Residual-based a posteriori error estimates for a nonconforming finite element discretization of the Stokes-Darcy coupled problem: Isotropic discretization

Wilfrid Houedanou, Bernardin Ahounou, Serge Nicaise

Université d'Abomey-Calavi (UAC), Bénin

Institut de Mathématiques et de Sciences Physiques (IMSP), Bénin

Laboratoire de Mathématiques et ses Applications de Valenciennes (LAMAV), France

email : wilfrid.houedanou@imsp-uac.org

Abstract

In this work we develop an a posteriori error analysis of a non-conforming mixed finite element method for the coupling of fluid flow with a porous medium . The approach utilizes the same non-conforming Crouzeix-Raviart element discretization on the entire domain [3].

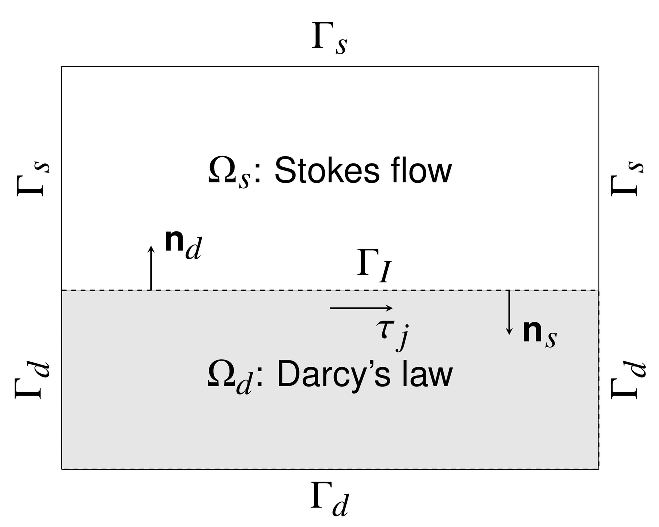
1. Introduction

The coupled Darcy-Stokes problem is a well-known and well-studied problem, with many important applications:

- Understand how beach are formed and how is the dynamic caused by the water.
- Simulate the effect of flooding in dry areas...

We refer to the nice overview [2] and the references therein for its physical background, modeling, and standard numerical methods.

2. Model



- $\Omega \subset \mathbb{R}^N$ is bounded domain, with $N = 2$ or $N = 3$.
- $\Omega_s, \Omega_d \subset \Omega$ such that $\Omega_s = \Omega \backslash \overline{\Omega}_d$. Let $\Omega_s^+ = \Omega_s \cup \Gamma_s$.
- In Ω, we denote by $\mathbf{u}$ the fluid velocity and by p the pressure.
  - ➢ **Stokes equations**: In Ω_s, $(\mathbf{u}, p)$ satisfy

$$\begin{cases} -2\mu \operatorname{div} \mathbf{D}(\mathbf{u}) + \nabla p = \mathbf{f} & \text{in} \quad \Omega_s, \\ \operatorname{div} \mathbf{u} = g & \text{in} \quad \Omega_s, \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \Gamma_s, \end{cases} \tag{1}$$

  - ➢ **Darcy equations**: In Ω_d, $(\mathbf{u}, p)$ satisfy

$$\begin{cases} \mu \mathbf{K}^{-1}\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega_d, \\ \operatorname{div} \mathbf{u} = g & \text{in} \quad \Omega_d, \\ \mathbf{u}\cdot\mathbf{n}_d = 0 & \text{on} \quad \Gamma_d. \end{cases} \tag{2}$$

- $\mu > 0$ is the fluid viscosity, $\mathbf{D}$ the deformation rate tensor defined by

$$\mathbf{D}(\psi)_{ij} := \frac{1}{2}\left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right), \ 1 \leqslant i, j \leqslant N,$$

- $\mathbf{K} : x \in \Omega_d \mapsto \mathbf{K}(x) \in \mathbb{R}^{N\times N}$ represents the rock permeability. $\mathbf{f} \in [L^2(\Omega)]^N$ is a term related to body forces and $g \in L^2(\Omega)$ a source or sink term satisfying the compatibility condition: $(g, 1)_\Omega \stackrel{\text{def}}{=} \int_\Omega g(x)\cdot 1 = 0.$
  - ➢ **Interface conditions on Γ_I** (Beavers-Joseph-Saffman law):

$$\mathbf{u}_s\cdot\mathbf{n}_s + \mathbf{u}_d\cdot\mathbf{n}_d = 0 \text{ (mass conservation)}, \tag{3}$$

$$p_s - 2\mu\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_s)\cdot\mathbf{n}_s = p_d \text{ (balance of normal forces)}, \tag{4}$$

$$\frac{\sqrt{\kappa_j}}{\alpha_1} 2\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_s)\cdot\tau_j = -\mathbf{u}_s\cdot\tau_j, \ j = 1,\ldots,N-1, \tag{5}$$

where $\kappa_j = \tau_j\cdot\mathbf{K}\cdot\tau_j$ and α_1 is a parameter determined by experimental evidence.

3. Weak formulation

➢ **Spaces:**

- $\mathbf{H} := \left\{\mathbf{v} \in \mathbf{H}(\operatorname{div}, \Omega) : \mathbf{v}_{|\Omega_s} \in [H^1(\Omega_s)]^N, \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_s\right.$ and $\mathbf{v}\cdot\mathbf{n}_d = 0$ on $\Gamma_d\}$, endowed with the norm $\| \mathbf{v} \|_{\mathbf{H}} := \left(|\mathbf{v}|^2_{1,\Omega_s} + \| \mathbf{v} \|^2_{L^2(\Omega_d)} + \| \operatorname{div} \mathbf{v} \|^2_{L^2(\Omega_d)}\right)^{1/2}$, where $|.|_{m,W}$ is the usual semi-norm of $H^m(W)$, $m \in \mathbb{N}$ and W is a bounded domain of $\mathbb{R}^N$.
- $Q = L^2_0(\Omega) := \left\{q \in L^2(\Omega) : \int_\Omega q(x)dx = 0\right\}$.

➢ **Forms:**

- $\mathbf{a}(\mathbf{u}, \mathbf{v}) \ := \ 2\mu(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_s + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} \left\langle \mathbf{u}_s\cdot\tau_j, \mathbf{v}_s\cdot\tau_j\right\rangle_{\Gamma_I} + \mu\left(\mathbf{K}^{-1}\mathbf{u}, \mathbf{v}\right)_d$ and $\mathbf{b}(\mathbf{v}, q) := -(q, \operatorname{div}\mathbf{v})_\Omega$;
- $L(\mathbf{v}) := (\mathbf{f}, \mathbf{v})_\Omega$ and $G(q) := -(g, q)_\Omega$.

➢ **Formulation:** Find $(\mathbf{u}, p) \in \mathbf{H}\times Q$ such that

$$\begin{cases} \mathbf{a}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, p) = L(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{H}, \\ \mathbf{b}(\mathbf{u}, q) = G(q), & \forall q \in Q. \end{cases} \tag{6}$$

Theorem 1 *If $\mathbf{f} \in [L^2(\Omega)]^N$ and $g \in L^2_o(\Omega)$, there exists a unique solution $(\mathbf{u}, p) \in \mathbf{H}\times Q$ to problem (6).*

4. Finite element discretization

We use a variant of the nonconforming Crouzeix-Raviart piecewise linear finite element approximation for the velocity and piecewise constant approximation for the pressure.

➢ **Mesh and notation:**

- $\{\mathcal{T}_h\}_h$: Family of regular triangulations of Ω with nondegenerate elements (i.e. triangles for $N = 2$ and tetrahedra for $N = 3$).
- h_T the diameter of T and ρ_T the diameter of the largest ball inscribed into T and set $h = \max_{T\in\mathcal{T}_h} h_T$, and $\sigma_h = \max_{T\in\mathcal{T}_h} \frac{h_T}{\rho_T}$.
- $\mathcal{E}(T) =$ set of edges/faces of T and $\mathcal{E}_h = \bigcup_{T\in\mathcal{T}_h} \mathcal{E}(T)$.
- For $\mathcal{A} \subset \overline{\Omega}$ we define $\mathcal{E}_h(\mathcal{A}) = \{E \in \mathcal{E}_h : E \subset \mathcal{A}\}$.
- For any $E \in \mathcal{E}_h$ and any piecewise continuous function φ, we denote by $[\varphi]_E$ its jump across E in the direction of $\mathbf{n}_E$: $[\varphi]_E(x) :=$

$$\begin{cases} \lim_{t\to 0+} \varphi(x + t\mathbf{n}_E) - \lim_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for } E \in \mathcal{E}_h(\Omega) \\ - \lim_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for } E \in \mathcal{E}_h(\partial\Omega) \ . \end{cases}$$

➢ **Discrete spaces:**

- $\mathbf{H}_h := \left\{\mathbf{v}_h : \mathbf{v}_{h|T} \in [\mathbb{P}^1(T)]^N \ \forall T \in \mathcal{T}_h, ([\mathbf{v}_h]_E, \mathbf{1})_E = 0, \right.$ $\forall E \in \mathcal{E}_h(\Omega_s^+), ([\mathbf{v}_h\cdot\mathbf{n}_E]_E, 1)_E = 0, \forall E \in \mathcal{E}_h(\Omega_d) \cup \mathcal{E}_h(\partial\Omega_d)\}$, endowed with the norm: $\| \mathbf{v} \|_h := \left(\sum_{T\in\mathcal{T}_h^s} |\mathbf{v}|^2_{1,T} + \sum_{j=1}^{N-1}\langle \mathbf{v}_s\cdot\tau_j, \mathbf{v}_s\cdot\tau_j\rangle_{\Gamma_I} + \| \mathbf{v} \|^2_d + \sum_{T\in\mathcal{T}_h^d} \| \operatorname{div}\mathbf{v} \|^2_T + \mathbf{J}(\mathbf{v}, \mathbf{v})\right)^{1/2}$, where $\mathbf{J}(\mathbf{u}, \mathbf{v}) = \mathbf{J}_{\Omega_s^+}(\mathbf{u}, \mathbf{v}) + \mathbf{J}_{\Omega_d}(\mathbf{u}, \mathbf{v}) + \mathbf{J}_{\partial\Omega_d}(\mathbf{u}, \mathbf{v})$ is a penalizing term.
- $Q_h := \left\{q_h \in L^2_0(\Omega) : q_{h|T} \in \mathbb{P}^o(T) \ \forall T \in \mathcal{T}_h\right\}$.

➢ **Approximation of forms:**

- $\mathbf{a}_h(\mathbf{u}, \mathbf{v}) := 2\mu \sum_{T\in\mathcal{T}_h^s}(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_T + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}}\langle \mathbf{u}_s\cdot\tau_j, \mathbf{v}_s\cdot\tau_j\rangle_{\Gamma_I} + \mu(\mathbf{K}^{-1}\mathbf{u}, \mathbf{v})_{\Omega_d}, \ \forall \mathbf{u}, \mathbf{v} \in \mathbf{H} + \mathbf{H}_h$
- $\mathbf{b}_h(\mathbf{v}, q) := - \sum_{T\in\mathcal{T}_h} (q, \operatorname{div}\mathbf{v})_T$.

➢ **Discrete problem:** Find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times Q_h$ such that

$$\begin{cases} \mathbf{a}_h(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{b}_h(\mathbf{v}_h, p_h) + \mathbf{J}(\mathbf{u}_h, \mathbf{v}_h) = L(\mathbf{v}_h), \forall \mathbf{v}_h \in \mathbf{H}_h, \\ \mathbf{b}_h(\mathbf{u}_h, q_h) = G(q_h), \forall q_h \in Q_h. \end{cases} \tag{7}$$

Theorem 2 (***Existence result and a priori error estimates***) *There exists a unique solution $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times Q_h$ to problem (7) and if the solution $(\mathbf{u}, p) \in \mathbf{H}\times Q$ of the continuous problem (6) is smooth enough, then we have*

$$\| \mathbf{u} - \mathbf{u}_h \|_h + \| p - p_h \| \lesssim h\left(|\mathbf{u}_s|_{2,\Omega_s} + |\mathbf{u}_d|_{2,\Omega_d} + |p_s|_{1,\Omega_s} + |p_d|_{1,\Omega_d}\right)$$

Here and below, the abbreviation $x \lesssim y$ stand for $x \leqslant cy$, with c a positive constant independent of x, y and $\mathcal{T}_h$.

5. Some technical results

Theorem 3 (***Helmholtz decomposition***): *Any $\mathbf{v} \in \mathbf{H}$ admits the Helmholtz type decomposition*

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \tag{8}$$

where $\mathbf{v}_0, \mathbf{v}_1 \in \mathbf{H}$ but satisfying $\mathbf{v}_0 \in H^1(\Omega)^N$,

$$\mathbf{v}_1 = \begin{cases} \mathbf{0} & in \quad \Omega_s, \\ \operatorname{curl}\psi & in \quad \Omega_d, \end{cases} \tag{9}$$

where $\psi \in H^1_0(\Omega_d)$ if $N = 2$, while $\psi \in H^1(\Omega_d)^3 \cap H_0(\operatorname{curl}, \Omega_d)$ if $N = 3$, with the estimate

$$\|\mathbf{v}_0\|_{1,\Omega} + \|\psi\|_{1,\Omega_d} \lesssim \|\mathbf{v}\|_{\mathbf{H}}. \tag{10}$$

Theorem 4 (***Regularity result***): *Let $(\mathbf{u}, p) \in \mathbf{H}\times Q$ be the unique solution of (6). If $\mathbf{f} \in H(\operatorname{curl}, \Omega_d)$ and $\mathbf{K} \in [C^{0,1}(\bar{\Omega}_d)]^{N\times N}$, then there exists $\epsilon > 0$ such that*

$$\mathbf{u} \in [H^{\frac{1}{2}+\epsilon}(\Omega_d)]^N.$$

Theorem 5 (***Estimation of the non conforming error***): *For any $\mathbf{u}_h \in \mathbf{H}_h$ we have*

$$\inf_{\mathbf{w}_h\in\mathbf{H}\cap\mathbf{H}_h} \|\mathbf{u}_h - \mathbf{w}_h\|^2_h \lesssim \mathbf{J}(\mathbf{u}_h, \mathbf{u}_h). \tag{11}$$

6. Error estimator

➢ The error estimator is expressed in terms of the residual elements,

$$\mathbf{r}_{s,T} = \mathbf{f}_T + 2\mu \operatorname{div}\mathbf{D}(\mathbf{u}_h) - \nabla p_h \text{ in } T \in \mathcal{T}_h^s, \tag{12}$$

$$\mathbf{r}_{d,T} = \mathbf{f}_T - \mu\mathbf{K}^{-1}\mathbf{u}_h - \nabla p_h \text{ in } T \in \mathcal{T}_h^d, \tag{13}$$

where $\mathbf{f}_T$ is a local approximation of $\mathbf{f}$.

➢ **Local estimator:** $\Theta_T := \left(\sum_{i=1}^9 \Theta^2_{i,T}\right)^{1/2}$, for each $T \in \mathcal{T}_h$, where $\Theta_{i,T}$, $i = 1, 2, 3$ are residual terms and $\Theta_{i,T}$, $i = 4, ...9$ are interface terms.

➢ **Global estimator:** $\Theta := \left(\sum_{T\in\mathcal{T}_h} \Theta^2_T\right)^{1/2}$,

7. A posteriori error analysis

Theorem 6 (***Lower error bound***): *Under the assumptions of Theorem 4, the following lower error bound holds:*

$$\Theta_T \lesssim \| \mathbf{u} - \mathbf{u}_h \|_{h,\tilde{\omega}_T} + \| p - p_h \|_{\tilde{\omega}_T} + \sum_{T'\subset\tilde{\omega}_T} \zeta_{T'}, \tag{14}$$

where $\tilde{\omega}_T$ is a finite union of neighboring elements of T and

$$\zeta_T := \begin{cases} h_T \| \mathbf{f} - \mathbf{f}_h \|_T, & \forall T \in \mathcal{T}_h^s, \\ h_T(\| \mathbf{f} - \mathbf{f}_h \|_T + \| \operatorname{curl}(\mathbf{f} - \mathbf{f}_h)\|_T), & \forall T \in \mathcal{T}_h^d, \end{cases}$$

➢ **Sketch of the proof:** In order to derive the local lower bounds, we proceed similarly as in [1] by applying inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions.

Theorem 7 (***Upper error bound***): *Under the assumptions of Theorem 4, the a posteriori error estimator Θ satisfies*

$$\| \mathbf{u} - \mathbf{u}_h \|_h + \| p - p_h \| \lesssim \Theta + \zeta. \tag{15}$$

where $\zeta := \left(\sum_{T\in\mathcal{T}_h} \zeta^2_T\right)^{1/2}$.

➢ **Sketch of the proof:** We set $\Theta^2 = \Theta_1^2 + \mathbf{J}(\mathbf{u}_h, \mathbf{u}_h)$, where $\Theta_1 := \left\{\sum_{T\in\mathcal{T}_h}\left(\sum_{i=1}^6 \Theta^2_{i,T}\right)\right\}^{1/2}$.
So it suffices to prove the estimations
$\| \mathbf{u} - \mathbf{u}_h \|_h + \| p - p_h \| \lesssim \Theta_1 + \zeta + \inf_{W_h\in\mathbf{H}\cap\mathbf{H}_h\times Q_h} |||U_h - W_h|||_h$,
$\inf_{W_h\in\mathbf{H}\cap\mathbf{H}_h\times Q_h} |||U_h - W_h|||^2_h \lesssim \inf_{v_h\in\mathbf{H}\cap\mathbf{H}_h} \| \mathbf{u}_h - v_h \|^2_h \leq \mathbf{J}(\mathbf{u}_h, \mathbf{u}_h)$.

References

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Perspective: We intend to extend our results to anisotropic meshes [4].

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