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Hamiltonian Mean Curvature Flow

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Abstract

Let (Σ, ω) be a compact Riemann surface with constant curvature c. In this work, we proved that the mean curvature flow of a given Hamiltonian diffeomorphism on Σ provides a smooth path in $Ham(\Sigma)$, the group of all Hamiltonian diffeomorphisms of Σ. This result gives a proof, in the case of graph of Hamiltonian diffeomorphisms to the conjecture of Thomas and Yau asserting that *the mean curvature flow of a compact embedded Lagrangian submanifold S with zero Maslov class in a Calabi-Yau manifolds M exists for all time and converges smoothly to a special Lagrangian submanifold in the Hamiltonian isotopy class of S.*

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1 Introduction

The deformation of maps between Riemannian manifolds has been intensively studied for a long time. The idea is to find a natural process to deform a map to a *canonical* one. The harmonic heat flow is probably the famous example although the Ricci flow and the mean curvature flow are also well used. The latter is an evolution process under which a submanifold of a given manifold evolves in the direction of its mean curvature vector. From the first variation formula for the volume functional, one easily observes that the mean curvature flow represents the most effective way to decrease the volume of a submanifold so that it is very useful when minimal submanifolds or volume minimizer submanifolds are sorted for. Several results have been found for the mean curvature flow in codimension one while the higher codimension case is still receiving attention of many researchers.

The simplest case of higher codimension is the mean curvature flow of surface in 4-dimensional manifolds. It compounds two important classes known as *symplectic mean curvature flow* and *Lagrangian mean curvature flow* ; since it was proven that being symplectic or Lagrangian is preserved along the flow. In the last decades, several works in geometric evolution equations research are devoted to these two classes (see e.g [3, 9, 11, 12]).

It is well known that the geometric structures of the ambient space plays a fundamental role when studying the existence and the properties of the mean curvature flow. For instance, to the authors knowledge, symplectic mean curvature flow exists only when the ambient manifold carries a (almost) Kähler-Einstein structure [12] ; or particularly when it is Calabi-Yau (target spaces for superstring compactification). Moreover Lagrangian and special (minimal) Lagrangian submanifolds of Calabi-Yau manifolds are considered as the cornerstones for understanding the mirror symmetry phenomenon between pairs of Calabi-Yau manifold both of the categorical point of view and from a physical-geometrical standpoint (see e.g [10]). Therefore one refers to the mean curvature flow while searching for these space.

Let us recall that being a graph and Lagrangian are preserved along the mean curvature flow (see e.g. [12], [13]) and these two properties together yield the mean curvature flow of symplectomorphism under the hypothesis that the universal covering of the ambient manifold is of the type $\mathbb{S}^2\times\mathbb{S}^2$, $\mathbb{R}^2\times\mathbb{R}^2$ or $\mathbb{H}^2\times\mathbb{H}^2$. Later on, in [15, Theorem 1.1], the author proved that the mean curvature flow $(\Sigma_t)_t$ of the graph Σ of a symplectomorphism $f : \Sigma_1 \to \Sigma_2$ between two homeomorphic compact Riemann surfaces of same constant curvature exist smoothly for all time, each Σ_t is the graph of a symplectomorphism f_t and Σ_t converges smoothly to a minimal Lagrangian submanifold of $M = \Sigma_1 \times \Sigma_2$ as t goes to ∞. This result tell us that the mean curvature flow of a symplectic diffeomorphism lies in the group of symplectomorphism. A natural question we address in this work is whether the mean curvature flow of a Hamiltonian result lies in the group of Hamiltonian diffeomorphisms.

There is a cohomology class attached to any given Lagrangian submanifold of a symplectic manifold : the *Maslov class*. This class can be represented by a closed 1-form expressed solely in term of the mean curvature of the submanifold and the symplectic form of the ambient manifold. Therefore one observes that minimal Lagrangian has zero Maslov class. In light of this fact we consider the deformation of Hamiltonian diffeomorphism by the mean curvature flow. We will call this flow, when it exists in the group of Hamiltonian diffeomorphisms, a *Hamiltonian mean curvature flow*. In account of this we proved that deforming Hamiltonian diffeomorphism by the mean curvature flow provides a path in the group of Hamiltonian diffeomorphisms. The result is stated as follows :

Theorem 1.1 *Let (Σ, ω) be a compact connected Riemann surface with non-negative constant curvature. Then any Hamiltonian diffeomorphism on Σ deforms through Hamiltonian diffeomorphisms under the mean curvature flow.*

In the sequel we set some of the needed materials in the approach of Hamiltonian mean curvature flow and recall the technical tools used for the study of such geometric evolution problem. We then discuss the Hamiltonian property of the time slice of the flow.

2 Preliminaries

Throughout this exposition, all manifolds are smooth and closed (compact without boundary) unless it is stated otherwise.

Definition 2.1 *Let M be a differentiable manifold. The mean curvature motion of $S \overset{F_0}{\hookrightarrow} M$ is a 1-parameter family of immersions of submanifolds S_t in M which admits a parametrization $F_t : S \longmapsto S_t \subset M$ over S with normal velocity equal to the mean curvature vector i.e.*

$$\begin{cases} \left(\frac{\partial}{\partial t} F_t(x)\right)^{\perp} & = & H(F_t(x)), \qquad x \in S, \\ F(\cdot, 0) & = & F_0. \end{cases}$$

The mean curvature motion is a non linear weakly parabolic system for F and is invariant under reparametrization of S. Indeed, by coupling with a diffeomorphism φ of S, the flow can be made into a normal direction, i.e.

$$\frac{\partial}{\partial t} F_t\big(\varphi(x)\big) = H\big(F_t(\varphi(x))\big). \tag{1}$$

For any smooth compact initial data, one can establish the short time existence solution for (1) and the uniqueness of the solution for suitable conditions on the initial data.

Let (Σ, ω) be a compact Riemann surface with a constant curvature c and let $f \in \mathcal{D}\mathrm{iff}(\Sigma, \omega)$ be a diffeomorphism of Σ. Put $M = \Sigma \times \Sigma$ and denote by S the graph of f.

The mean curvature flow of f is realized through the mean curvature motion of S. In fact, knowing that the mean curvature flow preserved the graph and Lagrangian properties, one obtains the flow in the group of symplectic diffeomorphisms $Symp(\Sigma,\omega)$.

Definition 2.2 *A diffeomorphism f on a symplectic manifold (Σ,ω) is said to be Hamiltonian if there exists a smooth function $G:\Sigma\longrightarrow\mathbb{R}$ such that $f\in\{f_s\}_s$, where $\{f_s\}_s$ is the Hamiltonian flow of the symplectic gradient $X=X_G$ of G, namely the family of diffeomorphisms obtained by solving the ordinary differential equation :*

$$\begin{cases} \frac{\partial}{\partial s}f_s(x) &= X(f_s(x)) \\ f_0(x) &= x. \end{cases}$$

The definition above is a classical definition of Hamiltonian diffeomorphism. In this definition, the vector field does not depend on time and is often refer to as autonomous vector field. The analog for time-dependent vector field is the characterization using the flux homomorphism. It is stated as follows :

Definition 2.3 *The time-one map of a symplectic isotopy (symplectic path to the identity) with zero flux is called a Hamiltonian diffeomorphism (see [1]).*

Therefore, the flux is an obstruction for symplectomorphism which is isotopic to identity to be a Hamiltonian diffeomorphism. Let us recall the definition of the flux homomorphism ; for more details, we refer to [2] where a comprehensive exposition is made. The flux homomorphism is defined as follows, [2, Theorem 3.1.1] :

$$\begin{array}{rccc} \widetilde{Flux}: & \widetilde{Symp}_0(\Sigma,\omega) & \longrightarrow & H^1(\Sigma,\mathbb{R}) \\ & \{\tilde{f}_s\} & \longmapsto & \left[\int_0^1 f_s^*(i_{X_s}\omega)\, ds\right] \end{array} \tag{2}$$

where $\widetilde{Symp}_0(\Sigma,\omega)$ is the universal covering of $Symp_0(\Sigma,\omega)$, the connected component of the identity in the group of symplectomorphisms, $\{\tilde{f}_s\}$ is a homotopy class of an isotopy $\{f_s\}_s$ generated by X_s, and $f_s^*(i_{X_s}\omega)$ stands for the pull-back of the form $i_{X_s}\omega$ (interior product of ω by X_s). Notice that the flux homomorphism descends to the group $Symp_0(\Sigma,\omega)$; for convenience, we will recall this definition in the following section.

Let $\{f_s\}_{0\le s\le 1}$ be a symplectic isotopy to f, generated by the time-dependent vector field X_s and denote $\mathcal{F}$, its flux form i.e.

$$\mathcal{F}=\int_0^1 f_s^* i_{X_s}\omega\, ds.$$

The cohomology class $[\mathcal{F}]$ depends only on the homotopy classes of the isotopy $\{f_s\}_{0\leq s\leq 1}$ relatively to fixed ends. The behavior of $\mathcal{F}$ under the mean curvature flow will be the major ingredient for the purpose of preserving Hamiltonian condition by the mean curvature flow since it is well known that the mean curvature flow of symplectomorphism exists smoothly for all time and converges (see e.g. [9, 12, 13, 14]).

In the sequel, we compute the evolution equation of $\mathcal{F}$ along the mean curvature flow and use it to find out under which hypothesis the Hamiltonian property is preserved along the flow.

3 Hamiltonian property along the flow

The group of Hamiltonian diffeomorphisms $Ham(\Sigma,\omega)$ is the kernel of an onto homomorphism (the flux homomorphism)

$$Flux : S\mathrm{ymp}_0(\Sigma,\omega) \longrightarrow H^1(\Sigma,\mathbb{R})/\Gamma_\omega$$

where the flux group $\Gamma_\omega = Flux\Big(\pi_1\Big(S\mathrm{ymp}_0(\Sigma,\omega)\Big)\Big)$ is finitely generated but is not known to be discrete in all cases. Hence the most one can say in general is that $Ham(\Sigma,\omega)$ sits inside the identity component $S\mathrm{ymp}_0(\Sigma,\omega)$ as the leaf of a foliation.

Let $f : \Sigma \longrightarrow \Sigma$ be a Hamiltonian diffeomorphism on a compact Riemann surface with constant curvature. Then, we have two different view point to consider f. Let us look at f with respect to the symplectic isotopy view point, meaning f is the end point of some symplectic isotopy $\{f_s\}_{0\leq s\leq 1}$ with zero flux. The mean curvature flow of f gives rise to a 2-parameter family of symplectomorphisms $\{f_{s,t}\}$ satisfying :

$$(A)\quad \begin{cases} f_{1,0} = f \\ f_{0,t} = id_\Sigma, \qquad f_{1,t} = f_t \qquad \text{for each t} \\ Flux\{f_{s,0}\} = \Big[\int_0^1 f^*_{s,0}(i_{X_{s,0}}\omega)ds\Big] = [\mathcal{F}_0] = 0 \\ \frac{\partial}{\partial s} f_{s,t} = X_{s,t} \\ \frac{\partial}{\partial t} f_{s,t} = H_{s,t} \end{cases}$$

where s is the isotopy parameter and t is the mean curvature flow's one ; $X_{s,t}$ is the isotopy vector field and $H_{s,t}$ is the mean curvature vector field. These vectors are symplectic and satisfy :

$$\frac{\partial}{\partial t}X_{s,t} = \frac{\partial}{\partial s}H_{s,t} - [H_{s,t}, X_{s,t}]$$

Lemma 3.1 *The flux form $\mathcal{F}_t$ of the isotopy $f_{s,t}$, satisfies the following equation :*

$$\frac{\partial}{\partial t}\mathcal{F}_t = f_t^* i_{H_t}\omega + dK_t \tag{3}$$

where $K_t = \int_0^1 f_{s,t}^*\omega(X_{s,t}, H_{s,t})ds$ and f_t (the time t-slice of the flow) is the time-one map of the isotopy $\{f_{s,t}\}_{0\leq s\leq 1}$.

Proof :

Recall that for any smooth 1-parameter family of maps $\varphi_t : M \longrightarrow N$ between two differentiable manifolds M and N and any smooth family of differentiable forms $(\omega_t)_t$ on N, $\varphi_t^*\omega_t$ is a smooth family of differentiable forms on M and the basic formula of differential calculus gives (see e.g [6]) :

$$\frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^* L_{X_t}\omega_t + \varphi_t^*\frac{d}{dt}\omega_t, \tag{4}$$

where X_t is the tangent vector field along φ_t.

From this calculus and taking into account the fact that $X_{s,t}$ and $H_{s,t}$ are symplectic vector fields, and using Definition 2.2, a direct computation yields

$$\begin{aligned}
\frac{\partial}{\partial t}\mathcal{F}_t &= \int_0^1 \frac{\partial}{\partial t}\Big(f_{s,t}^* i_{X_{s,t}}\omega\Big)ds \qquad (5)\\
&= \int_0^1 \left(f_{s,t}^* L_{H_{s,t}} i_{X_{s,t}}\omega + f_{s,t}^*\frac{\partial}{\partial t} i_{X_{s,t}}\omega\right)ds\\
&= \int_0^1 \left(f_{s,t}^* d i_{H_{s,t}} i_{X_{s,t}}\omega + f_{s,t}^*\frac{\partial}{\partial s} i_{H_{s,t}}\omega - f_{s,t}^* i_{[H_{s,t},X_{s,t}]}\omega\right)ds\\
&= \int_0^1 \left(d f_{s,t}^* i_{H_{s,t}} i_{X_{s,t}}\omega + f_{s,t}^*\frac{\partial}{\partial s} i_{H_{s,t}}\omega + f_{s,t}^* L_{X_{s,t}} i_{H_{s,t}}\omega\right)ds\\
&= \int_0^1 \left(d f_{s,t}^* i_{H_{s,t}} i_{X_{s,t}}\omega + \frac{\partial}{\partial s}\left(f_{s,t}^* i_{X_{s,t}}\omega\right)\right)ds\\
&= f_t^* i_{H_t}\omega + d\int_0^1 f_{s,t}^*\omega(X_{s,t}, H_{s,t})ds
\end{aligned}$$

□

It was discovered in 1965 by V. P. Maslov that there is a cohomology class which appears naturally in the resolution by the Hamilton-Jacobi method of the Schrödinger equation of quantum physics. In mathematics, it is a cohomology class attached to a given Lagrangian submanifold a symplectic manifold. This class is an important cohomology invariant and is called the Maslov class. Since J.-M. Morvan's work, it has been found possible to express this cohomology class solely in terms of the Riemmanian structure of the Lagrangian immersion associated to the Kähler metric on a symplectic manifold (see e.g [8]).

Definition 3.2 *Let (M, ω) be a Kähler-Einstein $2n$-dimensional manifold and $L \hookrightarrow M$ be an immersed Lagrangian submanifold of M. Then the Maslov class of L is defined by :*

$$\frac{n}{\pi}[i_H\omega], \tag{6}$$

where H is the mean curvature vector along L.

Therefore we obtain the following :

Lemma 3.3 *Let $(f_s)_{0\leq s\leq 1}$ be a symplectic isotopy to a Hamiltonian diffeomorphism f. Assume that the mean curvature flow f_t of f exists. Then the cohomology class of the flux form of the isotopy $f_{s,t}$ deforms to the Maslov class of $S =graphf$ under the mean curvature flow.*

Proof :
From equation (3) one gets :

$$\frac{\partial}{\partial t}[\mathcal{F}_t] = \left[\frac{\partial}{\partial t}\mathcal{F}_t\right] = [f_t^* i_{H_t}\omega]. \tag{7}$$

□

From which it follows :

Proposition 3.1
Let Σ be a compact connected Riemann surface with constant curvature and $f \in Ham(\Sigma)$. Suppose $S = graphf$ has zero Maslov class, then the flux of any symplectic isotopy to f is preserved along the mean curvature flow.

Proof :
Let $\{f_s\}_{0\leq s\leq 1}$ be a symplectic isotopy to f generated by the vector field X_s. The mean curvature flow of f is a 2-parameter family $\{f_{s,t}\}_{s,t}$ of symplectomorphisms. So using Lemma 3.3 and taking into account the fact that Σ is connected, one concludes that the cohomology class of the flux form is constant along the flow. □

We now state the main results of this work.

Theorem 3.4 *Let Σ be a compact connected Riemann surface with constant curvature and $f \in Ham(\Sigma)$. Assume that $S = graph f$ has zero Maslov class. Then any Hamiltonian diffeomorphism on Σ deforms through Hamiltonian diffeomorphisms by the mean curvature flow.*

Proof :

Let $f \in Ham(M, \omega)$. There exists a symplectic isotopy $\{f_s\}$ to f such that the mean curvature flow of f is a 2-parameter as in system (A). We know that the flow exists (see e.g. [13]). So using the Proposition 3.1, for each time t, the flux of the isotopy $\{f_{s,t}\}$ to f_t is zero. Therefore f_t is a Hamiltonian diffeomorphism. □

Thus we call *Hamiltonain mean curvature flow* the mean curvature flow for which any time slice of the flow (grphical mean curvature flow) is Hamiltonian or equivalently the mean curvature flow of Lagrangian graphs Hamiltonian isotopic to the diagonal (the graph of the identity).

3.1 Non-negative curvature case

In this section we assume that Σ has non-negative constant curvature and observe that the assumption of zero Maslov class can be removed. We have the following :

Theorem 3.5 *Let (Σ, ω) be a compact connected Riemann surface with constant curvature c and $f \in Ham(\Sigma)$. If c is non-negative, then f deforms through Hamiltonian diffeomorphisms under the mean curvature flow.*

Proof :

$M = \Sigma \times \Sigma$ is compact and its universal covering is either $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{R}^2 \times \mathbb{R}^2$. The submanifold S (graph of f) is Lagrangian w.r.t $\omega' = \omega \ominus \omega$ and symplectic w.r.t $\omega \oplus \omega$. Then f deforms through symplectomorphism [13]. What is left is to prove that each slice f_t of the flow is Hamiltonian, i.e a time one map of some symplectic isotopy $\{f_{s,t}\}_{0 \leq s \leq 1}$ with zero flux.

1. Suppose Σ is elliptic ($c > 0$), then $H^1(\Sigma, \mathbb{R})$ is trivial and so is $H^1(S, \mathbb{R})$; thus the flux is preserved. Since its initial value is zero, then each t-slice of the flow is Hamiltonian.

2. If $c = 0$, then M is Calabi-Yau. Let θ_t be the Lagrangian angle of S_t. The mean curvature form satisfies

$$i_{H_t} \omega = d\theta_t$$

which implies that the Maslov class vanishes. Thus, the flux is preserved along the flow and since its initial value is zero, we deduce that each f_t is Hamiltonian. □

Definition 3.6 *A diffeomorphism on* Σ *is called a minimal diffeomorphism if its graph is a minimal embedding in* $\Sigma \times \Sigma$.

As a consequence of Theorem 3.5, we obtain the following corollaries :

Corollary 3.7 *Let* (Σ, ω) *be a compact connected Riemann surface with non-negative constant curvature* c *and* $f \in Ham(\Sigma)$. *As* $t \longrightarrow \infty$, *a sequence of the mean curvature flow of the graph of* f *converges to a smooth minimal Hamiltonian graph.*

Corollary 3.8 *Let* (Σ, ω) *be a compact connected Riemann surface with strictly positive constant curvature* c *and* $f \in Ham(\Sigma)$. *Then the Hamiltonian mean curvature fow of* S=*graph*f *exists for all time* t, *each* S_t *can be written as a graph of a Hamitonian diffeomorphism* f_t. *The sequence of submanifolds* S_t *converges to the diagonal as* t *goes to infinity.*

The proof of these corollaries are the same as in [13] ; in addition with the preserving Hamiltonian property from Theorem 3.5.

A particular example of calibrated submanifolds was first introduced by Harvey and Lawson. These submanifolds are known as special Lagrangian (see definition bellow). It is not hard to check that calibrated submanifolds are volume minizer in their homology class so special Lagrangian are minimal submanifolds.

Definition 3.9 *A Lagrangian submanifold in a Calabi-Yau manifold* (M, Ω) *is called special if it has constant Lagrangian angle.*

The Theorem 3.5 and its Corollary 3.7 give the proof in the case of graph of Hamiltonian diffeomorphism to the conjecture of Thomas and Yau asserting that the mean curvature flows of a Lagrangian submanifold S with zero Maslov class exits for all time and converges to a special Lagrangian submanifold in the Hamiltonian isotopy class of S. We proved that the Hamiltonian isotopy is nothing else but the path obtained by the mean curvature flow.

Theorem 3.10 *Let* S *be a graph of some Hamiltonian diffeomorphism* f *on a flat torus* T^2. *Then the Hamiltonian mean curvature flow of* S *exists for all time and converges to a special Hamiltonian graph isotopic to* S.

Proof :

$Ham(T^2)$ is contractible, so every $f \in Ham(T^2)$ flows through Hamiltonian diffeormorphisms to the identity which graph (the diagonal in $T^2 \times T^2$) is a minimal surface. Then the mean curvature form is exact which infers that the Lagrangian angle is constant. Thus the limit is a special Lagrangian submanifold. □

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