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Nonconforming finite element methods for a Stokes/Biot fluid–poroelastic structure interaction model

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ABSTRACT

We analyze a strongly coupled mixed formulation of the problem defining the interaction between a free fluid and poroelastic structure. The free fluid is governed by the Stokes equations, while the flow in the poroelastic medium is modeled using the Biot poroelasticity system. Equilibrium and kinematic conditions are imposed on the interface. A stabilized mixed finite element method for solving the stationary coupled Stokes–Biot flows problem is formulated and analyzed. The approach utilizes the same nonconforming Crouzeix–Raviart (**C–R**) element discretization on the entire domain. Under a small data assumption, existence and uniqueness results are proved and an optimal a priori error estimate is derived.

1. Introduction

In this paper, we present an a priori error analysis for solving the interaction of a free incompressible viscous Newtonian fluid with a fluid within a poroelastic medium. This is a challenging multiphysics problem with applications to predicting and controlling processes arising in groundwater flow in fractured aquifers, oil and gas extraction, arterial flows, and industrial filters. In these applications, it is important to model properly the interaction between the free fluid with the fluid within the porous medium, and to take into account the effect of the deformation of the medium. For example, geomechanical effects play an important role in hydraulic fracturing, as well as in modeling phenomena such as subsidence and compaction.

We adopt the Stokes equations to model the free fluid and the Biot system [1] for the fluid in the poroelastic media. In the latter, the volumetric deformation of the elastic porous matrix is complemented with the Darcy equation that describes the average velocity of the fluid in the pores. The model features two different kinds of coupling across the interface: Stokes–Darcy coupling [2–10] and fluid–structure interaction (FSI) [11–15].

The well-posedness of the mathematical model based on the Stokes–Biot system for the coupling between a fluid and a poroelastic structure is studied in [16]. A numerical study of the problem, using a Navier–Stokes equations for the fluid, is presented in [11,17], utilizing a variational multiscale approach to stabilized the finite element spaces. The problem is solved using both a monolithic and a partitioned approach, with the latter requiring subiterations between the two problems.

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Nonphysical pressure oscillations are observed in finite element calculations of Biot's poroelastic equations in low-permeable media. These pressure oscillations may be understood as a failure of compatibility between the finite element spaces, rather than elastic locking. In [18], Joachim Berdal Haga et al. have presented evidence to support this view by comparing and contrasting the pressure oscillations in low-permeable porous media with those in low-compressible porous media. As a consequence, it is possible to use established families of stable mixed elements as candidates for choosing finite element spaces for Biot's equations. Through comparison with the displacement–solid pressure mixed formulation of linear elasticity, they identify the spurious pressure modes as a specific consequence of a vanishing Brezzi inf–sup constant. Since the Brezzi inf–sup condition for the poroelastic equations takes on a similar form as in, e.g., the mixed linear elasticity or Stokes problem, this identification opens up the field to a plethora of stable element candidates. These can be used directly for the basic solid displacement–fluid pressure two-field formulation of poroelasticity, or in combinations for the various three- and four-field formulations involving solid pressure and/or fluid velocity [18].

Finite element analysis of an arbitrary Lagrangian–Eulerian method for Stokes/parabolic moving interface problem with jump coefficients has been studied in [19]. The authors in [20] study a numerical solution of the coupled system of the time-dependent Stokes and fully dynamic Biot equations. They establish stability of the scheme and derive error estimates for the fully discrete coupled scheme. Numerical errors and convergence rates for smooth problems as well as tests on realistic material parameters have been presented. In [21], Jing Wen and Yinnian He consider a strongly conservative discretization for the rearranged Stokes–Biot model based on interior penalty discontinuous Galerkin method and mixed finite element method. The existence and uniqueness of solution of the numerical scheme have been presented. Then, the analysis of stability and priori error estimates have been derived. The numerical examples under uniform meshes, which well validate the analysis of convergence and the strong mass conservation are presented. A staggered finite element procedure for the coupled Stokes–Biot system with fluid entry resistance has been studied by Bergkamp et al. in [22] while Ambartsumyan et al. study in [23] flow and transport in fractured poroelastic media using Stokes flow in the fractures and the Biot model in the porous media. In [24], semidiscrete continuous-in-time approximation has been proposed for the weak coupled mixed formulation. For the discretization of the fluid velocity and pressure the authors have used the finite elements which include the MINI-elements, the Taylor–Hood elements and the conforming Crouzeix–Raviart elements. For the discretization of the porous medium problem they choose the spaces that include Raviart–Thomas and Brezzi–Douglas–Marini elements. An a priori error analysis is performed with some numerical tests confirming the convergence rates.

In this article, we study a stabilized nonconforming mixed finite element method using the Crouzeix–Raviart element for the Stokes–Biot problem. Considering mixed formulation of the Darcy problem, the fluid velocity and pressure are treated as functions defined in the entire domain. Existence, uniqueness of the finite element solution of the corresponding discrete problem and a priori estimates are shown. The proofs use the standard theory for mixed problems. The approach presented here is independent of the normal vectors of the interior edges in both regions, thus making the resulting finite element matrix sparser.

Indeed, to our best knowledge, there is no a priori error estimation for the strongly coupled mixed formulation (19) of the coupled Stokes–Biot problem where a nonconforming finite element method is used. The difference between our paper and the Refs. [19–24] is that our discretization is nonconforming in both the Stokes domain and Darcy domain. As a result, additional terms are included in the error estimators that measure the non-conformity of the method.

We use a nonconforming finite element method that has so many advantages for the velocities and piecewise constant for the pressures in both the Stokes and Biot regions, and apply a stabilization term penalizing the jumps over the element edges of the piecewise continuous velocities. Indeed, one can construct finite element methods where the incompressibility condition is exactly satisfied (cf. Fortin [25]) but this leads to the use of complex elements of limited applicability (e.g. oil and gas extraction for conforming case). Thus, in this paper, we shall construct and study finite element method using simpler elements where the incompressibility condition is only approximatively satisfied [cf. definition of operator div_h (36)].

On the other hand, we have found it very convenient to use nonconforming finite elements which violate the interelement continuity condition of the velocities. Thus, we shall develop in this paper nonconforming finite element method for solving the Stokes–Biot problem.

The outline of the rest of the paper follows. In Sections 2 and 3, the model problem, notations and strongly coupled mixed formulation are presented. In Section 4, we propose the unified finite element formulation with stabilization term added and derive the discrete inf–sup condition. In Section 5, we show that the method approximates both the true velocity and pressure to the optimal first order in the energy norm. We offer our conclusion and the further works in Section 6.

2. Model problem

We consider a multiphysics model problem for free fluid's interaction with a flow in a deformable porous media, where the simulation domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a union of non-overlapping regions Ω_f and Ω_p. Here Ω_f is a free fluid region with flow governed by the Stokes equations and Ω_p is a poroelastic material governed by the Biot system. For simplicity of notation, we assume that each region is connected. The extension to non-connected regions is straightforward. The two regions are separated by an interface $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$. Let $\Gamma_\star = \partial\Omega_\star \setminus \Gamma_{fp}$, $\star = f, p$. Each interface and boundary

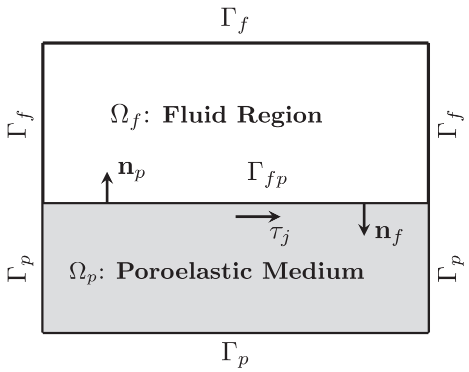

Fig. 1. Global domain Ω consisting of the fluid region Ω_f and the poroelastic media region Ω_p separated by the interface Γ_{fp}.

is assumed to be polygonal ($d = 2$) or polyhedral ($d = 3$). We denote by $\mathbf{n}_f$ (resp. $\mathbf{n}_p$) the unit outward normal vector along $\partial\Omega_f$ (resp. Ω_p). Note that on the interface Γ_{fp}, we have $\mathbf{n}_f = -\mathbf{n}_p$. Fig. 1 gives a schematic representation of the geometry. For any function v defined in Ω, since its restriction to Ω_f or Ω_p could play a different mathematical roles (for instance their traces on Γ_{fp}), we will set $v_f = v_{|\Omega_f}$ and $v_p = v_{|\Omega_p}$. In Ω, we denote by $\mathbf{u}$ the fluid velocity and by p the pressure, and let η_p be the displacement in Ω_p. Let $\mu > 0$ be the fluid viscosity, let $\mathbf{f} \in [L^2(\Omega)]^d$ be the body force terms, and let g be external source or sink terms satisfying the compatibility condition $\int_\Omega g(x)dx = 0$.

Let $\mathbf{D}(\mathbf{u})$ and $\sigma_f(\mathbf{u}, p)$ denote, respectively, the deformation rate tensor and the stress tensor:

$$\mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right), \text{ and } \sigma_f(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}).$$

In the free fluid region Ω_f, $(\mathbf{u}, p)$ satisfy the Stokes equations:

$$-\nabla\cdot\sigma_f(\mathbf{u}, p) = \mathbf{f} \text{ in } \Omega_f \tag{1}$$
$$\nabla\cdot\mathbf{u} = g \text{ in } \Omega_f \tag{2}$$
$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_f. \tag{3}$$

Let $\sigma_e(\eta_p)$ and $\sigma_p(\eta_p, p_p)$ be the elastic and poroelastic stress tensors, respectively:

$$\sigma_e(\eta_p) = \lambda_p\left(\nabla\cdot\eta_p\right)\mathbf{I} + 2\mu_p\mathbf{D}(\eta_p), \qquad \sigma_p(\eta_p, p_p) = \sigma_e(\eta_p) - \alpha p_p\mathbf{I}, \tag{4}$$

where $0 < \lambda_{\min} \le \lambda_p(\mathbf{x}) \le \lambda_{\max}$ and $0 < \mu_{\min} \le \mu_p(\mathbf{x}) \le \mu_{\max}$ are the Lamé parameters, and $0 < \alpha \le 1$ is the Biot–Willis constant. The poroelasticity region Ω_p is governed by the modified static Biot system [24]:

$$-\nabla\cdot\sigma_p(\eta_p, p_p) = \mathbf{f} \text{ in } \Omega_p \tag{5}$$
$$\mu\mathbf{K}^{-1}\mathbf{u} + \nabla p = 0 \text{ in } \Omega_p, \tag{6}$$
$$\alpha\nabla\cdot\eta_p + \nabla\cdot\mathbf{u} = g \text{ in } \Omega_p, \tag{7}$$
$$\mathbf{u}\cdot\mathbf{n}_d = 0 \text{ on } \Gamma_p \tag{8}$$
$$\eta_p = \mathbf{0} \text{ on } \Gamma_p. \tag{9}$$

$\mathbf{K}$ the symmetric and uniformly positive definite rock permeability tensor, satisfying, for some constants $0 < k_{\min} \le k_{\max}$,

$$\forall\xi \in \mathbb{R}^d, k_{\min}\xi^T\xi \le \xi^T\mathbf{K}(\mathbf{x})\xi \le k_{\max}\xi^T\xi, \forall\mathbf{x} \in \Omega_p.$$

Following [1], the interface conditions on the fluid–poroelasticity interface Γ_{fp} are mass conservation, balance of stresses, and the Beavers–Joseph–Saffman (BJS) condition [26] modeling slip with friction:

$$\mathbf{u}_f\cdot\mathbf{n}_f + \mathbf{u}_p\cdot\mathbf{n}_p = 0 \text{ on } \Gamma_{fp}, \tag{10}$$
$$\sigma_f\mathbf{n}_f + \sigma_p\mathbf{n}_p = 0 \text{ on } \Gamma_{fp} \tag{11}$$
$$-(\sigma_f\mathbf{n}_f)\cdot\mathbf{n}_f = p_p, \text{ on } \Gamma_{fp} \tag{12}$$
$$-(\sigma_f\mathbf{n}_f)\cdot\tau_{f,j} = \mu\alpha_{BJF}\sqrt{K_j^{-1}}\left(\mathbf{u}_f\right)\cdot\tau_{f,j} \text{ on } \Gamma_{fp}, \tag{13}$$

where $\tau_{f,j}$, $1 \le j \le d-1$, is an orthogonal system of unit tangent vectors on Γ_{fp}, $K_j = \left(\mathbf{K}\tau_{f,j}\right)\cdot\tau_{f,j}$, and $\alpha_{BJS} \ge 0$ is an experimentally determined friction coefficient. We note that continuity of flux constraints the normal velocity of the solid skeleton, while the BJS condition accounts for its tangential velocity.

Eqs. (1)–(13) consist of the model of the coupled Stokes and Biot flows problem that we will study below.

3. Strongly coupled weak formulation

We begin this subsection by introducing some useful notation. We first introduce some Sobolev spaces [27] and norms. If W is a bounded domain of $\mathbb{R}^d$ and m is a non negative integer, the Sobolev space $H^m(W) = W^{m,2}(W)$ is defined in the usual way with the usual norm $\|\cdot\|_{m,W}$ and semi-norm $|.|_{m,W}$. In particular, $H^0(W) = L^2(W)$ and we write $\|\cdot\|_W$ for $\|\cdot\|_{0,W}$. Similarly we denote by $(\cdot,\cdot)_W$ the $L^2(W)[L^2(W)]^d$ or $[L^2(W)]^{d\times d}$ inner product. For shortness if W is equal to Ω, we will drop the index Ω, while for any $m \geq 0$, $\|\cdot\|_{m,\star} = \|\cdot\|_{m,\Omega_\star}$, $|.|_{m,\star} = |.|_{m,\Omega_\star}$ and $(.,.)_\star = (\cdot,\cdot)_{\Omega_\star}$, for $\star = f, p$. The space $H_0^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Let $[H^m(\Omega)]^d$ be the space of vector valued functions $\mathbf{v} = (v_1, \ldots, v_d)$ with components v_i in $H^m(\Omega)$. The norm and the seminorm on $[H^m(\Omega)]^d$ are given by

$$\|\mathbf{v}\|_{m,\Omega} := \left(\sum_{i=0}^{d} \|v_i\|_{m,\Omega}^2\right)^{1/2} \quad \text{and} \quad |\mathbf{v}|_{m,\Omega} := \left(\sum_{i=0}^{d} |v_i|_{m,\Omega}^2\right)^{1/2}. \tag{14}$$

For a connected open subset of the boundary $E \subset \partial\Omega_f \cup \partial\Omega_p$, we write $\langle .,.\rangle_E$ for the $L^2(E)$ inner product (or duality pairing), that is, for scalar valued functions λ, σ one defines:

$$\langle \lambda, \sigma\rangle_E := \int_E \lambda\sigma\, ds \tag{15}$$

For a open subset F of the entire domain Ω, i.e. $F \subseteq \Omega$, we define the space $H(\mathrm{div}; F)$ by:

$$H(\mathrm{div}; F) := \left\{\mathbf{v} \in [L^2(F)]^d : \quad \mathrm{div}\,\mathbf{v} \in L^2(F)\right\}, \tag{16}$$

with a norm:

$$\|\mathbf{v}\|_{H(\mathrm{div};F)} := \left(\|\mathbf{v}\|_{[L^2(F)]^d}^2 + \|\,\mathrm{div}\,\mathbf{v}\|_{L^2(F)}^2\right)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in H(\mathrm{div}; F). \tag{17}$$

To present a variational form of the coupled problem we define the following three spaces for the velocity $\mathbf{u}$, the structure displacement η_p and the pressure:

$$\mathbf{H} := \left\{\mathbf{v} \in \mathbf{H}(\mathrm{div}; \Omega) : \mathbf{v}_f \in [H^1(\Omega_f)]^d, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_f, \mathbf{v}\cdot\mathbf{n}_p = 0 \text{ on } \Gamma_p\right\},$$

equipped with the norm:

$$\|\mathbf{v}\|_{\mathbf{H}} := \left(|\mathbf{v}|_{1,f}^2 + \|\mathbf{v}\|_{H(\mathrm{div};\Omega_p)}^2\right)^{\frac{1}{2}},$$

$$\mathbf{X}_p := \left\{\xi_p \in [H^1(\Omega_p)]^d : \xi_p = \mathbf{0} \text{ on } \Gamma_p\right\},$$

with the norm

$$\|\xi_p\|_{\mathbf{X}_p} := |\xi_p|_{1,p},$$

and

$$\mathbb{M} := L_0^2(\Omega) \times L_0^2(\Omega_p),$$

equipped with the norm $\|\mathbf{Q}\|_{\mathbb{M}} := \left(\|Q_1\|_{0,\Omega}^2 + \|Q_2\|_{0,\Omega_p}^2\right)^{1/2}$, $\forall \mathbf{Q} = (Q_1, Q_2) \in \mathbb{M}$.

Note that the vector valued functions in $\mathbf{H}$ have (weakly) continuous normal components on Γ_{fp} (consequence of Theorem I.2.5 of [28, p. 27]).

We set $\mathbb{H} = \mathbf{H} \times \mathbf{X}_p$ equipped with the product norm

$$\|\mathbf{V}\|_{\mathbb{H}} := \|\mathbf{v}\|_{\mathbf{H}} + \|\xi_p\|_{\mathbf{X}_p}, \quad \forall \mathbf{V} = (\mathbf{v}, \xi_p) \in \mathbb{H}. \tag{18}$$

Let us further introduce two bilinear forms:

$\mathbf{A} : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$, $(\mathbf{U}, \mathbf{V}) \mapsto \mathbf{A}(\mathbf{U}, \mathbf{V})$ define by,

$$\begin{aligned}\mathbf{A}(\mathbf{U}, \mathbf{V}) := {} & (2\mu\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_{\Omega_f} + \left(\mu\mathbf{K}^{-1}\mathbf{u}, \mathbf{v}\right)_{\Omega_p} \\ & + \left(2\mu_p\mathbf{D}(\eta_p), \mathbf{D}(\xi_p)\right)_{\Omega_p} + \left(\lambda_p\nabla\cdot\eta_p, \nabla\cdot\xi_p\right)_{\Omega_p} \\ & + \sum_{j=1}^{d-1}\langle \mu\alpha_{BJS}\sqrt{K_j^{-1}}\mathbf{u}_f\cdot\tau_{f,j}, \mathbf{v}_f\cdot\tau_{f,j}\rangle_{\Gamma_{fp}},\end{aligned}$$

$\mathbf{B} : \mathbb{H} \times \mathbb{M} \to \mathbb{R}$, $(\mathbf{V}, \mathbf{Q}) \mapsto \mathbf{B}(\mathbf{V}, \mathbf{Q})$ with

$$\mathbf{B}(\mathbf{V},\mathbf{Q}) := -(Q_1, \operatorname{div}\mathbf{v})_\Omega - \alpha(Q_2, \operatorname{div}\xi_p)_{\Omega_p}, \text{ where } \mathbf{Q} = (Q_1, Q_2),$$

and two linear forms

$$\mathbf{L} : \mathbb{H} \to \mathbb{R}, \mathbf{V} \mapsto \mathbf{L}(\mathbf{V}) := (\mathbf{f}, \mathbf{v})_\Omega$$

and

$$\mathbf{G} : \mathbb{M} \to \mathbb{R}, \mathbf{Q} = (Q_1, Q_2) \mapsto \mathbf{G}(\mathbf{Q}) := -(g, Q_1)_\Omega.$$

The weak formulation of the coupled problem (1)–(13) can be stated as follows: find $(\mathbf{U}, \mathbf{P}) \in \mathbb{H} \times \mathbb{M}$ with $\mathbf{U} = (\mathbf{u}, \eta_p)$ and $\mathbf{P} = (p, p_p)$ such that:

$$\left\{ \begin{array}{rcll} \mathbf{A}(\mathbf{U},\mathbf{V}) + \mathbf{B}(\mathbf{V},\mathbf{P}) & = & \mathbf{L}(\mathbf{V}) & \forall \mathbf{V} = (\mathbf{v}, \xi_p) \in \mathbb{H} \\ \mathbf{B}(\mathbf{U},\mathbf{Q}) & = & \mathbf{G}(\mathbf{Q}) & \forall \mathbf{Q} = (Q_1, Q_2) \in \mathbb{M}. \end{array} \right. \tag{19}$$

Note that if $\mathbf{f}$ and g are of mean zero, (19) directly implies that (1)–(11) hold (the differential equations being understood in the distributional sense), while the interface conditions (12) and (13) are imposed in a weak sense.

Now we will establish the existence and uniqueness of a weak solution by using the classical theory of mixed methods so-called Brezzi conditions (see, e.g., [28, Theorem and Corollary 4.1 in Chapter I]). But before, let us define:

$$\mathbf{H}_f = \left\{ \mathbf{v}_f \in H^1(\Omega_f)^d : \mathbf{v}_f = \mathbf{0} \text{ on } \Gamma_f \right\}, \qquad W_f = L_0^2(\Omega_f), \tag{20}$$

$$\mathbf{H}_p = \left\{ \mathbf{v}_p \in H(\operatorname{div}; \Omega_p) : \mathbf{v}_p \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p \right\}, \qquad W_p = L_0^2(\Omega_p), \tag{21}$$

and we set:

$$\begin{aligned} a_f(\mathbf{u}_f, \mathbf{v}_f) &:= \left(2\mu \mathbf{D}(\mathbf{u}_f), \mathbf{D}(\mathbf{v}_f)\right)_{\Omega_f}, \\ a_p^d(\mathbf{u}_p, \mathbf{v}_p) &:= \left(\mu \mathbf{K}^{-1}\mathbf{u}_p, \mathbf{v}_p\right)_{\Omega_p}, \\ a_p^e(\eta_p, \xi_p) &:= \left(2\mu_p \mathbf{D}(\eta_p), \mathbf{D}(\xi_p)\right)_{\Omega_p} + \left(\lambda_p \nabla \cdot \eta_p, \nabla \cdot \xi_p\right)_{\Omega_p} \end{aligned}$$

be the bilinear forms related to Stokes, Darcy and the elasticity operator, respectively.

The assumptions on the fluid viscosity μ and the material coefficients $\mathbf{K}$, λ_p, and μ_p imply that the bilinear forms $a_f(\cdot,\cdot)$, $a_p^d(\cdot,\cdot)$, and $a_p^e(\cdot,\cdot)$ are coercive and continuous in the appropriate norms. In particular, there exist positive constants c^f, c^p, c^e, C^f, C^p, C^e such that:

$$c^f \|\mathbf{v}_f\|^2_{H^1(\Omega_f)} \le a_f(\mathbf{v}_f, \mathbf{v}_f), \forall \mathbf{v}_f \in \mathbf{H}_f, \tag{22}$$

$$a_f(\mathbf{v}_f, \mathbf{u}_f) \le C^f \|\mathbf{v}_f\|_{H^1(\Omega_f)} \|\mathbf{u}_f\|_{H^1(\Omega_f)}, \quad \forall (\mathbf{v}_f, \mathbf{u}_f) \in \mathbf{H}_f^2 \tag{23}$$

$$c^p \|\mathbf{v}_p\|^2_{L^2(\Omega_p)} \le a_p^d(\mathbf{v}_p, \mathbf{v}_p), \forall \mathbf{v}_p \in \mathbf{H}_p \tag{24}$$

$$a_p^d(\mathbf{v}_p, \mathbf{u}_p) \le C^p \|\mathbf{v}_p\|_{L^2(\Omega_p)} \|\mathbf{u}_p\|_{L^2(\Omega_p)}, \forall (\mathbf{v}_p, \mathbf{u}_p) \in \mathbf{H}_p^2 \tag{25}$$

$$c^e \left(\|\xi_p\|^2_{[H^1(\Omega_p)]^d} + \|\operatorname{div}\xi_p\|^2_{L^2(\Omega_p)} \right) \le a_p^e(\xi_p, \xi_p), \quad \forall \xi_p \in \mathbf{X}_p \tag{26}$$

$$a_p^e(\xi_p, \zeta_p) \le C^e \|\xi_p\|_{H^1(\Omega_p)} \|\zeta_p\|_{H^1(\Omega_p)}, \forall (\xi_p, \zeta_p) \in \mathbf{X}_p^2, \tag{27}$$

where (22), (24) and (26) hold true thanks to Korn's inequality and Poincaré inequality while (23), (25) and (27) relies on the Cauchy–Schwarz inequality.

Theorem 3.1. *If $\mathbf{f} \in [L^2(\Omega)]^d$ and $g \in L_0^2(\Omega)$, then there exists a unique solution $(\mathbf{U}, P) \in \mathbb{H} \times \mathbb{M}$ to the problem (19).*

Proof. It is easy to prove that $\mathbf{A}$ and $\mathbf{B}$ are continuous. It is also clear that $\mathbf{F}$ and $\mathbf{G}$ are continuous and bounded. In summary, (23), (25) and (27) lead to: there exist four positive constants $(b_1, b_2, b_3, b_4) \in \mathbb{R}^4$ such that,

$$\begin{aligned} |\mathbf{A}(\mathbf{U},\mathbf{V})| &\le b_1 \|\mathbf{U}\|_\mathbb{H} \|\mathbf{V}\|_\mathbb{H}, \forall (\mathbf{U},\mathbf{V}) \in \mathbb{H}^2 \\ |\mathbf{B}(\mathbf{U},\mathbf{Q})| &\le b_2 \|\mathbf{U}\|_\mathbb{H} \|\mathbf{Q}\|_\mathbb{M}, \forall (\mathbf{U},\mathbf{Q}) \in \mathbb{H} \times \mathbb{M} \\ |\mathbf{L}(\mathbf{V})| &\le b_3 \|\mathbf{V}\|_\mathbb{H}, \forall \mathbf{V} \in \mathbb{H} \\ |\mathbf{G}(\mathbf{Q})| &\le b_4 \|\mathbf{Q}\|_\mathbb{M}, \forall \mathbf{Q} \in \mathbb{M}. \end{aligned}$$

- Now we define the null space of $(\operatorname{div}\mathbf{v}, Q)_\Omega$, i.e.

$$\mathbf{Z} := \left\{ \mathbf{V} = (\mathbf{v}, \xi_p) \in \mathbb{H} : (\operatorname{div}\mathbf{v}, Q)_\Omega = 0 \qquad \forall Q \in L_0^2(\Omega) \right\}.$$

Let $Q \in L^2(\Omega)$. Since $L^2(\Omega) = L^2_0(\Omega) \oplus \mathbb{R}$, then there exists unique $(Q_1, Q_0) \in L^2_0(\Omega) \times \mathbb{R}$ such that $Q = Q_1 + Q_0$. Thus, for all $\mathbf{V} = (\mathbf{v}, \xi_p) \in \mathbf{Z}$, we obtain

$$\begin{aligned}(\operatorname{div}\mathbf{v}, Q)_\Omega &= (\operatorname{div}\mathbf{v}, Q_1)_\Omega + (\operatorname{div}\mathbf{v}, Q_0)_\Omega = 0 + Q_0(\operatorname{div}\mathbf{v}, 1)_\Omega \\ &= Q_0 \int_\Omega \operatorname{div}\mathbf{v} \\ &= Q_0 \left[\int_{\Gamma_f} \mathbf{v}\cdot\mathbf{n}_f + \int_{\Gamma_p} \mathbf{v}\cdot\mathbf{n}_p \right] \\ &= 0.\end{aligned}$$

Hence from (22), (24) and (26), the coercivity of bilinear form $\mathbf{A}$ on $\mathbf{Z}$ follows, i.e., there exists positive constant $b_5 \in \mathbb{R}$ such that,

$$\mathbf{A}(\mathbf{V}, \mathbf{V}) \geq b_5 \|\mathbf{V}\|^2_{\mathbb{H}}, \quad \forall \mathbf{V} \in \mathbf{Z}.$$

• The second Brezzi condition that we need to verify is the inf–sup condition. Let $\mathbf{Q} = (Q_1, Q_2) \in L^2_0(\Omega) \times L^2_0(\Omega_p)$. Then there exists $(\mathbf{v}, \xi_p) \in \left\{[H^1_0(\Omega)]^d \setminus \{\mathbf{0}\}\right\} \times \left\{[H^1_0(\Omega_p)]^d \setminus \{\mathbf{0}\}\right\}$ such that:

$$\begin{cases} \nabla\cdot\mathbf{v} = -Q_1 \text{ in } \Omega \\ \nabla\cdot\xi_p = -Q_2 \text{ in } \Omega_p \end{cases} \tag{28}$$

with the estimations:

$$\|\mathbf{v}\|_{\mathbf{H}} \lesssim \|\mathbf{v}\|_{1,\Omega} \leq C' \|Q_1\|_{0,\Omega} \tag{29}$$

$$\|\xi_p\|_{1,\Omega_p} \leq C'' \|Q_2\|_{0,\Omega_p} \tag{30}$$

which imply

$$\|\mathbf{v}\|_{\mathbf{H}} + \|\xi_p\|_{1,\Omega_p} \lesssim \|Q_1\|_{0,\Omega} + \|Q_2\|_{0,\Omega_p} \leq \max\{C', C''\} \left(\|Q_1\|^2_{0,\Omega} + \|Q_2\|^2_{0,\Omega_p} \right)^{1/2},$$

i.e.,

$$\|\mathbf{V}\|_{\mathbb{H}} \leq \max\{C', C''\} \|\mathbf{Q}\|_{\mathbb{M}}. \tag{31}$$

Thus, from (28) we have,

$$\begin{aligned}\mathbf{B}(\mathbf{V}, \mathbf{Q}) &= -(Q_1, \nabla\cdot\mathbf{v})_\Omega - \alpha(Q_2, \nabla\cdot\xi_p)_{\Omega_p} \\ &= \|Q_1\|^2_{0,\Omega} + \alpha\|Q_2\|^2_{0,\Omega_p} \\ &\geq \min\{1, \alpha\} \left(\|Q_1\|^2_{0,\Omega} + \|Q_2\|^2_{0,\Omega_p} \right) = \min\{1, \alpha\} \|\mathbf{Q}\|^2_{\mathbb{M}}.\end{aligned} \tag{32}$$

Finally, (31) and (32) lead to inf–sup condition:

$$\sup_{\mathbf{V}\in\mathbb{H}\setminus\{\mathbf{0}\}} \frac{\mathbf{B}(\mathbf{V}, \mathbf{Q})}{\|\mathbf{V}\|_{\mathbb{H}}} \geq \left(\frac{\min\{1, \alpha\}}{\max\{C', C''\}} \right) \|\mathbf{Q}\|_{\mathbb{M}}, \qquad \forall \mathbf{Q} \in \mathbb{M}. \tag{33}$$

The proof is complete. □

4. Finite element discretization

In this section, we will use the nonconforming Crouzeix–Raviart piecewise linear finite element approximation for velocity and piecewise constant approximation for pressure and establish the existence and uniqueness of a finite element solution of the discrete problem.

Let $\mathcal{T}_h$ be a family of triangulations of $\bar{\Omega}$ with nondegenerate elements (i.e. triangles for $d = 2$ and tetrahedrons for $d = 3$). For any $K \in \mathcal{T}_h$, we denote by h_K the diameter of K and ρ_K the diameter of the largest ball inscribed into K.

We set:

$$h = \max_{K\in\mathcal{T}_h} h_K, \text{ and } \sigma_h = \max_{T\in\mathcal{T}_h} \frac{h_K}{\rho_K} \tag{34}$$

We assume that the family of triangulations is regular, in the sense that there exists $\sigma_0 > 0$ such that $\sigma_h \leqslant \sigma_0$, for all $h > 0$. We also assume that the triangulation is conforming with respect to the partition of Ω into Ω_f and Ω_p, namely each $K \in \mathcal{T}_h$ is either in Ω_f or in Ω_p (see Figs. 2, 3, 4 for illustration).

Let $\mathcal{T}^f_h$ and $\mathcal{T}^p_h$ be the corresponding induced triangulations of Ω_f and Ω_p. For any $K \in \mathcal{T}_h$, we denote by $\mathcal{E}(K)$ (resp. $\mathcal{N}(K)$) the set of its edges ($d = 2$) or faces ($d = 3$) (resp. vertices) and set $\mathcal{E}_h = \bigcup_{K\in\mathcal{T}_h} \mathcal{E}(K)$, $\mathcal{N}_h = \bigcup_{K\in\mathcal{T}_h} \mathcal{N}(K)$. For $\mathcal{A} \subset \overline{\Omega}$ we define

$$\mathcal{E}_h(\mathcal{A}) = \{E \in \mathcal{E}_h : E \subset \mathcal{A}\}.$$

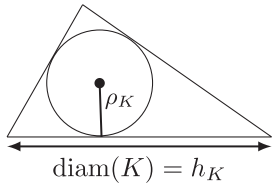

Fig. 2. Isotropic element K in $\mathbb{R}^2$.

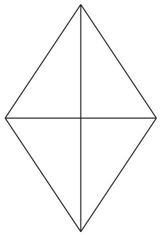

Fig. 3. Example of conforming mesh in $\mathbb{R}^2$.

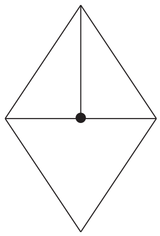

Fig. 4. Example of nonconforming mesh in $\mathbb{R}^2$.

Notice that $\mathcal{E}_h$ can be split up in the form

$$\mathcal{E}_h = \mathcal{E}_h(\Omega_f^+) \cup \mathcal{E}_h(\Omega_p) \cup \mathcal{E}_h(\partial\Omega_p) \tag{35}$$

where $\Omega_f^+ = \Omega_f \cup \Gamma_f$. Note that $\mathcal{E}_h(\Gamma_{fp})$ is included in $\mathcal{E}_h(\partial\Omega_p)$.

With every edges $E \in \mathcal{E}_h$, we associate a unit vector $\mathbf{n}_E$ such that $\mathbf{n}_E$ is orthogonal to E and equals to the unit exterior normal vector to $\partial\Omega$ if $E \subset \partial\Omega$. For any $E \in \mathcal{E}_h$ and any piecewise continuous function φ, we denote by $[\varphi]_E$ its jump across E in the direction of $\mathbf{n}_E$:

$$[\varphi]_E(x) := \begin{cases} \lim\limits_{t\to 0+} \varphi(x + t\mathbf{n}_E) - \lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for an interior edge/face } E, \\ -\lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for a boundary edge/face } E \end{cases}$$

Based on the above notations, we introduce a variant of the nonconforming Crouzeix–Raviart piecewise linear finite element space:

$$\begin{aligned} \mathbf{H}_h := \Big\{ \mathbf{v}_h \in [L^2(\Omega)]^d : \mathbf{v}_{h|K} \in [\mathbb{P}^1(K)]^d \quad \forall K \in \mathcal{T}_h, ([\mathbf{v}_h]_E, \mathbf{1})_E = 0 \quad \forall E \in \mathcal{E}_h(\Omega_f^+)\ , \\ ([\mathbf{v}_h \cdot \mathbf{n}_E]_E, 1)_E = 0 \ \ \forall E \in \mathcal{E}_h(\Omega_p) \cup \mathcal{E}_h(\partial\Omega_p) \Big\}\ , \end{aligned}$$

$$\mathbf{X}_{ph} := \left\{ \xi_{ph} \in [L^2(\Omega_p)]^d : \xi_{ph|K} \in [\mathbb{P}^1(K)]^d \quad \forall K \in \mathcal{T}_h^p, ([\xi_{ph}]_E, \mathbf{1})_E = 0 \quad \forall E \in \mathcal{E}_h(\bar{\Omega}_p) \right\}.$$

For $X \subseteq \Omega$, we set

$$E_h(X) := \left\{ q_h \in L_0^2(X) : q_{h|K} \in \mathbb{P}^0(K) \quad \forall K \subset X, K \in \mathcal{T}_h \right\}.$$

and we define

$$\begin{aligned} \mathbb{M}_h &:= E_h(\Omega) \times E_h(\Omega_p) \subset \mathbb{M} \\ \mathbb{H}_h &:= \mathbf{H}_h \times \mathbf{X}_{ph} \not\subseteq \mathbb{H}. \end{aligned}$$

Where $\mathbb{P}^m(K)$ is the space of the restrictions to K of all polynomials of degree less than or equal to m.

The space $\mathbb{M}_h$ is equipped with the norm $\|\cdot\|_{\mathbb{M}}$ while the norm on $\mathbb{H}_h$ will be specified later on. The choice of $\mathbf{H}_h$ is more natural since the space $\mathbf{H}_h$ approximates only $\mathbf{H}(\text{div};\Omega_p)$ and not $[H^1(\Omega_p)]^d$, while our a priori analysis is only valid in this larger space.

Let us introduce the discrete divergence operator $\text{div}_h \in \mathcal{L}(\mathbf{H}_h; E_h(\Omega)) \cap \mathcal{L}(\mathbf{H}; L^2_0(\Omega))$ by

$$(\text{div}_h\, \mathbf{v}_h)_{|K} = \text{div}(\mathbf{v}_{h|K}),\ \forall K \in \mathcal{T}_h, \tag{36}$$

or $\text{div}_h \in \mathcal{L}(\mathbf{X}_{ph}; E_h(\Omega_p)) \cap \mathcal{L}(\mathbf{X}_p; L^2_0(\Omega_p))$ by

$$(\text{div}_h\, \xi_{ph})_{|K} = \text{div}(\xi_{ph|K}),\ \forall K \in \mathcal{T}_h^p. \tag{37}$$

Then, for $\mathbf{U}_h = (\mathbf{u}_h, \eta_{ph}) \in \mathbb{H}_h$, $\mathbf{V}_h = (\mathbf{v}_h, \xi_{ph}) \in \mathbb{H}_h$ and $\mathbf{Q}_h = (Q_{1h}, Q_{2h}) \in \mathbb{M}_h$, we can introduce two bilinear forms:

$$\begin{aligned}\mathbf{A}_h(\mathbf{U}_h, \mathbf{V}_h) := &\sum_{K\in\mathcal{T}_h^f} (2\mu\mathbf{D}(\mathbf{u}_h), \mathbf{D}(\mathbf{v}_h))_K + \left(\mu\mathbf{K}^{-1}\mathbf{u}_h, \mathbf{v}_h\right)_{\Omega_p}\\ &+ \sum_{K\in\mathcal{T}_h^p} \left(2\mu_p\mathbf{D}(\eta_{ph}), \mathbf{D}(\xi_{ph})\right)_K + \left(\lambda_p\, \text{div}_h\, \eta_{ph}, \text{div}_h\, \xi_{ph}\right)_{\Omega_p}\\ &+ \sum_{j=1}^{d-1}\langle \mu\alpha_{BJS}\sqrt{K_j^{-1}}\mathbf{u}_{fh}\cdot\tau_{f,j}, \mathbf{v}_{fh}\cdot\tau_{f,j}\rangle_{\Gamma_{fp}},\end{aligned}$$

$$\mathbf{B}_h(\mathbf{V}_h, \mathbf{Q}_h) := -(Q_{1h}, \text{div}_h\, \mathbf{v}_h)_{\Omega} - \alpha(Q_{2h}, \text{div}_h\, \xi_{ph})_{\Omega_p}.$$

Then, we propose the following discrete problem: find $(\mathbf{U}_h, \mathbf{P}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ with $\mathbf{P}_h = (p_h, p_{ph})$ such that:

$$\begin{cases} \mathbf{A}_h(\mathbf{U}_h, \mathbf{V}_h) + \mathbf{B}_h(\mathbf{V}_h, \mathbf{P}_h) + \mathbf{J}(\mathbf{U}_h, \mathbf{V}_h) = \mathbf{L}(\mathbf{V}_h) & \forall \mathbf{V}_h \in \mathbb{H}_h\\ \mathbf{B}_h(\mathbf{U}_h, \mathbf{Q}_h) = \mathbf{G}(\mathbf{Q}_h) & \forall \mathbf{Q}_h \in \mathbb{M}_h.\end{cases} \tag{38}$$

This is the natural discretization of the weak formulation (19) only with the penalizing term $\mathbf{J}(\mathbf{U}_h, \mathbf{V}_h)$ added. We define the bilinear form $\mathbf{J}(\cdot,\cdot)$ following the decomposition of $\mathcal{E}_h$:

$$\mathbf{J}(\mathbf{U}_h, \mathbf{V}_h) = \mathbf{J}_{\Omega_f^+}(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{J}_{\Omega_p}(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{J}_{\partial\Omega_p}(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{J}_{\Omega_p^+}(\eta_{ph}, \xi_{ph}),$$

where

$$\begin{aligned}\mathbf{J}_{\Omega_f^+}(\mathbf{u}_h, \mathbf{v}_h) &:= (1+2\mu)\sum_{E\in\mathcal{E}_h(\Omega_f^+)} h_E^{-1}\int_E [\mathbf{u}_h]_E\cdot[\mathbf{v}_h]_E ds,\\ \mathbf{J}_{\Omega_p}(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_{E\in\mathcal{E}_h(\Omega_p)} h_E^{-1}\int_E [\mathbf{u}_h]_E\cdot[\mathbf{v}_h]_E ds,\\ \mathbf{J}_{\partial\Omega_p}(\mathbf{u}_h, \mathbf{v}_h) &:= \sum_{E\in\mathcal{E}_h(\partial\Omega_p)} h_E^{-1}\int_E [\mathbf{u}_h\cdot\mathbf{n}_E]_E[\mathbf{v}_h\cdot\mathbf{n}_E]_E ds \quad \text{and}\\ \mathbf{J}_{\Omega_p^+}(\eta_{ph}, \xi_{ph}) &:= \sum_{E\in\mathcal{E}_h(\Omega_p^+)} h_E^{-1}\int_E (1+2\mu_p)[\eta_{ph}]_E\cdot[\xi_{ph}]_E ds.\end{aligned}$$

Here, h_E is the length ($d = 2$) or diameter ($d = 3$) of E. Note that each element of $\mathcal{E}_h$ only contributes with one jump term in $\mathbf{J}(\mathbf{U}_h, \mathbf{V}_h)$.

We are now able to define the norm on $\mathbb{H}_h$:

$$\|\mathbf{V}_h\|_{\mathbb{H}_h} := \left[\|\mathbf{v}_h\|^2_{\mathbf{H}_h} + \|\xi_{ph}\|^2_{\mathbf{X}_{ph}} + \mathbf{J}(\mathbf{V}_h, \mathbf{V}_h)\right]^{1/2}, \tag{39}$$

where

$$\|\mathbf{v}_h\|_{\mathbf{H}_h} := \left(\sum_{K\in\mathcal{T}_h^f} |\mathbf{v}_h|^2_{1,K} + \sum_{j=1}^{d-1}\langle \mathbf{v}_{fh}\cdot\tau_j, \mathbf{v}_{fh}\cdot\tau_j\rangle_{\Gamma_{fp}} + \|\mathbf{v}_h\|^2_{\Omega_p} + \|\,\text{div}_h\, \mathbf{v}_h\|^2_{\Omega_p}\right)^{1/2},$$

and

$$\|\xi_{ph}\|_{\mathbf{X}_{ph}} := \left(\sum_{K\in\mathcal{T}_h^p} |\xi_{ph}|^2_{1,K}\right)^{1/2}$$

We define the subspace of $\mathbf{H}_h$:

$$\mathbf{Z}_h := \left\{ \mathbf{V}_h = (\mathbf{v}_h, \xi_{ph}) \in \mathbb{H}_h : (\mathrm{div}_h \, \mathbf{v}_h, Q_{1h})_\Omega = 0 \qquad \forall Q_{1h} \in E_h(\Omega) \right\}.$$

Lemma 4.1. *If $\mathbf{V}_h = (\mathbf{v}_h, \xi_{ph}) \in \mathbf{Z}_h$, then $\mathrm{div}_h \, \mathbf{v}_h = 0$.*

Proof. We recall $\mathbb{H}_h = \mathbf{H}_h \times \mathbb{M}_h$. Since $\mathrm{div}_h \, \mathbf{v}_h \in E_h(\Omega)$ for $\mathbf{v}_h \in \mathbf{H}_h$, we take $Q_{1h} = \mathrm{div}_h \, \mathbf{v}_h$, leading to $(\mathrm{div}_h \, \mathbf{v}_h, \mathrm{div}_h \, \mathbf{v}_h)_\Omega = 0$, and the lemma follows. □

In the sequel, we will define the bilinear form:

$$\mathbb{A}_h(\mathbf{U}_h, \mathbf{V}_h) = \mathbf{A}_h(\mathbf{U}_h, \mathbf{V}_h) + \mathbf{J}(\mathbf{U}_h, \mathbf{V}_h). \tag{40}$$

Lemma 4.2 (*Coercivity*). *$\mathbb{A}_h(\cdot, \cdot)$ is coercive on $\mathbf{Z}_h$: there is $\beta > 0$ such that*

$$\mathbb{A}_h(\mathbf{V}_h, \mathbf{V}_h) \geq \beta \|\mathbf{V}_h\|^2_{\mathbb{H}_h}, \qquad \forall \mathbf{V}_h \in \mathbf{Z}_h, \tag{41}$$

where β depends on σ_0, μ, α and $k_{\max}$.

Proof. Let $\mathbf{V}_h = (\mathbf{v}_h, \xi_{ph}) \in \mathbf{Z}_h \subset \mathbb{H}_h = \mathbf{H}_h \times \mathbf{X}_{ph}$.

$$\begin{aligned}
\mathbb{A}_h(\mathbf{V}_h, \mathbf{V}_h) &= \mathbf{A}(\mathbf{V}_h, \mathbf{V}_h) + \mathbf{J}(\mathbf{V}_h, \mathbf{V}_h) \\
&= \sum_{K \in \mathcal{T}_h^f} (2\mu \mathbf{D}(\mathbf{v}_h), \mathbf{D}(\mathbf{v}_h))_K + \left(\mu \mathbf{K}^{-1} \mathbf{v}_h, \mathbf{v}_h\right)_{\Omega_p} \\
&+ \sum_{K \in \mathcal{T}_h^p} \left(2\mu_p \mathbf{D}(\xi_{ph}), \mathbf{D}(\xi_{ph})\right)_K + \left(\lambda_p \, \mathrm{div}_h \, \xi_{ph}, \mathrm{div}_h \, \xi_{ph}\right)_{\Omega_p} \\
&+ \sum_{j=1}^{d-1} \langle \mu \alpha_{BJS} \sqrt{K_j^{-1}} \mathbf{v}_{fh} \cdot \tau_{f,j}, \mathbf{v}_{fh} \cdot \tau_{f,j} \rangle_{\Gamma_{fp}} \\
&+ (1 + 2\mu) \sum_{E \in \mathcal{E}_h(\Omega_f^+)} h_E^{-1} \int_E [\mathbf{v}_h]_E^2 ds \\
&+ \sum_{E \in \mathcal{E}_h(\Omega_p)} h_E^{-1} \int_E [\mathbf{v}_h]_E^2 ds \\
&+ \sum_{E \in \mathcal{E}_h(\partial \Omega_p)} h_E^{-1} \int_E [\mathbf{v}_h \cdot \mathbf{n}_E]_E^2 ds \\
&+ \sum_{E \in \mathcal{E}_h(\Omega_p^+)} h_E^{-1} \int_E (1 + 2\mu_p) [\xi_{ph}]_E^2 ds.
\end{aligned}$$

The Korn's inequality for piecewise H^1 vector fields [29] leads to:

$$2\mu \sum_{K \in \mathcal{T}_h^f} (\mathbf{D}(\mathbf{v}_h), \mathbf{D}(\mathbf{v}_h))_K + \mathbf{J}_{\Omega_f^+}(\mathbf{v}_h, \mathbf{v}_h) \geq C_1 \sum_{K \in \mathcal{T}_h^f} |\mathbf{v}_h|^2_{1,K}$$

$$\sum_{K \in \mathcal{T}_h^p} \left(2\mu_p \mathbf{D}(\xi_{ph}), \mathbf{D}(\xi_{ph})\right)_K + \mathbf{J}_{\Omega_p^+}(\xi_{ph}, \xi_{ph}) \geq C_2 \sum_{K \in \mathcal{T}_h^p} |\xi_{ph}|^2_{1,K},$$

where C_1 and C_2 are positive constants which depends only on σ_0 and μ.

In addition to these, we have $(\lambda_p \, \mathrm{div}_h \, \xi_{ph}, \mathrm{div}_h \, \xi_{ph})_{\Omega_p} \geq \lambda_{\min} \| \mathrm{div}_h \, \xi_{ph} \|^2_{0,\Omega_p} \geq 0$, and since $\mathbf{V}_h \in \mathbf{Z}_h$, then from Lemma 4.1, we obtain

$$\begin{aligned}
(\mu \mathbf{K}^{-1} \mathbf{v}_h, \mathbf{v}_h)_{\Omega_p} &\geq \mu k_{\max}^{-1} \|\mathbf{v}_h\|^2_{0,\Omega_p} = \mu k_{\max}^{-1} \|\mathbf{v}_h\|^2_{0,\Omega_p} + \| \mathrm{div}_h \, \mathbf{v}_h \|^2_{0,\Omega_p} \\
&\geq \min\{\mu k_{\max}^{-1}, 1\} \|\mathbf{v}_h\|^2_{H(\mathrm{div}_h, \Omega_p)}.
\end{aligned}$$

The proof is complete. □

From Hölder's inequality, we derive the boundedness of $\mathbb{A}_h(\cdot, \cdot)$ and $\mathbf{B}_h(\cdot, \cdot)$.

Lemma 4.3. *There holds:*

$$|\mathbb{A}_h(\mathbf{U}_h,\mathbf{V}_h)| \leq C_3\|\mathbf{U}_h\|_{\mathbb{H}_h}\|\mathbf{V}_h\|_{\mathbb{H}_h} \qquad \forall(\mathbf{U}_h,\mathbf{V}_h)\in\mathbb{H}_h^2, \tag{42}$$

$$|\mathbf{B}_h(\mathbf{V}_h,\mathbf{Q}_h)| \leq C_4\|\mathbf{V}_h\|_{\mathbb{H}_h}\|\mathbf{Q}_h\|_{\mathbb{M}_h} \qquad \forall(\mathbf{V}_h,\mathbf{Q}_h)\in\mathbb{H}_h\times\mathbb{M}_h, \tag{43}$$

where, C_3 and C_4 depend on μ, α and $\mathbf{K}$*.*

In order to verify the discrete inf–sup condition, we define the space

$$\mathbf{W}=\left\{\mathbf{v}\in\mathbf{H}: \quad \mathbf{v}_{|\Omega_p}\in[H^1(\Omega_p)]^d\right\}$$

and the Crouzeix–Raviart interpolation operators $\mathbf{r}_{1h}:\mathbf{W}\to\mathbf{H}_h$ and $\mathbf{r}_{2h}:\mathbf{X}_p\to\mathbf{X}_{ph}$ by:

$$\int_E(\mathbf{r}_{1h}\mathbf{v})_f\,ds=\int_E\mathbf{v}_f\,ds \qquad \forall E\in\mathcal{E}_h(\bar{\Omega}_f), \tag{44}$$

$$\int_E(\mathbf{r}_{1h}\mathbf{v})_p\,ds=\int_E\mathbf{v}_p\,ds \qquad \forall E\in\mathcal{E}_h(\bar{\Omega}_p), \tag{45}$$

$$\int_E\mathbf{r}_{2h}(\xi_p)\,ds=\int_E\xi_p\,ds \qquad \forall E\in\mathcal{E}_h(\bar{\Omega}_p); \tag{46}$$

Lemma 4.4. *The operators $\mathbf{r}_{1h}$ and $\mathbf{r}_{2h}$ are bounded: there are constants $C_5>0$, $C_6>0$ depending on σ_0, μ and d such that:*

$$\|\mathbf{r}_{1h}(\mathbf{v})\|_{\mathbf{H}_h} \leq C_5\left(\|\mathbf{v}\|_{1,f}^2+\|\mathbf{v}\|_{1,d}^2\right)^{1/2} \qquad \forall\mathbf{v}\in\mathbf{W}, \tag{47}$$

$$\|\mathbf{r}_{2h}(\xi_p)\|_{\mathbf{X}_{ph}} \leq C_6\|\xi_p\|_{\mathbf{X}_p} \qquad \forall\xi_p\in\mathbf{X}_p. \tag{48}$$

Proof. Similarly to [30, Lemma 4.5]. □

Then, we have the following result:

Lemma 4.5 (*Discrete inf–sup Condition*)**.** *There exists a positive constant β_0 depending on σ_0, μ, and d such that*

$$\sup_{\mathbf{V}_h\in\mathbb{H}_h\setminus\{\mathbf{0}\}}\frac{\mathbf{B}_h(\mathbf{V}_h,\mathbf{Q}_h)}{\|\mathbf{V}_h\|_{\mathbb{H}_h}}\geq\beta_0\|\mathbf{Q}_h\|_{\mathbb{M}_h} \qquad \forall\mathbf{Q}_h\in\mathbb{M}_h. \tag{49}$$

Proof. Let $\mathbf{Q}_h=(Q_{1h},Q_{2h})\in\mathbb{M}_h$. Then from [28] it is known that there exist vector valued functions $(\mathbf{v},\xi_p)\in[H^1(\Omega)]^d\times[H^1(\Omega_p)]^d$ and constants $C_8>0$, $C_9>0$, independent of $\mathbf{Q}_h$, such that: $\nabla\cdot\mathbf{v}=-Q_{1h}$, $\nabla\cdot\xi_p=-Q_{2h}$ and $\|\mathbf{v}\|_{\mathbf{H}}\leq C_8\|Q_{1h}\|_{0,\Omega}$, $\|\xi_p\|_{\mathbf{X}_p}\leq C_9\|Q_{2h}\|_{0,\Omega_p}$. We set $\mathbf{V}=(\mathbf{v},\xi_p)\in\mathbb{H}$ and we define the operator $\mathbf{R}_h(\mathbf{V})=(\mathbf{v}_h,\xi_{ph})$, with $\mathbf{v}_h=\mathbf{r}_{1h}(\mathbf{v})$ and $\xi_{ph}=\mathbf{r}_{2h}(\xi_p)$; and we take $\mathbf{V}_h=\mathbf{R}_h(\mathbf{V})\in\mathbb{H}_h$. Then we have from (44)–(46):

$$\begin{aligned}\mathbf{B}_h(\mathbf{V}-\mathbf{R}_h(\mathbf{V}),\mathbf{Q}_h) &= -(\mathrm{div}_h(\mathbf{v}-\mathbf{v}_h),Q_{1h})_\Omega-\alpha(\mathrm{div}_h(\xi_p-\xi_{ph}),Q_{2h})_{\Omega_p}\\ &= -\sum_{K\in\mathcal{T}_h}\int_K\nabla\cdot(\mathbf{v}-\mathbf{v}_h)Q_{1h}-\alpha\sum_{K\in\mathcal{T}_h^p}\int_K\nabla\cdot(\xi-\xi_{ph})Q_{2h}\\ &= -\sum_{E\mathcal{E}_h(\Omega)}(\mathbf{v}-\mathbf{r}_{1h}\mathbf{v})\cdot\mathbf{n}_EQ_{1h}-\alpha\sum_{E\mathcal{E}_h(\Omega_p)}(\xi_p-\mathbf{r}_{2h}\xi_p)\cdot\mathbf{n}_{E,p}Q_{2h}\\ &= 0.\end{aligned}$$

Then, Lemma 4.4 implies

$$\begin{aligned}\mathbf{B}_h(\mathbf{V}_h,\mathbf{Q}_h)=\mathbf{B}_h(\mathbf{V},\mathbf{Q}_h) &= \|Q_{1h}\|_{0,\Omega}^2+\alpha\|Q_{2h}\|_{0,\Omega_p}^2\\ &\geq \min\{1,\alpha\}\|\mathbf{Q}_h\|_{\mathbb{M}}^2\\ &\geq \left(\frac{\min\{1,\alpha\}}{\max\left\{C_8C_5^{-1},C_9C_6^{-1}\right\}}\right)\|\mathbf{Q}_h\|_{\mathbb{M}_h}\|\mathbf{V}_h\|_{\mathbb{H}_h},\end{aligned}$$

which completes the proof with $\beta_0=\frac{\min\{1,\alpha\}}{\max\left\{C_8C_5^{-1},C_9C_6^{-1}\right\}}>0$. □

Lemmas 4.2, 4.3 and 4.5, together with the abstract theory of mixed problem [28], immediately imply the following theorem:

Theorem 4.1. *There exists a unique solution $(\mathbf{U}_h,\mathbf{P}_h)\in\mathbb{H}_h\times\mathbb{M}_h$ to the problem (38).*

5. A convergence analysis

We now present an a priori analysis of the approximation error. The use of nonconforming finite element leads to $\mathbb{H}_h \not\subseteq \mathbb{H}$, so the approximation error contains some extra consistency error terms. In fact, the abstract error estimates from [28] give the following result.

Lemma 5.1. *Let $(\mathbf{U}, \mathbf{P})$ be the solution of problem (19) and $(\mathbf{U}_h, \mathbf{P}_h)$ be the solution of the discrete problem (38). Then, there exists a $C_{10} > 0$ depending on α, C_3, d and β_0 such that:*

$$\begin{aligned} \|\mathbf{U}-\mathbf{U}_h\|_{\mathbb{H}_h\cup\mathbb{H}} &+ \|\mathbf{P}-\mathbf{P}_h\|_{\mathbb{M}} \\ &\le C_{10}\left(\inf_{\mathbf{v}_h\in\mathbf{H}_h}\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{H}_h\cup\mathbf{H}} + \inf_{\xi_{ph}\in\mathbf{X}_{ph}}\|\eta_p-\xi_{ph}\|_{\mathbf{X}_{ph}\cup\mathbf{X}_p}\right. \\ &\left. + \inf_{Q_{1h}\in E_h(\Omega)}\|p-Q_{1h}\|_{0,\Omega} + \inf_{Q_{2h}\in E_h(\Omega_p)}\|p_p-Q_{2h}\|_{0,\Omega_p} + M_{1h}+M_{2h}\right), \end{aligned} \tag{50}$$

where

$$M_{1h} = \sup_{\mathbf{V}_h\in\mathbb{H}_h\setminus\{\mathbf{0}\}}\frac{|\mathbb{A}_h(\mathbf{U},\mathbf{V}_h)+\mathbf{B}_h(\mathbf{V}_h,\mathbf{P})-\mathbf{L}(\mathbf{V}_h)|}{\|\mathbf{V}_h\|_{\mathbb{H}_h}}$$

$$M_{2h} = \sup_{\mathbf{Q}_h\in\mathbb{M}_h\setminus\{\mathbf{0}\}}\frac{|\mathbf{B}_h(\mathbf{U},\mathbf{Q}_h)-\mathbf{G}(\mathbf{Q}_h)|}{\|\mathbf{Q}_h\|_{\mathbb{M}_h}}.$$

are the consistency error terms.

Note that $\mathbf{B}_h(\mathbf{U},\mathbf{Q}_h) = \mathbf{B}(\mathbf{U},\mathbf{Q}_h)$, thus $M_{2h} = 0$.

For estimating the approximation error, we assume that the solution $(\mathbf{U}, \mathbf{P})$ of problem (19) satisfies the smoothness assumptions:

(i): $\mathbf{u} \in \mathbf{H}$, $\mathbf{u}_f \in [H^2(\Omega_f)]^d$, $\mathbf{u}_p \in [H^2(\Omega_p)]^d$
(ii): $\eta_p \in \mathbf{X}_p$, $\eta_p \in [H^2(\Omega_p)]^d$
(iii): $p \in L^2_0(\Omega)$, $p_f \in H^1(\Omega_f)$, $p_p \in H^1(\Omega_p)$.

We begin with an estimate for the terms $\inf_{\mathbf{v}_h\in\mathbf{H}_h}\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{H}_h\cup\mathbf{H}}$ and $\inf_{\xi_{ph}\in\mathbf{X}_{ph}}\|\eta_p-\xi_{ph}\|_{\mathbf{X}_p\cup\mathbf{X}_{ph}}$:

Lemma 5.2. *There holds:*

$$\inf_{\mathbf{v}_h\in\mathbf{H}_h}\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{H}_h\cup\mathbf{H}} \le C_{11}h\left(|\mathbf{u}|_{2,f}+|\mathbf{u}|_{2,p}\right) \tag{51}$$

$$\inf_{\xi_{ph}\in\mathbf{X}_{ph}}\|\eta_p-\xi_{ph}\|_{\mathbf{X}_p\cup\mathbf{X}_{ph}} \le C_{12}h|\eta_p|_{2,p}, \tag{52}$$

where C_{11} and C_{12} are constants depending only σ_0 and d.

Proof. Since $\mathbf{r}_{1h}(\mathbf{u}) \in \mathbf{H}_h$ and $\mathbf{r}_{2h}(\eta_p) \in \mathbf{X}_{ph}$, we only bound the errors $\|\mathbf{u}-\mathbf{r}_{1h}(\mathbf{u})\|_{\mathbf{H}_h\cup\mathbf{H}}$ and $\|\eta_p-\mathbf{r}_{2h}(\eta_p)\|_{\mathbf{X}_p\cup\mathbf{X}_{ph}}$. From the interpolation theory for $\mathbf{r}_{ih}$, $i \in \{1,2\}$ [31] and the trace inequality $\|\varphi\|^2_{0,\partial K} \le C_{13}\left(h_K^{-1}\|\varphi\|^2_{0,K}+h_K|\varphi|^2_{1,K}\right) \quad \forall\varphi \in H^1(K)$, $K \in \mathcal{T}_h$, the lemma follows. □

Next, for evaluating the terms $\inf_{Q_{1h}\in E_h(\Omega)}\|p-Q_{1h}\|_{0,\Omega}$ and $\inf_{Q_{2h}\in E_h(\Omega_p)}\|p_p-Q_{2h}\|_{0,\Omega_p}$ appearing in (51), we define the projection operators $\pi_1 : L^2_0(\Omega) \to E_h(\Omega)$ and $\pi_2 : L^2_0(\Omega_p) \to E_h(\Omega_p)$ by:

$$\int_K(\pi_1Q_1-Q_1)dx = 0\forall K \in \mathcal{T}_h \tag{53}$$

$$\int_K(\pi_2Q_2-Q_2)dx = 0\forall K \in \mathcal{T}_h^p. \tag{54}$$

Then, we have

$$\|Q_i-\pi_iQ_i\|_{0,K} = \min_{Q_i\in P^0(K)}\|Q_i-\pi_iQ_i\|_{0,K} \le \alpha_ih|Q_i|_{1,K}, \forall K,$$

with $\alpha_i > 0$ depending only on σ_0 and d, so that

$$\inf_{Q_{1h}\in E_h(\Omega)}\|p-Q_{1h}\|_{0,\Omega} \le \|p-\pi_1(p)\|_{0,\Omega} \le \alpha_1h\left(|p|_{1,f}+|p|_{1,p}\right) \tag{55}$$

$$\inf_{Q_{2h}\in E_h(\Omega_p)}\|p_p-Q_{2h}\|_{0,\Omega_p} \le \|p_p-\pi_2(p_p)\|_{0,\Omega_p} \le \alpha_2h|p|_{1,p}. \tag{56}$$

Finally, let us consider the term $\mathbb{A}_h(\mathbf{U},\mathbf{V}_h)+\mathbf{B}_h(\mathbf{V}_h,\mathbf{P})-\mathbf{L}(\mathbf{V}_h)$. The smoothness assumption of $\mathbf{u}$ and η_p implies $\mathbf{J}(\mathbf{U},\mathbf{V}_h)=0$, thus

$$\mathbb{A}_h(\mathbf{U},\mathbf{V}_h)=\mathbf{A}_h(\mathbf{U},\mathbf{V}_h)\qquad \forall \mathbf{V}_h\in\mathbb{H}_h.$$

Clearly for $\mathbf{V}_h=(\mathbf{v}_h,\xi_{ph})\in\mathbb{H}_h$, we have:

$$\begin{aligned}-\mathbf{L}(\mathbf{V}_h) &= -(\mathbf{f},\mathbf{v}_h)_\Omega\\ &= -(\mathbf{f},\mathbf{v}_h)_{\Omega_f}-(\mathbf{f},\xi_{ph})_{\Omega_p}\\ &= (\nabla\cdot\sigma_f(\mathbf{u},p),\mathbf{v}_h)_{\Omega_f}+(\nabla\cdot\sigma_p(\eta_p,p_p),\xi_{ph})_{\Omega_p}-\mu(\mathbf{K}^{-1}\mathbf{u},\mathbf{v}_h)_{\Omega_p}-(\nabla p,\mathbf{v}_h)_{\Omega_p},\end{aligned}$$

and, by using Green's formula on each $K\in\mathcal{T}_h$ and conditions (10)–(13), we obtain

$$\begin{aligned}-(\mathbf{f},\mathbf{v}_h)_\Omega &= -\mathbf{A}_h(\mathbf{U},\mathbf{V}_h)-\mathbf{B}_h(\mathbf{V}_h,\mathbf{P})-2\mu\sum_{E\in\mathcal{E}_h(\Omega_f^+)}\int_E \mathbf{n}_E\cdot\mathbf{D}(\mathbf{u})\cdot[\mathbf{v}_h]_E ds\\ &\quad+\sum_{E\in\mathcal{E}_h(\Omega_f^+\cup\Omega_p)}\int_E p[\mathbf{v}_h\cdot\mathbf{n}_E]_E ds+\sum_{E\in\mathcal{E}_h(\partial\Omega_p)}\int_E p_p[\mathbf{v}_h\cdot\mathbf{n}_E]_E ds\\ &\quad-\sum_{E\in\mathcal{E}_h(\Omega_p^+)}\int_E 2\mu_p\mathbf{n}_E\cdot\mathbf{D}(\eta_p)\cdot[\xi_{ph}]_E ds+\alpha\sum_{E\in\mathcal{E}_h(\bar\Omega_p)}\int_E p_p[\xi_{ph}\cdot\mathbf{n}_E]_E ds\\ &\quad+\sum_{E\in\mathcal{E}_h(\bar\Omega_p)}\int_E \lambda_p\nabla\cdot\eta_p[\xi_{ph}\cdot\mathbf{n}_E]ds\end{aligned}\tag{57}$$

We denote by $R_1(\mathbf{V}_h)$, $R_2(\mathbf{V}_h)$, $R_3(\mathbf{V}_h)$, $R_4(\mathbf{V}_h)$, $R_5(\mathbf{V}_h)$ and $R_6(\mathbf{V}_h)$ the last six terms in the right-hand side of Eq. (57), so that

$$\mathbb{A}_h(\mathbf{U},\mathbf{V}_h)+\mathbf{B}_h(\mathbf{V}_h,\mathbf{P})-\mathbf{L}(\mathbf{V}_h)=\sum_{i=1}^{6}R_i(\mathbf{V}_h).\tag{58}$$

In order to evaluate the six face integrals, let us introduce two projection operators in the following.

For any $K\in\mathcal{T}_h$ and $E\in\mathcal{E}(K)$, denote by $P_0(E)$ the constant space of the restrictions to E and π_E the projection operator from $L^2(E)$ onto $P_0(E)$ such that,

$$\int_E \pi_E(v)ds=\int_E v ds.\tag{59}$$

The operator π_E has the property [31]:

$$\|v-\pi_E(v)\|_{0,E}\le C_{14}h_E^{1/2}|v|_{1,K},\qquad \forall v\in H^1(K),\tag{60}$$

where C_{14} depends only on σ_0 and d.

For any $\mathbf{v}\in[L^2(E)]^d$, we let $\Pi_E(\mathbf{v})$ be the function in $[P_0(E)]^d$ such that $(\Pi_E(\mathbf{v}))_i:=\pi_E(v_i)$, $1\le i\le d$. Using inequality (60), we obtain:

$$\|\mathbf{v}-\Pi_E(\mathbf{v})\|_{0,E}\le C_{14}h_E^{1/2}|\mathbf{v}|_{1,K}\forall \mathbf{v}\in[H^1(K)]^d.\tag{61}$$

Then we have the following lemma.

Lemma 5.3. *There holds*

$$\begin{aligned}&|\mathbb{A}_h(\mathbf{U},\mathbf{V}_h)+\mathbf{B}_h(\mathbf{V}_h,\mathbf{P})-\mathbf{L}(\mathbf{V}_h)|\\ &\qquad\le C_{15}h\left(|\mathbf{u}|_{2,\Omega_f}+|\eta_p|_{2,\Omega_p}+|p|_{1,\Omega_f}+|p|_{1,\Omega_p}\right),\forall \mathbf{V}_h\in\mathbb{H}_h,\end{aligned}$$

where C_{15} is a constant depending only on d, σ_0 and μ.

Proof. Let $\mathbf{V}_h\in\mathbb{H}_h$. We begin with an estimate for the first term $R_1(\mathbf{V}_h)$. For any $E\in\mathcal{E}_h(\Omega_f^+)$, there exists at last one elements $K\in\mathcal{T}_h^f$ such that $E\in\mathcal{E}_h(K)$. Then, from condition (59), Hölder's inequality and inequality (61), it follows that:

$$\begin{aligned}\int_E \mathbf{n}_E\cdot\mathbf{D}(\mathbf{u})\cdot[\mathbf{v}_h]_E ds &= \int_E(\mathbf{n}_E\cdot\mathbf{D}(\mathbf{u})-\Pi_E(\mathbf{n}_E\cdot\mathbf{D}(\mathbf{u})))\cdot[\mathbf{v}_h]_E ds\\ &\le \|h_E^{1/2}((\mathbf{I}-\Pi_E)(\mathbf{n}_E\cdot\mathbf{D}(\mathbf{u}))\|_{0,E}\|h_E^{-1/2}[\mathbf{v}_h]_E\|_{0,E}\\ &\le \sqrt{d}C_{14}h_E|\mathbf{u}|_{2,K}\|h_E^{-1/2}[\mathbf{v}_h]_E\|_{0,E}\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}|R_1(\mathbf{V}_h)| &\leq 2\mu\sqrt{d}C_{14}\left(\sum_{E\in\mathcal{E}_h(\Omega_f^+)} h_E^2|\mathbf{u}|_{2,E}^2\right)^{1/2}\left(\sum_{E\in\mathcal{E}_h(\Omega_f^+)} \|h_E^{-1/2}[\mathbf{v}_h]_E\|_{0,E}^2\right)^{1/2}\\ &\leq 2\mu\sqrt{d(d+1)}C_{14}\left(\sum_{K\in\mathcal{T}_h^f} h_K^2|\mathbf{u}|_{2,K}^2\right)^{1/2}(\mathbf{J}_{\Omega_f^+}(\mathbf{v}_h,\mathbf{v}_h))^{1/2}\\ &\leq 2\mu\sqrt{d(d+1)}C_{14}h|\mathbf{u}|_{2,\Omega_f}\|\mathbf{v}_h\|_{\mathbf{H}_h}. \end{aligned} \tag{62}$$

For the terms $R_i(\mathbf{V}_h)$, $i\in\{2,3,4,5,6\}$, use the same techniques as in the proof of the bound for $R_1(\mathbf{V}_h)$ and we have:

$$|R_2(\mathbf{V}_h)| \leq \sqrt{d+1}C_{14}h\left(|p|_{1,\Omega_f}+|p|_{1,\Omega_p}\right)\|\mathbf{v}_h\|_{\mathbf{H}_h} \tag{63}$$

$$|R_3(\mathbf{V}_h)| \leq \sqrt{d+1}C_{14}h|p|_{1,\Omega_p}\|\mathbf{v}_h\|_{\mathbf{H}_h} \tag{64}$$

$$|R_4(\mathbf{V}_h)| \leq 2\mu_{\max}\sqrt{d(d+1)}C_{14}h|\eta_p|_{2,\Omega_p}\|\xi_{ph}\|_{\mathbf{X}_{ph}} \tag{65}$$

$$|R_5(\mathbf{V}_h) \leq \alpha\sqrt{d+1}C_{14}h|p|_{1,\Omega_p}\|\xi_{ph}\|_{\mathbf{X}_{ph}} \tag{66}$$

$$|R_6(\mathbf{V}_h)| \leq \lambda_{\max}\sqrt{d+1}C_{14}h|\eta_p|_{2,\Omega_p}\|\xi_{ph}\|_{\mathbf{X}_{ph}}. \tag{67}$$

Then, the lemma follows by combining inequalities (62)–(67). □

From Lemmas 5.1 and 5.2, inequalities (55)–(56) and Lemma 5.3, now we derive the following convergence theorem:

Theorem 5.1 (*A-priori Error Estimation*)**.** *Let the solution* $(\mathbf{U},\mathbf{P})$ *of problem* (19) *satisfies the smoothness assumption* $(i)-(iii)$. *Let* $(\mathbf{U}_h,\mathbf{P}_h)$ *be the solution of the discrete problem* (38)*. Then, there exists a positive constant C depending on d,* μ, α, σ_0, $k_{\max}$ *and* $k_{\min}$ *such that:*

$$\|\mathbf{U}-\mathbf{U}_h\|_{\mathbb{H}_h\cup\mathbb{H}}+\|\mathbf{P}-\mathbf{P}_h\|_{\mathbb{M}_h\cup\mathbb{M}} \leq Ch\left(|\mathbf{u}|_{2,\Omega_f}+|\mathbf{u}|_{2,\Omega_p}+|\eta_p|_{2,\Omega_p}+|p|_{1,\Omega_f}+|p|_{1,\Omega_p}\right).$$

6. Summary

We have introduced and analyzed a-priori error estimation for nonconforming approximations of the Stokes/Biot fluid–poroelastic structure interaction model on isotropic meshes. Under a small data assumption, existence and uniqueness results have been proved, and an optimal a-priori error estimate has been derived. We end this paper by mentioning that further developments, including computational aspects of (38), a-posteriori error analysis, adaptivity and numerical results, will be reported in separate works.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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