

$\theta(\hat{x}, \hat{p})$ -deformation of the Harmonic Oscillator in a 2D-phase Space

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Abstract. This work addresses a $\theta(\hat{x}, \hat{p})$ -deformation of the harmonic oscillator in a 2D-phase space. Specifically, it concerns a quantum mechanics of the harmonic oscillator based on a noncanonical commutation relation depending on the phase space coordinates. A reformulation of this deformation is considered in terms of a q -deformation allowing to easily deduce the energy spectrum of the induced deformed harmonic oscillator. Then, it is proved that the deformed position and momentum operators admit a one-parameter family of self-adjoint extensions. These operators engender new families of deformed Hermite polynomials generalizing usual q -Hermite polynomials. Relevant matrix elements are computed. Finally, a $su(2)$ -algebra representation of the considered deformation is investigated and discussed.

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1. Introduction

Consider a 2D phase space $\mathcal{P} \subset \mathbb{R}^2$. Coordinates of position and momentum are denoted by x and p . Corresponding Hilbert space quantum operators \hat{x} and \hat{p} satisfy the following commutation relation

$$[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = i\theta(\hat{x}, \hat{p}) \quad (1)$$

where θ is the deformation parameter encoding the noncommutativity of the phase space: $\theta(x, p) = 1 + \alpha x^2 + \beta p^2$, $\alpha, \beta \in \mathbb{R}_+$ with the uncertainty relation:

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{1}{2} (1 + \alpha(\Delta\hat{x})^2 + \beta(\Delta\hat{p})^2)$$

where the parameters α and β are real positive numbers.

The motivations for this study derive from a series of works devoted to the relation (1). Indeed, already in [1] Kempf *et al.* investigated (1) for the particular case $\alpha = 0$ with

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{1}{2}(1 + \beta(\Delta\hat{p})^2)$$

and led to the conclusion that the energy levels of a given system can deviate significantly from the usual quantum mechanical case once energy scales become comparable to the scale $\sqrt{\beta}$. Although the onset of this scale is an empirical question, it is presumably set by quantum gravitational effects. In another work [2], Kempf, for the same model with $\alpha = 0$, led to the conclusion that the anomalies observed with fields over unsharp coordinates might be testable if the onset of strong gravity effects is not too far above the currently experimentally accessible scale about $10^{-18}m$, rather than at the Planck scale of $10^{-35}m$. More recently, in [3], it was shown that similar relation can be applied to discrete models of matter or space-time, including loop quantum cosmology. For more motivations, see [4], [5] and [6], but also [7], [8], [9] and [10] and references therein.

In this work, we investigate how such a deformation may affect main properties, e.g., energy spectrum, of a simple physical system like a harmonic oscillator.

The paper is organized as follows. In Section 2, we introduce a reformulation of the $\theta(\hat{x}, \hat{p})$ -deformation in terms of a q -deformation allowing to easily deduce the energy spectrum of the induced deformed harmonic oscillator. Then it is proved that the deformed position and momentum operators admit a one-parameter family of self-adjoint extensions. These operators engender new families of deformed Hermite polynomials generalizing usual q -Hermite polynomials. Section 3 is devoted to the matrix element computation. Finally, in Section 4, we provide a $su(2)$ -algebra representation of the considered deformation. Section 5 deals with concluding remarks.

2. q -like realization

It is worth noticing that such a $\theta(x, p)$ -deformation (1) admits an interesting q -like realization via the following re-parameterization of deformed creation and annihilation operators:

$$\hat{b}^\dagger = \frac{1}{2}(m_\alpha\hat{x} - im_\beta\hat{p}), \quad \hat{b} = \frac{1}{2}(m_\alpha\hat{x} + im_\beta\hat{p})$$

satisfying the peculiar q -Heisenberg commutation relation:

$$\hat{b}\hat{b}^\dagger - q\hat{b}^\dagger\hat{b} = 1$$

where the parameter q is written in the form

$$q = \frac{1 + \sqrt{\alpha\beta}}{1 - \sqrt{\alpha\beta}} \geq 1$$

and the quantities m_α and m_β are given by

$$m_\alpha = \sqrt{2\alpha \left(\frac{1}{\sqrt{\alpha\beta}} - 1 \right)}, \quad m_\beta = \sqrt{2\beta \left(\frac{1}{\sqrt{\alpha\beta}} - 1 \right)}.$$

With this consideration, the spectrum of the induced harmonic oscillator Hamiltonian $\hat{H} = \hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b}$ is given by

$$\mathcal{E}_n = \frac{1}{2} ([n]_q + [n+1]_q)$$

where the q -number $[n]_q$ is defined by $[n]_q = \frac{1-q^n}{1-q}$. Let \mathcal{F} be a q -deformed Fock space and $\{|n, q\rangle \mid n \in \mathbb{N} \cup \{0\}\}$ be its orthonormal basis. The actions of \hat{b} , \hat{b}^\dagger on \mathcal{F} are given by

$$\hat{b}|n, q\rangle = \sqrt{[n]_q}|n-1, q\rangle, \quad \text{and} \quad \hat{b}^\dagger|n, q\rangle = \sqrt{[n+1]_q}|n+1, q\rangle,$$

where $|0, q\rangle$ is a normalized vacuum:

$$\hat{b}|0, q\rangle = 0, \quad \langle q, 0|0, q\rangle = 1.$$

The Hamiltonian operator \hat{H} acts on the states $|n, q\rangle$ to give: $\hat{H}|n, q\rangle = \mathcal{E}_n|n, q\rangle$.

Theorem 1. *The position operator \hat{x} and momentum operator \hat{p} , defined on the Fock space \mathcal{F} , are not essentially self-adjoint, but have a one-parameter family of self-adjoint extensions.*

Proof. Consider the following matrix elements of the position operator \hat{x} and momentum operator \hat{p} :

$$\begin{aligned} \langle m, q|\hat{x}|n, q\rangle &:= \left\langle m, q \left| \frac{1}{m_\alpha} (\hat{b}^\dagger + \hat{b}) \right| n, q \right\rangle \\ &= \frac{1}{m_\alpha} \sqrt{[n+1]_q} \delta_{m, n+1} + \frac{1}{m_\alpha} \sqrt{[n]_q} \delta_{m, n-1} \\ \langle m, q|\hat{p}|n, q\rangle &:= \left\langle m, q \left| \frac{i}{m_\beta} (\hat{b}^\dagger - \hat{b}) \right| n, q \right\rangle \\ &= \frac{i}{m_\beta} \sqrt{[n+1]_q} \delta_{m, n+1} - \frac{i}{m_\beta} \sqrt{[n]_q} \delta_{m, n-1}. \end{aligned}$$

Setting $x_{n,\alpha} = \frac{1}{m_\alpha} \sqrt{[n]_q}$ and $x_{n,\beta} = \frac{1}{m_\beta} \sqrt{[n]_q}$, then the position operator \hat{x} and momentum operator \hat{p} can be represented by the two following symmetric Jacobi matrices, respectively:

$$M_{\hat{x}, q, \alpha} = \begin{pmatrix} 0 & x_{1,\alpha} & 0 & 0 & 0 & \cdots \\ x_{1,\alpha} & 0 & x_{2,\alpha} & 0 & 0 & \cdots \\ 0 & x_{2,\alpha} & 0 & x_{3,\alpha} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \quad (2)$$

and

$$M_{\hat{p},q,\beta} = \begin{pmatrix} 0 & -ix_{1,\beta} & 0 & 0 & 0 & \cdots \\ ix_{1,\beta} & 0 & -ix_{2,\beta} & 0 & 0 & \cdots \\ 0 & ix_{2,\beta} & 0 & -ix_{3,\beta} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \quad (3)$$

The quantity $|x_{n,\alpha}| = \frac{1}{m_\alpha} \left| \frac{1-q^n}{1-q} \right|^{1/2}$ is not bounded since $q > 1$ by definition, and $\lim_{n \rightarrow \infty} |x_{n,\alpha}| = \infty$. Considering the series $y_\alpha = \sum_{n=0}^{\infty} 1/x_{n,\alpha}$, we get

$$\overline{\lim}_{n \rightarrow \infty} \left(\frac{1/x_{n+1}}{1/x_n} \right) = q^{-1/2} < 1$$

and, hence, the series y_α converges. Besides, as the quantity $q^{-1} + q > 2$,

$$0 < x_{n+1,\alpha} x_{n-1,\alpha} = \frac{1}{m_\alpha^2 (1-q)} [1 - q^n (q^{-1} + q) + q^{2n}]^{1/2} < x_{n,\alpha}^2$$

Hence, the Jacobi matrices in (2) and (3) are not self-adjoint (Theorem 1.5., Chapter VII in Ref. [11]). The deficiency indices of these operators are equal to (1, 1). One concludes that the position operator \hat{x} and the momentum operator \hat{p} are no longer essentially self-adjoint but have each a one-parameter family of self-adjoint extensions instead. This means that their deficiency subspaces are one-dimensional. \square

Besides, in this case, the deficiency subspaces N_z , $Imz \neq 0$, are defined by the generalized vectors $|z\rangle = \sum_0^\infty P_n(z)|n, q\rangle$ such that [11], [12]:

$$\sqrt{[n]_q} P_{n-1}(z) + \sqrt{[n+1]_q} P_{n+1}(z) = z P_n(z) \quad (4)$$

with the initial conditions $P_{-1}(z) = 0$, $P_0(z) = 1$.

- In the position representation, the states $|x, q\rangle$ such that

$$\hat{x}|x, q\rangle = x|x, q\rangle, \text{ and } |x, q\rangle = \sum_{n=0}^{\infty} P_{n,q}(x)|n, q\rangle,$$

transforms the relation (4) into

$$\begin{aligned} m_\alpha x P_{n,q}(x) &= \sqrt{[n+1]_q} P_{n+1,q}(x) + \sqrt{[n]_q} P_{n-1,q}(x) \\ n &= 0, 1, \dots; P_{-1,q}(x) = 0, P_{0,q}(x) = 1 \end{aligned}$$

giving

$$\begin{aligned} 2\gamma(x, q) P_{n,q} \left(\frac{2\gamma(x, q)}{(1-q)^{1/2} m_\alpha} \right) &= (1 - q^{n+1})^{\frac{1}{2}} P_{n+1,q} \left(\frac{2\gamma(x)}{(1-q)^{1/2} m_\alpha} \right) \\ &+ (1 - q^n)^{\frac{1}{2}} P_{n-1,q} \left(\frac{2\gamma(x, q)}{(1-q)^{1/2} m_\alpha} \right) \end{aligned} \quad (5)$$

where $2\gamma(x, q) = (1 - q)^{1/2} m_\alpha x$.

Setting $\tilde{P}_{n,q}(\gamma(x, q)) = P_{n,q}\left(\frac{2\gamma(x, q)}{(1-q)^{1/2}m_\alpha}\right)$, the equation (5) can be re-expressed as

$$2\gamma(x, q)\tilde{P}_{n,q}(\gamma(x, q)) = (1 - q^{n+1})^{\frac{1}{2}}\tilde{P}_{n+1,q}(\gamma(x, q)) + (1 - q^n)^{\frac{1}{2}}\tilde{P}_{n-1,q}(\gamma(x, q)). \quad (6)$$

Finally, putting $(q; q)_n^{1/2}\tilde{P}_{n,q}(\gamma(x, q)) = H_{n,q}(x)$, the formula (6) recalls the recurrence relation satisfied by q -Hermite polynomials:

$$2xH_{n,q}(x) = H_{n+1,q}(x) + (1 - q^n)H_{n-1,q}(x)$$

where $(q; q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)$.

- In the momentum representation, the state $|p, q\rangle$ such that

$$\hat{p}|p, q\rangle = p|p, q\rangle, \quad \text{and} \quad |p, q\rangle = \sum_{n=0}^{\infty} Q_{n,q}(p)|n, q\rangle$$

leads to the following recurrence relation between functions $Q_n(x, q)$:

$$im_\beta p Q_{n,q}(p) = \sqrt{[n+1]_q} Q_{n+1,q}(p) - \sqrt{[n]_q} Q_{n-1,q}(p) \\ n = 0, 1, \dots; \quad Q_{-1,q}(p) = 0, \quad Q_{0,q}(p) = 1.$$

This equation can be also re-expressed as

$$\tilde{\gamma}(\tilde{p}, q)Q_{n,q}(p) = (1 - q^{n+1})^{\frac{1}{2}}Q_{n+1,q}(p) - (1 - q^n)^{\frac{1}{2}}Q_{n-1,q}(p) \quad (7)$$

where $\tilde{\gamma}(\tilde{p}, q) = (1 - q)^{1/2}m_\beta\tilde{p}$, $\tilde{p} = ip$.

Setting $\tilde{Q}_{n,q}(\tilde{\gamma}(\tilde{p}, q)) = Q_{n,q}\left(\frac{2\tilde{\gamma}(\tilde{p}, q)}{(1-q)^{1/2}m_\alpha}\right)$, then equation (7) yields

$$2\tilde{\gamma}(\tilde{p}, q)\tilde{Q}_{n,q}(\tilde{\gamma}(\tilde{p}, q)) = (1 - q^{n+1})^{\frac{1}{2}}\tilde{Q}_{n+1,q}(\tilde{\gamma}(\tilde{p}, q)) - (1 - q^n)^{\frac{1}{2}}\tilde{Q}_{n-1,q}(\tilde{\gamma}(\tilde{p}, q)).$$

Letting $(q; q)_n^{1/2}\tilde{Q}_{n,q}(\tilde{\gamma}(\tilde{p}, q)) = H_{n,q}(ip)$, we arrive at the recurrence relation satisfied by the complex q -Hermite polynomials given by

$$2ipH_{n,q}(ip) = H_{n+1,q}(ip) - (1 - q^n)H_{n-1,q}(ip).$$

Remark 1. The following is worthy of attention:

- (i) In the x -space where the momentum operator is defined by the relation

$$\hat{p} := -i\theta(\hat{x}, \hat{p})\partial_x, \quad (8)$$

any function $\Psi_q(x)$ in x -representation can be expressed in terms of its analog $\Psi_q(p)$ in p -representation by the relation

$$\Psi_q(x) = \int_{-\infty}^{\infty} dp \exp\left(\frac{ip}{\alpha\sigma(p)} \arctan \frac{x}{\sigma(p)}\right) \Psi_q(p),$$

where $\sigma(p) = \sqrt{p^2 + \frac{1}{\alpha}}$. Defining the Hilbert space inner product as

$$\langle f, g \rangle = \int \frac{dx}{\theta(\hat{x}, \hat{p})} \bar{f}_q(x) g_q(x)$$

one can readily prove that \hat{p} reverts the property of a Hermitian operator.

(ii) Analogously, in the p -space

$$\hat{x} := i\theta(\hat{x}, \hat{p})\partial_p \quad (9)$$

and

$$\Psi_q(p) = \int_{-\infty}^{\infty} dx \exp\left(\frac{-ix}{\alpha\sigma(x)} \arctan \frac{p}{\sigma(x)}\right) \Psi_q(x).$$

The appropriate inner product, in the momentum space, rendering the operator \hat{x} Hermitian is defined as

$$\langle f, g \rangle = \int \frac{dp}{\theta(\hat{x}, \hat{p})} \bar{f}_q(p) g_q(p)$$

with the condition $\lim_{x \rightarrow -\infty} \Psi_q(x) = \lim_{x \rightarrow \infty} \Psi_q(x) = 0$.

3. Matrix elements

From the natural actions of q -deformed position operator \hat{x} and momentum operator \hat{p} on the basis vectors $|n, q\rangle \in \mathcal{F}$:

$$\hat{x}|n, q\rangle = \frac{1}{m_\alpha}(\hat{b} + \hat{b}^\dagger)|n, q\rangle, \quad \hat{p}|n, q\rangle = \frac{i}{m_\beta}(\hat{b}^\dagger - \hat{b})|n, q\rangle$$

we immediately deduce the matrix elements

$$\begin{aligned} \langle m, q | \hat{b}^{\dagger l} \hat{b}^r | n, q \rangle &= \sqrt{\frac{\Gamma_q(n+1)\Gamma_q(n-r+l+1)}{\Gamma_q(n-r+1)\Gamma_q(n-r+1)}} \delta_{m, n-r+l} \\ \langle m, q | \hat{b}^r \hat{b}^{\dagger l} | n, q \rangle &= \sqrt{\frac{\Gamma_q(n+l+1)\Gamma_q(n+l+1)}{\Gamma_q(n+1)\Gamma_q(n-r+l+1)}} \delta_{m, n-r+l} \end{aligned} \quad (10)$$

where $\Gamma_q(\cdot)$ is the q -Gamma function. Denoting by $::$ the normal ordering, then the expectation value of normal ordering product of $\hat{x}^l \hat{p}^r$ can be computed by the following relation:

$$\langle m, q | : \hat{x}^l \hat{p}^r : | n, q \rangle = \frac{i^r}{m_\alpha^l m_\beta^r} \sum_{s=0}^l \sum_{t=0}^r C_i^s C_r^t \langle m, q | \hat{b}^{\dagger l-s+t} \hat{b}^{s+r-t} | n, q \rangle$$

which can be given explicitly by using relation (10). Then it becomes a matter of computation to determine the basis operators in terms of \hat{b} and \hat{b}^\dagger as follows:

$$|m, q\rangle \langle n, q| =: \frac{\hat{b}^{\dagger m}}{\sqrt{[m]_q!}} e^{-\hat{b}^\dagger \hat{b}} \frac{\hat{b}^n}{\sqrt{[n]_q!}} :.$$

4. $su(2)$ -algebra representation

Turning back to the standard expression of the harmonic oscillator Hamiltonian operator, i.e., $\hat{H} = \hat{a}^\dagger \hat{a}$, such that

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}),$$

we get explicitly

$$\hat{H} = \frac{1}{2} [(1 + \alpha)\hat{x}^2 + (1 + \beta)\hat{p}^2 + 1]$$

giving the simpler form $\hat{H} = \frac{1}{2}$ when $\alpha = -1$ and $\beta = -1$.

From (8) and (9), the Hamiltonian H can be considered as non-local and we can define

$$\hat{H}^{\text{loc}} := \hat{H}(\theta, \partial_x \theta, \partial_x^2 \theta, \dots, x, \partial_x, \partial_x^2, \dots, \alpha, \beta)$$

with

$$\theta, \partial_x \theta, \dots = f(\theta, \partial_x \theta, \partial_x^2 \theta, \dots, x, \partial_x, \partial_x^2, \dots, \alpha, \beta).$$

adding some nonlinearity to the Hamiltonian operator nonlocality.

Assume the parameters α and β satisfy the condition: $|\alpha| \ll 1, |\beta| \ll 1$ and put $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = -\beta$. Then \hat{x}, \hat{p}, θ can be viewed as the elements of $su(2)$ -algebra, i.e.,

$$[\hat{x}, \hat{p}] = i\theta, \quad [\hat{p}, \theta] = i\alpha\{\hat{x}, \theta\} = i\tilde{\alpha}\hat{x}, \quad [\theta, \hat{x}] = -i\beta\{\hat{p}, \theta\} = i\tilde{\beta}\hat{p}.$$

Let $\vec{\hat{J}} := (\hat{x}, \hat{p}, \theta)$ be the angular momentum such that there exist states $|j, m\rangle \in \mathcal{F}$ satisfying the condition $\langle j, m | j, m' \rangle = \delta_{mm'}$. Define the operators \hat{J}_+ and \hat{J}_- by

$$\hat{J}_+ := \frac{1}{\beta}\hat{x} + \frac{i}{\alpha}\hat{p}, \quad \hat{J}_- := \frac{1}{\beta}\hat{x} - \frac{i}{\alpha}\hat{p}.$$

Proposition 1. *There exists an arbitrary number ν such that*

$$\begin{aligned} \hat{J}_- |j, m\rangle &= C_-(m, j) |j, m - \nu\rangle, \quad \hat{J}_+ |j, m\rangle = C_+(m, j) |j, m + \nu\rangle, \\ \theta |j, m\rangle &= f(m, j) |j, m\rangle \end{aligned} \quad (11)$$

where $C_-(m, j)$, $C_+(m, j)$ and $f(m, j)$ are three constants depending on j and m .

The parameters j and m depend on α and β . The unitary representation of $su(2)$ -algebra, $\{|j, m\rangle, j \in \mathbb{N}, -j \leq m \leq j\}$, is infinite dimensional. The operators $\{\hat{x}, \hat{p}, \theta\}$ act on the Fock space $\mathcal{H} = \{|j, m\rangle / m \in \mathbb{N} \cup \{0\}\}$ following (11). Note that θ and $\vec{\hat{J}}^2 = (1 + 2\alpha)\hat{x}^2 + (1 + 2\beta)\hat{p}^2 + 1$ commute. Therefore, $\vec{\hat{J}}^2$ and \hat{H} commute too. Besides, the following commutation relations are in order:

$$[\theta, \hat{J}_+] = \hat{x} + i\hat{p}, \quad [\theta, \hat{J}_-] = -(\hat{x} - i\hat{p}). \quad (12)$$

In the interesting particular case where $\alpha = \beta$, the relations (12) are reduced to

$$[\theta, \hat{J}_+] = \alpha\hat{J}_+, \quad [\theta, \hat{J}_-] = -\alpha\hat{J}_-, \quad [\hat{J}_+, \hat{J}_-] = 2\alpha^{-2}\theta.$$

Taking $f(m, j) = m$ yields the condition

$$\theta\hat{J}_+ |j, m\rangle = (m + \alpha)\hat{J}_+ |j, m\rangle, \quad \theta\hat{J}_- |j, m\rangle = (m - \alpha)\hat{J}_- |j, m\rangle.$$

Besides, we have

$$\hat{J}_+|j, m\rangle = C_+|j, m + \alpha\rangle, \quad \hat{J}_-|j, m\rangle = C_-|j, m - \alpha\rangle$$

where

$$C_+ = \sqrt{(j - m)(j + m + \alpha)}, \quad C_- = \sqrt{(j + m)(j - m + \alpha)}.$$

The eigenfunctions of the Hamiltonian \hat{H} in the position and momentum representations are given, respectively, by

$$\Psi_{j,m}(x) = \langle x|j, m\rangle, \quad \Psi_{j,m}(p) = \langle p|j, m\rangle$$

solution of the equation

$$\hat{H}\Psi_{j,m}(x) = \frac{\alpha^2}{2} \sqrt{(j + m)(j - m)(j + m + \alpha)(j - m + \alpha)}\Psi_{j,m}(x)$$

easily obtainable by solving the eigenvalue problem for the Casimir operator $\hat{J}_+\hat{J}_-$. Furthermore, we get

$$\hat{J}^2\Psi_{j,m}(x) = 2\alpha^2 \sqrt{(j + m)(j - m)(j + m + 2\alpha)(j - m + 2\alpha)}\Psi_{j,m}(x).$$

5. Concluding remarks

In work, we have introduced a reformulation of the $\theta(\hat{x}, \hat{p})$ -deformation in terms of a q -deformation allowing to easily deduce the energy spectrum of the induced deformed harmonic oscillator. Then we have proved that the deformed position and momentum operators admit a one-parameter family of self-adjoint extensions. These operators have engendered new families of deformed Hermite polynomials generalizing usual q -Hermite polynomials. We have also computed relevant matrix elements. Finally, a $su(2)$ -algebra representation of the considered deformation has been investigated and discussed.

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