

Maximum Principle for Nonlinear Cooperative Elliptic Systems on \mathbb{R}^N

LEADI Liamidi* and MARCOS Aboubacar

*Institut de Mathématiques et de Sciences Physiques, Université d'Abomey Calavi,
01 BP : 613 Porto-Novo, Bénin, West Africa.*

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Abstract. We investigate in this work necessary and sufficient conditions for having a Maximum Principle for a cooperative elliptic system on the whole \mathbb{R}^N . Moreover, we prove the existence of solutions by an approximation method for the considered system.

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1 Introduction

Let us consider the following nonlinear cooperative elliptic system

$$(S) \begin{cases} -\Delta_p u = am(x)|u|^{p-2}u + bm_1(x)h(u,v) + f & \text{in } \mathbb{R}^N, \\ -\Delta_q v = dn(x)|v|^{q-2}v + cn_1(x)k(u,v) + g & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0, v(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty. \end{cases} \quad (1.1)$$

Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < +\infty$, is the so-called p-Laplacian operator. The parameters a, b, c, d are nonnegative real parameters. The functions h and $k: \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous and the weight functions m, m_1, n, n_1 , have some properties which will be specified later.

Under suitable conditions on the functions h and k , we investigate necessary and sufficient Maximum Principle conditions and existence results for problem (1.1). The

*Corresponding author. *Email addresses:* leadiare@imsp-uac.org, leadiare@yahoo.com (L. Leadi), abmarcos@imspuac.org (A. Marcos)

Maximum Principle is intended in the sense that, if $f, g \geq 0$ a.e. in \mathbb{R}^N then $u, v \geq 0$ a.e. in \mathbb{R}^N for any solution (u, v) of (1.1).

It is well known that Maximum Principle plays an important role in the theory of nonlinear equations (cf. [1, 2],... for a survey) and in the literature, many works have been devoted to the investigation of Maximum Principle for linear and nonlinear cooperative elliptic systems (cf. [3–9]). For specific interest for the present investigation are the previous works in [10] and [9] where the Maximum Principle for problem (1.1) has been dealt in the framework of some two variables functions of the type $h(s, t) = |s|^\alpha |t|^\beta t$ and $k(s, t) = |s|^\beta s |t|^\alpha$ or of some one variable functions $h(s) = |s|^\beta s$ and $k(s) = |s|^\alpha s$, α and β being nonnegative real parameters satisfying appropriate conditions.

More specifically, our purpose in the present work is to generalize those previous results to a more wide class of functions h and k .

The paper is organized as follows: In the preliminary Section 2, we specify the required assumptions on the data of our problem and we collect some need results relative to the principal positive eigenvalue of the p -Laplacian operator. In Section 3, we give our result on the Maximum Principle; this result paved the way to yield an existence result for (1.1) in Section 4, by using an approximation method. In Section 5 we present some related results to this work and an example of functions h, k for which our result applies.

2 Preliminaries

Throughout this work, we will assume that $1 < p, q < N$ and

$$(H_1) \quad m, n > 0; m \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N) \text{ and } n \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^{N/q}(\mathbb{R}^N).$$

$$(H_2) \quad 0 < m_1(x), n_1(x) \leq [m(x)]^{\frac{\alpha+1}{p}} [n(x)]^{\frac{\beta+1}{q}}.$$

$$(H_3) \quad f \geq 0 \text{ and } f \in L^{(p^*)'}(\mathbb{R}^N); g \geq 0 \text{ and } g \in L^{(q^*)'}(\mathbb{R}^N).$$

$$(H_4) \quad b, c \geq 0; \alpha, \beta \geq 0; \frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

(H5) The functions h and k satisfy the sign conditions

$$t \cdot h(s, t) \geq 0 \quad \text{and} \quad s \cdot k(s, t) \geq 0 \quad \text{for all } (s, t) \in \mathbb{R}^2,$$

and there exists a constant $\Gamma > 0$ such that

$$\begin{cases} h(s, -t) \leq -h(s, t) & \text{for } t \geq 0 \text{ and for all } s \in \mathbb{R}, \\ h(s, t) = \Gamma^{\alpha+\beta+2-p} |s|^\alpha |t|^\beta t & \text{for } t \leq 0 \text{ and for all } s \in \mathbb{R}, \\ k(-s, t) \leq -k(s, t) & \text{for } s \geq 0 \text{ and for all } t \in \mathbb{R}, \\ k(s, t) = \Gamma^{\alpha+\beta+2-q} |s|^\alpha s |t|^\beta & \text{for all } s \leq 0, t \in \mathbb{R}. \end{cases}$$

Here $p^* = Np/(N-p)$ and $q^* = Nq/(N-q)$ denote the critical Sobolev exponents of p and q respectively; p' is the Hölder conjugate of p .

We denote by $D^{1,s}(\mathbb{R}^N)$ (with $1 < s < N$) the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,s}(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |\nabla u|^s \right)^{\frac{1}{s}}.$$

$D^{1,s}(\mathbb{R}^N)$ is a reflexive Banach space and it can be shown (cf [11]) that

$$D^{1,s}(\mathbb{R}^N) = \{u \in L^{s^*}(\mathbb{R}^N) : \nabla u \in (L^s(\mathbb{R}^N))^N\}$$

and for any positive weight $w \in L_{loc}^\infty(\mathbb{R}^N) \cap L^{N/s}(\mathbb{R}^N)$ the following embeddings hold (cf. [12–14])

$$D^{1,s}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N) \quad \text{and} \quad D^{1,s}(\mathbb{R}^N) \hookrightarrow L^s(w, \mathbb{R}^N),$$

where $L^s(w, \mathbb{R}^N)$ is the L^s space on \mathbb{R}^N with the weight w .

By solution (u, v) of (1.1), we mean a weak solution, i.e $(u, v) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ with

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla w = \int_{\mathbb{R}^N} [am(x)|u|^{p-2}u + bm_1(x)h(u, v) + f] w, \\ \int_{\mathbb{R}^N} |\nabla v|^{q-2} \nabla v \cdot \nabla z = \int_{\mathbb{R}^N} [cn_1(x)k(u, v) + dn(x)|v|^{q-2}v + g] z \end{cases} \quad (2.1)$$

for all $(w, z) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$.

Note that by the above embeddings, every integral in (2.1) is well-defined. Regularity results from [15, 16] on general quasilinear equations imply that such a weak solution (u, v) belong to $C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$. It is also known that a weak solution of (1.1) decays to zero at infinity (cf. [12, 17]).

To conclude this preliminaries section, let us briefly recall some properties of the spectrum of $-\Delta_p$ with weight to be used later (cf. [13, 18]). We denote by

$$\lambda_1(m, p) := \min \left\{ \int_{\mathbb{R}^N} |\nabla u|^p : u \in D^{1,p}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} m|u|^p = 1 \right\} \quad (2.2)$$

the unique principal eigenvalue of

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 \text{ as } |x| \rightarrow +\infty; u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (2.3)$$

and by $\varphi_1(m, p) \in D^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ the associated positive eigenvalue such that

$$\int_{\mathbb{R}^N} m|\varphi_1(m, p)|^p = 1.$$

It is well known that $\lambda_1(m,p)$ is simple and isolated. It is also known that any eigenfunction of (2.3) associated to a positive eigenvalue different from $\lambda_1(m,p)$ changes sign in \mathbb{R}^N .

Here and henceforth, the Lebesgue norm in $L^r(\mathbb{R}^N)$ (for $r > 1$) will be denoted by $\|\cdot\|_r$ and the usual norm of $D^{1,p}(\mathbb{R}^N)$ by $\|\cdot\|$. The weak convergence in $D^{1,p}(\mathbb{R}^N)$ is denoted by \rightharpoonup . We will denote by $\Phi = \varphi_1(m,p)$ (respectively by $\Psi = \varphi_1(n,q)$) the positive eigenfunction associated to $\lambda_1(m,p)$ (respectively $\lambda_1(n,q)$) and normalized by

$$\int_{\mathbb{R}^N} m\Phi(x)^p = \int_{\mathbb{R}^N} n\Psi(x)^q = 1. \tag{2.4}$$

The positive and negative parts of a function u are respectively defined as $u^+ = \max\{u,0\}$ and $u^- = \max\{-u,0\}$. Equalities (and inequalities) between two functions must be understood a.e. in \mathbb{R}^N .

3 Maximum principle

We assume that $1 < p,q < N$ and also that hypothesis (H_1) is satisfied. We begin by consider the problem

$$\begin{cases} -\Delta_p u = \mu m(x)|u|^{p-2}u + h(x) & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty. \end{cases} \tag{3.1}$$

The following result was proved in [12,13].

Proposition 3.1. (1) Let $h \in L^{(p^*)'}(\mathbb{R}^N)$. If $\mu < \lambda_1(m,p)$ then (3.1) admits a solution in $D^{1,p}(\mathbb{R}^N)$.

(2) Let $h \in L^{(p^*)'}(\mathbb{R}^N)$ with $h \geq 0$ a.e. in \mathbb{R}^N and $h \not\equiv 0$.

- (a) If $\mu \in [0, \lambda_1(m,p)[$, then any solution u of (3.1) is positive in \mathbb{R}^N .
- (b) If $\mu = \lambda_1(m,p)$ then (3.1) has no solution.
- (c) If $\mu > \lambda_1(m,p)$ then (3.1) has no positive solution.

Using [15,16], one also has the following regularity result.

Proposition 3.2. For all $r > 0$, any solution u of (2.3) belongs to $C^{1,\gamma}(B_r)$, where $\gamma = \gamma(r) \in]0,1[$ and B_r is the ball of radius r centred at the origin.

Let

$$a_1(r) := \inf_{B_r} k_1(x) \quad \text{and} \quad a_2(r) := \sup_{B_r} k_2(x), \tag{3.2}$$

where

$$k_1(x) := \left[\frac{n_1(x)}{n(x)} \right]^{\frac{\beta+1}{q}} \left[\frac{\Phi(x)^p}{\Psi(x)^q} \right]^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}, \quad k_2(x) := \left[\frac{m(x)}{m_1(x)} \right]^{\frac{\alpha+1}{q}} \left[\frac{\Phi(x)^p}{\Psi(x)^q} \right]^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}.$$

We denote by $a_{1\infty} = \lim_{r \rightarrow +\infty} a_1(r)$ and $a_{2\infty} = \lim_{r \rightarrow +\infty} a_2(r)$. Let set

$$\Theta = \frac{a_{1\infty}}{a_{2\infty}}. \tag{3.3}$$

The following inequalities can easily be proved.

$$\Theta \leq \frac{a_1(r)}{a_2(r)} \quad \text{for all } r > 0 \quad \text{and} \quad 0 \leq \Theta \leq 1. \tag{3.4}$$

We now turn to our first main result, i.e the validity of the Maximum Principle (in short **(MP)**) which is stated as follows.

Theorem 3.1. *Assume that hypothesis (H_1) – (H_5) are satisfied. Then the (MP) holds for (1.1) if*

(C1) $\lambda_1(m,p) > a,$

(C2) $\lambda_1(n,q) > d,$

(C3) $[\lambda_1(m,p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n,q) - d]^{\frac{\beta+1}{q}} > b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}}.$

Conversely, if the (MP) holds, then (C1), (C2) and (C4) are satisfied, where

(C4) $[\lambda_1(m,p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n,q) - d]^{\frac{\beta+1}{q}} > \Theta b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}}.$

Corollary 3.1. *If $p = q$ and $m \equiv n$ a.e in \mathbb{R}^N , then the (MP) holds for (1.1) if only if (C1), (C2) and (C3) are satisfied.*

Proof of Theorem 3.1. The proof borrows ideas from [3, 9, 10]. We give it below.

The condition is necessary.

Assume that the **(MP)** holds for system (S). If $\lambda_1(m,p) \leq a$ then the functions $f := (a - \lambda_1(m,p))m(x)\Phi^{p-1}$ and $g := 0$ are nonnegative, however $(-\Phi, 0)$ is a solution of (1.1), a contradiction with the **(MP)**.

Similarly, if $\lambda_1(n,q) \leq d$ then the functions

$$f := 0, \quad g := (d - \lambda_1(n,q))n(x)\Psi^{q-1}$$

are nonnegative and $(0, -\Psi)$ is a solution of (1.1), which is a contradiction with the **(MP)**.

So, we assume that $\lambda_1(m,p) > a$ and $\lambda_1(n,q) > d$. If one of the coefficients Θ, b or c vanishes, then (C4) is satisfied. We will then assume that $\Theta \neq 0, b \neq 0, c \neq 0$ and that (C4) does not hold, i.e.,

$$[\lambda_1(m,p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n,q) - d]^{\frac{\beta+1}{q}} \leq \Theta b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}}. \tag{3.5}$$

Set

$$A = \left(\frac{\lambda_1(m,p) - a}{b} \right)^{\frac{\alpha+1}{p}} \quad \text{and} \quad B = \left(\frac{\lambda_1(n,q) - d}{c} \right)^{\frac{\beta+1}{q}}.$$

Then, by (3.5), one has $AB \leq \Theta$, which clear implies that $Aa_{2\infty} \leq \frac{1}{B}a_{1\infty}$. One deduces that there exists $\zeta \in \mathbb{R}_+^*$ such that

$$Aa_{2\infty} \leq \zeta \leq (1/B)a_{1\infty}.$$

Since the function $a_1(r)$ (respectively $a_2(r)$) is decreasing (respectively increasing) on \mathbb{R}_+^* , one has

$$Aa_2(r) \leq Aa_{2\infty} \leq \zeta \leq \frac{1}{B}a_{1\infty} \leq \frac{1}{B}a_1(r) \quad \text{for all } r > 0.$$

But for any $x \in \mathbb{R}^N$, there exists $r > 0$ such that

$$Ak_2(x) \leq Aa_2(r) \quad \text{and} \quad \frac{1}{B}a_1(r) \leq \frac{1}{B}k_1(x).$$

Consequently, choose ζ such that

$$Ak_2(x) \leq Aa_2(r) \leq \zeta \leq \frac{1}{B}a_1(r) \leq \frac{1}{B}k_1(x)$$

for all $x \in \mathbb{R}^N$, i.e.,

$$Ak_2(x) \leq \zeta, \quad \forall x \in \mathbb{R}^N, \tag{3.6}$$

$$\frac{B}{k_1(x)} \leq \frac{1}{\zeta}, \quad \forall x \in \mathbb{R}^N. \tag{3.7}$$

Next, let us set

$$\zeta = \left(\frac{c_1^q \Gamma^q}{c_2^p \Gamma^p} \right)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}},$$

where c_1 and c_2 are positive constants.

From (3.6) and (H_4) , we have

$$[\lambda_1(m,p) - a]m(x)[c_2\Phi(x)]^{p-1} \leq \Gamma^{\alpha+\beta+2-p}bm_1(x)[c_2\Phi(x)]^\alpha [c_1\Psi(x)]^{\beta+1}$$

for all $x \in \mathbb{R}^N$, which implies (by using (H_5))

$$[\lambda_1(m,p) - a]m(x)[c_2\Phi(x)]^{p-1} \leq -bm_1(x)h(-c_2\Phi, -c_1\Psi)$$

for all $x \in \mathbb{R}^N$. Similarly, from (3.7) and (H_4) , one has

$$[\lambda_1(n,q) - d]n(x)[c_1\Psi(x)]^{q-1} \leq \Gamma^{\alpha+\beta+2-q}cn_1(x)[c_2\Phi(x)]^{\alpha+1} [c_1\Psi(x)]^\beta$$

for all $x \in \mathbb{R}^N$, and consequently (using (H_5)), we get

$$[\lambda_1(n,q) - d]n(x)[c_1\Psi(x)]^{q-1} \leq -cn_1(x)k(-c_2\Phi, -c_1\Psi)$$

for all $x \in \mathbb{R}^N$. Hence

$$f(x) = -[\lambda_1(m,p) - a]m(x)[c_2\Phi(x)]^{p-1} - bm_1(x)h(-c_2\Phi, -c_1\Psi) \geq 0$$

for all $x \in \mathbb{R}^N$ and

$$g(x) = -[\lambda_1(n, q) - d]n(x)[c_1\Psi(x)]^{q-1} - cn_1(x)k(-c_2\Phi, -c_1\Psi) \geq 0$$

for all $x \in \mathbb{R}^N$, are nonnegative functions and $(u, v) = (-c_2\Phi, -c_1\Psi)$ is a solution of (1.1). This is a contradiction with the **(MP)**.

The condition is sufficient

Assume that the conditions (C1), (C2) and (C3) are satisfied. Let (u, v) be a solution of (1.1) for $f \geq 0, g \geq 0$ and suppose that $u^- \not\equiv 0$ and $v^- \not\equiv 0$.

Multiplying the first equation in (1.1) by u^- and the second one by v^- and integrating over \mathbb{R}^N , we get

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u^-|^p &= a \int_{\mathbb{R}^N} m|u^-|^p - b \int_{\mathbb{R}^N} m_1(x)h(u, v)u^- - \int_{\mathbb{R}^N} fu^-, \\ \int_{\mathbb{R}^N} |\nabla v^-|^q &= d \int_{\mathbb{R}^N} n|v^-|^q - c \int_{\mathbb{R}^N} n_1(x)k(u, v)v^- - \int_{\mathbb{R}^N} gv^-. \end{aligned}$$

Then

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u^-|^p &\leq a \int_{\mathbb{R}^N} m|u^-|^p - b \int_{\mathbb{R}^N} m_1(x)h(u, -v^-)u^-, \\ \int_{\mathbb{R}^N} |\nabla v^-|^q &\leq d \int_{\mathbb{R}^N} n|v^-|^q - c \int_{\mathbb{R}^N} n_1(x)k(-u^-, v)v^-. \end{aligned}$$

Using (H_5) , one has

$$\begin{aligned} h(u, -v^-)u^- &= -\Gamma^{\alpha+\beta+2-p}(u^-)^{\alpha+1}(v^-)^{\beta+1}, \\ k(-u^-, v)v^- &= -\Gamma^{\alpha+\beta+2-q}(u^-)^{\alpha+1}(v^-)^{\beta+1}. \end{aligned}$$

Consequently, we get

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u^-|^p &\leq a \int_{\mathbb{R}^N} m(x)|u^-|^p + b\Gamma^{\alpha+\beta+2-p} \int_{\mathbb{R}^N} m_1(x)(u^-)^{\alpha+1}(v^-)^{\beta+1}, \\ \int_{\mathbb{R}^N} |\nabla v^-|^q &\leq d \int_{\mathbb{R}^N} n(x)|v^-|^q + c\Gamma^{\alpha+\beta+2-q} \int_{\mathbb{R}^N} n_1(x)(u^-)^{\alpha+1}(v^-)^{\beta+1}. \end{aligned}$$

Combining the variational characterization of $\lambda_1(m, p)$ and $\lambda_1(n, q)$ with Hölder inequality and assumptions $(H_2), (H_4)$, we have

$$\begin{aligned} [\lambda_1(m, p) - a] \int_{\mathbb{R}^N} m(x)|u^-|^p &\leq b\Gamma^{\alpha+\beta+2-p} \left(\int_{\mathbb{R}^N} m(x)|u^-|^p \right)^{\frac{\alpha+1}{p}} \left(\int_{\mathbb{R}^N} n(x)|v^-|^q \right)^{\frac{\beta+1}{q}}, \\ [\lambda_1(n, q) - d] \int_{\mathbb{R}^N} n(x)|v^-|^q &\leq c\Gamma^{\alpha+\beta+2-q} \left(\int_{\mathbb{R}^N} m(x)|u^-|^p \right)^{\frac{\alpha+1}{p}} \left(\int_{\mathbb{R}^N} n(x)|v^-|^q \right)^{\frac{\beta+1}{q}}, \end{aligned} \tag{3.8}$$

Let us show that we have necessarily $u^- = v^- \equiv 0$, which will yield a contradiction.

- If $\int_{\mathbb{R}^N} m(x)|u^-|^p = 0$ or $\int_{\mathbb{R}^N} n(x)|v^-|^q = 0$ then, using the fact that $m, n > 0$, and (3.8), we get $u^- = v^- \equiv 0$, and the conclusion follows.
- If $\int_{\mathbb{R}^N} m(x)|u^-|^p \neq 0$ and $\int_{\mathbb{R}^N} n(x)|v^-|^q \neq 0$, then from (3.8) and using the fact that $(\alpha + 1)/p$ and $(\beta + 1)/q$ are Hölder conjugate exponents we have

$$\begin{aligned} [\lambda_1(m, p) - a] \left(\int_{\mathbb{R}^N} m(x)|u^-|^p \right)^{\frac{\beta+1}{q}} &\leq b \Gamma^{\alpha+\beta+2-p} \left(\int_{\mathbb{R}^N} n(x)|v^-|^q \right)^{\frac{\beta+1}{q}}, \\ [\lambda_1(n, q) - d] \left(\int_{\mathbb{R}^N} n(x)|v^-|^q \right)^{\frac{\alpha+1}{p}} &\leq c \Gamma^{\alpha+\beta+2-q} \left(\int_{\mathbb{R}^N} m(x)|u^-|^p \right)^{\frac{\alpha+1}{p}}, \end{aligned} \quad (3.9)$$

which implies

$$\begin{aligned} [\lambda_1(m, p) - a]^{\frac{\alpha+1}{p}} \left(\int_{\mathbb{R}^N} m(x)|u^-|^p \right)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}} &\leq b^{\frac{\alpha+1}{p}} \Gamma^{(\alpha+\beta+2-p)\frac{\alpha+1}{p}} \left(\int_{\mathbb{R}^N} n(x)|v^-|^q \right)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}, \\ [\lambda_1(n, q) - d]^{\frac{\beta+1}{q}} \left(\int_{\mathbb{R}^N} n(x)|v^-|^q \right)^{\frac{\beta+1}{q} \frac{\alpha+1}{p}} &\leq c^{\frac{\beta+1}{q}} \Gamma^{(\alpha+\beta+2-q)\frac{\beta+1}{q}} \left(\int_{\mathbb{R}^N} m(x)|u^-|^p \right)^{\frac{\beta+1}{q} \frac{\alpha+1}{p}}. \end{aligned} \quad (3.10)$$

From assumption (H_4) , one easily derives that

$$\begin{aligned} &(\alpha + \beta + 2 - p) \frac{\alpha + 1}{p} + (\alpha + \beta + 2 - q) \frac{\beta + 1}{q} \\ &= (\alpha + \beta + 2) \left(\frac{\alpha + 1}{p} + \frac{\beta + 1}{q} \right) - (\alpha + 1) - (\beta + 1) = 0. \end{aligned} \quad (3.11)$$

Then multiplying both inequalities in (3.10), one has:

$$\begin{aligned} &[\lambda_1(m, p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n, q) - d]^{\frac{\beta+1}{q}} \left[\left(\int_{\mathbb{R}^N} m(x)|u^-|^p \right) \left(\int_{\mathbb{R}^N} n(x)|v^-|^q \right) \right]^{\frac{\alpha+1}{p} \frac{\beta+1}{q}} \\ &\leq b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}} \left[\left(\int_{\mathbb{R}^N} m(x)|u^-|^p \right) \left(\int_{\mathbb{R}^N} n(x)|v^-|^q \right) \right]^{\frac{\alpha+1}{p} \frac{\beta+1}{q}} \end{aligned}$$

and consequently

$$\begin{aligned} &\left[[\lambda_1(m, p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n, q) - d]^{\frac{\beta+1}{q}} - b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}} \right] \\ &\quad \times \left[\left(\int_{\mathbb{R}^N} m(x)|u^-|^p \right) \left(\int_{\mathbb{R}^N} n(x)|v^-|^q \right) \right]^{\frac{\alpha+1}{p} \frac{\beta+1}{q}} \leq 0. \end{aligned} \quad (3.12)$$

Since assumptions $(C1)$, $(C2)$ and $(C3)$ hold, and $m, n > 0$, one deduces that $u^- = v^- \equiv 0$, a contradiction.

By applying regularity results of [15, 16] and the maximum principle of [19], one has in fact $u > 0$ and $v > 0$ a.e. in \mathbb{R}^N . \square

4 Existence of positive solution

In this section, we prove the existence of positive solutions for (1.1) under conditions (C1), (C2) and (C3). The technique used here have been previously developed in [3, 10, 20]. It combines Leray-Schauder fixed point argument with an approximation method.

So, for $\epsilon \in]0, 1[$, we define the following expression

$$\begin{aligned} X_n &:= m(x) \frac{|u_n|^{p-2} u_n}{1 + |\epsilon^{1/p} u_n|^{p-1}}, & X &:= m(x) \frac{|u|^{p-2} u}{1 + |\epsilon^{1/p} u|^{p-1}}, \\ Y_n &:= m_1(x) \frac{h(u_n, v_n)}{1 + \epsilon |h(u_n, v_n)|}, & Y &:= m_1(x) \frac{h(u, v)}{1 + \epsilon |h(u, v)|}, \\ X'_n &:= n(x) \frac{|v_n|^{q-2} v_n}{1 + |\epsilon^{1/q} v_n|^{q-1}}, & X' &:= n(x) \frac{|v|^{q-2} v}{1 + |\epsilon^{1/q} v|^{q-1}}, \\ Y'_n &:= n_1(x) \frac{k(u_n, v_n)}{1 + \epsilon |k(u_n, v_n)|}, & Y' &:= n_1(x) \frac{k(u, v)}{1 + \epsilon |k(u, v)|}. \end{aligned}$$

One has the following result which will be useful later.

Lemma 4.1. *If (u_n, v_n) converges to (u, v) in $L^{p^*}(\mathbb{R}^N) \times L^{q^*}(\mathbb{R}^N)$ then*

(i) $X_n \rightarrow X$ in $L^{p^{*'}}(\mathbb{R}^N)$, $Y_n \rightarrow Y$ in $L^{p^{*'}}(\mathbb{R}^N)$, where $p^{*' } = \frac{pN}{N(p-1)+p}$.

(ii) $X'_n \rightarrow X'$ in $L^{q^{*'}}(\mathbb{R}^N)$, $Y'_n \rightarrow Y'$ in $L^{q^{*'}}(\mathbb{R}^N)$, where $q^{*' } = \frac{qN}{N(q-1)+q}$.

Proof. We give the proof for (i), the same arguments hold for the proof of (ii).

Since $u_n \rightarrow u$ in $L^{p^*}(\mathbb{R}^N)$, then there exists a subsequence denoted (u_n) such that $u_n \rightarrow u$ a.e on \mathbb{R}^N and $|u_n(x)| \leq |l_1(x)|$ a.e. on \mathbb{R}^N for some $l_1 \in L^{p^*}(\mathbb{R}^N)$. Hence

$$X_n(x) \rightarrow X(x) \quad \text{a.e. in } \mathbb{R}^N, \quad \text{as } n \rightarrow +\infty$$

and

$$|X_n(x)| \leq m(x) |u_n(x)|^{p-1} \leq m(x) |l_1(x)|^{p-1} \quad \text{in } L^{p^{*'}}(\mathbb{R}^N),$$

which implies, the by dominated convergence theorem, that $X_n \rightarrow X$ in $L^{p^{*'}}(\mathbb{R}^N)$.

Moreover, since $v_n \rightarrow v$ in $L^{q^*}(\Omega)$ there exists a subsequence still denoted (v_n) such that

$$\begin{cases} v_n(x) \rightarrow v(x) & \text{a.e. on } \mathbb{R}^N, \\ |v_n(x)| \leq |l_2(x)| & \text{a.e. on } \mathbb{R}^N \text{ with } l_2 \in L^{q^*}(\mathbb{R}^N). \end{cases} \tag{4.1}$$

The function h being continuous, one has

$$Y_n(x) \rightarrow Y(x) \quad \text{a.e. in } \mathbb{R}^N, \quad \text{as } n \rightarrow +\infty$$

and

$$|Y_n(x)| \leq m_1(x) |h(u_n, v_n)| \leq D m^{\frac{\alpha+1}{p}} n^{\frac{\beta+1}{q}} |l_1|^\alpha |l_2|^{\beta+1} \quad \text{in } L^{p^*}'(\mathbb{R}^N),$$

for some constant D which depends only on Γ (cf. (H_5)), and the conclusion follows by using the dominated convergence theorem. \square

We are now in position to give the main result of this section

Theorem 4.1. *Assume that $(H_1) - (H_5)$, $(C1)$, $(C2)$, $(C3)$ are satisfied. Then for all $f \in L^{(p^*)}'(\mathbb{R}^N)$ and $g \in L^{(q^*)}'(\mathbb{R}^N)$, the system (1.1) has at least one solution $(u, v) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$.*

The proof will be given in several steps. It is partly adapted from [3,10,20]. We choose $r > 0$ such that $a+r > 0$ and $d+r > 0$. The system (1.1) is then equivalent to

$$\begin{cases} -\Delta_p u + r m(x) |u|^{p-2} u = (a+r) m(x) |u|^{p-2} u + b n_1(x) h(u, v) + f & \text{in } \mathbb{R}^N, \\ -\Delta_q v + r n(x) |v|^{q-2} v = c n_1 k(u, v) + (d+r) n(x) |v|^{q-2} v + g & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0, v(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty. \end{cases} \quad (4.2)$$

For $\epsilon \in]0, 1[$, let us introduce the system :

$$\begin{cases} -\Delta_p u_\epsilon + r m(x) |u_\epsilon|^{p-2} u_\epsilon = m(x) H(u_\epsilon) + m_1(x) \hat{H}(u_\epsilon, v_\epsilon) + f & \text{in } \mathbb{R}^N, \\ -\Delta_q v_\epsilon + r n(x) |v_\epsilon|^{q-2} v_\epsilon = n(x) K(v_\epsilon) + n_1(x) \hat{K}(u_\epsilon, v_\epsilon) + g & \text{in } \mathbb{R}^N, \\ u_\epsilon(x) \rightarrow 0, v_\epsilon(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases} \quad (4.3)$$

where

$$H(s) = (a+r) |s|^{p-2} s (1 + \epsilon^{\frac{1}{p}} |s|^{p-1})^{-1}, \quad \hat{H}(s, t) = b h(s, t) (1 + \epsilon |h(s, t)|)^{-1},$$

$$K(s) = (d+r) |s|^{q-2} s (1 + \epsilon^{\frac{1}{q}} |s|^{q-1})^{-1}, \quad \hat{K}(s, t) = c k(s, t) (1 + \epsilon |k(s, t)|)^{-1}.$$

Lemma 4.2. *Under the hypothesis of Theorem 4.1, system (4.3) admits at least a solution (u, v) in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$.*

Proof. The arguments are similar to those in the proof of Lemma 4.3 in [9, 10]. We will only indicate below the main steps.

One starts a first step by constructing a couple of sub-super solution for (4.3). Indeed the functions H, \hat{H}, K and \hat{K} are bounded, so there exists a constant $M > 0$ such that

$$|H(u)| \leq M, \quad |\hat{H}(u, v)| \leq M, \quad |K(v)| \leq M, \quad |\hat{K}(u, v)| \leq M$$

for all $(u, v) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$. Let $\zeta^0 \in D^{1,p}(\mathbb{R}^N)$ (respectively $\eta^0 \in D^{1,q}(\mathbb{R}^N)$) be a solution of

$$-\Delta_p u + r m(x) |u|^{p-2} u = (m(x) + m_1(x)) M + f$$

(respectively $-\Delta_q v + rm(x)|v|^{q-2}v = (n(x) + n_1(x))M + g$), and let $\xi_0 \in D^{1,p}(\mathbb{R}^N)$ (respectively $\eta_0 \in D^{1,q}(\mathbb{R}^N)$) be solution of

$$-\Delta_p u + rm(x)|u|^{p-2}u = -(m(x) + m_1(x))M + f$$

(respectively $-\Delta_q v + rm(x)|v|^{q-2}v = -(n(x) + n_1(x))M + g$). The existence of (ξ^0, η^0) and (ξ_0, η_0) is proved in [21]. Moreover (ξ^0, η^0) (respectively (ξ_0, η_0)) is a super solution (respectively sub solution) of system (4.3) since

$$\begin{aligned} -\Delta_p \xi^0 + rm(x)|\xi^0|^{p-2}\xi^0 - m(x)H(\xi^0) - m_1(x)\hat{H}(\xi^0, v) - f &\geq 0, \\ -\Delta_q \eta^0 + rn(x)|\eta^0|^{q-2}\eta^0 - n_1(x)\hat{K}(u, \xi) - n(x)K(\eta^0) - g &\geq 0, \\ -\Delta_p \xi_0 + rm(x)|\xi_0|^{p-2}\xi_0 - m(x)H(\xi_0) - m_1(x)\hat{H}(\xi_0, v) - f &\leq 0, \\ -\Delta_q \eta_0 + rn(x)|\eta_0|^{q-2}\eta_0 - n_1(x)\hat{K}(u, \xi) - n(x)K(\eta_0) - g &\leq 0, \end{aligned}$$

for $\forall v \in [\eta_0, \eta^0], \forall u \in [\xi_0, \xi^0]$. In a second step, let us define an operator T on the set $K = [\xi_0, \xi^0] \times [\eta_0, \eta^0]$ by $T: (u, v) \mapsto (w, z)$ such that

$$\begin{cases} -\Delta_p w + rm(x)|w|^{p-2}w = m(x)H(u) + m_1(x)\hat{H}(u, v) + f & \text{in } \mathbb{R}^N, \\ -\Delta_q z + rn(x)|z|^{q-2}z = n_1(x)\hat{K}(u, v) + n(x)K(v) + g & \text{in } \mathbb{R}^N, \\ w(x) \rightarrow 0, z(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty. \end{cases} \tag{4.4}$$

From Lemma 4.1 and arguing similarly as in [9,10], one can easily show that $T(K) \subset K$ and T is continuous and compact. In order to conclude, we observe that the set K is convex, bounded and closed in $D^{1,q}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$, so that one can apply the Schauder's fixed point theorem to T to get existence of solution for system (4.3). So Lemma 4.2 is proved. \square

Proof of Theorem 4.1. The proof will be given in three steps.

Step 1. Let's prove that (u_ϵ, v_ϵ) is bounded in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$. Indeed assume that $\|u_\epsilon\| \rightarrow \infty$ or $\|v_\epsilon\| \rightarrow \infty$ when $\epsilon \rightarrow 0$. Let

$$t_\epsilon = \max\{\|u_\epsilon\|; \|v_\epsilon\|\}$$

and

$$w_\epsilon = t_\epsilon^{-\frac{1}{p}} u_\epsilon, \quad z_\epsilon = t_\epsilon^{-\frac{1}{q}} v_\epsilon.$$

We have $\|w_\epsilon\| \leq 1$ and $\|z_\epsilon\| \leq 1$ with either $\|w_\epsilon\| = 1$ or $\|z_\epsilon\| = 1$. Dividing the first equation in (4.3) by $t_\epsilon^{\frac{1}{p}}$ we get

$$\begin{aligned} -\Delta_p w_\epsilon + rm(x)|w_\epsilon|^{p-2}w_\epsilon &= (a+r)m(x)|w_\epsilon|^{p-2}w_\epsilon(1 + |\epsilon^{\frac{1}{p}}u_\epsilon|^{p-1})^{-1} \\ &+ t_\epsilon^{-\frac{1}{p'}} bm_1(x)h(t_\epsilon^{\frac{1}{p}}w_\epsilon, t_\epsilon^{\frac{1}{q}}z_\epsilon)(1 + \epsilon|h(u_\epsilon, v_\epsilon)|)^{-1} + t_\epsilon^{-\frac{1}{p'}} f. \end{aligned} \tag{4.5}$$

Similarly dividing the second equation in (4.3) by $(t_\epsilon)^{1/q'}$ we get

$$\begin{aligned}
 -\Delta_q z_\epsilon + rn(x)|z_\epsilon|^{q-2}w_\epsilon &= (d+r)n(x)|w_\epsilon|^\alpha w_\epsilon(1+|\epsilon^{\frac{1}{q}}u_\epsilon|^{\alpha+1})^{-1} \\
 &+ t_\epsilon^{\frac{-1}{q'}} cn_1(x)k(t_\epsilon^{\frac{1}{p}}w_\epsilon, t_\epsilon^{\frac{1}{q}}z_\epsilon)(1+\epsilon|k(u_\epsilon, v_\epsilon)|)^{-1} + t_\epsilon^{\frac{-1}{q'}} g.
 \end{aligned}
 \tag{4.6}$$

Testing equation (4.5) by w_ϵ and using (H_5) , we get

$$\begin{aligned}
 \int_\Omega |\nabla w_\epsilon|^p &\leq a \int_\Omega m(x)|w_\epsilon|^p + b\Gamma^{\alpha+\beta+2-p} \int_\Omega m(x)^{\frac{\alpha+1}{p}} |w_\epsilon|^{\alpha+1} n(x)^{\frac{\beta+1}{q}} |z_\epsilon|^{\beta+1} \\
 &+ (t_\epsilon)^{\frac{-1}{p'}} \int_\Omega |f||w_\epsilon|.
 \end{aligned}$$

This implies, using the Hölder inequality,

$$\begin{aligned}
 \int_{\mathbb{R}^N} |\nabla w_\epsilon|^p &\leq a \int_{\mathbb{R}^N} m|w_\epsilon|^p + b\Gamma^{\alpha+\beta+2-p} \left(\int_{\mathbb{R}^N} m|w_\epsilon|^p \right)^{\frac{(\alpha+1)}{p}} \\
 &\left(\int_{\mathbb{R}^N} n|w_\epsilon|^q \right)^{\frac{(\beta+1)}{q}} + (t_\epsilon)^{\frac{-1}{p'}} \|f\|_{(p^*)'} \|z_\epsilon\|_{p^*}.
 \end{aligned}$$

Using the variational characterization of $\lambda_1(m, p)$ and the imbedding of $D^{1,p}(\mathbb{R}^N)$ in $L^{p^*}(\mathbb{R}^N)$ one has

$$\begin{aligned}
 \|w_\epsilon\|^p &\leq \frac{a}{\lambda_1(m, p)} \|w_\epsilon\|^p + b\Gamma^{\alpha+\beta+2-p} \frac{\|w_\epsilon\|^{\alpha+1}}{[\lambda_1(m, p)]^{\frac{(\alpha+1)}{p}}} \frac{\|z_\epsilon\|^{\beta+1}}{[\lambda_1(n, q)]^{\frac{(\beta+1)}{q}}} \\
 &+ c(p, N)(t_\epsilon)^{\frac{-1}{p'}} \|f\|_{(p^*)'} \|z_\epsilon\|_{p^*},
 \end{aligned}$$

where $c(p, N)$ is the constant of this imbedding. So, one gets

$$\begin{aligned}
 &(\lambda_1(m, p) - a) \frac{(\|w_\epsilon\|^p)^{\frac{\beta+1}{q}}}{\lambda_1(m, p)} \\
 &\leq \frac{b\Gamma^{\alpha+\beta+2-p} (\|z_\epsilon\|^q)^{\frac{\beta+1}{q}}}{\lambda_1(m, p)^{\frac{\alpha+1}{p}} \lambda_1(n, q)^{\frac{\beta+1}{q}}} + t_\epsilon^{\frac{-1}{p'}} \left(\int_\Omega |f|^{p'} \right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla w_\epsilon|^p \right)^{\frac{-\alpha}{p}},
 \end{aligned}
 \tag{4.7}$$

and then

$$(\lambda_1(m, p) - a) \frac{(\overline{\lim} \|w_\epsilon\|^p)^{\frac{\beta+1}{q}}}{\lambda_1(m, p)} \leq \frac{b\Gamma^{\alpha+\beta+2-p} (\overline{\lim} \|z_\epsilon\|^q)^{\frac{\beta+1}{q}}}{\lambda_1(m, p)^{\frac{\alpha+1}{p}} \lambda_1(n, q)^{\frac{\beta+1}{q}}}.$$

Hence

$$(\lambda_1(m, p) - a)^{\frac{\alpha+1}{p}} \frac{(\overline{\lim} \|w_\epsilon\|^p)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}}{\lambda_1(m, p)^{\frac{\alpha+1}{p}}} \leq b^{\frac{\alpha+1}{p}} \frac{\Gamma^{(\alpha+\beta+2-p)(\frac{\alpha+1}{p})} (\overline{\lim} \|z_\epsilon\|^q)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}}{\lambda_1(m, p)^{(\frac{\alpha+1}{p})^2} \lambda_1(n, q)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}}.
 \tag{4.8}$$

In a similar way with Eq. (4.6) and z_ϵ as test function, we get

$$(\lambda_1(n,q) - d)^{\frac{\beta+1}{q}} \frac{(\overline{\lim} \|z_\epsilon\|^q)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}}{\lambda_1(n,q)^{\frac{\beta+1}{q}}} \leq c^{\frac{\beta+1}{q}} \frac{\Gamma^{(\alpha+\beta+2-q)(\frac{\beta+1}{q})} (\overline{\lim} \|w_\epsilon\|^p)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}}{\lambda_1(n,q)^{(\frac{\beta+1}{q})^2} \lambda_1(m,p)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}}. \tag{4.9}$$

Multiplying term by term the expressions in (4.8) and (4.9), and using (H_5) we get

$$\left[(\lambda_1(m,p) - a)^{\frac{\alpha+1}{p}} (\lambda_1(n,q) - d)^{\frac{\beta+1}{q}} - b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}} \right] \frac{(\overline{\lim} \|w_\epsilon\|^p)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}} (\overline{\lim} \|z_\epsilon\|^p)^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}}{\lambda_1(m,p)^{\frac{\alpha+1}{p}} \lambda_1(n,q)^{\frac{\beta+1}{q}}} \leq 0.$$

Since conditions $(C_1), (C_2)$ and (C_3) hold, one has:

$$\overline{\lim} \|w_\epsilon\|^p = \overline{\lim} \|z_\epsilon\|^p = 0.$$

This yields a contradiction since $\|w_\epsilon\|=1$ or $\|z_\epsilon\|=1$, and consequently (u_ϵ, v_ϵ) is bounded in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$.

Step 2. Let's now prove that (u_ϵ, v_ϵ) converges strongly in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ when ϵ goes to 0.

From Step 1, we have that (u_ϵ, v_ϵ) is bounded in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ and consequently $(u_\epsilon, v_\epsilon) \rightharpoonup (u_0, v_0)$ in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$. So up to a subsequence $(u_\epsilon, v_\epsilon) \rightarrow (u_0, v_0)$ in $L^p(m, \mathbb{R}^N) \times L^q(n, \mathbb{R}^N)$, that is $(m^{1/p} u_\epsilon, n^{1/q} v_\epsilon) \rightarrow (m^{1/p} u_0, n^{1/q} v_0)$ in $L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N)$ and consequently, there exists a function $\eta \in L^p(\mathbb{R}^N), \zeta \in L^q(\mathbb{R}^N)$ such that

$$\begin{cases} m^{\frac{1}{p}}(x)u_\epsilon(x) \rightarrow m^{\frac{1}{p}}(x)u_0(x) & \text{a.e. in } \mathbb{R}^N, \text{ when } \epsilon \rightarrow 0, \\ n^{\frac{1}{q}}(x)v_\epsilon(x) \rightarrow n^{\frac{1}{q}}(x)v_0(x) & \text{a.e. in } \mathbb{R}^N, \text{ when } \epsilon \rightarrow 0, \\ |m^{\frac{1}{p}}u_\epsilon| \leq \eta \in L^p(\mathbb{R}^N), \\ |n^{\frac{1}{q}}v_\epsilon| \leq \zeta \in L^q(\mathbb{R}^N). \end{cases} \tag{4.10}$$

Since $m, n > 0$, we have

$$\begin{cases} u_\epsilon(x) \rightarrow u_0(x) & \text{a.e. in } \mathbb{R}^N, \text{ when } \epsilon \rightarrow 0, \\ v_\epsilon(x) \rightarrow v_0(x) & \text{a.e. in } \mathbb{R}^N, \text{ when } \epsilon \rightarrow 0. \end{cases} \tag{4.11}$$

Clearly $\epsilon^{\frac{1}{p}} u_\epsilon \rightarrow 0, \epsilon^{\frac{1}{q}} v_\epsilon \rightarrow 0$ a.e. in \mathbb{R}^N as $\epsilon \rightarrow 0$, and combining this fact with (4.10) and (4.11), one obtains

$$\begin{aligned} m|H(u_\epsilon)| &= m||u_\epsilon|^{p-2}u_\epsilon(1 + \epsilon^{\frac{1}{p}}|u_\epsilon|^{p-1})^{-1}| \leq m^{\frac{1}{p}}|m^{\frac{1}{p}}u_\epsilon|^{p-1} \leq m^{\frac{1}{p}}\eta^{p-1} \in L^{p^*}(\mathbb{R}^N). \\ n|K(v_\epsilon)| &= n||v_\epsilon|^{q-2}v_\epsilon(1 + \epsilon^{\frac{1}{q}}|v_\epsilon|^{q-1})^{-1}| \leq n^{\frac{1}{q}}|n^{\frac{1}{q}}v_\epsilon|^{q-1} \leq n^{\frac{1}{q}}\zeta^{q-1} \in L^{q^*}(\mathbb{R}^N), \end{aligned}$$

and

$$\begin{aligned} m(x)H(u_\epsilon(x)) &\rightarrow m(x)|u_0(x)|^{p-2}u_0(x) && \text{a.e. in } \mathbb{R}^N, \\ n(x)k(v_\epsilon(x)) &\rightarrow n(x)|v_0(x)|^{q-2}v_0(x) && \text{a.e. in } \mathbb{R}^N, \end{aligned}$$

as $\epsilon \rightarrow 0$. So, applying the dominated convergence theorem we have:

$$\begin{aligned} mH(u_\epsilon) &\rightarrow m|u_0|^{p-2}u_0 && \text{in } L^{p^*}(\mathbb{R}^N) \text{ when } \epsilon \rightarrow 0, \\ nk(v_\epsilon) &\rightarrow n|v_0|^{q-2}v_0 && \text{in } L^{q^*}(\mathbb{R}^N) \text{ when } \epsilon \rightarrow 0. \end{aligned}$$

Similarly we have

$$m_1(x)\hat{H}(u_\epsilon(x), v_\epsilon(x)) = \frac{m_1(x)h(u_\epsilon(x), v_\epsilon(x))}{1 + \epsilon|h(u_\epsilon(x), v_\epsilon(x))|} \rightarrow m_1(x)h(u_0(x), v_0(x)) \quad \text{a.e. in } \mathbb{R}^N,$$

and

$$n_1(x)\hat{K}(u_\epsilon(x), v_\epsilon(x)) = \frac{n_1(x)k(u_\epsilon(x), v_\epsilon(x))}{1 + \epsilon|k(u_\epsilon(x), v_\epsilon(x))|} \rightarrow n_1(x)k(u_0(x), v_0(x)) \quad \text{a.e. in } \mathbb{R}^N,$$

when $\epsilon \rightarrow 0$ and

$$\begin{aligned} m_1|\hat{H}(u_\epsilon, v_\epsilon)| &\leq Dm^{\frac{1}{p}}|m^{\frac{1}{p}}u_\epsilon|^\alpha|n^{\frac{1}{q}}v_\epsilon|^{\beta+1} \leq Dm^{\frac{1}{p}}|\eta|^\alpha|\zeta|^{\beta+1} \in L^{p^*}(\mathbb{R}^N), \\ n_1|\hat{K}(u_\epsilon, v_\epsilon)| &\leq Dn^{\frac{1}{q}}|m^{\frac{1}{p}}u_\epsilon|^{\alpha+1}|n^{\frac{1}{q}}v_\epsilon|^\beta \leq D|\eta|^{\alpha+1}|\zeta|^\beta \in L^{q^*}(\mathbb{R}^N), \end{aligned}$$

where D is a constant depending only on Γ . Her again from the dominated convergence theorem, we deduce that

$$m_1\hat{H}(u_\epsilon, v_\epsilon) \rightarrow m_1h(u_0, v_0) \quad \text{in } L^{p'}(\mathbb{R}^N), \text{ as } \epsilon \rightarrow 0,$$

and

$$n_1\hat{K}(u_\epsilon, v_\epsilon) \rightarrow n_1k(u_0, v_0) \quad \text{in } L^{q'}(\mathbb{R}^N), \text{ as } \epsilon \rightarrow 0.$$

In order to conclude to the strong convergence of u_ϵ in $D^{1,p}(\mathbb{R}^N)$, let us use $(u_\epsilon - u_0)$ as a test function in the first equation in (4.3). Then we get

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - |\nabla u_0|^{p-2}u_0)(\nabla u_\epsilon - \nabla u_0) \\ &\leq \int_{\mathbb{R}^N} (|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - |\nabla u_0|^{p-2}u_0)(\nabla u_\epsilon - \nabla u_0) + r \int_{\mathbb{R}^N} m(x)(|u_\epsilon|^{p-2}u_\epsilon - |u_0|^{p-2}u_0)(u_\epsilon - u_0) \\ &= (a+r) \int_{\mathbb{R}^N} m(H(u_\epsilon) - |u_0|^{p-2}u_0)(u_\epsilon - u_0) + b \int_{\mathbb{R}^N} m_1\hat{H}(u_\epsilon, v_\epsilon) - \hat{H}(u_0, v_0))(u_\epsilon - u_0). \end{aligned}$$

Using then the inequality

$$\|x - y\|^p \leq c[(\|x\|^{p-2}x - \|y\|^{p-2}y)(x - y)]^{\frac{p}{2}}[\|x\|^p + \|y\|^p]^{1-\frac{p}{2}},$$

where $x, y \in \mathbb{R}^N$, $c = c(p) > 0$ and $s = 2$ if $p \geq 2$, $s = p$ if $1 < p < 2$ (cf. e.g. [22]), one easily obtains that $u_\epsilon \rightarrow u_0$ in $D^{1,p}(\mathbb{R}^N)$. In a similar way we can show that $v_\epsilon \rightarrow v_0$ in $D^{1,q}(\mathbb{R}^N)$.

Step 3. From the strong convergence of (u_ϵ, v_ϵ) in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ and a classical result in functional analysis (cf. [21]), we derive that

$$\begin{cases} -\Delta_p u_0 + rm(x)|u_0|^{p-2}u_0 = (a+r)m(x)|u_0|^{p-2}u_0 + bm_1(x)h(u_0, v_0) + f & \text{in } \mathbb{R}^N, \\ -\Delta_q v_0 + rn(x)|v_0|^{q-2}v_0 = (d+r)n(x)|v_0|^{q-2}v_0 + cn_1(x)k(u_0, v_0) + g & \text{in } \mathbb{R}^N, \\ u_0(x) \rightarrow 0, v_0(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases}$$

which can be written again as

$$\begin{cases} -\Delta_p u_0 = am(x)|u_0|^{p-2}u_0 + bm_1(x)h(u_0, v_0) + f & \text{in } \mathbb{R}^N, \\ -\Delta_q v_0 = dn(x)|v_0|^{q-2}v_0 + cn_1(x)k(u_0, v_0) + g & \text{in } \mathbb{R}^N, \\ u_0(x) \rightarrow 0, v_0(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty. \end{cases} \tag{4.12}$$

This achieves the proof of Theorem 4.1. □

5 Related results and example

5.1 Related results

1. The results obtained here are still valid when the second part of assumption (H_5) has changed as follows

$$\begin{cases} h(s, t) = \Gamma^{\alpha+\beta+2-p}|s|^\alpha|t|^{\beta+1} & \text{for } t \geq 0 \text{ and for all } s \in \mathbb{R}, \\ h(s, -t) \leq -h(s, t) & \text{for } t \leq 0 \text{ and for all } s \in \mathbb{R}, \\ k(s, t) = \Gamma^{\alpha+\beta+2-q}(s)^{\alpha+1}|t|^\beta & \text{for } s \geq 0 \text{ and for all } t \in \mathbb{R}, \\ k(-s, t) \leq -k(s, t) & \text{for } s \leq 0 \text{ and for all } t \in \mathbb{R}. \end{cases}$$

2. Results obtained above can be established in a similar way for the following problem

$$\begin{cases} -\Delta_p u = am(x)|u|^{p-2}u + bm_1(x)h(v) + f & \text{in } \mathbb{R}^N, \\ -\Delta_q v = cn_1(x)k(u) + dn(x)|v|^{q-2}v + g & \text{in } \mathbb{R}^N, \\ u(x) \rightarrow 0, v(x) \rightarrow 0 & \text{as } |x| \rightarrow +\infty, \end{cases} \tag{S'}$$

under the assumptions that

$$(H_1) \quad m, n > 0; m \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N) \text{ and } n \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap L^{N/q}(\mathbb{R}^N).$$

$$(H_2) \quad 0 < m_1(x), n_1(x) \leq [m(x)]^{\frac{\alpha+1}{p}} [n(x)]^{\frac{\beta+1}{q}}.$$

(H₃) $f \geq 0$ and $f \in L^{(p^*)'}(\mathbb{R}^N)$; $g \geq 0$ and $g \in L^{(q^*)'}(\mathbb{R}^N)$.

(H₄) $b, c \geq 0$; $\alpha, \beta \geq 0$; $\frac{1}{p} + \frac{\beta+1}{q} = \frac{\alpha+1}{p} + \frac{1}{q} = 1$.

(H₅) The functions h and k are continuous and satisfy the sign conditions

$$s \cdot h(s) \geq 0 \quad \text{and} \quad s \cdot k(s) \geq 0 \quad \text{for all } s \in \mathbb{R},$$

and there exists a constant $\Gamma > 0$ such that

$$\begin{cases} h(-s) \leq -h(s) & \text{for } s \geq 0, \\ h(s) = \Gamma^{\beta+2-p} |s|^\beta s & \text{for } s \leq 0, \end{cases} \quad \begin{cases} k(-s) \leq -k(s) & \text{for } s \geq 0, \\ k(s) = \Gamma^{\alpha+2-q} |s|^\alpha s & \text{for } s \leq 0. \end{cases}$$

5.2 Example

Many illustrative examples for the results obtained in this paper can be easily constructed. We just give one below

$$h(s, t) = \begin{cases} |\sin s|^\alpha |\arctan t|^\beta t & \text{for } t \geq 0 \text{ and for all } s \in \mathbb{R}, \\ |s|^\alpha |t|^\beta t & \text{for } t \leq 0 \text{ and for all } s \in \mathbb{R}. \end{cases}$$

$$k(s, t) = \begin{cases} s^{\alpha+1} e^{-\alpha s} |\arctan t|^\beta & \text{for } s \geq 0 \text{ and for all } t \in \mathbb{R}, \\ |s|^\alpha s |t|^\beta & \text{for } s \leq 0 \text{ and for all } t \in \mathbb{R}. \end{cases}$$

It is easy to see that the functions h and k are continuous and satisfy condition H₅ as well, so that the results obtained in this work can be applied.

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