Topology and its Applications 322 (2022) 108301

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

C^0-transport of flux geometry

Stephane Tchuiaga [a], Franck Houenou [b,c,*], Carole Madengko [c], Ancille Nguedakumana [d]

[a] *Department of Mathematics, University of Buea, South West Region, Cameroon*
[b] *Department of Mathematics, University of Abomey Calavi, Abomey Calavi, Benin*
[c] *Institut de Mathématiques et de Sciences Physiques, University of Abomey Calavi, Dangbo, Benin*
[d] *Ecole Normale Supérieure de Bujumbura, Burundi*

A R T I C L E   I N F O

Article history:
Received 30 November 2021
Received in revised form 7 October 2022
Accepted 7 October 2022
Available online 12 October 2022

MSC:
53C24
53D05
57S05

Keywords:
Rigidity results
Symplectic manifolds general
Topological properties of groups of homeomorphisms or diffeomorphisms

A B S T R A C T

The goal of this paper is to study, in a large scale point of view, the flux geometry of a closed symplectic manifold (M, ω): namely, the topological counterpart of the flux homomorphism. Using metrics arising from the decomposition of closed 1-forms with respect to an arbitrary linear section $\mathcal{S}$, we generalize the construction of the group of strong symplectic homeomorphisms. The flux homomorphism for symplectomorphisms is extended to a surjective group homomorphism S^0_ω on the group of $\mathcal{S}$-homeomorphisms. We prove that the kernel of S^0_ω is path connected, coincides with the subgroup $Hameo(M, \omega)$ of all Hamiltonian homeomorphisms and investigate the discreteness of the corresponding flux group $\mathcal{S}\Gamma_\omega$. Later on, without appealing to any lifting map, we give an alternative proof of a result from the classical flux geometry saying that any smooth symplectic isotopy in $Ham(M, \omega)$ is a Hamiltonian isotopy. Furthermore under some hypothesis, we prove that any $\mathcal{S}$-topological isotopy in $Hameo(M, \omega)$ is a continuous Hamiltonian isotopy. We also proved that any $\mathcal{S}$-topological isotopy with trivial flux is homotopic to a continuous Hamiltonian isotopy, relatively to fixed endpoints.

0. Introduction

The structures and the dynamics of the group of diffeomorphisms of a symplectic manifold have been studied quite long time by several authors. In particular, inspired by the works of Epstein [5], Herman [7], Mather [8] and Thurston [19], Banyaga [1] studied the group $G_\omega(M)$: the identity component with respect to the C^∞-compact-open topology in the group $\mathrm{Symp}(M, \omega)$ consisted of all symplectic diffeomorphisms of a symplectic manifold (M, ω).

* Corresponding author at: Department of Mathematics, University of Abomey Calavi, Abomey Calavi, Benin.
E-mail addresses: tchuiagas@gmail.com (S. Tchuiaga), rdjeam@gmail.com (F. Houenou), carolemadengko@gmail.com (C. Madengko), nancille@yahoo.fr (A. Nguedakumana).

https://doi.org/10.1016/j.topol.2022.108301
0166-8641/

Although the structures of the groups of volume-preserving diffeomorphisms, symplectic diffeomorphisms, and strictly contact diffeomorphisms of a given oriented manifold are well understood, it is still a big challenge to perform similar studies for the corresponding group of homeomorphisms of the same manifold. This is probably due to the lack of an adapted differentiable structure on the group of homeomorphisms. The latter led researchers to various characterizations, or approximations of homeomorphisms of a compact oriented manifold [4,11].

In view of constructing a framework in which a topological counterpart of the usual flux geometry could be studied, the present work reformulates the definition of the flux homomorphism for symplectomorphisms and extends it to a subgroup of symplectic homeomorphisms of a closed connected symplectic manifold (M,ω). To this end, we consider an arbitrary linear section $\mathcal{S}$ of the natural projection $\pi : \mathcal{Z}^1(M) \to H^1(M,\mathbb{R})$ from the space of closed 1-forms onto the first de Rham's group of a closed symplectic manifold (M,ω) of dimension $2n$. We use the linear section to define the group $\mathcal{S}G_\omega(M)$ of $\mathcal{S}$-homeomorphisms, such that

$$Hameo(M,\omega) \subset \mathcal{S}G_\omega(M) \subset Homeo(M,\Omega),$$

where $\Omega := \dfrac{\omega^n}{n!}$ and $Hameo(M,\omega)$ stands for the group of all Hamiltonian homeomorphisms of (M,ω) [11].

This work extends to C^0-context some results obtained in [1] by studying the group $\mathcal{S}G_\omega(M)$ (no matter the choice of the linear section $\mathcal{S}$). Specifically, we realize the subgroup $Hameo(M,\omega)$ as the kernel of a surjective group homomorphism S^0_ω which extends the usual flux homomorphism to $\mathcal{S}G_\omega(M)$. Of course, the price to pay here resides in how to go round the lack of smoothness, which is crucial in the study of the flux homomorphism in the differentiable context.

In spite of this difficulty, we mainly brought into light the following results on closed connected Lefschetz type symplectic manifolds.

Theorem A. *Let (M,ω) be a closed symplectic manifold of Lefschetz type. Any $\mathcal{S}$-topological isotopy in $Hameo(M,\omega)$ is a continuous Hamiltonian isotopy. In particular, any smooth isotopy in $Hameo(M,\omega)$ is a Hamiltonian flow.*

The last sentence of Theorem A follows from Theorem 2-[13]. This result implies that the usual subgroup $Ham(M,\omega)$ of all Hamiltonian diffeomorphisms is rigid inside $Hameo(M,\omega)$.

Theorem B. *On a closed symplectic manifold of Lefschetz type, any $\mathcal{S}$-topological isotopy with a trivial flux is homotopic to a continuous Hamiltonian isotopy relatively to fixed endpoints.*

Theorem C. *Let (M,ω) be a closed connected symplectic manifold.*

1. *For each $z \in M$, the orbit $\mathcal{O}^H_z$ of any continuous hamiltonian isotopy $H = \{h^t\}_t$ with time-one map identity is null-homologous in $H_1(M,\mathbb{R})$.*
2. *Assume M is of Lefschetz type. Let H be a $\mathcal{S}$-topological isotopy with time one map identity and $z \in M$. If the orbit $\mathcal{O}^H_z$ is null-homologous in $H_1(M,\mathbb{R})$ then H is a continuous hamiltonian isotopy with time-one map identity.*

Theorem C is a generalization of a result found by McDuff-Salamon [9] in the smooth case and its proof does not appeal to the Floer theory.

Theorem D. *Let (M,ω) be a closed symplectic manifold of Lefschetz type. The flux group $\mathcal{S}\Gamma_\omega$ is discrete.*

In the sequel, we recall in Section 1, a relationship between symplectic isotopies and the splitting of closed 1-forms namely, the $\mathcal{S}$-decomposition of symplectic isotopies which includes the Hodge decomposition of symplectic isotopies and $\mathcal{S}$-like geometry. Section 2 briefly deals with the theory of smooth flux geometry by proposing an alternative and easy proof of a rigidity theorem found by Banyaga [1], while Section 3 is devoted to the study of topological flux geometry: after extending the flux in the C^0-context, some properties of its kernel and the topological flux group have been given. Section 4 studies the invariance of homotopies and gives a non-lifting path procedure in C^0-context which is used to prove some of our main results.

1. Preliminaries

Let (M, ω) be a closed symplectic manifold and fix a riemannian metric g on M.

Splitting of closed 1-forms and generators of symplectic isotopies

Consider a linear section $\mathcal{S} : H^1(M, \mathbb{R}) \to \mathcal{Z}^1(M)$ of the natural projection $\pi : \mathcal{Z}^1(M) \to H^1(M, \mathbb{R})$. Each $\alpha \in \mathcal{Z}^1(M)$ splits as:

$$\alpha = \mathcal{S}(\pi(\alpha)) + (\alpha - \mathcal{S}(\pi(\alpha))) \tag{1.1}$$

where $\mathcal{S}(\pi(\alpha))$ is the $\mathcal{S}$-form and $\alpha - \mathcal{S}(\pi(\alpha))$ is the exact part of α.

Let $\mathbb{H}^1(M, \mathcal{S})$ denote the space of all $\mathcal{S}$-forms. Then we have the following direct sum:

$$\mathcal{Z}^1(M) = \mathbb{H}^1(M, \mathcal{S}) \oplus B^1(M),$$

where $B^1(M)$ is the set of all exact forms on M and $\dim(\mathbb{H}^1(M, \mathcal{S})) = \dim(H^1(M, \mathbb{R})) < \infty$, for each linear section $\mathcal{S}$ (see [17]).

Given any symplectic isotopy $\Phi = \{\phi^t\}_t$ the 1-form $\imath_{\dot{\phi}^t}\omega$ splits as $\imath_{\dot{\phi}^t}\omega = dU^t + \mathcal{H}^t$ where U^t is a smooth function and $\mathcal{H}^t$ is a $\mathcal{S}$-form. Let $\mathcal{N}([0, 1] \times M, \mathbb{R})$ denote the vector space of all smooth functions F defined on $[0, 1] \times M$ such that $\int_M F^t \omega^n = 0$, for all $t \in [0, 1]$, and denote by $\mathcal{P}\mathbb{H}^1(M, \mathcal{S})$ the space of all smooth mappings $\mathcal{H} : [0, 1] \longrightarrow \mathbb{H}^1(M, \mathcal{S})$. It is shown in [17], that for each linear section $\mathcal{S}$, the Cartesian product

$$\mathcal{N}([0, 1] \times M, \mathbb{R}) \times \mathcal{P}\mathbb{H}^1(M, \mathcal{S}) =: \mathfrak{T}(M, \omega, \mathcal{S}), \tag{1.2}$$

forms a group called the group of $\mathcal{S}$-generators of elements in $Iso(M, \omega)$, the space of all symplectic isotopies of (M, ω). The product rule on $\mathfrak{T}(M, \omega, \mathcal{S})$ is given by,

$$(U, \mathcal{H}) \bowtie_{\mathcal{S}} (V, \mathcal{K}) = (U + V \circ \phi^{-1}_{(U,\mathcal{H})} + \widetilde{\Delta}(\mathcal{K}, \phi^{-1}_{(U,\mathcal{H})}), \mathcal{H} + \mathcal{K}), \tag{1.3}$$

and the inverse of $(U, \mathcal{H})$, that we denote $\overline{(U, \mathcal{H})}$, is given by

$$\overline{(U, \mathcal{H})} = (-U \circ \phi_{(U,\mathcal{H})} - \widetilde{\Delta}(\mathcal{H}, \phi_{(U,\mathcal{H})}), -\mathcal{H}) \tag{1.4}$$

where $\widetilde{\Delta}_t(\mathcal{H}, \psi_{(\mathcal{U},\mathcal{H})}) = \int_0^t \mathcal{H}^t(\dot{\psi}^s_{(\mathcal{U},\mathcal{H})}) \circ \psi^s_{(\mathcal{U},\mathcal{H})} ds - \dfrac{\int_M \left(\int_0^t \mathcal{H}^t(\dot{\psi}^s_{(\mathcal{U},\mathcal{H})}) \circ \psi^s_{(\mathcal{U},\mathcal{H})} ds \right) \omega^n}{\int_M \omega^n}$ for all $t \in [0, 1]$ (for more details see [4,17]).

L^∞-Hofer-like length and L^∞-Hofer-like distance

Given a symplectic isotopy $\Phi = \{\phi^t\}_t$ generated by $(U, \mathcal{H})$, the L^∞-version of Hofer-like length of Φ is defined by

$$l^\infty_{\kappa,\mathcal{S}}(\Phi) = \max_{t\in[0,1]}(\nu^B(dU^t) + \kappa\|\mathcal{H}^t\|_{L^2}), \tag{1.5}$$

where ν^B is any norm on $B^1(M)$, the coefficient κ is a positive real number and $\|\cdot\|_{L^2}$ is the L^2-norm on $H^1(M,\mathbb{R})$. For more details we refer to [17].

In the sequel of this paper, we take the norm ν^B to be the oscillation norm.

Proposition 1.1. *Consider a closed symplectic manifold (M,ω). Then, for each positive real number κ, different choices of linear section $\mathcal{S}$ result in versions of $l^\infty_{\kappa,\mathcal{S}}$ that are uniformly equivalent.*

Proof. Let $\mathcal{S}$ and $\mathcal{T}$ be two linear sections of the projection $\pi : \mathcal{Z}^1(M) \longrightarrow H^1(M,\mathbb{R})$. Take any symplectic vector field X with $\mathcal{S}$-decomposition $\imath_X\omega = \mathcal{S} + dU$, and $\mathcal{T}$-decomposition $\imath_X\omega = \mathcal{T} + dV$. We will show that there exists $C_1 > 0$ and $C_2 > 0$ such that

$$C_1(osc(V) + \kappa\|\mathcal{T}\|_{L^2}) \leqslant osc(U) + \kappa\|\mathcal{S}\|_{L^2} \leqslant C_2(osc(V) + \kappa\|\mathcal{T}\|_{L^2}).$$

For this end, let us equip $\mathbb{H}^1(M,\mathcal{S})$ (resp. $\mathbb{H}^1(M,\mathcal{T})$ with a basis $\mathbf{B}$ (resp. $\mathbf{B}'$) and denote by $\|\cdot\|^1_{\mathbf{B}}$ (resp. $\|\cdot\|^1_{\mathbf{B}'}$) the norm on $\mathbb{H}^1(M,\mathcal{S})$ (resp. $\mathbb{H}^1(M,\mathcal{T})$). So, we only have to show that

$$C_1(osc(V) + \kappa\|\mathcal{T}\|^1_{\mathbf{B}'}) \leqslant (osc(U) + \kappa\|\mathcal{S}\|^1_{\mathbf{B}}) \leqslant C_2(osc(V) + \kappa\|\mathcal{T}\|^1_{\mathbf{B}'}). \tag{1.6}$$

The inequalities (1.6) follow from similar arguments to those used in [2] with respect to Hodge's decomposition. Note here that the uniqueness of the harmonic part in the Hodge's decomposition is replaced by the fact that $\mathbb{H}^1(M,\mathcal{S}) \cap B^1(M) = \{0\}$ (resp. $\mathbb{H}^1(M,\mathcal{T}) \cap B^1(M) = \{0\}$). □

Remark 1.1. The above Hofer-like lengths induce metrics $D^\infty_{\kappa,\mathcal{S}}$ on the group of symplectic isotopies:

$$D^\infty_{\kappa,\mathcal{S}}((U,\mathcal{H}),(V,\mathcal{K})) := \frac{D_0^{\kappa,\mathcal{S}}((U,\mathcal{H}),(V,\mathcal{K})) + D_0^{\kappa,\mathcal{S}}(\overline{(U,\mathcal{H})},\overline{(V,\mathcal{K})})}{2},$$

where

$$D_0^{\kappa,\mathcal{S}}((U,\mathcal{H}),(V,\mathcal{K})) := \max_{t\in[0,1]}(osc(U^t - V^t) + \kappa\|\mathcal{H}^t - \mathcal{K}^t\|_{L^2}) \qquad \forall\, \kappa > 0.$$

When $\kappa = 1$, the map $D_0^{1,\mathcal{S}}$ (resp., the metric $D^\infty_{1,\mathcal{S}}$) will be simply denoted $D_0^{\mathcal{S}}$ (resp., $D^\infty_{\mathcal{S}}$) and the Hofer-like lengths $l^\infty_{1,\mathcal{S}}$ will be denoted $l^\infty_{\mathcal{S}}$ (no matter the choice of the linear section $\mathcal{S}$) [17].

From now, we will always assume that $\kappa = 1$.

C^0-metric and $(C^0 + L^\infty)$-metric

Let (M,ω) be equipped with a riemannian metric g. We recall the C^0-metric on the group of homeomorphisms $Homeo(M)$, the $(C^0 + L^\infty)$-metric on the set of symplectic isotopies and the uniform norm of closed 1-forms.

- Let d_g denote the induced distance by the riemannian metric g on M. Consider on $Homeo(M)$ the distance

$$d_0(f,h) = \max\left\{ \sup_{x\in M} d_g(f(x),h(x)), \sup_{x\in M} d_g(f^{-1}(x),h^{-1}(x))\right\}, \quad \forall\, f,h \in Homeo(M)$$

and on $\mathcal{P}(Homeo(M), id_M)$, the space of paths in $Homeo(M)$ starting from id_M, the distance

$$\bar{d}(\lambda,\mu) = \max_{t\in[0,1]} d_0(\lambda(t),\mu(t)), \qquad \forall\,\lambda,\ \mu \in \mathcal{P}(Homeo(M), id_M).$$

The topology induced by the distance d_0 (resp. $\bar{d}$) is called the compact-open topology (C^0-topology).

- The $(C^0 + L^\infty)$-topology on $Iso(M,\omega)$ is the topology induced by the distance

$$d_{(C^0+L^\infty)}(\Phi,\Psi) := D^\infty_{\mathcal{S}}((U,\mathcal{H}),(V,\mathcal{K})) + \bar{d}(\Phi,\Psi), \tag{1.7}$$

for any Φ and Ψ generated by $(U,\mathcal{H})$ and $(V,\mathcal{K})$ respectively.

- Any differential form α induces a linear map $\alpha_x : T_xM \longrightarrow \mathbb{R}$, for each $x \in M$. The norm of α_x is given by

$$\|\alpha_x\|^g := \sup\left\{|\alpha_x(X)|, X \in T_xM,\ \|X\|_g = 1\right\}$$

where $\|\cdot\|_g$ is the norm induced on each tangent space by the riemannian metric g. Thus the uniform norm of α is defined as

$$\|\alpha\|_\infty := \sup_{x\in M} \|\alpha_x\|^g.$$

2. Classical flux geometry

Let $\widetilde{G_\omega(M)}$ be the universal cover of $G_\omega(M)$. There is a surjective group homomorphism

$$\begin{aligned} \widetilde{S}_\omega : \widetilde{G_\omega(M)} &\longrightarrow H^1(M,\mathbb{R}) \\ \Phi &\longmapsto \left[\int_0^1 \mathcal{H}^t dt\right], \end{aligned}$$

called the flux homomorphism (see [1]), where $\mathcal{H}$ is the $\mathcal{S}$-form in the $\mathcal{S}$-decomposition of Φ. The image of the first homotopy group $\pi_1(G_\omega(M))$ under $\widetilde{S}_\omega$ is denoted by Γ_ω. It is known that Γ_ω is a discrete group (see [10]). The homomorphism $\widetilde{S}_\omega$ induces an epimorphism S_ω from $G_\omega(M)$ onto $H^1(M,\mathbb{R})/_{\Gamma_\omega}$.

An advantage of redefining the flux homomorphism as above is the fact that we can give an alternative proof of the following well-known result from the theory of classical flux geometry without appealing to any lifting map from $\widetilde{G_\omega(M)}$ onto $G_\omega(M)$, but just using C^0-arguments.

Lemma 2.1 *(Smooth non-lifting paths procedure). Let $\Phi = \{\phi^t\}_t$ be a smooth symplectic isotopy generated by $(0,\mathcal{H})$ and consider for each t the reparametrized isotopy $\bar{\phi}^t = \{\phi^{st}\}_s$. Then, the following map is continuous*

$$\begin{aligned} [0,1] &\longrightarrow (Iso(M,\omega), d_{(C^0+L^\infty)}) \\ t &\longmapsto \bar{\phi}^t. \end{aligned} \tag{2.1}$$

Proof. Following some approximation lemmas in [4,17] one can find a positive finite constant C^Φ such that for all u, and v in $[0,1]$

$$d_{(C^0+L^\infty)}(\bar{\phi}^u, \bar{\phi}^v) \leqslant C^\Phi |u-v|. \tag{2.2}$$

Indeed, from the proof of Lemma 3.1-[4], it follows that there exists a positive constant C_1^{Φ} such that, for all u, and v

$$\bar{d}(\bar{\phi}^u, \bar{\phi}^v) \leqslant C_1^{\Phi}|u - v|. \tag{2.3}$$

Besides, since the path $\bar{\phi}^u = \{\phi^{su}\}_s$ is generated by the smooth family $\mathcal{K}^u : s \longmapsto u\mathcal{H}^{(su)}$, then

$$\begin{aligned}\|\mathcal{K}^u(s) - \mathcal{K}^v(s)\|_{L^2} &\leqslant \|u\mathcal{H}^{(su)} - v\mathcal{H}^{(su)}\|_{L^2} + \|v\mathcal{H}^{(su)} - v\mathcal{H}^{(sv)}\|_{L^2}, \\ &\leqslant \|u\mathcal{H}^{(su)} - v\mathcal{H}^{(su)}\|_{L^2} + vsLip(\mathcal{H}^{(\cdot)})|u - v|, \qquad \forall\, u, v \in [0,1]\end{aligned} \tag{2.4}$$

where $Lip(\mathcal{H}^{(\cdot)})$ is the Lipschitz constant of the smooth map $t \longmapsto \mathcal{H}^t$. Thus, for all u, and v

$$\max_s \|\mathcal{K}^u(s) - \mathcal{K}^v(s)\|_{L^2} \leqslant \left(\max_s \|\mathcal{H}^s\|_{L^2} + Lip(\mathcal{H}^{(\cdot)})\right)|u - v|. \quad \square \tag{2.5}$$

Using Lemma 2.1, one can give an alternative proof of the following well-known result (see [1]) from classical flux geometry without appealing to any lifting map from $\widetilde{G_\omega(M)}$ onto $G_\omega(M)$.

Theorem 2.1. *[1]*

Let (M, ω) be a closed symplectic manifold and $Ham(M, \omega)$ be its group of Hamiltonian diffeomorphisms. Any symplectic isotopy in $Ham(M, \omega)$ is a Hamiltonian isotopy.

Proof. Consider $\Phi = \{\phi^t\}_t$ as a symplectic isotopy in $Ham(M, \omega)$ with generator $(U, \mathcal{H})$.

Step (1/2): Let $\Phi = \rho \circ \nu$ be the $\mathcal{S}$-decomposition of Φ, where $\rho = \{\rho^t\}_t$ is an $\mathcal{S}$-isotopy and $\nu = \{\nu^t\}_t$ is a Hamiltonian isotopy [17]. For each t, the diffeomorphisms ν^t and $\phi^t = \rho^t \circ \nu^t$ are Hamiltonian, therefore $\rho^t \in Ham(M, \omega)$ for all t. It follows from the assumption that for each fixed t, the path $\bar{\rho}^t : s \longmapsto \rho^{(st)}$ has its flux in Γ_ω because Φ is a smooth path in $Ham(M, \omega)$. Combining (2.2) with the continuity of the usual flux homomorphism with respect to the metric $d_{(C^0 + L^\infty)}$, leads to the following continuous maps

$$\begin{aligned}[0,1] &\longrightarrow (Iso(M, \omega), d_{(C^0+L^\infty)}) \longrightarrow (H^1(M, \mathbb{R}), \|\cdot\|_{L^2}), \\ t &\longmapsto \qquad\qquad \bar{\phi}^t \qquad\qquad \longmapsto \qquad \widetilde{S}_\omega(\bar{\phi}^t).\end{aligned} \tag{2.6}$$

Thus, the map $t \longmapsto \widetilde{S}_\omega(\bar{\rho}^t)$ is continuous from the connected topological space $[0, 1]$ into the discrete topological space Γ_ω. Hence, the latter map is constant, namely, $\widetilde{S}_\omega(\bar{\rho}^t) = \widetilde{S}_\omega(\bar{\rho}^0) = 0$ for all t. That is, $\left[\int_0^t \mathcal{H}^s ds\right] = 0$ for all t.

Step (2/2): From $\left[\int_0^t \mathcal{H}^s ds\right] = 0$, we derive that for each t there exists a smooth mean-zero real-valued function f^t on M such that $\int_0^t \mathcal{H}^s ds = df^t$. Therefore the map $t \longmapsto f^t$ is smooth. The permutation of derivatives yields

$$\mathcal{H}^t = \frac{d}{dt}(df^t) = d\left(\frac{d}{dt} f^t\right) \in B^1(M), \qquad \forall\, t. \tag{2.7}$$

Formula (2.7) tells us that $\mathcal{H}^t \in \left(\mathbb{H}^1(M, \mathcal{S}) \cap B^1(M)\right)$ for all t, while the latter intersection is the trivial set. Hence, we must have $\mathcal{H}^t = 0$, for each t. Therefore, the generator of Φ is of the form $(U, 0)$, i.e., Φ is a Hamiltonian isotopy. $\square$

3. Topological flux geometry

In this section we define and study the analogue of the classical flux homomorphism $\widetilde{S}_\omega$ on $\mathcal{P}(Homeo(M), id_M)$. This is motivated by the intention to extend classical flux geometry to the set of isotopies through continuous maps on (M, ω). The following theorem is a core of the motivation.

Theorem 3.1. *(C^0-flux transport). Let (M, ω) be a closed symplectic manifold of dimension $2n$ with associated normalized volume form $\Omega = \frac{\omega^n}{n!}$. Let $(\Phi_i)_i = (\{\phi_i^t\}_t)_i$, be a sequence of symplectic isotopies, $\Phi = \{\phi^t\}_t$ a symplectic isotopy and $\Psi = \{\psi^t\}_t$ be an arbitrary isotopy. Assume that,*

- $\Phi_i \longrightarrow \Psi$ *uniformly*
- $\widetilde{S}_\omega(\Phi_i) \xrightarrow{\|\cdot\|_{L^2}} \widetilde{S}_\omega(\Phi)$.

Then, we have for all closed 1-form α,

$$\int_M \left(\int_{\mathcal{O}^\Psi_{(.)}} \alpha \right) \omega^n = -\frac{1}{(n-1)!} \langle \widetilde{S}_\omega(\Phi), [\omega^{n-1} \wedge \alpha] \rangle \tag{3.1}$$

where $\int_{\mathcal{O}^\Psi_{(.)}} \alpha : z \longmapsto \int_{\mathcal{O}^\Psi_z} \alpha$ and $\mathcal{O}^\Psi_z$ is the orbit of z under Ψ.

Proof. Equip M with a riemannian metric g, let $r(g)$ be its injectivity radius, and assume that $\bar{d}(\Psi, \Phi_i) \leqslant \frac{r(g)}{2}$ for i sufficiently large. From the latter, for each $z \in M$, the points $\psi^t(z)$ and $\phi_i^t(z)$ can be connected through a unique minimizing geodesic $\chi_z^{i,t}$: that is, the curves $\mathcal{O}^\Psi_z$, $\mathcal{O}^{\Phi_i}_z$, and $\chi_z^{i,1}$ form the boundary of a 2-chain $\Sigma_z^{(\Phi_i, \Psi)}$. We derive from Stokes' theorem that

$$0 = \int_{\Sigma_z^{(\Phi_i,\Psi)}} d\alpha = \int_{\partial\Sigma_z^{(\Phi_i,\Psi)}} \alpha, \qquad \forall\, \alpha \in \mathcal{Z}^1(M),$$

which implies that for i sufficiently large

$$\left| \int_{\mathcal{O}^\Psi_z} \alpha - \int_{\mathcal{O}^{\Phi_i}_z} \alpha \right| = \left| \int_{\chi_z^{i,1}} \alpha \right| \leqslant \|\alpha\|_\infty \bar{d}(\Psi, \Phi_i), \qquad \forall\, \alpha \in \mathcal{Z}^1(M),\ \forall\, z \in M.$$

Note that to obtain the above inequality, use have been made of the fact that the riemannian length of any minimizing geodesic is bounded from above by the riemannian distance between its endpoints. Thus, it follows that for i sufficiently large

$$\left| \int_M \left(\int_{\mathcal{O}^\Psi_{(.)}} \alpha \right) \frac{\omega^n}{n!} - \int_M \left(\int_{\mathcal{O}^{\Phi_i}_{(.)}} \alpha \right) \frac{\omega^n}{n!} \right| \leqslant \|\alpha\|_\infty \bar{d}(\Psi, \Phi_i) \int_M \frac{\omega^n}{n!}, \qquad \forall\, \alpha \in \mathcal{Z}^1(M). \tag{3.2}$$

For each i, define the function

$$\begin{aligned}\mathcal{F}_\alpha^{\Phi_i} : [0,1] &\longrightarrow C(M,\mathbb{R})\\ t &\longmapsto \mathcal{F}_\alpha^{\Phi_i}(t) = \int_0^t (\phi_i^s)^*(\imath_{\dot\phi_i^s}\alpha)ds\end{aligned}$$

For any closed 1-form α and any $z \in M$ one has $\int_{\mathcal{O}_z^\Phi} \alpha = \mathcal{F}_\alpha^\Phi(1)(z)$, therefore

$$\int_M \mathcal{F}_\alpha^{\Phi_i}(1)\Omega = \int_M \left(\int_0^1 (\phi_i^t)^*(\imath_{\dot\phi_i^t}\alpha)dt\right)\frac{\omega^n}{n!} = \frac{1}{n!}\int_M\int_0^1 (\phi_i^t)^*\big((\imath_{\dot\phi_i^t}\alpha)\omega^n\big)dt. \tag{3.3}$$

Since $\alpha\wedge\omega^n = 0$ then $0 = (\phi_i^t)^*\big(\imath_{\dot\phi_i^t}(\alpha\wedge\omega^n)\big) = (\phi_i^t)^*\big((\imath_{\dot\phi_i^t}\alpha)\omega^n\big) + n(\phi_i^t)^*\big((\imath_{\dot\phi^t}\omega)\wedge\alpha\wedge\omega^{n-1}\big)$.

Thus

$$\begin{aligned}\int_M \mathcal{F}_\alpha^{\Phi_i}(1)\Omega &= -\frac{1}{(n-1)!}\int_M \left(\int_0^1 (\phi_i^t)^*\big((\imath_{\dot\phi_i^t}\omega)\big)\wedge(\phi_i^t)^*\big(\alpha\wedge\omega^{n-1}\big)dt\right)\\ &= -\frac{1}{(n-1)!}\int_M \left(\int_0^1 (\phi_i^t)^*\big((\imath_{\dot\phi_i^t}\omega)\big)\wedge(\phi_i^t)^*\big(\alpha\big)dt\right)\wedge\omega^{n-1}\\ &= -\frac{1}{(n-1)!}\int_M \left(\int_0^1 (\phi_i^t)^*\big((\imath_{\dot\phi_i^t}\omega)\big)\wedge\big(\alpha + d\mathcal{F}_\alpha^{\Phi_i}(t)\big)dt\right)\wedge\omega^{n-1}\end{aligned}$$

Using the fact that $(\phi_i^t)^*\alpha - \alpha = d\mathcal{F}_\alpha^{\Phi_i}(t)$ one has

$$\begin{aligned}\int_M \mathcal{F}_\alpha^{\Phi_i}(1)\Omega &= -\frac{1}{(n-1)!}\int_M \left(\int_0^1 (\phi_i^t)^*\big((\imath_{\dot\phi_i^t}\omega)\big)\wedge d\mathcal{F}_\alpha^{\Phi_i}(t)ds\right)\\ &\qquad\wedge\omega^{n-1} - \frac{1}{(n-1)!}\int_M\left(\int_0^1 (\phi_i^t)^*\big((\imath_{\dot\phi_i^t}\omega)\big)dt\right)\wedge\alpha\wedge\omega^{n-1}\\ &= \frac{1}{(n-1)!}\int_M d\left(\int_0^1 \mathcal{F}_\alpha^{\Phi_i}(t)(\phi_i^t)^*(\imath_{\dot\phi_i^t}\omega)dt\right)\\ &\qquad\wedge\omega^{n-1} - \frac{1}{(n-1)!}\int_M\left(\int_0^1 (\phi_i^t)^*\big((\imath_{\dot\phi_i^t}\omega)\big)dt\right)\wedge\alpha\wedge\omega^{n-1}\\ &= 0 - \frac{1}{(n-1)!}\langle\widetilde{S}_\omega(\Phi_i), [\alpha\wedge\omega^{n-1}]\rangle \quad \text{by Stoke's theorem.}\end{aligned} \tag{3.4}$$

Combining (3.2) and (3.4) yields

$$\begin{aligned}\int_M\left(\int_{\mathcal{O}_{(\cdot)}^\Psi}\alpha\right)\frac{\omega^n}{n!} &= \lim_{i\longrightarrow\infty}\int_M\left(\int_{\mathcal{O}_{(\cdot)}^{\Phi_i}}\alpha\right)\frac{\omega^n}{n!} = \lim_{i\longrightarrow\infty} -\frac{1}{(n-1)!}\langle\widetilde{S}_\omega(\Phi_i), [\alpha\wedge\omega^{n-1}]\rangle\\ &= -\frac{1}{(n-1)!}\langle\widetilde{S}_\omega(\Phi), [\alpha\wedge\omega^{n-1}]\rangle. \quad\square\end{aligned} \tag{3.5}$$

In Theorem 3.1, the de Rham cohomology class $\widetilde{S}_\omega(\Phi)$ is uniquely determined by Ψ. Furthermore, if in Theorem 3.1, one only assumes that the sequence $\left(\widetilde{S}_\omega(\Phi_i)\right)_i$ is just Cauchy in the complete metric space $H^1(M,\mathbb{R})$, then Ψ will still uniquely determine a de Rham cohomology class $\beta(\Psi) \in H^1(M,\mathbb{R})$: this seems to tell us that one can transport flux geometry under the $(C^0 + L^\infty)$-topology.

3.1. $\mathcal{S}$-homeomorphisms and $\mathcal{S}$-topological isotopies

Definition 3.1. A continuous map $\xi : [0,1] \longrightarrow Homeo(M)$ with $\xi(0) = id_M$ is called an $\mathcal{S}$-topological isotopy if there exists a $D^\infty_{\mathcal{S}}$-Cauchy sequence $(F_i, \mathcal{G}_i)_i \subset \mathfrak{T}(M,\omega,\mathcal{S})$ generating a sequence $\{\Phi_i\}_i \subset Iso(M,\omega)$ such that $\Phi_i \xrightarrow{C^0} \xi$.

Denote by $\mathcal{PSG}_\omega(M)$ the group of all $\mathcal{S}$-topological isotopies of a closed symplectic manifold (M,ω), and introduce the group $\mathcal{SG}_\omega(M)$, consisted of $\mathcal{S}$-homeomorphisms of (M,ω) defined as the time-one maps of $\mathcal{S}$-topological isotopies: $\mathcal{SG}_\omega(M) := ev_1(\mathcal{PSG}_\omega(M))$. Note that if the linear section $\mathcal{S}$ is trivial, the $\mathcal{S}$-forms are harmonic forms, therefore the corresponding splitting is the usual Hodge decomposition and the group $\mathcal{SG}_\omega(M)$ coincides with the group $G^0_\omega(M)$ of all strong symplectic homeomorphisms of (M,ω) [3,4,16,18].

In particular, when a continuous map $\xi : [0,1] \longrightarrow Homeo(M)$ with $\xi(0) = id_M$ satisfies Definition 3.1 with respect to a sequence of $\mathcal{S}$-generators $\{(U_i, 0)\} \subset \mathfrak{T}(M,\omega,\mathcal{S})$, then ξ is called continuous Hamiltonian isotopy and its time-one map a Hamiltonian homeomorphism. The set of all Hamiltonian homeomorphisms is a group denoted $Hameo(M,\omega)$ [11].

Remark 3.1.

1. Note that by Proposition 1.1, the group $\mathcal{SG}_\omega(M)$ is independent of the choice of a linear section $\mathcal{S}$.
2. One can show that in Definition 3.1, the sequence $\{(U_i, \mathcal{H}_i)\}_i \subset \mathfrak{T}(M,\omega,\mathcal{S})$ converges in a certain complete metric space to an element $(U, \mathcal{H})$, called the generator of the corresponding $\mathcal{S}$-topological isotopy [3,16,18].

The following uniqueness results have been proved in [3,18] with respect to the Hodge decomposition theorem. Moreover, the uniqueness result found in [3] has been generalized in [16]. Here, we state similar results with respect to an arbitrary linear section $\mathcal{S}$; their proofs are similar to those in [3,18], where the harmonic forms in the Hodge decomposition are replaced by $\mathcal{S}$-forms.

Theorem 3.2. *Let (M,ω) be a closed symplectic manifold.*

1. *Any $\mathcal{S}$-topological isotopy determines a unique generator.*
2. *Any generator determines a unique $\mathcal{S}$-topological isotopy.*

Based on the above uniqueness results, one writes any $\mathcal{S}$-topological isotopy as $\aleph_{(U,\mathcal{H})}$ to mean that its generator is $(U, \mathcal{H})$: observe that U is a continuous family of normalized continuous functions on M, while $\mathcal{H}$ is a continuous path in $\mathbb{H}^1(M,\mathcal{S})$. Furthermore, the $\mathcal{S}$-decomposition of symplectic isotopies [17] extends to $\mathcal{S}$-topological isotopies [3].

Remark 3.2. Given any $\mathcal{S}$-topological isotopy $\aleph_{(U,\mathcal{H})}$, we write $\aleph_{(U,\mathcal{H})} = \lim_{(C^0+L^\infty)}(\phi_{(U_i,\mathcal{H}_i)})$, to mean that $\phi_{(U_i,\mathcal{H}_i)} \xrightarrow{C^0} \aleph_{(U,\mathcal{H})}$ and $(U_i,\mathcal{H}_i) \xrightarrow{L^\infty} (U,\mathcal{H})$ namely, $U_i \xrightarrow{L^\infty\text{-Hofer norm}} U$ and $\mathcal{H}_i \xrightarrow{L^2-\text{norm}} \mathcal{H}$.

Proposition 3.1. *[11] For each fixed linear section $\mathcal{S}$, the group $Hameo(M,\omega)$ is a normal subgroup of $\mathcal{SG}_\omega(M)$.*

By Proposition 3.1, it follows from a result of classical group theory that $Hameo(M,\omega)$ can be realized as the kernel of a non-trivial group homomorphism defined on $\mathcal{S}G_\omega(M)$.

3.2. An extension of the flux homomorphism

According to the uniqueness results (Theorem 3.2), to any $\mathcal{S}$-topological isotopy $\aleph_{(U,\mathcal{H})}$, corresponds a well-defined de Rham cohomology class $\left[\int_0^1 \mathcal{H}^t dt\right] \in H^1(M,\mathbb{R})$, namely a surjective group homomorphism

$$\begin{aligned} \widetilde{S}^0_\omega : \mathcal{PS}G_\omega(M) &\longrightarrow H^1(M,\mathbb{R}) \\ \aleph_{(U,\mathcal{H})} &\longmapsto \left[\int_0^1 \mathcal{H}^t dt\right]. \end{aligned}$$

The restriction of $\widetilde{S}^0_\omega$ to the group of all smooth symplectic isotopies coincides with the usual flux homomorphism. Let $\mathcal{V}_{id}(\mathcal{S}G_\omega(M))$ denote the set of $\mathcal{S}$-topological isotopies with time-one map identity and set $\mathcal{S}\Gamma_\omega := \widetilde{S}^0_\omega\big(\mathcal{V}_{id}(\mathcal{S}G_\omega(M))\big)$. The mapping $\widetilde{S}^0_\omega$ induces a surjective homomorphism S^0_ω from $\mathcal{S}G_\omega(M)$ onto $H^1(M,\mathbb{R})/_{\mathcal{S}\Gamma_\omega}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{PS}G_\omega(M) & \xrightarrow{\widetilde{S}^0_\omega} & H^1(M,\mathbb{R}) \\ ev_1 \downarrow & & \downarrow Q \\ \mathcal{S}G_\omega(M) & \xrightarrow{S^0_\omega} & H^1(M,\mathbb{R})/_{\mathcal{S}\Gamma_\omega}, \end{array} \qquad (I)$$

where Q is the quotient map and ev_1 is the time-one evaluation map.

The set of loops in $\mathcal{S}G_\omega(M)$ is non-empty because it contains any loop in $G_\omega(M)$ and the following example shows that this inclusion is non-trivial.

Example 3.1. Pick $\beta \in \Gamma_\omega$ to be a non-trivial element, then by definition of the flux group, there exists a non-trivial loop $\Psi := \{\psi^t\}_t$ in $\pi_1(G_\omega(M))$ such that $\widetilde{S}_\omega(\Psi) = \beta$. Besides, let $\mathfrak{H} := \{h^t\}_t$ be a non smooth continuous Hamiltonian isotopy. Consider the boundary flat smooth functions $\mu_1 : [0, \frac{1}{3}] \longrightarrow [0,1]$ with $\mu_1 = 0$ near 0 and $\mu_1 = 1$ near $\frac{1}{3}$; $\mu_2 : [\frac{1}{3}, \frac{2}{3}] \longrightarrow [0,1]$ with $\mu_2 = 0$ near $\frac{1}{3}$ and $\mu_2 = 1$ near $\frac{2}{3}$; and $\mu_3 : [\frac{2}{3}, 1] \longrightarrow [0,1]$ with $\mu_3 = 0$, near $\frac{2}{3}$; and $\mu_3 = 1$ near 1, which are used to define the path $\mathfrak{R} = \{r^t\}_t$ as follows:

$$r^t = \begin{cases} \psi^{\mu_1(t)} & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{3}, \\ h^{\mu_2(t)} & \text{if} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ h^1 \circ (h^{\mu_3(t)})^{-1} & \text{if} \quad \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

The path $\mathfrak{R} = \{r^t\}_t$ is continuous, satisfies $r^0 = \psi^0 = id_M$ and $r^1 = h^1 \circ (h^1)^{-1} = id_M$. Assume that $\mathfrak{H} := \{h^t\}_t$ has a generating function U and is the C^0-limit of a sequence $(\mathfrak{H}_i)_i = (\{h^t_i\}_t)_i$ with the sequence of generators $(U_i)_i$ such that $U = \lim_{Hofer} U_i$. Assume also that Ψ has generator $(V, \mathcal{K})$. Then, the sequence of symplectic isotopies $(\mathfrak{R}_i)_i = (\{r^t_i\}_t)_i$:

$$r^t_i = \begin{cases} \psi^{\mu_1(t)} & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{3}, \\ h_i^{\mu_2(t)} & \text{if} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ h^1_i \circ (h_i^{\mu_3(t)})^{-1} & \text{if} \quad \frac{2}{3} \leqslant t \leqslant 1, \end{cases}$$

converges uniformly to $\Re$, and for each i, the generator $(U_i, \mathcal{K}_i)_i$ of $\Re_i$ is defined by

$$(U_i^t, \mathcal{K}_i^t) = \begin{cases} \dot{\mu}_1(t)\big(V^{\mu_1(t)}, \mathcal{K}^{\mu_1(t)}\big) & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{3}, \\ \dot{\mu}_2(t)\big(U_i^{\mu_2(t)}, 0\big) & \text{if} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ \dot{\mu}_3(t)\big(-U_i^{\mu_2(t)} \circ h_i^{\mu_3(t)} \circ (h_i^1)^{-1}, 0\big) & \text{if} \quad \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

Since the functions μ_j, $j = 1, 2, 3$ are all boundary flat, then one can assure the continuity at the juxtaposition points. The sequence of generators $(U_i, \mathcal{K}_i)_i$ uniformly converges to the generator $(U, \mathcal{K})$ given by

$$(U^t, \mathcal{K}^t) = \lim_{Hofer-like} (U_i^t, \mathcal{K}_i^t) = \begin{cases} \dot{\mu}_1(t)(V^{\mu_1(t)}, \mathcal{K}^{\mu_1(t)}) & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{3}, \\ \dot{\mu}_2(t)(U^{\mu_2(t)}, 0) & \text{if} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ \dot{\mu}_3(t)(-U^{\mu_2(t)} \circ h^{\mu_3(t)} \circ (h^1)^{-1}, 0) & \text{if} \quad \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

Thus, $\Re$ is a non-smooth loop in $\mathcal{S}G_\omega(M)$ so that $\Re \notin \pi_1(G_\omega(M))$. Observe that the flux of $\Re$ is the same as that of Ψ.

3.3. Some properties of $\ker S_\omega^0$ *and* $\mathcal{S}\Gamma_\omega$

Hereafter we highlight a consequence of diagram (I) and present some properties of $\ker S_\omega^0$ and $\mathcal{S}\Gamma_\omega$.

Corollary 3.1. *Let (M, ω) be a closed symplectic manifold and $H = \{h^t\}_t$ be an $\mathcal{S}$-topological isotopy. Then, $h^1 \in \ker S_\omega^0$ if and only if $\widetilde{S}_\omega^0(H) \in \mathcal{S}\Gamma_\omega$.*

Proof. Let $H = \{h^t\}_t$ be an $\mathcal{S}$-topological isotopy such that $h^1 \in \ker S_\omega^0$. From the commutation of the diagram (I), one has $\left(Q \circ \widetilde{S}_\omega^0\right)(H) = \left(S_\omega^0 \circ ev_1\right)(H) = 0$, that is $\widetilde{S}_\omega^0(H) \in \mathcal{S}\Gamma_\omega$. Conversely, if $\widetilde{S}_\omega^0(H) \in \mathcal{S}\Gamma_\omega$, then there exists a loop Ψ such that $\widetilde{S}_\omega^0(H \circ \Psi^{-1}) = 0$. That is, $0 = \left(Q \circ \widetilde{S}_\omega^0\right)(H \circ \Psi^{-1}) = \left(S_\omega^0 \circ ev_1\right)(H \circ \Psi^{-1})$, i.e., $S_\omega^0(h^1 \circ id_M) = 0$, namely $h^1 \in \ker S_\omega^0$. □

We now prove that the kernel of S_ω^0 is path connected and coincides with $Hameo(M, \omega)$.

Proposition 3.2. *Let (M, ω) be a closed symplectic manifold. Then, the subgroup $\ker S_\omega^0$ is path connected.*

Proof. Let $h \in \ker S_\omega^0$. From the characterization of $\ker S_\omega^0$, there is an $\mathcal{S}$-topological isotopy $H = \{h^t\}_t$ from the identity map to h such that $\widetilde{S}_\omega^0(H) \in \mathcal{S}\Gamma_\omega$, that is, one can compose H with a loop at the identity and assume without loss of generality that $\widetilde{S}_\omega^0(H) = 0$. Assume that H is generated by $(U, \mathcal{H})$ and for each t, let $h^t = \rho^t \circ \psi^t$ be the $\mathcal{S}$-decomposition of h^t where $\rho : t \longmapsto \rho^t$ is generated by $(0, \mathcal{H})$. It is enough to show that ρ^1 can be connected to the identity through a path lying entirely in $\ker S_\omega^0$. By assumption we have, $\widetilde{S}_\omega^0(\rho) = \widetilde{S}_\omega^0(H) = 0$, i.e., $\int_0^1 \mathcal{H}^t dt = 0$. Consider the following 2-parameters family of $\mathcal{S}$-forms, $\mathcal{K}^t(s) = t\mathcal{H}^{(st)} - 2s \int_0^t \mathcal{H}^u du$, for all s, t. It is clear that for each fixed t, the element $(0, \mathcal{K}^t)$ generates an $\mathcal{S}$-topological isotopy $\eta^t = \{\eta_s^t\}_s$. Furthermore, from the uniqueness result of the generators of $\mathcal{S}$-topological isotopies, we have that both paths ρ and η^1 must agree everywhere since $(0, \mathcal{K}^1) = (0, \mathcal{H})$. For each fixed t, compute

$$\begin{aligned}
\widetilde{S}^0_\omega(\eta^t) &= \left[\int_0^1 \left(t\mathcal{H}^{(st)} - 2s\int_0^t \mathcal{H}^u du\right) ds\right] \\
&= \left[\int_0^1 t\mathcal{H}^{(st)} ds\right] - \left(\int_0^1 2s ds\right)\left[\int_0^t \mathcal{H}^u du\right] \\
&= \left[\int_0^t \mathcal{H}^u du\right] - \left[\int_0^t \mathcal{H}^u du\right] = 0. \qquad (3.6)
\end{aligned}$$

Thus, for each fixed t, the time-one map η_1^t of the isotopy $s \longmapsto \eta_s^t$ belongs to $\ker S^0_\omega$. Note that the map $t \longmapsto \eta_1^t$ is a path in $\ker S^0_\omega$ such that $\eta_0^0 = id_M$ and $\eta_1^1 = \rho^1$. □

Lemma 3.1. *Let (M, ω) be a closed symplectic manifold of Lefschetz type. Then $Hameo(M, \omega) = \ker S^0_\omega$.*

Proof. By definition, $Hameo(M, \omega) \subseteq \ker S^0_\omega$. Let $h \in \ker S^0_\omega$ then there is an $\mathcal{S}$-topological isotopy $H = \{h^t\}_t$ ending at h such that $\widetilde{S}^0_\omega(H) \in \mathcal{S}\Gamma_\omega$, therefore, one can compose H with a loop at the identity and assume without loss of generality that $\widetilde{S}^0_\omega(H) = 0$. Thus, from Theorem B, we derive that the path H is homotopic relatively to fixed endpoints to a continuous Hamiltonian isotopy. That is, $h \in Hameo(M, \omega)$. □

Proof of Theorem D. Let (M, ω) be a closed symplectic manifold of Lefschetz type, μ the measure associated to the symplectic volume form and let $\tilde{\mathfrak{F}} : \widetilde{Homeo_0(M, \mu)} \to H_1(M, \mathbb{R})$ be the Fathi's mass flow defined on the universal cover $Homeo_0(M, \mu)$ of the identity component in the group of measure preserving homeomorphisms. From the inclusion $\mathcal{PSG}_\omega(M) \subseteq \widetilde{Homeo_0(M, \mu)}$, Tchuiaga [16] showed that for any $\Phi \in \mathcal{PSG}_\omega(M)$,

$$\tilde{\mathfrak{F}}(\Phi)(f) = \frac{1}{(n-1)!}\left\langle \widetilde{S^0_\omega}(\Phi), [\omega^{n-1} \wedge f^*\sigma]\right\rangle,$$

for all continuous mapping f from M to the circle $\mathbb{S}^1$, where σ is the volume form on $\mathbb{S}^1$. Since M is of Lefschetz type, the map $\omega^{n-1} : H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})$ is an isomorphism, and the following diagram commutes:

$$\begin{array}{ccccc}
\mathcal{PSG}_\omega(M) & \xrightarrow{\widetilde{S^0_\omega}} & H^1(M, \mathbb{R}) & & \\
\downarrow \iota & & & \searrow^{\omega^{n-1}} & \\
\widetilde{Homeo_0(M, \mu)} & \xrightarrow{\tilde{\mathfrak{F}}} & H_1(M, \mathbb{R}) & \xrightarrow{I} & H^{2n-1}(M, \mathbb{R})
\end{array} \qquad (II)$$

where I is an isomorphism. Set $\mathcal{N}(M, \mu) = \left\{\{h^t\}_t \in \widetilde{Homeo_0(M, \mu)} : h^1 = id\right\}$. Fathi [6] showed that $\Gamma = \tilde{\mathfrak{F}}(\mathcal{N}(M, \mu))$ is discrete. Therefore, $\tilde{\mathfrak{F}}(\mathcal{V}_{id}(\mathcal{SG}_\omega(M)))$ is discrete as a subgroup of Γ. Hence from the commutativity of diagram (II), $\mathcal{S}\Gamma_\omega = \widetilde{S^0_\omega}(\pi_1(\mathcal{SG}_\omega(M)))$ is isomorphic to a discrete group $I(\tilde{\mathfrak{F}}(\mathcal{V}_{id}(\mathcal{SG}_\omega(M))))$. That is $\mathcal{S}\Gamma_\omega$ is discrete. □

4. Homotopic invariance and the topological flux group

In this subsection, we generalize Proposition 2.7-[17] and prove some of our main results. Consider $H \in \mathcal{PSG}_\omega(M)$ and $\alpha \in \mathcal{Z}^1(M)$. Let $z \in M$, the orbit $\mathcal{O}_z^H$ is not smooth so one cannot generally integrate over it. However, for any continuous curve γ and closed 1-form α we will still use the notation $\int_\gamma \alpha$ to mean the integral $\int_{\gamma'} \alpha$ for any piecewise smooth path γ' that is homotopic relatively to fixed ends to γ. Stoke's theorem implies that the integral is independent of the choice of γ' because α is closed. Since $H = \lim_{C^0+L^\infty} \Phi_i$ one has $\mathcal{O}_z^H = \lim_{C^0} \mathcal{O}_z^{\Phi_i}$. Therefore, for i sufficiently large, $\mathcal{O}_z^H$ and $\mathcal{O}_z^{\Phi_i}$, can be connected through a minimizing geodesic say $\chi_{i,t}$ such that $\mathcal{O}_z^H$, $\mathcal{O}_z^{\Phi_i}$ and $\chi_{i,1}$ forms the boundary of a 2-chain. Thus

$$\left| \int_{\mathcal{O}_z^{\Phi_i}} \alpha - \int_{\mathcal{O}_z^H} \alpha \right| = \left| \int_{\chi_{i,1}} \alpha \right| \leqslant d_0(\phi_i^1, h^1)$$

that is, $\int_{\mathcal{O}_z^H} \alpha = \lim_{i\to\infty} \int_{\mathcal{O}_z^{\Phi_i}} \alpha$.

$$\left| \int_{\mathcal{O}_z^{\Phi_i}} \alpha - \int_{\mathcal{O}_z^H} \alpha \right| = \left| \int_{\chi_{i,1}} \alpha \right| \leqslant \|\alpha\|_\infty d_0(\phi_i^1, h^1)$$

that is, $\int_{\mathcal{O}_z^H} \alpha = \lim_{i\to\infty} \int_{\mathcal{O}_z^{\Phi_i}} \alpha$.

Remark 4.1. [15] Let M be a connected manifold. Given two isotopies $\Phi = \{\phi^t\}_t$ and $\Psi = \{\psi^t\}_t$ with $\phi^1 = \psi^1$, then for each fixed $x_0 \in M$ and for any closed 1-form α, one has for all $z \in M$

$$\int_{\mathcal{O}_z^\Phi} \alpha - \int_{\mathcal{O}_{x_0}^\Phi} \alpha = \int_{\mathcal{O}_z^\Psi} \alpha - \int_{\mathcal{O}_{x_0}^\Psi} \alpha. \tag{4.1}$$

Proposition 4.1. *Let (M, ω) be a closed connected symplectic manifold and $\Omega = \frac{\omega^n}{n!}$ its associated volume form. If $H \in \mathcal{V}_{id}(\mathcal{S}G_\omega(M))$, then for each closed 1-form α and for all $x \in M$ we have*

$$\int_{\mathcal{O}_x^H} \alpha = -\frac{1}{(n-1)! Vol_\Omega(M)} \langle \widetilde{S}_\omega^0(H), [\alpha \wedge \omega^{n-1}] \rangle. \tag{4.2}$$

Proof. Let $\phi \in G_\omega(M)$, $\alpha \in \mathcal{Z}^1(M)$. For each fixed $x_0 \in M$, integrating $\int_{\mathcal{O}_x^\Phi} \alpha - \int_{\mathcal{O}_{x_0}^\Phi} \alpha$ over M leads to the function

$$\Delta_\alpha^{x_0} : \phi \longmapsto \left(\int_M \mathcal{F}_\alpha^\Phi(1)\Omega \right) - Vol_\Omega(M)\mathcal{F}_\alpha^\Phi(1)(x_0) \tag{4.3}$$

which is independent of the choice of a symplectic isotopy Φ with endpoint ϕ. Therefore from (3.4), one obtains

$$\Delta_\alpha^{x_0}(\phi) = -\frac{1}{(n-1)!} \langle \widetilde{S}_\omega(\Phi), [\alpha \wedge \omega^{n-1}] \rangle - Vol_\Omega(M)\mathcal{F}_\alpha^\Phi(1)(x_0). \tag{4.4}$$

Furthermore the function defined in (4.3) is continuous with respect to the C^0-topology (see Lemma 4.1-[15]). In fact, given a sequence of isotopies $\{\Phi_i\}_i = \{\phi_{(\mathcal{U}_i,\mathcal{H}_i)}\}_i$ with $d_0(\Phi_i(1), id_M) \longrightarrow 0$ when $i \longrightarrow \infty$ then $\lim_{C^0} \Delta^x_\alpha(\Phi_i(1)) = \Delta^x_\alpha(id_M) = 0$, for all $x \in M$, since id_M is smooth. Besides, observe that the $(C^0 + L^\infty)$-topology is finer than the C^0-topology and if necessary regarding $\Phi_i(1)$ as the constant path $t \longmapsto \Phi_i(1)$, we may assume as well that $\lim_{(C^0+L^\infty)} (\Delta^x_\alpha(\Phi_i(1))) = 0$. Note that the $(C^0 + L^\infty)$-metric on the space of constant paths coincides with the C^0-metric. Hence, we have for each fixed $x \in M$:

$$0 = \lim_{(C^0+L^\infty)} \Delta^x_\alpha(\Phi_i(1)) = \lim_{(C^0+L^\infty)} \left(-\frac{1}{(n-1)!}\langle \widetilde{S}_\omega(\Phi_i), [\alpha \wedge \omega^{n-1}]\rangle - Vol_\Omega(M) \int_{\mathcal{O}^{\Phi_i}_x} \alpha \right)$$
$$= -\frac{1}{(n-1)!}\langle \widetilde{S}^0_\omega(H), [\alpha \wedge \omega^{n-1}]\rangle - Vol_\Omega(M) \int_{\mathcal{O}^H_x} \alpha$$

Thus

$$\int_{\mathcal{O}^H_x} \alpha = -\frac{1}{(n-1)!Vol_\Omega(M)}\langle \widetilde{S}^0_\omega(H), [\alpha \wedge \omega^{n-1}]\rangle. \quad \square \tag{4.5}$$

In order to understand the dynamics of the solution curves of ODE's, one should travel along the manifold and thoroughly study the behavior of the trajectories in different regions [12]. This procedure can be delicate depending on the topology of the manifold. However, Proposition 4.1 tells us that given any element in $\mathcal{V}_{id}\big(\mathcal{S}G_\omega(M)\big)$, the topology of its orbits in M is determined by its flux geometry: the geometric properties of loops in $\mathcal{S}G_\omega(M)$ reflect the dynamics of their orbits on M.

Proof of Theorem C. 1. Let H be a continuous Hamiltonian isotopy with time-one map identity, by definition, we have that $\widetilde{S}^0_\omega(H) = 0$. By Proposition 4.1, we have for all closed 1-forms α

$$\int_{\mathcal{O}^H_x} \alpha = -\frac{1}{(n-1)!Vol_\Omega(M)}\langle \widetilde{S}^0_\omega(H), [\alpha \wedge \omega^{n-1}]\rangle = 0 \quad \text{since } \widetilde{S}^0_\omega(H) = 0.$$

Therefore, $\int_{\mathcal{O}^H_x} \alpha = 0$ i.e. $\mathcal{O}^H_x$ is null-homologous in $H_1(M,\mathbb{R})$.

2. Let H be in $\mathcal{V}_{id}(\mathcal{S}G_\omega(M))$ whose orbit is null-homologous in $H_1(M,\mathbb{R})$. Therefore, for all closed 1-forms α

$$0 = \int_{\mathcal{O}^H_x} \alpha = -\frac{1}{(n-1)!Vol_\Omega(M)}\langle \widetilde{S}^0_\omega(H), [\alpha \wedge \omega^{n-1}]\rangle \quad \text{by Proposition 4.1.}$$

Since M is of Lefschetz type we have $\widetilde{S}^0_\omega(H) = 0$ i.e. H is a continuous isotopy with time one map identity. $\square$

The next result claims that the flux homomorphism $\widetilde{S}^0_\omega$ preserves the homotopy class of $\mathcal{S}$-topological isotopies.

Theorem 4.1. *Let (M, ω) be a closed connected symplectic manifold of Lefschetz type. If H and L are two $\mathcal{S}$-topological isotopies homotopic relatively to fixed endpoints in $\mathcal{S}G_\omega(M)$, then $\widetilde{S}^0_\omega(H) = \widetilde{S}^0_\omega(L)$.*

Proof. Let $\{h^t\}_t =: H = \lim_{(C^0+L^\infty)}(\phi_{(U_i,\mathcal{H}_i)})$ and $\{l^t\}_t =: L = \lim_{(C^0+L^\infty)}(\phi_{(V_i,\mathcal{K}_i)})$, be two $\mathcal{S}$-topological isotopies homotopic relatively to fixed endpoints in $\mathcal{S}G_\omega(M)$. Fix a sufficiently large integer $\mathbf{p}$. From the C^0-convergence $(\Phi^i(t))_i := \{\phi^t_{(U_i,\mathcal{H}_i)}\}_t \xrightarrow{C^0+L^\infty} \{h^t\}_t$, we may assume that for each $z \in M$, there exists a minimizing geodesic $\zeta_z^{\mathbf{p}}$ from $h^1(z)$ to $(\Phi^{\mathbf{p}}(1))(z)$ such that the curves $\zeta_z^{\mathbf{p}}$, $\mathcal{O}_z^H$, and $\mathcal{O}_z^{\Phi^{\mathbf{p}}}$ delimit a 2-chain in M. Hence, for all closed 1-form α, we derive from Stokes' theorem that

$$\int_{\mathcal{O}_z^H} \alpha = \int_{\mathcal{O}_z^{\Phi^{\mathbf{p}}}} \alpha + \int_{\zeta_z^{\mathbf{p}}} \alpha. \tag{4.6}$$

Similarly, since $(\Psi^i(t))_i := \{\phi^t_{(V_i,\mathcal{K}_i)}\}_t \xrightarrow{C^0+L^\infty} \{l^t\}_t$, then we may assume that for each $z \in M$, there exists a minimizing geodesic $\xi_z^{\mathbf{p}}$ from $l^1(z)$ to $(\Psi^{\mathbf{p}}(1))(z)$ such that the curves $\xi_z^{\mathbf{p}}$, $\mathcal{O}_z^L$, and $\mathcal{O}_z^{\Psi^{\mathbf{p}}}$ delimit a 2-chain in M, and then

$$\int_{\mathcal{O}_z^L} \alpha = \int_{\mathcal{O}_z^{\Psi^{\mathbf{p}}}} \alpha + \int_{\xi_z^{\mathbf{p}}} \alpha, \qquad \forall\, \alpha \in \mathcal{Z}^1(M). \tag{4.7}$$

The paths H and L being homotopic relatively to fixed endpoints, we derive from (4.6) and (4.7) that

$$\int_{\mathcal{O}_z^{\Phi^{\mathbf{p}}}} \alpha + \int_{\zeta_z^{\mathbf{p}}} \alpha = \int_{\mathcal{O}_z^{\Psi^{\mathbf{p}}}} \alpha + \int_{\xi_z^{\mathbf{p}}} \alpha, \qquad \forall\ \alpha \in \mathcal{Z}^1(M) \tag{4.8}$$

i.e.,

$$\int_{\mathcal{O}_z^{\Phi^{\mathbf{p}}}} \alpha - \int_{\mathcal{O}_z^{\Psi^{\mathbf{p}}}} \alpha = \int_{\xi_z^{\mathbf{p}}} \alpha - \int_{\zeta_z^{\mathbf{p}}} \alpha, \qquad \forall\ \alpha \in \mathcal{Z}^1(M). \tag{4.9}$$

Therefore, integrating (4.9) over M and applying (3.4) one obtains

$$n\langle \widetilde{S}_\omega(\Psi^{\mathbf{p}}) - \widetilde{S}_\omega(\Phi^{\mathbf{p}}), [\alpha \wedge \omega^{n-1}]\rangle = \int_M \left(\int_{\xi^{\mathbf{p}}_{(\cdot)}} \alpha - \int_{\zeta^{\mathbf{p}}_{(\cdot)}} \alpha \right) \omega^n, \qquad \forall\ \alpha \in \mathcal{Z}^1(M) \tag{4.10}$$

where

$$\begin{aligned} \int_{\zeta^{\mathbf{p}}_{(\cdot)}} \alpha : M &\longrightarrow \mathbb{R} \\ z &\longmapsto \int_{\zeta^{\mathbf{p}}_z} \alpha \end{aligned} \quad \text{and} \quad \begin{aligned} \int_{\xi^{\mathbf{p}}_{(\cdot)}} \alpha : M &\longrightarrow \mathbb{R} \\ z &\longmapsto \int_{\xi^{\mathbf{p}}_z} \alpha. \end{aligned}$$

The length of a minimizing geodesic being bounded from above by the distance between its endpoints, for all $\alpha \in \mathcal{Z}^1(M)$ we have

$$\int_M \left(\int_{\xi^{\mathbf{p}}_{(\cdot)}} \alpha - \int_{\zeta^{\mathbf{p}}_{(\cdot)}} \alpha \right) \omega^n \leqslant \|\alpha\|_\infty \left(\int_M \omega^n \right) \left(d_0(h^1, \Phi^{\mathbf{p}}(1)) + d_0(l^1, \Psi^{\mathbf{p}}(1)) \right). \tag{4.11}$$

Hence, combining (4.10) and (4.11) we derive by passing to the $(C^0 + L^\infty)$-limit that,

$$\langle \lim_{L^\infty} \left(\widetilde{S}_\omega(\Psi^{\mathbf{P}}) - \widetilde{S}_\omega(\Phi^{\mathbf{P}}) \right), [\alpha \wedge \omega^{n-1}] \rangle = 0 \qquad \forall\, \alpha \in \mathcal{Z}^1(M), \tag{4.12}$$

i.e.,

$$\langle (\widetilde{S}^0_\omega(H) - \widetilde{S}^0_\omega(L)), [\alpha \wedge \omega^{n-1}] \rangle = 0 \qquad \forall\, \alpha \in \mathcal{Z}^1(M). \tag{4.13}$$

Since M is of Lefschetz type, then (4.13) implies $\widetilde{S}^0_\omega(H) = \widetilde{S}^0_\omega(L)$. $\square$

In order to extend Theorem 2.1 to $\mathcal{S}$-topological isotopies, let's prove the following lemma.

Lemma 4.1 *(Topological non-lifting procedure). Let $H = \{h^t\}_t$ be an $\mathcal{S}$-topological isotopy whose generator is of the form $(0, \mathcal{H})$. For each fixed t, consider H^t, the path from identity to h^t as the reparametrized path $s \longmapsto h^{st}$. Then, the map*

$$\begin{array}{ccc} [0,1] & \longrightarrow & \mathcal{PSG}_\omega(M) \\ t & \longmapsto & H^t \end{array}$$

is continuous with respect to the $(C^0 + L^\infty)$-topology on $\mathcal{PSG}_\omega(M)$.

Proof. Let $H = \{h^t\}_t$ be an $\mathcal{S}$-topological isotopy with generator $(0, \mathcal{H})$. By definition of H there is a sequence $(\Phi_j)_j = \left(\{\phi^t_j\}_t\right)_j$ of symplectic isotopies generated by $\left((0, \mathcal{H}_j)\right)_j$ such that $H = \lim_{(C^0+L^\infty)} \Phi_j$. Consider the sequence of reparametrized isotopies $\left(\{\bar{\phi}^t_j\}_t\right)_j$ where for each j and each fixed t, the isotopy $\bar{\phi}^t_j = \{\phi_j^{(st)}\}_s$ is generated by $\left(0, t\mathcal{H}_j^{(st)}\right)$.

For each j observe that $\{\bar{\phi}^t_j\}_t$ is an element of $\mathcal{P}([0,1], \mathcal{P}(Homeo(M), id_M))$. Since the space $\mathcal{P}(Homeo(M), id_M)$ is a complete metric space with respect to the metric $\bar{d}$, so is $\mathcal{P}([0,1], \mathcal{P}(Homeo(M), id_M))$ with respect to the metric:

$$\bar{\bar{d}}(\{\bar{\phi}^t\}_t, \{\bar{\psi}^t\}_t) = \max_t \bar{d}(\bar{\phi}^t, \bar{\psi}^t), \qquad \forall\, \{\bar{\phi}^t\}_t, \{\bar{\psi}^t\}_t \in \mathcal{P}([0,1], \mathcal{P}(Homeo(M), id_M)). \tag{4.14}$$

Therefore

$$\bar{\bar{d}}(\{\bar{\phi}^t_j\}_t, \{\bar{\phi}^t_{j+k}\}_t) = \max_t \bar{d}(\bar{\phi}^t_j, \bar{\phi}^t_{j+k}) = \bar{d}(\Phi_j, \Phi_{j+k}) \longrightarrow 0 \text{ as } \; j \longrightarrow \infty.$$

That is, the sequence $(\{\bar{\phi}^t_j\}_t)_j$ is Cauchy in a complete metric space and hence converges to $H^t = \{h^{(st)}\}_s$. It follows that the map $t \longmapsto H^t$ is continuous with respect to the C^0-metric.

Besides for each j, the path $\{\mathcal{H}^t_j\}_t$ is an element of $\mathcal{P}([0,1], \mathcal{P}\mathbb{H}^1(M, \mathcal{S}))$ where $\mathcal{H}^t_j = \{t\mathcal{H}_j^{(st)}\}_s$ for each fixed t. Since the space $\mathcal{P}\mathbb{H}^1(M, \mathcal{S})$ is a complete metric space with respect to the supremum norm

$$\|\{\mathcal{K}^t\}_t\|_0 := \max_t \|\mathcal{K}^t\|_{L^2}, \quad \forall\, \{\mathcal{K}^t\}_t \in \mathcal{P}\mathbb{H}^1(M, \mathcal{S})$$

so is $\mathcal{P}\left([0,1], \mathcal{P}\mathbb{H}^1(M, \mathcal{S})\right)$ with respect to the supremum norm

$$|||\{\mathcal{L}^u\}_u|||_0 := \max_u \|\mathcal{L}^u\|_0, \quad \forall\, \{\mathcal{L}^u\}_u \in \mathcal{P}\left([0,1], \mathcal{P}\mathbb{H}^1(M, \mathcal{S})\right). \tag{4.15}$$

Therefore

$$\begin{aligned} |||\{\mathcal{H}^t_j\}_t - \{\mathcal{H}^t_{j+k}\}_t|||_0 &:= \max_t \|\mathcal{H}^t_j - \mathcal{H}^t_{j+k}\|_0 = \max_{s,t} \|t\mathcal{H}_j^{(st)} - t\mathcal{H}_{j+k}^{(st)}\|_{L^2} \\ &\leqslant \max_u \|\mathcal{H}^u_j - \mathcal{H}^u_{j+k}\|_{L^2} \longrightarrow 0, \qquad j \longrightarrow \infty. \end{aligned} \tag{4.16}$$

That is $(\{\mathcal{H}^t_j\}_t)_j$ is a Cauchy sequence in a complete metric space. Hence, it converges to $\{\mathcal{H}^t\}_t$ where $\mathcal{H}^t = \{t\mathcal{H}^{(st)}\}_s$. It follows that the map $t \longmapsto \mathcal{H}^t$ is continuous in the L^2-norm. Thus the map $t \longmapsto H^t$ is continuous in $(C^0 + L^\infty)$-topology. □

To prove Theorem A, let us recall the following inequality: given a family of closed 1-forms $(\alpha^t)_t$ together with two isotopies $\Phi = \{\phi^t\}_t$ and $\Psi = \{\psi^t\}_t$ such that $\bar{d}(\Phi, \Psi) \leqslant \dfrac{r(g)}{2}$, where $r(g)$ is the injectivity radius of the riemannian manifold (M, g), one has (see [17])

$$\max_{t\in[0,1]} osc\left(\left(\left(\int_{\mathcal{O}^{\bar{\phi}^t}_{(\cdot)}} \alpha^t\right) - \left(\int_{\mathcal{O}^{\bar{\psi}^t}_{(\cdot)}} \alpha^t\right)\right)\right) \leqslant 4 \max_{t\in[0,1]} \|\alpha^t\|_\infty \,\bar{d}(\Phi, \Psi). \tag{4.17}$$

Proof of Theorem A. Let $H = \{h^t\}_t$ be an $\mathcal{S}$-topological isotopy in $Hameo(M, \omega)$ with generator $(U, \mathcal{H})$.

- Step (1/2): Let $H = \rho \circ \nu$ be the $\mathcal{S}$-decomposition of H where $\rho : t \longmapsto \rho^t$ is a continuous isotopy in $G_\omega(M)$, and $\nu : t \longmapsto \nu^t$ is a continuous Hamiltonian isotopy (see [3,14,17]). For each t, since $h^t \in Hameo(M, \omega)$, we use the above $\mathcal{S}$-decomposition to derive that $\rho^t \in Hameo(M, \omega)$ for all t. Thus, it follows from Corollary 3.1 that for each fixed t, the path $\bar{\rho}^t : s \longmapsto \rho^{(st)}$ has its flux in $\mathcal{S}\Gamma_\omega$ (since M is of Lefschetz type).
- Step (2/2): By formula (4.17) the map $\Psi \longmapsto \widetilde{S}^0_\omega(\Psi)$ is $(C^0 + L^\infty)$-continuous and by Lemma 4.1 the map $t \longmapsto \bar{\rho}^t$ is continuous with respect to the $(C^0 + L^\infty)$-topology on $\mathcal{PSG}_\omega(M)$. Therefore, the map $t \longmapsto \widetilde{S}^0_\omega(\bar{\rho}^t)$ is continuous from the connected topological space $[0, 1]$ into the discrete space $\mathcal{S}\Gamma_\omega$ (because (M, ω) is of Lefschetz type). Thus, the latter map is constant, i.e., $\widetilde{S}^0_\omega(\bar{\rho}^t) = \widetilde{S}^0_\omega(\bar{\rho}^0) = 0$. That is, $\left[\int_0^t \mathcal{H}^s ds\right] = 0$ for all t, and we derive as in the proof of Theorem 2.1 that $\mathcal{H}^t = 0$, for each t. We have proved that the generator of the isotopy H is of the form $(U, 0)$, i.e., H is in fact a continuous Hamiltonian isotopy. □

Proof of Theorem B. Let $\aleph = \lim_{(C^0+L^\infty)} \phi_{(U_i,\mathcal{H}_i)}$ be an $\mathcal{S}$-topological isotopy such that $\widetilde{S}^0_\omega(\aleph) = 0$ and $\aleph = \rho \circ \nu$ be the $\mathcal{S}$-decomposition of $\aleph$. The path ρ is a continuous path in $G_\omega(M)$ (see [14,16]). Therefore $\rho = \{\rho^t\}_t$ is homotopic relatively to fixed endpoints to a smooth isotopy Ψ in $G_\omega(M)$ via a homotopy mapping $\Upsilon_{\rho,\Psi}$ such that $0 = \widetilde{S}^0_\omega(\aleph) = \widetilde{S}^0_\omega(\rho) = \widetilde{S}_\omega(\Psi)$ (by Theorem 4.1). Besides, it follows from a well-known result proved by McDuff-Salamon [9] and Banyaga [1] that $\Psi = \{\psi^t\}_t$ is homotopic relatively to fixed endpoints to a Hamiltonian isotopy $\Phi = \{\phi^t\}_t$. Let $\Upsilon_{\Psi,\Phi}$ be the homotopy between Ψ and Φ, and derive that the mappings $\Upsilon_{\rho,\Psi}$ and $\Upsilon_{\Psi,\Phi}$ induce a homotopy $\Upsilon_{\rho,\Phi}$ between ρ and Φ. Then, the following mapping induces a homotopy between $\aleph$ and a continuous Hamiltonian isotopy $\{\theta^t\}_t = \Theta := \Phi \circ \nu$ as follows:

$$\begin{aligned} \Upsilon_{\aleph,\Theta} : [0,1] \times [0,1] &\longrightarrow \mathcal{S}G_\omega(M) \\ (s,t) &\longmapsto \Upsilon^{(s,t)}_{\rho,\Phi} \circ \nu^t, \end{aligned}$$

with $\Upsilon^{(0,t)}_{\aleph,\Theta} = \Upsilon^{(0,t)}_{\rho,\Phi} \circ \nu^t = \rho^t \circ \nu^t = \aleph^t$, $\Upsilon^{(1,t)}_{\aleph,\Theta} = \Upsilon^{(1,t)}_{\rho,\Phi} \circ \nu^t = \phi^t \circ \nu^t = \theta^t$, for each t $\Upsilon^{(s,0)}_{\aleph,\Theta} = \Upsilon^{(s,0)}_{\rho,\Phi} \circ id_M = id_M$ and $\Upsilon^{(s,1)}_{\aleph,\Theta} = \Upsilon^{(s,1)}_{\rho,\Phi} \circ \nu^1 = \phi^1 \circ \nu^1 = \theta^1$, for each s. □

Acknowledgements

The authors are thankful to the anonymous reviewers for their profound and contribuable comments and to the editor for considering their work. The first author would like to thank the University of Buea for the financial support of his works through the faculty of science research grant. The second and third authors express their gratitude to CEA-SMIA for their financial support.

References

[1] A. Banyaga, The Structure of Classical Diffeomorphisms Groups, Mathematics and Its Applications, vol. 400, Kluwer Academic Publisher's Group, Dordrecht, the Netherlands, 1997.
[2] A. Banyaga, On the group of strong symplectic homeomorphisms, C. R. Math. Acad. Sci. Paris 346 (15–16) (2008) 867–872.
[3] A. Banyaga, S. Tchuiaga, Uniqueness of generators of strong symplectic isotopies, IMHOTEP-Math. J. 1 (2016) 7–23.
[4] A. Banyaga, S. Tchuiaga, The group of strong symplectic homeomorphisms in L^∞-metric, Adv. Geom. 14 (3) (2014) 523–539.
[5] D.B.A. Epstein, The simplicity of certain groups of homeomorphisms, Compos. Math. 22 (2) (1969) 93–100.
[6] A. Fathi, Structure of the group of homeomorphism preserving a good measure on a compact manifold, Ann. Sci. Éc. Norm. Supér. 13 (1980) 45–93.
[7] M.R. Herman, Sur le groupe des difféomorphismes du tore, Ann. Inst. Fourier 23 (2) (1973) 75–86.
[8] J.N. Mather, Commutator of diffeomorphisms I, Comment. Math. Helv. 49 (1974) 512–528; Commutator of diffeomorphisms II, Comment. Math. Helv. 50 (1975) 33–40.
[9] D. McDuff, D. Salamon, Introduction to Symplectic Topology, second ed., Oxford Mathematical Monographs, Oxford University Press, New York, 1998.
[10] K. Ono, Floer-Novikov cohomology and the flux conjecture, Geom. Funct. Anal. 16 (5) (2006) 981–1020.
[11] Y-G. Oh, S. Müller, The group of Hamiltonian homeomorphisms and C^0-symplectic topology, J. Symplectic Geom. 5 (2007) 167–225.
[12] L. Polterovich, The Geometry of the Group of Symplectic Diffeomorphism, Lecture in Mathematics ETH Zürich, Birkhäuser Verlag, Basel-Boston, 2001.
[13] S. Seyfadini, A note on C^0-rigidity of Hamiltonian isotopies, J. Symplectic Geom. 11 (3) (2013) 489–496.
[14] S. Tchuiaga, Some structures of the group of strong symplectic homeomorphisms, Glob. J. Adv. Res. Class. Mod. Geom. 2 (1) (2013) 36–49.
[15] S. Tchuiaga, Hofer-like geometry and flux theory, J. Dyn. Syst. Geom. Theories 19 (2) (2021) 227–270, https://doi.org/10.1080/172603X.2021.2011110.
[16] S. Tchuiaga, C^0-symplectic geometry under displacements, J. Dyn. Syst. Geom. Theories 179 (1) (2019) 109–129, https://doi.org/10.1080/1726037X.2018.1551717.
[17] S. Tchuiaga, On symplectic dynamics, Differ. Geom. Appl. 61 (2018) 170–196.
[18] S. Tchuiaga, et al., On topological symplectic dynamical systems, CUBO 19 (2) (2017) 49–71.
[19] W. Thurston, Foliations and group of diffeomorphisms, Bull. Am. Math. Soc. 80 (1974) 304–307.