressure pair in Ω_* , $* = f, p$, and let η_p be the displacement in Ω_p . Let $\mu > 0$ be the fluid viscosity, let \mathbf{f}_* be the body force terms, and let q_* be external source or sink terms. Let $\mathbf{D}(\mathbf{u}_f)$ and $\sigma_f(\mathbf{u}_f, p_f)$ denote, respectively, the deformation rate tensor and the stress tensor:

$$\mathbf{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla \mathbf{u}_f + \nabla \mathbf{u}_f^T), \quad (1)$$

$$\sigma_f(\mathbf{u}_f, p_f) = -p_f \mathbf{I} + 2\mu \mathbf{D}(\mathbf{u}_f).$$

In the free fluid region Ω_f , (\mathbf{u}_f, p_f) satisfies the Stokes equations:

$$-\nabla \cdot \sigma_f(\mathbf{u}_f, p_f) = \mathbf{f}_f, \quad \text{in } \Omega_f \times (0, T], \quad (2)$$

$$\nabla \cdot \mathbf{u}_f = q_f, \quad \text{in } \Omega_f \times (0, T], \quad (3)$$

where $T > 0$ is the final time. Let $\sigma_e(\eta_p)$ and $\sigma_p(\eta_p, p_p)$ be the elastic and poroelastic stress tensors, respectively:

$$\sigma_e(\eta_p) = \lambda_p(\nabla \cdot \eta_p) \mathbf{I} + 2\mu_p \mathbf{D}(\eta_p), \quad (4)$$

$$\sigma_p(\eta_p, p_p) = \sigma_e(\eta_p) - \alpha p_p \mathbf{I},$$

where $0 < \lambda_{\min} \leq \lambda_p(\mathbf{x}) \leq \lambda_{\max}$ and $0 < \mu_{\min} \leq \mu_p(\mathbf{x}) \leq \mu_{\max}$ are the Lamé parameters and $0 < \alpha \leq 1$ is the Biot–Willis constant. The poroelasticity region Ω_p is governed by the quasi-static Biot system [23]:

$$-\nabla \cdot \sigma_p(\eta_p, p_p) = \mathbf{f}_p, \quad (5)$$

$$\mu K^{-1} \mathbf{u}_p + \nabla p_p = 0, \quad \text{in } \Omega_p \times (0, T],$$

$$\frac{\partial}{\partial t}(s_0 p_p + \alpha \nabla \cdot \eta_p) + \nabla \cdot \mathbf{u}_p = q_p, \quad \text{in } \Omega_p \times (0, T], \quad (6)$$

where $s_0 \geq 0$ is a storage coefficient and K is the symmetric and uniformly positive definite rock permeability tensor, satisfying, for some constants $0 < k_{\min} \leq k_{\max}$:

$$\begin{aligned} \forall \xi \in \mathbb{R}^d, \\ k_{\min} \xi^T \xi \leq \xi^T K(\mathbf{x}) \xi \leq k_{\max} \xi^T \xi, \quad \forall \mathbf{x} \in \Omega_p. \end{aligned} \quad (7)$$

Following [1], the interface conditions on the fluid–poroelasticity interface Γ_{fp} are mass conservation, balance of stresses, and the Beavers–Joseph–Saffman (BJS) condition [31] modeling slip with friction:

$$\begin{aligned} \mathbf{V} &= \{\mathbf{v} = (\mathbf{v}_f, \mathbf{v}_p) \in \mathbf{V}_f \times \mathbf{V}_p\}, \\ W &= \{w = (w_f, w_p) \in W_f \times W_p\}, \end{aligned} \quad (18)$$

with norms

$$\begin{aligned} \|\mathbf{v}\|_{\mathbf{V}}^2 &:= \|\mathbf{v}_f\|_{\Omega_f}^2 + \|\mathbf{v}_p\|_{H(\text{div}; \Omega_p)}^2, \\ \|w\|_W^2 &= \|w_f\|_{\Omega_f}^2 + \|w_p\|_{\Omega_p}^2. \end{aligned} \quad (19)$$

The weak formulation is obtained by multiplying the equations in each region by suitable test functions, integrating by parts the second-order terms in space, and utilizing the interface and boundary conditions.

Let

$$\begin{aligned} a_f(\mathbf{u}_f, \mathbf{v}_f) &:= (2\mu \mathbf{D}(\mathbf{u}_f), \mathbf{D}(\mathbf{v}_f))_{\Omega_f}, \\ a_p^D(\mathbf{u}_p, \mathbf{v}_p) &:= (\mu K^{-1} \mathbf{u}_p, \mathbf{v}_p)_{\Omega_p}, \\ a_p^e(\eta_p, \xi_p) &:= (2\mu_p \mathbf{D}(\eta_p), \mathbf{D}(\xi_p))_{\Omega_p} + (\lambda_p \nabla \cdot \eta_p, \nabla \cdot \xi_p)_{\Omega_p}, \end{aligned} \quad (20)$$

be the bilinear forms related to Stokes, Darcy, and the elasticity operator, respectively. Let

$$b_*(\mathbf{v}, w) := -(\nabla \cdot \mathbf{v}, w)_{\Omega_*}. \quad (21)$$

Integration by parts in (2) and the two equations in (5) leads to the interface term as follows:

$$I_{\Gamma_{fp}} = -\langle \sigma_f \mathbf{n}_f, \mathbf{v}_f \rangle_{\Gamma_{fp}} - \langle \sigma_p \mathbf{n}_p, \xi_p \rangle_{\Gamma_{fp}} + \langle p_p, \mathbf{v}_p \cdot \mathbf{n}_p \rangle_{\Gamma_{fp}}. \quad (22)$$

Using the first condition for balance of normal stress in (9), we set

$$\lambda = -(\sigma_f \mathbf{n}_f) \cdot \mathbf{n}_f = p_p, \quad \text{on } \Gamma_{fp}, \quad (23)$$

which will be used as a Lagrange multiplier to impose the mass conservation interface condition (8). Utilizing the BJS condition (10) and the second condition for balance of stresses in (9), we obtain

$$I_{\Gamma_{fp}} = a_{\text{BJS}}(\mathbf{u}_f, \partial_t \eta_p; \mathbf{v}_f, \xi_p) + b_{\Gamma}(\mathbf{v}_f, \mathbf{v}_p, \xi_p, \lambda), \quad (24)$$

where

$$\begin{aligned} a_{\text{BJS}}(\mathbf{u}_f, \eta_p; \mathbf{v}_f, \xi_p) &= \sum_{j=1}^{d-1} \langle \mu \alpha_{\text{BJS}} \sqrt{K_j^{-1}} (\mathbf{u}_f - \eta_p) \\ &\quad \cdot \boldsymbol{\tau}_{f,j}, (\mathbf{v}_f - \xi_p) \cdot \boldsymbol{\tau}_{f,j} \rangle_{\Gamma_{fp}}, \\ b_{\Gamma}(\mathbf{v}_f, \mathbf{v}_p, \xi_p; \mu) &= \langle \mathbf{v}_f \cdot \mathbf{n}_f + (\xi_p + \mathbf{v}_p) \cdot \mathbf{n}_p, \mu \rangle_{\Gamma_{fp}}. \end{aligned} \quad (25)$$

For the well-posedness of b_{Γ} , we require that $\lambda \in \Lambda = (\mathbf{V}_p \cdot \mathbf{n}_p|_{\Gamma_{fp}})^T$. According to the normal trace theorem, since $\mathbf{v}_p \in \mathbf{V}_p \subset H(\text{div}; \Omega_p)$, then $\mathbf{v}_p \cdot \mathbf{n}_p \in H^{-1/2}(\partial\Omega_p)$. Furthermore, since $\mathbf{v}_p \cdot \mathbf{n}_p = 0$ on Γ_p^N and $\text{dis}(\Gamma_p^D, \Gamma_{fp}) \geq s > 0$, then $\mathbf{v}_p \cdot \mathbf{n}_p \in H^{-1/2}(\Gamma_{fp})$, (see, e.g., [33]). Therefore, we take $\Lambda = H^{1/2}(\Gamma_{fp})$.

The Lagrange multiplier variational formulation is for $t \in (0, T]$, find $\mathbf{u}_f(t) \in \mathbf{V}_f$, $p_f(t) \in W_f$, $\mathbf{u}_p(t) \in \mathbf{V}_p$, $p_p(t) \in W_p$, $\eta_p(t) \in \mathbf{X}_p$, and $\lambda(t) \in \Lambda$, such that $p_p(0) = p_{p,0}$ and $\eta_p(0) = \eta_{p,0}$, and for all $\mathbf{v}_f \in \mathbf{V}_f$, $w_f \in W_f$, $\mathbf{v}_p \in \mathbf{V}_p$, $w_p \in W_p$, $\xi_p \in \mathbf{X}_p$, and $\mu \in \Lambda$,

$$\begin{aligned} a_f(\mathbf{u}_f, \mathbf{v}_f) + a_p^D(\mathbf{u}_p, \mathbf{v}_p) + a_p^e(\eta_p, \xi_p) + a_{\text{BJS}}(\mathbf{u}_f, \partial_t \eta_p; \mathbf{v}_f, \xi_p) \\ + b_f(\mathbf{v}_f, p_f) + b_p(\mathbf{v}_p, p_p) + \alpha b_p(\xi_p, p_p) + b_{\Gamma}(\mathbf{v}_f, \mathbf{v}_p, \xi_p; \lambda) = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f} + (\mathbf{f}_p, \xi_p)_{\Omega_p}, \end{aligned} \quad (26)$$

$$\begin{aligned} (s_0 \partial_t p_p, w_p)_{\Omega_p} - \alpha b_p(\partial_t \eta_p, w_p) - b_p(\mathbf{u}_p, w_p) - b_f(\mathbf{u}_f, w_f) \\ = (q_f, w_f)_{\Omega_f} + (q_p, w_p)_{\Omega_p}, \end{aligned} \quad (27)$$

$$b_{\Gamma}(\mathbf{u}_f, \mathbf{u}_p, \partial_t \eta_p; \mu) = 0, \quad (28)$$

where we used the notation $\partial_t = (\partial/\partial t)$.

The assumptions on the fluid viscosity μ and the material coefficients K , λ_p , and μ_p imply that the bilinear forms $a_f(\cdot, \cdot)$, $a_p^D(\cdot, \cdot)$, and $a_p^e(\cdot, \cdot)$ are coercive and continuous in the appropriate norms. In particular, there exist positive constants c^f , c^p , c^e , C^f , C^p , and C^e such that

$$c^f \|\mathbf{v}_f\|_{H^1(\Omega_f)}^2 \leq a_f(\mathbf{v}_f, \mathbf{v}_f), \quad \forall \mathbf{v}_f \in \mathbf{V}_f, \quad (29)$$

$$a_f(\mathbf{v}_f, \mathbf{q}_f) \leq C^f \|\mathbf{v}_f\|_{H^1(\Omega_f)} \|\mathbf{q}_f\|_{H^1(\Omega_f)}, \quad \forall \mathbf{v}_f, \mathbf{q}_f \in \mathbf{V}_f, \quad (30)$$

$$c^p \|\mathbf{v}_p\|_{L^2(\Omega_p)}^2 \leq a_p^D(\mathbf{v}_p, \mathbf{v}_p), \quad \forall \mathbf{v}_p \in \mathbf{V}_p, \quad (31)$$

$$a_p^D(\mathbf{v}_p, \mathbf{q}_p) \leq C^p \|\mathbf{v}_p\|_{L^2(\Omega_p)} \|\mathbf{q}_p\|_{L^2(\Omega_p)}, \quad \forall \mathbf{v}_p, \mathbf{q}_p \in \mathbf{V}_p, \quad (32)$$

$$c^e \|\xi_p\|_{H^1(\Omega_p)}^2 \leq a_p^e(\xi_p, \xi_p), \quad \forall \xi_p \in \mathbf{X}_p, \quad (33)$$

$$a_p^e(\xi_p, \zeta_p) \leq C^e \|\xi_p\|_{H^1(\Omega_p)} \|\zeta_p\|_{H^1(\Omega_p)}, \quad \forall \xi_p, \zeta_p \in \mathbf{X}_p, \quad (34)$$

where (29), (30), (33), and (34) hold true thanks to Poincaré inequality and (33) and (34) also relies on Korn's inequality.

In summary, from Corollary 3.1 in [23] (p. 7), the following result holds.

Theorem 1. *There exists a unique solution $(\mathbf{u}_f, p_f, \mathbf{u}_p, p_p, \eta_p, \lambda) \in L^\infty(0, T; \mathbf{V}_f) \times L^\infty(0, T; W_f) \times L^\infty(0, T; \mathbf{V}_p) \times W^{1,\infty}(0, T; W_p) \times W^{1,\infty}(0, T; \mathbf{X}_p) \times L^\infty(0, T; \Lambda)$ to problems (26)–(28).*

2.3. Finite Element Discretization. Let \mathcal{T}_h^f and \mathcal{T}_h^p be shape-regular and quasi-uniform partition of Ω_f and Ω_p , respectively, both consisting of affine elements with maximal element diameter h . The two partitions may be nonmatching

at the interface Γ_{fp} . For the discretization of the fluid velocity and pressure, we choose finite element spaces $\mathbf{V}_{f,h} \subset \mathbf{V}_f$ and $W_{f,h} \subset W_f$, which are assumed to be inf-sup stable. Examples of such spaces include the mini-elements, the Taylor–Hood elements, and the conforming Crouzeix–Raviart elements. For the discretization of the porous medium problem, we choose $\mathbf{V}_{p,h} \subset \mathbf{V}_p$ and $W_{p,h} \subset W_p$ to be any of well-known inf-sup stable mixed finite element spaces, such as the Raviart–Thomas or the Brezzi–Douglas–Marini spaces. The global spaces are as follows:

$$\begin{aligned} \mathbf{V}_h &:= \{\mathbf{v}_h = (\mathbf{v}_{f,h}, \mathbf{v}_{p,h}) \in \mathbf{V}_{f,h} \times \mathbf{V}_{p,h}\}, \\ W_h &:= \{w_h = (w_{f,h}, w_{p,h}) \in W_{f,h} \times W_{p,h}\}. \end{aligned} \quad (35)$$

We employ a conforming Lagrangian finite element space $\mathbf{X}_{p,h} \subset \mathbf{X}_p$ to approximate the structure displacement. Note that the finite element spaces $\mathbf{V}_{f,h}$, $\mathbf{V}_{p,h}$, and $\mathbf{X}_{p,h}$ satisfy the prescribed homogeneous boundary conditions on the external boundaries. For the discrete Lagrange multiplier space, we take

$$\Lambda_h = \mathbf{V}_{p,h} \cdot \mathbf{n}_p|_{\Gamma_{fp}}. \quad (36)$$

The semidiscrete continuous-in-time problem reads that given $p_{p,h}(0)$ and $\eta_{p,h}(0)$, for $t \in (0, T]$, find $\mathbf{u}_{f,h}(t) \in \mathbf{V}_{f,h}$, $p_{f,h}(t) \in W_{f,h}$, $\mathbf{u}_{p,h}(t) \in \mathbf{V}_{p,h}$, $p_{p,h}(t) \in W_{p,h}$, $\eta_{p,h}(t) \in \mathbf{X}_{p,h}$, and $\lambda_h(t) \in \Lambda_h$ such that for all $\mathbf{v}_{f,h} \in \mathbf{V}_{f,h}$, $w_{f,h} \in W_{f,h}$, $\mathbf{v}_{p,h} \in \mathbf{V}_{p,h}$, $w_{p,h} \in W_{p,h}$, $\xi_{p,h} \in \mathbf{X}_{p,h}$, and $\mu_h \in \Lambda_h$,

$$\begin{aligned} a_f(\mathbf{u}_{f,h}, \mathbf{v}_{f,h}) + a_p^D(\mathbf{u}_{p,h}, \mathbf{v}_{p,h}) + a_p^e(\eta_{p,h}, \xi_{p,h}) \\ + a_{\text{BJS}}(\mathbf{u}_{f,h}, \partial_t \eta_{p,h}; \mathbf{v}_{f,h}, \xi_{p,h}) + b_f(\mathbf{v}_{f,h}, p_{f,h}) \\ + b_p(\mathbf{v}_{p,h}, p_{p,h}) + \alpha b_p(\xi_{p,h}, p_{p,h}) + b_\Gamma(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, \xi_{p,h}; \lambda_h) = (\mathbf{f}_{f,h}, \mathbf{v}_{f,h})_{\Omega_f} + (\mathbf{f}_{p,h}, \xi_{p,h})_{\Omega_p}, \end{aligned} \quad (37)$$

$$\begin{aligned} (s_0 \partial_t p_{p,h}, w_{p,h})_{\Omega_p} - \alpha b_p(\partial_t \eta_{p,h}, w_{p,h}) - b_p(\mathbf{u}_{p,h}, w_{p,h}) - b_f(\mathbf{u}_{f,h}, w_{f,h}) \\ = (q_f, w_{f,h})_{\Omega_f} + (q_p, w_{p,h})_{\Omega_p}, \end{aligned} \quad (38)$$

$$b_\Gamma(\mathbf{u}_{f,h}, \mathbf{u}_{p,h}, \partial_t \eta_{p,h}; \mu_h) = 0. \quad (39)$$

We will take $p_{p,h}(0)$ and $\eta_{p,h}(0)$ to be suitable projections of the initial data $p_{p,0}$ and $\eta_{p,0}$.

We introduce the errors for all variables as follows:

$$\begin{aligned} \mathbf{e}_f &:= \mathbf{u}_f - \mathbf{u}_{f,h}, \\ \mathbf{e}_p &:= \mathbf{u}_p - \mathbf{u}_{p,h}, \\ \mathbf{e}_s &:= \eta_p - \eta_{p,h}, \\ e_{fp} &:= p_f - p_{f,h}, \\ e_{pp} &:= p_p - p_{p,h}, \\ e_\lambda &:= \lambda - \lambda_h. \end{aligned} \quad (40)$$

The following results hold cf. [23].

Theorem 2 (a priori error estimation). *There exists a unique solution $(\mathbf{u}_{f,h}, p_{f,h}, \mathbf{u}_{p,h}, p_{p,h}, \eta_{p,h}, \lambda_h)$ in $L^\infty(0, T; \mathbf{V}_{f,h}) \times L^\infty(0, T; W_{f,h}) \times L^\infty(0, T; \mathbf{V}_{p,h}) \times W^{1,\infty}(0, T; W_{p,h}) \times W^{1,\infty}(0, T; \mathbf{X}_{p,h}) \times L^\infty(0, T; \Lambda_h)$ of the discrete problems (37)–(39), and if the solution $(\mathbf{u}_f, p_f, \mathbf{u}_p, p_p, \eta_p, \lambda) \in \mathbf{V}_f \times W_f \times \mathbf{V}_p \times W_p \times \mathbf{X}_p \times \Lambda$ of the continuous problems (26)–(28) is smooth enough, then we have*

$$\begin{aligned} \|\mathbf{e}_f\|_{L^2(0, T; H^1(\Omega_f))} + \|\mathbf{e}_p\|_{L^2(0, T; L^2(\Omega_p))} \\ + \|\mathbf{e}_s\|_{L^\infty(0, T; H^1(\Omega_p))} + \|e_{pp}\|_{L^\infty(0, T; L^2(\Omega_p))} \\ + \|e_{fp}\|_{L^2(0, T; L^2(\Omega_f))} + \|e_\lambda\|_{L^2(0, T; \Lambda_h)} \\ \leq C(\mathbf{u}_f, p_f, \mathbf{u}_p, p_p, \eta_p, \lambda) h^r, \quad r > 0. \end{aligned} \quad (41)$$

Remark 1. For $\mathbf{W}_h = (\mathbf{v}_{f,h}, w_{f,h}, \mathbf{v}_{p,h}, w_{p,h}, \xi_{p,h}, \lambda_h) \in L^\infty(0, T; \mathbf{V}_{f,h}) \times L^\infty(0, T; W_{f,h}) \times L^\infty(0, T; \mathbf{V}_{p,h}) \times W^{1,\infty}(0, T; W_{p,h}) \times W^{1,\infty}(0, T; \mathbf{X}_{p,h}) \times L^\infty(0, T; \Lambda_h)$, we can subtract

(37)–(39) to (26)–(28) to obtain the Galerkin orthogonality relation for all $t \in]0, T]$:

$$\begin{aligned} & a_f(\mathbf{e}_f(t), \mathbf{v}_{f,h}(t)) + a_p^D(\mathbf{e}_s(t), \xi_{p,h}(t)) + a_{\text{BJS}}(\mathbf{e}_f(t), \partial_t \mathbf{e}_s(t); \mathbf{v}_{f,h}(t), \xi_{p,h}(t)) \\ & + b_f(\mathbf{v}_{f,h}(t), e_f(t)) + b_p(\mathbf{v}_{p,h}(t), e_p(t)) + \alpha b_p(\xi_p(t), e_p(t)) \\ & + b_\Gamma(\mathbf{v}_f(t), \mathbf{v}_p(t), \xi_p(t), e_\lambda(t)) + (s_0 \partial_t e_p(t), w_{p,h}(t)) \\ & - \alpha b_p(\partial_t(\mathbf{e}_s(t), w_{p,h}(t)) - b_p(\mathbf{e}_p(t), w_{p,h}(t)) - b_f(\mathbf{e}_f(t), w_{f,h}(t)) = 0. \end{aligned} \quad (42)$$

2.4. Fully Discrete Formulation. For the time discretization, we employ the backward Euler method. Let τ be the time step, $T = N\tau$, and let $t_n = n\tau$, $0 \leq n \leq N$. Let $\delta_\tau u^n := (u^n - u^{n-1})/\tau$ be the first-order (backward) discrete time derivation, where $u^n := u(t_n)$. Then, the fully discrete model reads

that given $p_{p,h}^0 = p_{p,h}(0)$ and $\eta_{p,h}^0 = \eta_{p,h}(0)$, find $\mathbf{u}_{f,h}^n \in \mathbf{V}_{f,h}$, $p_{f,h}^n \in W_{f,h}$, $\mathbf{u}_{p,h}^n \in \mathbf{V}_{p,h}$, $p_{p,h}^n \in W_{p,h}$, $\eta_{p,h}^n \in \mathbf{X}_{p,h}$, and $\lambda_h^n \in \Lambda_h$, $1 \leq n \leq N$, such that for all $\mathbf{v}_{f,h} \in \mathbf{V}_{f,h}$, $w_{f,h} \in W_{f,h}$, $\mathbf{v}_{p,h} \in \mathbf{V}_{p,h}$, $w_{p,h} \in W_{p,h}$, $\xi_{p,h} \in \mathbf{X}_{p,h}$, and $\mu_h \in \Lambda_h$,

$$\begin{aligned} & a_f(\mathbf{u}_{f,h}^n, \mathbf{v}_{f,h}) + a_p^D(\mathbf{u}_{p,h}^n, \mathbf{v}_{p,h}) + a_p^e(\eta_{p,h}^n, \xi_{p,h}) \\ & + a_{\text{BJS}}(\mathbf{u}_{f,h}^n, \delta_\tau \eta_{p,h}^n; \mathbf{v}_{f,h}, \xi_{p,h}) + b_f(\mathbf{v}_{f,h}, p_{f,h}^n) \\ & + b_p(\mathbf{v}_{p,h}, p_{p,h}^n) + \alpha b_p(\xi_{p,h}, p_{p,h}^n) + b_\Gamma(\mathbf{v}_{f,h}, \mathbf{v}_{p,h}, \xi_{p,h}; \lambda_h^n) = (\mathbf{f}_f^n, \mathbf{v}_{f,h})_{\Omega_f} + (\mathbf{f}_p^n, \xi_{p,h})_{\Omega_p}, \end{aligned} \quad (43)$$

$$(s_0 \delta_\tau p_{p,h}, w_{p,h})_{\Omega_p} - \alpha b_p(\delta_\tau \eta_{p,h}^n, w_{p,h}) - b_p(\mathbf{u}_{p,h}^n, w_{p,h}) - b_f(\mathbf{u}_{f,h}^n, w_{f,h}) = (q_f^n, w_{f,h})_{\Omega_f} + (q_p^n, w_{p,h})_{\Omega_p}, \quad (44)$$

$$b_\Gamma(\mathbf{u}_{f,h}^n, \mathbf{u}_{p,h}^n, \delta_\tau \eta_{p,h}^n; \mu_h) = 0. \quad (45)$$

We introduce the discrete-in-time norms as follows:

$$\|\phi\|_{L^2(0,T;X)}^2 := \left(\tau \sum_{n=0}^N \|\phi^n\|_X^2 \right)^{1/2}, \quad (46)$$

$$\|\phi\|_{L^\infty(0,T;X)} := \max_{0 \leq n \leq N} \|\phi^n\|_X,$$

and the errors for all variables-in-time as follows:

$$\begin{aligned} \mathbf{e}_f^n &:= \mathbf{u}_f - \mathbf{u}_{f,h}^n, \\ \mathbf{e}_p^n &:= \mathbf{u}_p - \mathbf{u}_{p,h}^n, \\ \mathbf{e}_s^n &:= \eta_p - \eta_{p,h}^n, \\ e_{fp}^n &:= p_f - p_{f,h}^n, \\ e_{pp}^n &:= p_p - p_{p,h}^n, \\ e_\lambda^n &:= \lambda - \lambda_h^n. \end{aligned} \quad (47)$$

Remark 2. For $n \in \{0, \dots, N\}$ and $\mathbf{W}_h^n = (\mathbf{v}_{f,h}^n, w_{f,h}^n, \mathbf{v}_{p,h}^n, w_{p,h}^n, \xi_{p,h}^n, \mu_h^n) \in \mathbf{V}_{f,h} \times W_{f,h} \times \mathbf{V}_{p,h} \times W_{p,h} \times \mathbf{X}_{p,h} \times \Lambda_h$, we can subtract (37)–(39) to (43) and (44) to obtain the Galerkin orthogonality relation:

$$\begin{aligned} & a_f(\mathbf{e}_f^n, \mathbf{v}_{f,h}^n) + a_p^D(\mathbf{e}_s^n, \xi_{p,h}^n) + a_{\text{BJS}}(\mathbf{e}_f^n, \delta_\tau \mathbf{e}_s^n; \mathbf{v}_{f,h}^n, \xi_{p,h}^n) \\ & + b_f(\mathbf{v}_{f,h}^n, e_f^n) + b_p(\mathbf{v}_{p,h}^n, e_p^n) + \alpha b_p(\xi_p^n, e_p^n) \\ & + b_\Gamma(\mathbf{v}_f^n, \mathbf{v}_p^n, \xi_p^n, e_\lambda^n) + (s_0 \delta_\tau e_p^n, w_{p,h}^n) \\ & - \alpha b_p(\delta_\tau \mathbf{e}_s^n, w_{p,h}^n) - b_p(\mathbf{e}_p^n, w_{p,h}^n) - b_f(\mathbf{e}_f^n, w_{f,h}^n) = 0. \end{aligned} \quad (48)$$

3. Error Estimation

In order to solve the Stokes–Biot model problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriated indicators. In this section, we first define the local

and global indicators (Section 3.1) and then the lower and upper error bounds are derived (Sections 3.4 and 3.5).

3.1. Residual Error Estimators. The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier and that involve the data at hand. To this end, define the exact element residuals.

Definition 1 (exact element residuals). Let $n \in \{0, \dots, N\}$ and $\mathbf{W}_h^n = (\mathbf{v}_{f,h}^n, p_{f,h}^n, \mathbf{v}_{p,h}^n, w_{p,h}^n, \xi_{p,h}^n, \mu_h^n) \in \mathbf{V}_{f,h} \times W_{f,h} \times \mathbf{V}_{p,h} \times W_{p,h} \times \mathbf{X}_{p,h} \times \Lambda_h$ be an arbitrary finite element function. The exact element residuals over a triangle or tetrahedra $K \in \mathcal{T}_h$ and over $E \in \mathcal{E}_h(\Gamma_{f,p})$ are defined by

$$\begin{aligned}
\mathbf{R}_{f,K}(\mathbf{W}_h^n) &= \mathbf{f}_f^n + \nabla \cdot \sigma_f(\mathbf{v}_{f,h}^n, w_{f,h}^n), \\
\mathbf{R}_{p,K,1}(\mathbf{W}_h^n) &= \mathbf{f}_p^n + \nabla \cdot \sigma_p(\xi_{p,h}^n, w_{p,h}^n), \\
\mathbf{R}_{p,K,2}(\mathbf{W}_h^n) &= \mu K^{-1} \mathbf{v}_{p,h}^n + \nabla w_{p,h}^n, \\
R_{f,K}(\mathbf{W}_h^n) &= q_f^n - \nabla \cdot \mathbf{v}_{f,h}^n, \\
R_{p,K}(\mathbf{W}_h^n) &= q_p^n - \delta_\tau (s_0 w_{p,h}^n + \alpha \nabla \cdot \xi_{p,h}^n) + \nabla \cdot \mathbf{v}_{p,h}^n, \\
R_{E,pf,1}(\mathbf{W}_h^n) &= \mathbf{v}_{f,h}^n \cdot \mathbf{n}_{f,E} + (\delta_\tau \xi_p^n + \mathbf{v}_{p,h}^n) \cdot \mathbf{n}_{p,E}, \\
R_{E,pf,2}(\mathbf{W}_h^n) &= w_{p,h}^n + (\sigma_f(\mathbf{v}_{f,h}^n, w_{f,h}^n) \mathbf{n}_{f,E}) \cdot \mathbf{n}_{f,E}, \\
R_{E,pf,3}(\mathbf{W}_h^n) &= \sigma_f(\mathbf{v}_{f,h}^n, w_{f,h}^n) \mathbf{n}_{f,E} + \sigma_p(\xi_{p,h}^n, w_{p,h}^n) \mathbf{n}_{p,E}, \\
R_{E,pf,4}(\mathbf{W}_h^n)(j) &= (\sigma_f(\mathbf{v}_{f,h}^n, w_{f,h}^n)) \tau_{f,E,j} + \mu \alpha_{\text{BJS}} \sqrt{|K_j^{-1}|} (\mathbf{v}_{f,h}^n - \delta_\tau \xi_{p,h}^n) \cdot \tau_{f,E,j}.
\end{aligned} \tag{49}$$

As it is common, these exact residuals are replaced by some finite-dimensional approximation called approximate element residual $\mathbf{r}_{*,K}, r_{*,K}, * \in \{f, p\}, R_{E,pf,l}, l \in \{1, 2, 3, 4\}$. This approximation is here achieved by projecting \mathbf{f}_f and q_f

on the space of piecewise constant functions in Ω_f and piecewise \mathbb{P}^1 functions in Ω_p ; more precisely for all $K \in \mathcal{T}_h^f$, we take

$$\begin{aligned}
\mathbf{f}_{f,K}^n &= P_K(\mathbf{f}_f^n), \\
q_{f,K}^n &= P_K(q_f^n), \quad \text{with } P_K: L^1(K) \longrightarrow \mathbb{R} \text{ such that } P_K(\phi) = \frac{1}{|K|} \int_K \phi(x) dx.
\end{aligned} \tag{50}$$

While for all $K \in \mathcal{T}_h^p$, we take $\mathbf{f}_{p,K}^n$ and q_p^n as the unique element of $[\mathbb{P}^1(K)]^d$, respectively, $\mathbb{P}^1(K)$ such that

$$\int_K \mathbf{f}_{p,K}^n(x) \cdot \mathbf{q}(x) dx = \int_K \mathbf{f}_p^n(x) \mathbf{q}(x) dx, \quad \mathbf{q} \in [\mathbb{P}^1(K)]^d, \tag{51}$$

respectively,

$$\int_K q_{p,K}^n(x) q(x) dx = \int_K q_p^n(x) q(x) dx, \quad \forall q \in \mathbb{P}^1(K). \tag{52}$$

Thereby, we define the approximate element residuals.

Definition 2 (approximate element residuals). Let $n \in \{0, \dots, N\}$ and $\mathbf{W}_h^n = (\mathbf{v}_{f,h}^n, p_{f,h}^n, \mathbf{v}_{p,h}^n, w_{p,h}^n, \xi_{p,h}^n, \mu_h^n) \in \mathbf{V}_{f,h} \times W_{f,h} \times \mathbf{V}_{p,h} \times W_{p,h} \times \mathbf{X}_{p,h} \times \Lambda_h$ be an arbitrary finite element function. Then, the approximate element residuals are defined by

$$\begin{aligned}
\mathbf{r}_{f,K}(\mathbf{W}_h^n) &= \mathbf{f}_{f,K}^n + \nabla \cdot \sigma_f(\mathbf{v}_{f,h}^n, w_{f,h}^n), \\
\mathbf{r}_{p,K,1}(\mathbf{W}_h^n) &= \mathbf{f}_{p,K}^n + \nabla \cdot \sigma_p(\xi_{p,h}^n, w_{p,h}^n), \\
\mathbf{r}_{p,K,2}(\mathbf{W}_h^n) &= \mu K^{-1} \mathbf{v}_{p,h}^n + \nabla w_{p,h}^n, \\
r_{f,K}(\mathbf{W}_h^n) &= q_{f,K}^n - \nabla \cdot \mathbf{v}_{f,h}^n, \\
r_{p,K}(\mathbf{W}_h^n) &= q_{p,K}^n - \delta_\tau (s_0 w_{p,h}^n + \alpha \nabla \cdot \xi_{p,h}^n) + \nabla \cdot \mathbf{v}_{p,h}^n.
\end{aligned} \tag{53}$$

Next, introduce the gradient jump in normal direction by

$$\begin{cases} \mathbf{J}_{E,\mathbf{n}_E,f}(\mathbf{U}_h^n) := [(2\mu \mathbf{D}(\mathbf{u}_{f,h}^n) - p_{f,h}^n \mathbf{I}) \cdot \mathbf{n}_E]_E, & \text{if } E \in \mathcal{E}_h(\Omega_f), \\ \mathbf{J}_{E,\mathbf{n}_E,p}(\mathbf{U}_h^n) := [(2\mu \mathbf{D}(\eta_{p,h}^n) - p_{p,h}^n \mathbf{I}) \cdot \mathbf{n}_E]_E, & \text{if } E \in \mathcal{E}_h(\Omega_p), \end{cases} \tag{54}$$

where \mathbf{I} is the identity matrix of $\mathbb{R}^{d \times d}$.

Definition 3 (residual error estimators). Let $n \in \{0, \dots, N\}$ and $\mathbf{U}_h^n = (\mathbf{u}_{f,h}^n, p_{f,h}^n, \mathbf{u}_{p,h}^n, p_{p,h}^n, \eta_{p,h}^n, \lambda_h^n)$ be the finite element solution of the fully discrete problems (43)–(45) in $\mathbf{V}_{f,h} \times W_{f,h} \times \mathbf{V}_{p,h} \times W_{p,h} \times \mathbf{X}_{p,h} \times \Lambda_h$.

We define

$$\Theta_K^n := [\Theta_{K,f}^2(\mathbf{U}_h^n) + \Theta_{K,p}^2(\mathbf{U}_h^n) + \Theta_{K,pf}^2(\mathbf{U}_h^n)]^{(1/2)}, \quad (55)$$

where

$$\begin{aligned} \Theta_{K,f}^2(\mathbf{U}_h^n) &:= h_K^2 \|\mathbf{r}_{f,K}(\mathbf{U}_h^n)\|_{L^2(K)}^2 + \|r_{f,K}(\mathbf{U}_h^n)\|_{L^2(K)}^2 \\ &+ \sum_{E \in \mathcal{E}_h(\Omega_f)} h_E \|\mathbf{J}_{E,\mathbf{n}_{E,f}}(\mathbf{U}_h^n)\|_{L^2(E)}^2, \end{aligned} \quad (56)$$

$$\begin{aligned} \Theta_{K,p}^2(\mathbf{U}_h^n) &:= h_K^2 \left(\|\mathbf{r}_{p,K,1}(\mathbf{U}_h^n)\|_{L^2(K)}^2 + \|\mathbf{r}_{p,K,2}(\mathbf{U}_h^n)\|_{L^2(K)}^2 \right) \\ &+ h_K^2 \|\operatorname{curl}[\mathbf{r}_{p,K,2}(\mathbf{U}_h^n)]\|_{L^2(K)}^2 + \|r_{p,K}(\mathbf{U}_h^n)\|_{L^2(K)}^2 \\ &+ \sum_{E \in \mathcal{E}_h(\Omega_p)} h_E \|\mathbf{J}_{E,\mathbf{n}_{E,p}}(\mathbf{U}_h^n)\|_{L^2(E)}^2, \end{aligned} \quad (57)$$

and

$$\begin{aligned} \Theta_{K,pf}^2(\mathbf{U}_h^n) &:= \sum_{E \in \mathcal{E}_h(\Gamma_{pf})} h_E \|\mathbf{R}_{E,pf,1}(\mathbf{U}_h^n)\|_{L^2(E)}^2 \\ &+ \sum_{E \in \mathcal{E}_h(\Gamma_{pf})} h_E \|\mathbf{R}_{E,pf,2}(\mathbf{U}_h^n)\|_{L^2(E)}^2 \\ &+ \sum_{E \in \mathcal{E}_h(\Gamma_{pf})} h_E \|\mathbf{R}_{E,pf,3}(\mathbf{U}_h^n)\|_{L^2(E)}^2 \\ &+ \sum_{E \in \mathcal{E}_h(\Gamma_{pf})} h_E \left\{ \sum_{j=1}^{d-1} \|\mathbf{R}_{E,pf,4}(\mathbf{U}_h^n)(j)\|_{L^2(E)}^2 \right\}. \end{aligned} \quad (58)$$

Then, the residual error estimator is locally defined by

$$\Theta_K := \min_{0 \leq n \leq N} \Theta_K^n, \quad \forall K \in \mathcal{T}_h. \quad (59)$$

The global residual error estimator is given by

$$\Theta := \left(\sum_{K \in \mathcal{T}_h} \Theta_K^2 \right)^{1/2}. \quad (60)$$

Furthermore, denote the local approximation terms by

$$\zeta_K^n := [\zeta_{K,f}(\mathbf{U}_h^n)^2 + \zeta_{K,p}(\mathbf{U}_h^n)^2]^{1/2}, \quad (61)$$

where

$$\begin{aligned} \zeta_{K,f}(\mathbf{U}_h^n)^2 &:= h_K^2 \|\mathbf{R}_{f,K}(\mathbf{U}_h^n) - \mathbf{r}_{f,K}(\mathbf{U}_h^n)\|_{L^2(K)}^2 \\ &+ h_K^2 \|R_{f,K}(\mathbf{U}_h^n) - r_{f,K}(\mathbf{U}_h^n)\|_{L^2(K)}^2, \quad \text{if } K \in \mathcal{T}_h^f, \\ \zeta_{K,p}(\mathbf{U}_h^n)^2 &:= h_K^2 \|\mathbf{R}_{p,K,1}(\mathbf{U}_h^n) - \mathbf{r}_{p,K,1}(\mathbf{U}_h^n)\|_{L^2(K)}^2 \\ &+ h_K^2 \|R_{p,K}(\mathbf{U}_h^n) - r_{p,K}(\mathbf{U}_h^n)\|_{L^2(K)}^2, \quad \text{if } K \in \mathcal{T}_h^p. \end{aligned} \quad (62)$$

We set

$$\begin{aligned} \zeta_K &:= \min_{0 \leq n \leq N} \zeta_K^n, \\ \zeta &:= \left(\sum_{K \in \mathcal{T}_h} \zeta_K^2 \right)^{1/2}. \end{aligned} \quad (63)$$

Remark 3. The residual character of each term on the right-hand sides of (55)–(58) is quite clear since if \mathbf{U}_h would be the exact solution of (2)–(10), then they would vanish.

3.2. Analytical Tools

3.2.1. Inverse Inequalities. In order to derive the lower error bounds, we proceed similarly as in [30, 34] (see also [35]), by applying inverse inequalities and the localization technique based on simplex-bubble and face-bubble functions. To this end, we recall some notation and introduce further preliminary results. Given $K \in \mathcal{T}_h$ and $E \in \mathcal{E}(K)$, we let b_K and b_E be the usual simplex-bubble and face-bubble functions (see (1.5) and (1.6) in [29]), respectively. In particular, b_K satisfies $b_K \in \mathbb{P}^3(K)$, $\operatorname{supp}(b_K) \subseteq K$, $b_K = 0$, on ∂K , and $0 \leq b_K \leq 1$, on K . Similarly, $b_E \in \mathbb{P}^2(K)$, $\operatorname{supp}(b_E) \subseteq \omega_E := \{K' \in \mathcal{T}_h: E \in \mathcal{E}(K')\}$, $b_E = 0$, on $\partial K \setminus E$ and $0 \leq b_E \leq 1$, in ω_E . We also recall from [36] that, given $k \in \mathbb{N}$, there exists an extension operator $L: C(E) \rightarrow C(K)$ that satisfies $L(p) \in \mathbb{P}^k(K)$ and $L(p)|_E = p$, $\forall p \in \mathbb{P}^k(E)$. A corresponding vectorial version of L , that is, the component-wise application of L , is denoted by \mathbf{L} . Additional properties of b_K , b_E , and L are collected in the following lemma (see [36]).

Lemma 1. *Given $k \in \mathbb{N}^*$, there exist positive constants depending only on k and shape-regularity of the triangulations (minimum angle condition), such that for each simplex K and $E \in \mathcal{E}(K)$, there hold*

$$\begin{aligned} \|\phi\|_K &\leq \|\phi b_K^{1/2}\|_K \leq \|\phi\|_K, \quad \forall \phi \in \mathbb{P}^k(K), \\ |\phi b_K|_{1,K} &\leq h_K^{-1} \|\phi\|_K, \quad \forall \phi \in \mathbb{P}^k(K), \\ \|\psi\|_E &\leq \|\mathbf{b}_E^{1/2} \psi\|_E \leq \|\psi\|_E, \quad \forall \psi \in \mathbb{P}^k(E), \\ \|L(\psi)\|_K + h_E |L(\psi)|_{1,K} &\leq h_E^{1/2} \|\psi\|_E, \quad \forall \psi \in \mathbb{P}^k(E). \end{aligned} \quad (64)$$

Lemma 2 (continuous trace inequality). *There exists a positive constant $\beta_1 > 0$ depending only on σ_0 such that*

$$\|\mathbf{v}\|_{\partial K}^2 \leq \beta_1 \|\mathbf{v}\|_K \|\mathbf{v}\|_{1,K}, \quad \forall K \in \mathcal{T}_h, \forall \mathbf{v} \in [H^1(K)]^d. \quad (65)$$

3.2.2. Clément Interpolation Operator. In order to derive the upper error bounds, we introduce the Clément interpolation

$$\mathcal{P}_c^b(\mathcal{T}_h) := \{v \in C^0(\bar{\Omega}); v|_K \in \mathbb{P}^1(K), \forall K \in \mathcal{T}_h \text{ and } v = 0, \text{ on } \partial\Omega\}. \quad (66)$$

In addition, we will make use of a vector valued version of \mathbf{I}_{Cl}^0 , that is, $\mathbf{I}_{\text{Cl}}^0: [H_0^1(\Omega)]^d \rightarrow [\mathcal{P}_c^b(\mathcal{T}_h)]^d$, which is defined component-wise by \mathbf{I}_{Cl}^0 . The following lemma establishes the local approximation properties of \mathbf{I}_{Cl}^0 (and hence of \mathbf{I}_{Cl}^0), for a proof (see Section 3 in [37]).

Lemma 3. *There exist constants $C_1, C_2 > 0$, independent of h , such that for all $v \in H_0^1(\Omega)$, there hold*

$$\begin{aligned} \|v - \mathbf{I}_{\text{Cl}}^0(v)\|_K &\leq C_1 h_K \|v\|_{1,\Delta(K)}, \quad \forall K \in \mathcal{T}_h, 0.2 \text{ cm}, \\ \|v - \mathbf{I}_{\text{Cl}}^0(v)\|_E &\leq C_2 h_E^{1/2} \|v\|_{1,\Delta(E)}, \quad \forall E \in \mathcal{E}_h, \end{aligned} \quad (67)$$

where $\Delta(K) := \cup\{K' \in \mathcal{T}_h: K' \cap K \neq \emptyset\}$ and $\Delta(E) := \cup\{K' \in \mathcal{T}_h: K' \cap E \neq \emptyset\}$.

3.2.3. Helmholtz Decomposition

operator $\mathbf{I}_{\text{Cl}}^0: H_0^1(\Omega) \rightarrow \mathcal{P}_c^b(\mathcal{T}_h)$ that approximates optimally nonsmooth functions by continuous piecewise linear functions:

Lemma 4 (see [7]). *There exists $C_p > 0$ such that every $\mathbf{v}_p \in \mathbf{H}(\text{div}; \Omega_p)$ can be decomposed as $\mathbf{v}_p = \mathbf{w} + \text{curl } \beta$, where $\mathbf{w} \in [H^1(\Omega_p)]^d$, $\beta \in H^1(\Omega_p)$, $\int_{\Omega_p} \beta(x) dx = 0$, and*

$$\|\mathbf{w}\|_{1,\Omega_p} + \|\beta\|_{1,\Omega_p} \leq C_p \|\mathbf{v}_p\|_{\mathbf{H}(\text{div}; \Omega_p)}. \quad (68)$$

3.3. Error Equation. We set $\mathbf{H} = \mathbf{V}_f \times W_f \times \mathbf{V}_p \times W_p \times \mathbf{X}_p \times \Lambda$ and $\mathbf{H}_h = \mathbf{V}_{f,h} \times W_{f,h} \times \mathbf{V}_{p,h} \times W_{p,h} \times \mathbf{X}_{p,h} \times \Lambda_h$. For $n \in \{0, \dots, N\}$, let $\mathbf{U}^n = (\mathbf{u}_f^n, p_f^n, \mathbf{u}_p^n, p_p^n, \eta_p^n, \lambda^n) \in \mathbf{H}$ and $\mathbf{W} = (\mathbf{v}_f^n, w_f^n, \mathbf{v}_p^n, w_p^n, \xi_p^n, \mu^n) \in \mathbf{H}$. We define now the operator \mathbf{A} by

$$\begin{aligned} \mathbf{A}(\mathbf{U}^n, \mathbf{W}^n) &:= a_f(\mathbf{u}_f^n, \mathbf{v}_f^n) + a_p^D(\mathbf{u}_p^n, \mathbf{v}_p^n) + b_p(\mathbf{v}_p^n, p_p^n) + \alpha b_p(\xi_p^n, p_p^n) \\ &\quad + b_\Gamma(\mathbf{v}_f^n, \mathbf{v}_p^n, \xi_p^n, \lambda^n) + (s_0 \delta_\tau p_p^n, w_p^n)_{\Omega_p} - \alpha b_p(\delta_\tau \eta_p^n, w_p^n) \\ &\quad - b_p(\mathbf{u}_p^n, w_p^n) - b_f(\mathbf{u}_f^n, w_f^n) + b_\Gamma(\mathbf{u}_f^n, \mathbf{u}_p^n, \delta_\tau \eta_p^n; \mu^n), \\ \mathbf{F}(\mathbf{W}^n) &:= (\mathbf{f}_f^n, \mathbf{v}_f^n)_{\Omega_f} + (\mathbf{f}_p^n, \xi_p^n) + (q_f^n, w_f^n)_{\Omega_f} + (q_p^n, w_p^n)_{\Omega_p}. \end{aligned} \quad (69)$$

Then, the continuous problems (26)–(28) are equivalent to the following: for $n \in \{0, \dots, N\}$, find $\mathbf{U}^n \in \mathbf{H}$ such that we have

$$\mathbf{A}(\mathbf{U}^n, \mathbf{W}^n) = \mathbf{F}(\mathbf{W}^n), \quad \forall \mathbf{W}^n \in \mathbf{H}. \quad (70)$$

We define the discrete version by the same way: find $\mathbf{U}_h^n \in \mathbf{H}_h$ such that

$$\mathbf{A}(\mathbf{U}_h^n, \mathbf{W}_h^n) = \mathbf{F}(\mathbf{W}_h^n), \quad \forall \mathbf{W}_h^n \in \mathbf{H}_h. \quad (71)$$

Since for all $n \in \{0, \dots, N\}$ and $\mathbf{W}_h^n \in \mathbf{H}_h$, $\mathbf{A}(\mathbf{U}^n - \mathbf{U}_h^n, \mathbf{W}_h^n) = 0$, then from (70), we obtain

$$\begin{aligned} \mathbf{A}(\mathbf{U}^n - \mathbf{U}_h^n, \mathbf{W}^n) &= \mathbf{A}(\mathbf{U}^n - \mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n) \\ &= \mathbf{A}(\mathbf{U}^n, \mathbf{W}^n - \mathbf{W}_h^n) - \mathbf{A}(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n) \\ &= \mathbf{F}(\mathbf{W}^n - \mathbf{W}_h^n) - \mathbf{A}(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n) \\ &= (\mathbf{f}_f^n, \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f} + (\mathbf{f}_p^n, \xi_p^n - \xi_{p,h}^n)_{\Omega_p} \\ &\quad + (q_f^n, w_f^n - w_{f,h}^n)_{\Omega_f} + (q_p^n, w_p^n - w_{p,h}^n)_{\Omega_p} - \mathbf{A}(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n) \\ &= (\mathbf{f}_f^n - \mathbf{f}_{f,h}^n, \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f} + (\mathbf{f}_p^n - \mathbf{f}_{p,h}^n, \xi_p^n - \xi_{p,h}^n)_{\Omega_p} \\ &\quad + (q_f^n - q_{f,h}^n, w_f^n - w_{f,h}^n)_{\Omega_f} + (q_p^n - q_{p,h}^n, w_p^n - w_{p,h}^n)_{\Omega_p} \\ &\quad + (\mathbf{f}_{f,h}^n, \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f} + (\mathbf{f}_{p,h}^n, \xi_p^n - \xi_{p,h}^n)_{\Omega_p} + (q_{f,h}^n, w_f^n - w_{f,h}^n)_{\Omega_f} + (q_{p,h}^n, w_p^n - w_{p,h}^n)_{\Omega_p} \\ &\quad - \mathbf{A}(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n). \end{aligned} \quad (72)$$

Now, we set

$$\mathbf{A}(\mathbf{U}^n - \mathbf{U}_h^n, \mathbf{W}^n) = \sum_{K \in \mathcal{T}_h} \mathbf{A}_K(\mathbf{U}^n - \mathbf{U}_h^n, \mathbf{W}^n). \quad (73)$$

Then, from (72), we can write

$$\begin{aligned} \mathbf{A}_K(\mathbf{U}^n - \mathbf{U}_h^n, \mathbf{W}^n) &:= (\mathbf{f}_f^n - \mathbf{f}_{f,h}^n, \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f \cap K} + (\mathbf{f}_p^n - \mathbf{f}_{p,h}^n, \xi_p^n - \xi_{p,h}^n)_{\Omega_p \cap K} \\ &\quad + (q_f^n - q_{f,h}^n, w_f^n - w_{f,h}^n)_{\Omega_f \cap K} + (q_p^n - q_{p,h}^n, w_p^n - w_{p,h}^n)_{\Omega_p \cap K} \\ &\quad + (\mathbf{f}_{f,h}^n, \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f \cap K} + (\mathbf{f}_{p,h}^n, \xi_p^n - \xi_{p,h}^n)_{\Omega_p \cap K} \\ &\quad + (q_{f,h}^n, w_f^n - w_{f,h}^n)_{\Omega_f \cap K} + (q_{p,h}^n, w_p^n - w_{p,h}^n)_{\Omega_p \cap K} - \mathbf{A}_K(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n) \\ &= (\mathbf{R}_{f,K}(\mathbf{U}_h^n) - \mathbf{r}_{f,K}(\mathbf{U}_h^n), \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f \cap K} + (\mathbf{R}_{f,K}(\mathbf{U}_h^n) - r_{f,K}(\mathbf{U}_h^n), w_f^n - w_{f,h}^n)_{\Omega_f \cap K} \\ &\quad + (\mathbf{R}_{p,K,1}(\mathbf{U}_h^n) - \mathbf{r}_{p,K,1}(\mathbf{U}_h^n), \xi_p^n - \xi_{p,h}^n)_{\Omega_p \cap K} \\ &\quad + (\mathbf{R}_{p,K}(\mathbf{U}_h^n) - r_{p,K}(\mathbf{U}_h^n), w_p^n - w_{p,h}^n)_{\Omega_p \cap K} + \mathbf{B}_K(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n), \end{aligned} \quad (74)$$

where

$$\begin{aligned} \mathbf{B}_K(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n) &= (\mathbf{f}_{f,K}^n, \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f \cap K} + (\mathbf{f}_{p,K}^n, \xi_p^n - \xi_{p,h}^n)_{\Omega_p \cap K} \\ &\quad + (q_{f,K}^n, w_f^n - w_{f,h}^n)_{\Omega_f \cap K} + (q_{p,K}^n, w_p^n - w_{p,h}^n)_{\Omega_p \cap K} - \mathbf{A}_K(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n). \end{aligned} \quad (75)$$

3.4. Reliability of the A Posteriori Error Estimator. The first main result is given by the following theorem.

Theorem 3 (upper error bound). *Let $\mathbf{U} = (\mathbf{U}^n)_{n=0}^N$ such that $\mathbf{U}^n = (\mathbf{u}_f^n, \mathbf{p}_f^n, \mathbf{u}_p^n, \mathbf{p}_p^n, \eta_p^n, \lambda^n) \in \mathbf{H}$ be the exact solution and*

$\mathbf{U}_h = (\mathbf{U}_h^n)_{n=0}^N$ with $\mathbf{U}_h^n = (\mathbf{u}_{f,h}^n, \mathbf{p}_{f,h}^n, \mathbf{u}_{p,h}^n, \mathbf{p}_{p,h}^n, \eta_{p,h}^n, \lambda_h^n) \in \mathbf{H}_h$ be the finite element solution. Then, there exist a positive constant C_{upper} such that the error is bounded globally from above by

$$\|\mathbf{U} - \mathbf{U}_h\|_{L^\infty(0,T;\mathbf{H}_h)} \leq C_{\text{upper}} (\Theta + \zeta). \quad (76)$$

Proof. Let $n \in \{0, \dots, N\}$. We consider the residual equation (73), and we set $\mathbf{W}^n = (\mathbf{v}_f^n, w_f^n, \mathbf{v}_p^n, w_p^n, \xi_p^n, \mu^n)$. We take

$$\begin{aligned} \mathbf{B}_K(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n) &= (\mathbf{f}_{f,K}^n, \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f \cap K} + (\mathbf{f}_{p,K}^n, \xi_p^n - \xi_{p,h}^n)_{\Omega_p \cap K} \\ &\quad + (q_{f,K}^n, w_f^n - w_{f,h}^n)_{\Omega_f \cap K} + (q_{p,K}^n, w_p^n - w_{p,h}^n)_{\Omega_p \cap K} - \mathbf{A}_K(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n). \end{aligned} \quad (77)$$

Therefore, integrating by parts element by element, we may write

$$\mathbf{B}_K(\mathbf{U}_h^n, \mathbf{W}^n - \mathbf{W}_h^n) = \sum_{i=1}^{12} \Upsilon_{i,K}, \quad (78)$$

where

$$\begin{aligned} \Upsilon_{1,K} &:= (\mathbf{r}_{f,K}(\mathbf{U}_h^n), \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f \cap K}, \\ \Upsilon_{2,K} &:= (\mathbf{r}_{p,K,1}(\mathbf{U}_h^n), \xi_p^n - \xi_{p,h}^n)_{\Omega_p \cap K}, \\ \Upsilon_{3,K} &:= -(\mathbf{r}_{p,K,2}(\mathbf{U}_h^n), \mathbf{w}_p^n - \mathbf{w}_{p,h}^n)_{\Omega_p \cap K}, \\ \Upsilon_{4,K} &:= -(\text{curl } \mathbf{r}_{p,K,2}(\mathbf{U}_h^n), \beta_p^n - \beta_{p,h}^n)_{\Omega_p \cap K}, \\ \Upsilon_{5,K} &:= (r_{f,K}(\mathbf{U}_h^n), w_f^n)_{\Omega_f \cap K}, \\ \Upsilon_{6,K} &:= (r_{p,K}(\mathbf{U}_h^n), w_p^n)_{\Omega_p \cap K}, \\ \Upsilon_{7,K} &:= - \sum_{E \in \mathcal{E}_h(\Omega_f \cap K)} (\mathbf{J}_{E,\mathbf{n}_{E,f}}(\mathbf{U}_h^n), \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_E, \\ \Upsilon_{8,K} &:= - \sum_{E \in \mathcal{E}_h(\Omega_p \cap K)} (\mathbf{J}_{E,\mathbf{n}_{E,p}}(\mathbf{U}_h^n), \xi_p^n - \xi_{p,h}^n)_E, \\ \Upsilon_{9,K} &:= \sum_{E \in \mathcal{E}_h(\Gamma_{fp})} (R_{E,pf,1}(\mathbf{U}_h^n), (\mathbf{v}_f^n - \mathbf{v}_{f,h}^n) \cdot \mathbf{n}_{f,E})_E, \\ \Upsilon_{10,K} &:= \sum_{E \in \mathcal{E}_h(\Gamma_{fp})} (R_{E,pf,2}(\mathbf{U}_h^n), (\mathbf{v}_f^n - \mathbf{v}_{f,h}^n) \cdot \mathbf{n}_{f,E})_E, \\ \Upsilon_{11,K} &:= \sum_{E \in \mathcal{E}_h(\Gamma_{fp})} (R_{E,pf,3}(\mathbf{U}_h^n), (\xi_p^n - \xi_{p,h}^n))_E, \\ \Upsilon_{12,K} &:= - \sum_{E \in \mathcal{E}_h(\Gamma_{fp})} \sum_{j=1}^{d-1} (R_{E,pf,4}(\mathbf{U}_h^n)(j), \mathbf{v}_f^n - \mathbf{v}_{f,h}^n \cdot \boldsymbol{\tau}_j)_E. \end{aligned} \quad (79)$$

Thus, from equations (74) and (78), we obtain

$\mathbf{W}_h^n = (\mathbf{v}_{f,h}^n, 0, \mathbf{v}_{p,h}^n, 0, \xi_{p,h}^n, 0)$ with $\mathbf{v}_{f,h}^n = \mathbf{I}_{\text{Cl}}^0(\mathbf{v}_f^n)$ and $\xi_{p,h}^n = \mathbf{I}_{\text{Cl}}^0(\xi_p^n)$. As $\mathbf{v}_p^n \in \mathbf{H}(\text{div}; \Omega_p)$, then by Lemma 4, \mathbf{v}_p^n admits the decomposition $\mathbf{v}_p^n = \mathbf{w}_p^n + \text{curl } \beta^n$ where $\mathbf{w}_p^n \in [H^1(\Omega_p)]^d$ and $\beta^n \in H^1(\Omega_p)$ with $\int_{\Omega} \beta^n(x) dx = 0$ and $\|\mathbf{w}_p^n\|_{[H^1(\Omega_p)]^d} + \|\beta^n\|_{H^1(\Omega_p)} \leq C_p \|\mathbf{v}_p^n\|_{\mathbf{V}_p}$. We consider $\mathbf{v}_{p,h}^n = \mathbf{w}_{p,h}^n + \text{curl } \beta_{p,h}^n$ with $\mathbf{w}_{p,h}^n = \mathbf{I}_{\text{Cl}}(\mathbf{w}_p^n)$ and $\beta_{p,h}^n = \mathbf{I}_{\text{Cl}}(\beta_p^n)$. Thus, $\mathbf{v}_p^n - \mathbf{v}_{p,h}^n = (\mathbf{w}_p^n - \mathbf{w}_{p,h}^n) + \text{curl } (\beta_p^n - \beta_{p,h}^n)$. Now, we recall

$$\begin{aligned} \mathbf{A}_K(\mathbf{U}^n - \mathbf{U}_h^n, \mathbf{W}^n) &:= (\mathbf{r}_{f,K}(\mathbf{U}_h^n) - \mathbf{r}_{f,K}(\mathbf{U}_h^n), \mathbf{v}_f^n - \mathbf{v}_{f,h}^n)_{\Omega_f \cap K} \\ &\quad + (R_{f,K}(\mathbf{U}_h^n) - r_{f,K}(\mathbf{U}_h^n), w_f^n - w_{f,h}^n)_{\Omega_f \cap K} \\ &\quad + (\mathbf{r}_{p,K,1}(\mathbf{U}_h^n) - \mathbf{r}_{p,K,1}(\mathbf{U}_h^n), \xi_p^n - \xi_{p,h}^n)_{\Omega_p \cap K} \\ &\quad + (R_{p,K}(\mathbf{U}_h^n) - r_{p,K}(\mathbf{U}_h^n), w_p^n - w_{p,h}^n)_{\Omega_p \cap K} \\ &\quad + \sum_{i=1}^{12} \eta_{i,K}. \end{aligned} \quad (80)$$

Coercivity of operator \mathbf{A} leads to inf-sup condition as follows:

$$\|\mathbf{U}^n - \mathbf{U}_h^n\|_{\mathbf{H}_h} \leq \sup_{\mathbf{W}^n \in \mathbf{H}} \frac{|\mathbf{A}(\mathbf{U}^n - \mathbf{U}_h^n, \mathbf{W}^n)|}{\|\mathbf{W}^n\|_{\mathbf{H}}}. \quad (81)$$

By consequently, identities (73) and (80), inf-sup condition of operator \mathbf{A} (81), Cauchy-Schwarz inequality, estimation of Lemma 4, and the approximation properties of Lemma 3 imply the required estimate and finish the proof. \square

3.5. Efficiency of the A Posteriori Error Estimator. For $K \in \mathcal{T}_h$, we set

$$\begin{aligned} \mathbf{H}_h(K) &= \mathbf{V}_{f,h}(K) \times W_{f,h}(K) \times \mathbf{V}_{p,h}(K) \\ &\quad \times W_{p,h}(K) \times \mathbf{X}_{p,h}(K) \times \Lambda_h(K), \end{aligned} \quad (82)$$

where

$$\begin{aligned} \mathbf{V}_{f,h}(K) &:= \{\mathbf{v}_{f,h|K} : \mathbf{v}_{f,h} \in \mathbf{V}_{f,h}\}, \\ W_{f,h}(K) &:= \{w_{f,h|K} : w_{f,h} \in W_{f,h}\}, \\ \mathbf{V}_{p,h}(K) &:= \{\mathbf{v}_{p,h|K} : \mathbf{v}_{p,h} \in \mathbf{V}_{p,h}\}, \\ W_{p,h}(K) &:= \{w_{p,h|K} : w_{p,h} \in W_{p,h}\}, \\ \mathbf{X}_{p,h}(K) &:= \{\xi_{p,h|K} : \xi_{p,h} \in \mathbf{X}_{p,h}\}, \\ \Lambda_h(K) &:= \{\lambda_{h|K} : \lambda_h \in \Lambda_h\}. \end{aligned} \quad (83)$$

The family estimator $(\Theta_K)_{K \in \mathcal{T}_h}$ is considered efficient if it satisfies the following theorem.

Theorem 4 (lower error bound). *Let $\mathbf{U} = (\mathbf{U}^n)_{n=0}^N$ such that $\mathbf{U}^n = (\mathbf{u}_f^n, \mathbf{p}_f^n, \mathbf{u}_p^n, \mathbf{p}_p^n, \eta_p^n, \lambda^n) \in \mathbf{H}$ be the exact solution and $\mathbf{U}_h = (\mathbf{U}_h^n)_{n=0}^N$ with $\mathbf{U}_h^n = (\mathbf{u}_{f,h}^n, \mathbf{p}_{f,h}^n, \mathbf{u}_{p,h}^n, \mathbf{p}_{p,h}^n, \eta_{p,h}^n, \lambda_h^n) \in \mathbf{H}_h$ be the finite element solution. Then, there exist a positive constant C_{low} such that the error is bounded locally from below for all $K \in \mathcal{T}_h$ by*

$$\Theta_K \leq C_{\text{low}} \left[\|\mathbf{U} - \mathbf{U}_h\|_{L^\infty(0,T; \mathbf{H}_h(K))} + \sum_{K' \subset \tilde{w}_K} \zeta_{K'} \right], \quad (84)$$

where \tilde{w}_K is a finite union of neighbouring elements of K and $\|\cdot\|_{\mathbf{H}_h(K)}$ is the product norm.

Proof. The lower bound is proved using the standard elementwise integration by parts, namely, error equation of Section 3.3 (i.e., identity (73) and equation (80)) and some inverse estimates of Lemma 1 (cf. [38] for details). \square

4. Discussion

In this paper, we have discussed a posteriori error estimates for a finite element approximation of the Stokes–Biot system where homogeneous boundary conditions are employed. The approach utilizes a fully discrete conforming finite element method. A residual type a posteriori error estimator is provided, that is, both reliable and efficient.

In a future paper, we will study the influence of non-homogeneous boundary conditions on the a posteriori error indicators presented in this work. Further, it is well known that an internal layer appears at the interface Γ_{fp} as the permeability tensor degenerates; in that case, anisotropic meshes have to be used in this layer (see, for instance, [10]). Hence, we intend to extend our results to such anisotropic meshes.

5. Nomenclatures

- (i) $\Omega \subset \mathbb{R}^d, d \in \{2, 3\}$: bounded domain
- (ii) Ω_p : the poroelastic medium domain
- (iii) $\Omega_f = \Omega \setminus \bar{\Omega}_d$
- (iv) $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$
- (v) $\Gamma_* = \partial\Omega_* \setminus \Gamma_{fp}, * = f, p$
- (vi) \mathbf{n}_f (respectively, \mathbf{n}_p): the unit outward normal vector along $\partial\Omega_f$ (respectively, $\partial\Omega_p$)
- (vii) \mathbf{u}_f : the fluid velocity in Ω_f
- (viii) p_f : the fluid pressure in Ω_f
- (ix) \mathbf{u}_p, η_p : the fluid velocities in Ω_p
- (x) p_p : the fluid pressure in Ω_p
- (xi) In 2 D , the curl of a scalar function w is given as usual by $\text{curl } w := ((\partial w / \partial x_2), -(\partial w / \partial x_1))^T$
- (xii) In 3 D , the curl of a vector function $\mathbf{w} = (w_1, w_2, w_3)$ is given as usual by $\text{curl } \mathbf{w} := \nabla \times \mathbf{w}$,

namely, $\text{curl } \mathbf{w} := ((\partial w_3 / \partial x_2) - (\partial w_2 / \partial x_3), (\partial w_1 / \partial x_3) - (\partial w_3 / \partial x_1), (\partial w_2 / \partial x_1) - (\partial w_1 / \partial x_2))$

- (xiii) \mathbb{P}^k : the space of polynomials of total degree not larger than k
- (xiv) \mathcal{T}_h : triangulation of Ω
- (xv) \mathcal{T}_h^* : the corresponding induced triangulation of Ω_* , $* \in \{f, p\}$
- (xvi) For any $K \in \mathcal{T}_h$, h_K is the diameter of K and $\rho_K = 2r_K$ is the diameter of the largest ball inscribed into K
- (xvii) $h := \max_{K \in \mathcal{T}_h} h_K$ and $\sigma_h := \max_{K \in \mathcal{T}_h} (h_K / \rho_K)$
- (xviii) \mathcal{E}_h : the set of all the edges or faces of the triangulation
- (xix) $\mathcal{E}(K)$: the set of all the edges ($N = 2$) or faces ($N = 3$) of an element K
- (xx) $\mathcal{E}_h := \cup_{K \in \mathcal{T}_h} \mathcal{E}(K)$
- (xxi) $\mathcal{V}(K)$: the set of all the vertices of an element K
- (xxii) $\mathcal{N}_h := \cup_{K \in \mathcal{T}_h} \mathcal{V}(K)$
- (xxiii) For $\mathcal{A} \subset \bar{\Omega}$, $\mathcal{E}_h(\mathcal{A}) := \{E \in \mathcal{E}_h : E \subset \mathcal{A}\}$
- (xxiv) For $E \in \mathcal{E}_h(\Omega_*)$, we associate a unit vector $\mathbf{n}_{E,*}$ such that $\mathbf{n}_{E,*}$ is orthogonal to E and equals to the unit exterior normal vector to $\partial\Omega_*$, $* \in \{f, p\}$
- (xxv) For $E \in \mathcal{E}_h(\Omega_*)$, $[\phi]_E$ is the jump across E in the direction of $\mathbf{n}_{E,*}$
- (xxvi) In order to avoid excessive use of constants, the abbreviations $x \lesssim y$ and $x \sim y$ stand for $x \leq c y$ and $c_1 x \leq y \leq c_2 x$, respectively, with positive constants independent of x, y or \mathcal{T}_h
- (xxvii) $\partial_t = (\partial / \partial t)$
- (xxviii) $\|\phi\|_{L^2(0,T;X)} := [\int_0^T \|\phi(s)\|_X^2 ds]^{1/2}$
- (xxix) $\|\phi\|_{L^\infty(0,T;X)} := \text{esssup}_{0 \leq t \leq T} \|\phi(t)\|_X$
- (xxx) $\|\phi\|_{W^{1,\infty}(0,T;X)} := \text{esssup}_{0 \leq t \leq T} \{\|\phi(t)\|_X, \|\partial_t \phi(t)\|_X\}$
- (xxxi) $\|\phi\|_{L^\infty(0,T;X)}^2 := \max_{0 \leq n \leq N} \|\phi^n\|_X^2$
- (xxxii) $\|\phi\|_{L^2(0,T;X)} := (\tau \sum_{n=0}^N \|\phi^n\|_X^2)^{1/2}$
- (xxxiii) $\|\cdot\|_{\mathbf{H}_h}$: product norm on \mathbf{H}_h
- (xxxiv) $\|\cdot\|_{\mathbf{H}_h(K)}$: local product norm on $\mathbf{H}_h(K)$.

Data Availability

There are no data underlying the findings in this paper to be shared.

Disclosure

The results presented in this paper constitute a continuation of our work posted on arxiv at the following link: <https://arxiv.org/abs/2004.10676>.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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