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Research Article

A Posteriori Error Estimates for a Nonconforming Finite Element Discretization of the Stokes–Biot System

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This paper presents an a posteriori error estimator for a (piecewise linear) nonconforming finite element approximation of the problem defining the interaction between a free fluid and poroelastic structure. The free fluid is governed by the Stokes equations, while the flow in the poroelastic medium is modeled using the Biot poroelasticity system. Equilibrium and kinematic conditions are imposed on the interface. The approach utilizes the same nonconforming Crouzeix–Raviart element discretization on the entire domain. For this discretization, we derive a residual indicator based on the jumps of the normal derivative of the nonconforming approximation. Lower and upper bounds form the main results with minimal assumptions on the mesh.

1. Introduction

In this paper, we develop an a posteriori error analysis for solving the interaction of a free incompressible viscous Newtonian fluid with a fluid within a poroelastic medium. This is a challenging multiphysics problem with applications to predicting and controlling processes arising in groundwater flow in fractured aquifers, oil and gas extraction, arterial flows, and industrial filters. In these applications, it is important to model properly the interaction between the free fluid with the fluid within the porous medium and to take into account the effect of the deformation of the medium. For example, geomechanical effects play an important role in hydraulic fracturing, as well as in modeling phenomena such as subsidence and compaction.

We adopt the Stokes equations to model the free fluid and the Biot system [1] for the fluid in the poroelastic media. In the latter, the volumetric deformation of the elastic porous matrix is complemented with the Darcy equation that describes the average velocity of the fluid in the pores. The model features two different kinds of coupling across the interface: Stokes–Darcy coupling [2–10] and fluid-structure interaction (FSI) [11–15].

The well-posedness of the mathematical model based on the Stokes–Biot system for the coupling between a fluid and a poroelastic structure is studied in [16–19]. A numerical study of the problem, using Navier–Stokes equations for the fluid, is presented in [11, 20], utilizing a variational multiscale approach to stabilize the finite element spaces. The problem is solved using both a monolithic and a partitioned approach, with the latter requiring subiterations between the two problems.

Nonphysical pressure oscillations are observed in finite element calculations of Biot's poroelastic equations in low-permeable media. These pressure oscillations may be understood as a failure of compatibility between the finite element spaces, rather than elastic locking. In [21], Joachim Berdal Haga et al. have presented evidence to support this view by comparing and contrasting the pressure oscillations in low-permeable porous media with those in low-compressible porous media. As a consequence, it is possible to use established families of stable mixed elements as candidates for choosing finite element spaces for Biot's equations. Through comparison with the displacement-solid pressure mixed formulation of linear elasticity, they identify the spurious pressure modes as a specific consequence of a vanishing Brezzi inf-sup constant. Since the Brezzi inf-sup condition for the poroelastic equations takes on a similar form as in, e.g., the mixed linear elasticity or Stokes problem, this identification opens up the field to a plethora of stable element candidates. These can be used directly for the basic solid displacement-fluid pressure two-field formulation of poroelasticity, or in combinations for the various three- and four-field formulations involving solid pressure and/or fluid velocity [21].

Finite element analysis of an arbitrary Lagrangian–Eulerian method for Stokes/parabolic moving interface problem with jump coefficients has been studied in [22]. The authors in [23] study a numerical solution of the coupled system of the time-dependent Stokes and fully dynamic Biot equations. They establish the stability of the scheme and derive error estimates for the fully discrete coupled scheme. Numerical errors and convergence rates for smooth problems as well as tests on realistic material parameters have been presented. In [24], Jing Wen and Yinnian He consider a strongly conservative discretization for the rearranged Stokes–Biot model based on interior penalty discontinuous Galerkin method and mixed finite element method. The existence and uniqueness of a solution of the numerical scheme have been presented. Then, the analysis of stability and priori error estimates have been derived. The numerical examples under uniform meshes, which will validate the analysis of convergence and the strong mass conservation are presented. A staggered finite element procedure for the coupled Stokes–Biot system with fluid entry resistance has been studied by Bergkamp et al. in [25], while Ambartsumyan et al. study in [26] flow and transport in fractured poroelastic media using Stokes flow in the fractures and the Biot model in the porous media. In [27], semidiscrete continuous-in-time approximation has been proposed for the weak coupled mixed formulation. For the discretization of the fluid velocity and pressure, the authors have used the finite elements which include the MINI-elements, the Taylor-Hood elements, and the conforming Crouzeix–Raviart elements. For the discretization of the porous medium problem, they choose spaces that include Raviart–Thomas and Brezzi–Douglas–Marini elements. An a priori error analysis is performed with some numerical tests confirming the convergence rates.

A posteriori error estimators are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. They are essential to design adaptive mesh refinement algorithms which equi-distribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuška and Rheinboldt [28–31], adaptive finite element methods based on a posteriori error estimates have been extensively investigated.

In the article [32], the author studies a stabilized nonconforming mixed finite element method using the Crouzeix–Raviart element for the Stokes–Biot problem. Considering the mixed formulation of the Darcy problem, the fluid velocity and pressure are treated as functions defined in the entire domain. Existence, uniqueness of the finite element solution of the corresponding discrete problem, and a priori estimates have been shown. The proofs use the standard theory for mixed problems. The approach presented is independent of the normal vectors of the interior edges in both regions, thus making the resulting finite element matrix sparser.

We use a nonconforming finite element method that has so many advantages for the velocities and piecewise constant for the pressures in both the Stokes and Biot regions and apply a stabilization term penalizing the jumps over the element edges of the piecewise continuous velocities. Indeed, one can construct finite element methods where the incompressibility condition is exactly satisfied (cf. Fortin [33]), but this leads to the use of complex elements of limited applicability (e.g., oil and gas extraction for conforming case). Thus, in the work [32], Houédanou has constructed and studied the finite element method using simpler elements where the incompressibility condition is only approximatively satisfied (cf. definition of an operator div_h (22)).

So, in this paper, we have found it very convenient to use nonconforming finite elements which violate the interelement continuity condition of the velocities.

In the work [34, 35], we develop an a posteriori error analysis of a conforming mixed finite element method for solving the coupled problem arising in the interaction between a free fluid and a fluid in a poroelastic medium on isotropic meshes in $\mathbb{R}^d$. The approach utilizes a Lagrange multiplier method to impose weakly the interface conditions. To our best knowledge, there is no a posteriori error estimation for the strongly coupled mixed formulation (19) ([32], Section 3) of the coupled Stokes–Biot problem where a nonconforming finite element method is used. Here, we develop such a posteriori error analysis. The technique used to derive the a posteriori error estimator is the residue-based approach developed by Verfürth [36] in the late 80s and 90s. This paper could have several applications in dynamical systems, Wavelet analysis, and generalized harmonic analysis [37–39].

This article is mainly composed of six sections. Some preliminaries and notations are given in Section 2. In Section 3, the a posteriori error estimators are derived. The efficiency result is derived using the technique of bubble function introduced by Verfürth [36] and used in a similar context by Carstensen [40, 41]. These analytical tools are recalled in Section 4. The main results are given in Section 5. We offer our conclusion and the further works in Section 6.

2. Preliminaries and Notation

2.1. Model Problem. We consider a multiphysics model problem for free fluid's interaction with a flow in a deformable porous media, where the simulation domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a union of nonoverlapping regions Ω_f and Ω_p. Here, Ω_f is a free fluid region with flow governed by the Stokes equations and Ω_p is a poroelastic material governed by the Biot system. For simplicity of notation, we assume that each region is connected. The extension to nonconnected regions is straightforward. The two regions are separated by an interface $\Gamma_{fp} = \partial\Omega_f \cap \partial\Omega_p$. Let $\Gamma_\star = \partial\Omega_\star \setminus \Gamma_{fp}$, $\star = f, p$. Each interface and boundary is assumed to be polygonal ($d = 2$) or polyhedral ($d = 3$). We denote by n_f (resp., n_p) the unit outward normal vector along $\partial\Omega_f$ (resp., Ω_p). Note that, on the interface Γ_{fp}, we have $n_f = -n_p$. Figure 1 gives a schematic representation of the geometry.

For any function v defined in Ω, since its restriction to Ω_f or Ω_p could play a different mathematical role (for instance, their traces on Γ_{fp}), we will set $v_f = v_{|\Omega_f}$ and $v_p = v_{|\Omega_p}$. In Ω, we denote by u the fluid velocity and by p the pressure and let η_p be the displacement in Ω_p. Let $\mu > 0$ be the fluid viscosity, let $f \in [L^2(\Omega)]^d$ be the body force terms, and let g be the external source or sink terms satisfying the compatibility condition $\int_\Omega g(x)dx = 0$.

Let $D(u)$ and $\sigma_f(u, p)$ denote, respectively, the deformation rate tensor and the stress tensor:

$$D(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right), \text{and } \sigma_f(u, p) = -pI + 2\mu D(u). \quad (1)$$

Equations (1)–(13) consist of the model of the coupled Stokes and Biot flow problem that we will study in this paper.

In the free fluid region Ω_f, (u, p) satisfy the Stokes equations:

$$-\nabla \cdot \sigma_f(u, p) = f \text{ in } \Omega_f, \quad (2)$$

$$\nabla \cdot u = g \text{ in } \Omega_f \quad (3)$$

$$u = 0 \text{ on } \Gamma_f. \quad (4)$$

Let $\sigma_e(\eta_p)$ and $\sigma_p(\eta_p, p_p)$ be the elastic and poroelastic stress tensors, respectively:

$$\sigma_e(\eta_p) = \lambda_p(\nabla \cdot \eta_p)I + 2\mu_p D(\eta_p), \ \sigma_p(\eta_p, p_p) = \sigma_e(\eta_p) - \alpha p_p I, \quad (5)$$

where $0 < \lambda_{\min} \le \lambda_p(x) \le \lambda_{\max}$ and $0 < \mu_{\min} \le \mu_p(x) \le \mu_{\max}$ are the Lamé parameters, and $0 < \alpha \le 1$ is the Biot–Willis constant. The poroelasticity region Ω_p is governed by the modified static Biot system [27]:

$$-\nabla \cdot \sigma_p(\eta_p, p_p) = f \text{ in } \Omega_p, \quad (6)$$

$$\mu K^{-1} u + \nabla p = 0 \text{ in } \Omega_p, \quad (7)$$

$$\alpha \nabla \cdot \eta_p + \nabla \cdot u = g \text{ in } \Omega_p, \quad (8)$$

$$u \cdot n_d = 0 \text{ on } \Gamma_p, \quad (9)$$

$$\eta_p = 0 \text{ on } \Gamma_p. \quad (10)$$

K is the symmetric and uniformly positive definite rock permeability tensor, satisfying, for some constants $0 < k_{\min} \le k_{\max}$,

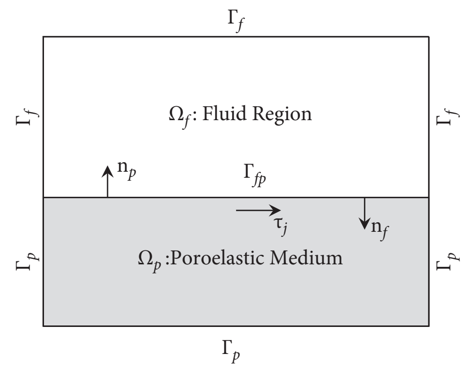

Figure 1: Global domain Ω consisting of the fluid region Ω_f and the poroelastic media region Ω_p separated by the interface Γ_{fp}.

$$\forall \xi \in \mathbb{R}^d, k_{\min} \xi^T \xi \le \xi^T K(x) \xi \le k_{\max} \xi^T \xi, \forall x \in \Omega_p. \quad (11)$$

Following [1], the interface conditions on the fluid-poroelasticity interface Γ_{fp} are mass conservation, the balance of stresses, and the Beavers–Joseph–Saffman (BJS) condition [42] modeling slip with friction are as follows:

$$u_f \cdot n_f + u_p \cdot n_p = 0 \text{ on } \Gamma_{fp}, \quad (12)$$

$$\sigma_f n_f + \sigma_p n_p = 0 \text{ on } \Gamma_{fp}, \quad (13)$$

$$-\left(\sigma_f n_f\right) \cdot n_f = p_p, \text{ on } \Gamma_{fp}, \quad (14)$$

$$-\left(\sigma_f n_f\right) \cdot \tau_{f,j} = \mu \alpha_{BJF} \sqrt{K_j^{-1}}\left(u_f\right) \cdot \tau_{f,j} \text{ on } \Gamma_{fp}, \quad (15)$$

where $\tau_{f,j}$, $1 \le j \le d-1$, is an orthogonal system of unit tangent vectors on Γ_{fp}, $K_j = (K\tau_{f,j}) \cdot \tau_{f,j}$, and $\alpha_{BJS} \ge 0$ is an experimentally determined friction coefficient. We note that continuity of flux constraints the normal velocity of the solid skeleton, while the BJS condition accounts for its tangential velocity.

2.2. Strongly Coupled Weak Formulation. We begin this section by introducing some useful notations. We first introduce some Sobolev spaces [43] and norms. If W is a bounded domain of $\mathbb{R}^d$ and m is a nonnegative integer, the Sobolev space $H^m(W) = W^{m,2}(W)$ is defined in the usual way with the usual norm $\|\cdot\|_{m,W}$ and seminorm $|.|_{m,W}$. In particular, $H^0(W) = L^2(W)$ and we write $\|\cdot\|_W$ for $\|\cdot\|_{0,W}$. Similarly, we denote by $(\cdot,\cdot)_W$ the $L^2(W)[L^2(W)]^d$ or $[L^2(W)]^{d\times d}$ inner product. For shortness, if W is equal to Ω, we will drop the index Ω, while for any $m \ge 0$, $\|\cdot\|_{m,\star} = \|\cdot\|_{m,\Omega_\star}$, $|.|_{m,\star} = |.|_{m,\Omega_\star}$ and $(.,.)_\star = (.,.)_{\Omega_\star}$, for $\star = f, p$. The space $H_0^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Let $[H^m(\Omega)]^d$ be the space of vector-valued functions $v = (v_1, \ldots, v_d)$ with components v_i in $H^m(\Omega)$. The norm and the semi-norm on $[H^m(\Omega)]^d$ are given by

$$\|v\|_{m,\Omega} := \left(\sum_{i=0}^{d} \|v_i\|_{m,\Omega}^2\right)^{(1/2)},$$
$$|v|_{m,\Omega} := \left(\sum_{i=0}^{d} |v_i|_{m,\Omega}^2\right)^{(1/2)}. \tag{16}$$

For a connected open subset of the boundary $E \subset \partial\Omega_f \cup \partial\Omega_p$, we write $\langle .,. \rangle_E$ for the $L^2(E)$ inner product (or duality pairing); that is, for scalar-valued functions λ, σ one defines

$$\langle \lambda, \sigma \rangle_E := \int_E \lambda\sigma ds. \tag{17}$$

For an open subset F of the entire domain Ω, i.e., $F \subseteq \Omega$, we define the space $H(\text{div}; F)$ by

$$H(\text{div}; F) := \left\{ v \in \left[L^2(F)\right]^d : \text{div} v \in L^2(F) \right\}, \tag{18}$$

with a norm

$$\|v\|_{H(\text{div};F)} := \left(\|v\|_{[L^2(F)]^d}^2 + \|\text{div} v\|_{L^2(F)}^2 \right)^{(1/2)}, \quad \forall v \in H(\text{div}; F). \tag{19}$$

To present a variational form of the coupled problem, we define the following three spaces for the velocity u, the structure displacement η_p, and the pressure:

$$H := \left\{ v \in H(\text{div}; \Omega) : v_f \in \left[H^1(\Omega_f)\right]^d, v = 0 \text{on} \Gamma_f, v \cdot n_p = 0 \text{on} \Gamma_p \right\}, \tag{20}$$

equipped with the norm:

$$\|v\|_H := \left(|v|_{1,f}^2 + \|v\|_{H(\text{div};\Omega_p)}^2 \right)^{(1/2)},$$
$$X_p := \left\{ \xi_p \in \left[H^1(\Omega_p)\right]^d : \xi_p = 0 \text{on} \Gamma_p \right\}, \tag{21}$$

with the norm

$$\left\|\xi_p\right\|_{X_p} := \left|\xi_p\right|_{1,p}, \tag{22}$$

$$\mathbb{M} := {}_0^2(\Omega) \times L_0^2(\Omega_p), \tag{23}$$

equipped with the norm $\|Q\|_{\mathbb{M}} := (\|Q_1\|_{0,\Omega}^2 + \|Q_2\|_{0,\Omega_p}^2)^{(1/2)}$, $\forall Q = (Q_1, Q_2) \in \mathbb{M}$.

Note that the vector-valued functions in H have (weakly) continuous normal components on Γ_{fp} (a consequence of Theorem 1.2.5 of [44], p. 27).

We set $\mathbb{H} = H \times X_p$ equipped with the product norm

$$\|V\|_{\mathbb{H}} := \|v\|_H + \left\|\xi_p\right\|_{X_p}, \forall V = (v, \xi_p) \in \mathbb{H}. \tag{24}$$

Let us further introduce two bilinear forms:

$$\begin{aligned}
&A: \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{R}, (U, V) \mapsto A(U, V) \text{defineby},\\
&A(U, V) := (2\mu D(u), D(v))_{\Omega_f} + \left(\mu K^{-1} u, v\right)_{\Omega_p} + \left(2\mu_p D(\eta_p), D(\xi_p)\right)_{\Omega_p}\\
&\qquad + \left(\lambda_p \nabla \cdot \eta_p, \nabla \cdot \xi_p\right)_{\Omega_p} + \sum_{j=1}^{d-1} \langle \mu\alpha_{BJS} \sqrt{K_j^{-1}} u_f \cdot \tau_{f,j}, v_f \cdot \tau_{f,j} \rangle_{\Gamma_{fp}},\\
&B: \mathbb{H} \times \mathbb{M} \longrightarrow \mathbb{R}, (V, Q) \mapsto B(V, Q) \text{with},\\
&B(V, Q) := -(Q_1, div v)_\Omega - \alpha\left(Q_2, div \xi_p\right)_{\Omega_p}, \text{where} Q = (Q_1, Q_2),
\end{aligned} \tag{25}$$

and two linear forms

$$L: \mathbb{H} \longrightarrow \mathbb{R}, V \mapsto L(V) := (f, v)_\Omega, \tag{26}$$

$$G: \mathbb{M} \longrightarrow \mathbb{R}, Q = (Q_1, Q_2) \mapsto G(Q) := -(g, Q_1)_\Omega. \tag{27}$$

The weak formulation of the coupled problem (1)–(13) can be stated as follows: find $(U, P) \in \mathbb{H} \times \mathbb{M}$ with $U = (u, \eta_p)$ and $P = (p, p_p)$ such that

$$\begin{cases} A(U, V) + B(V, P) = L(V) \forall V = (v, \xi_p) \in \mathbb{H} \\ B(U, Q) = G(Q) \forall Q = (Q_1, Q_2) \in \mathbb{M}. \end{cases} \tag{28}$$

Note that, if $\mathbf{f}$ and g are of mean zero, (28) directly implies that (1)–(11) hold (the differential equations being understood in the distributional sense), while the interface conditions (12) and (13) are imposed in a weak sense.

This problem has a unique solution as proved in ([32], Theorem 3.1).

Theorem 1. *If $f \in [L^2(\Omega)]^d$ and $g \in L_0^2(\Omega)$, then there exists a unique solution $(U, P) \in \mathbb{H} \times \mathbb{M}$ to problem (19).*

2.3. Discontinuous Galerkin Discretization. In this section, we will use the nonconforming Crouzeix–Raviart piecewise linear finite element approximation for velocity and piecewise constant approximation for pressure and establish the existence and uniqueness of a finite element solution of the discrete problem.

Let $\mathcal{T}_h$ be a family of triangulations of $\overline{\Omega}$ with nondegenerate elements (i.e. triangles for $d = 2$ and tetrahedrons for $d = 3$). For any $K \in \mathcal{T}_h$, we denote by h_K the diameter of K and ρ_K the diameter of the largest ball inscribed into K.

We set

$$h = \max_{K\in\mathcal{T}_h} h_K, \text{and} \sigma_h = \max_{T\in\mathcal{T}_h} \frac{h_K}{\rho_K}. \tag{29}$$

We assume that the family of triangulations is regular, in the sense that there exists $\sigma_0 > 0$ such that $\sigma_h \leqslant \sigma_0$, for all $h > 0$. We also assume that the triangulation is conforming with respect to the partition of Ω into Ω_f and Ω_p, namely, each $K \in \mathcal{T}_h$ is either in Ω_f or in Ω_p (see Figures 2–4 for illustration).

Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be the corresponding induced triangulations of Ω_f and Ω_p. For any $K \in \mathcal{T}_h$, we denote by $E(K)$

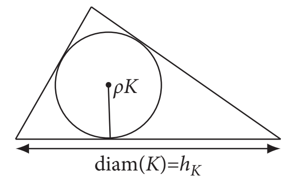

Figure 2: Isotropic element K in $\mathbb{R}^2$.

(resp. $\mathcal{N}(K)$) the set of its edges ($d = 2$) or faces ($d = 3$) (resp. vertices) and set $\mathrm{E}_h = \cup_{K\in\mathcal{T}_h} E(K)$, $\mathcal{N}_h = \cup_{K\in\mathcal{T}_h}\mathcal{N}(K)$. For $\mathcal{A} \subset \overline{\Omega}$, we define

$$\mathrm{E}_h(\mathcal{A}) = \{E \in \mathrm{E}_h \colon E \subset \mathcal{A}\}. \tag{30}$$

Notice that $\mathcal{E}_h$ can be split up in the form

$$\mathcal{E}_h = \mathcal{E}_h\left(\Omega_f^+\right) \cup \mathcal{E}_h\left(\Omega_p\right) \cup \mathcal{E}_h\left(\partial\Omega_p\right), \tag{31}$$

where $\Omega_f^+ = \Omega_f \cup \Gamma_f$. Note that $\mathcal{E}_h(\Gamma_{fp})$ is included in $\mathcal{E}_h(\partial\Omega_p)$.

With every edge $E \in \mathcal{E}_h$, we associate a unit vector n_E such that n_E is orthogonal to E and equals to the unit exterior normal vector to $\partial\Omega$ if $E \subset \partial\Omega$. For any $E \in \mathcal{E}_h$ and any piecewise continuous function φ, we denote by $[\varphi]_E$ its jump across E in the direction of n_E:

$$[\varphi]_E(x) \coloneqq \begin{cases} \lim_{t\longrightarrow 0+}\varphi(x + tn_E) - \lim_{t\longrightarrow 0+}\varphi(x - tn_E), & \text{for an interior (edge/face } E), \\ -\lim_{t\longrightarrow 0+}\varphi(x - tn_E), & \text{for a boundary (edge/face } E). \end{cases} \tag{32}$$

Based on the above notations, we introduce a variant of the nonconforming Crouzeix–Raviart piecewise linear finite element space:

$$\begin{aligned} H_h &\coloneqq \left\{v_h \in \left[L^2(\Omega)\right]^d \colon v_{h|K} \in \left[\mathbb{P}^1(K)\right]^d \forall K \in \mathcal{T}_h, \left([v_h]_E, 1\right)_E = 0 \forall E \in \mathrm{E}_h\left(\Omega_f^+\right), \right. \\ &\left. \left([v_h \cdot n_E]_E, 1\right)_E = 00.2cm \forall E \in \mathrm{E}_h\left(\Omega_p\right) \cup \mathrm{E}_h\left(\partial\Omega_p\right)\right\}, \\ X_{ph} &\coloneqq \left\{\xi_{ph} \in \left[L^2\left(\Omega_p\right)\right]^d \colon \xi_{ph|K} \in \left[\mathbb{P}^1(K)\right]^d \forall K \in \mathcal{T}_h^p, \left(\left[\xi_{ph}\right]_E, 1\right)_E = 0 \forall E \in \mathrm{E}_h\left(\overline{\Omega}_p\right)\right\}. \end{aligned} \tag{33}$$

For $X \subseteq \Omega$, we set

$$E_h(X) \coloneqq \left\{q_h \in L_0^2(X) \colon q_{h|K} \in \mathbb{P}^0(K) \forall K \subset X, K \in \mathcal{T}_h\right\}, \tag{34}$$

and we define

$$\begin{aligned} \mathbb{M}_h &\coloneqq E_h(\Omega) \times E_h\left(\Omega_p\right) \subset \mathbb{M}, \\ \mathbb{H}_h &\coloneqq H_h \times X_{ph} \not\subset \mathbb{H}, \end{aligned} \tag{35}$$

where $\mathbb{P}^m(K)$ is the space of the restrictions to K of all polynomials of degree less than or equal to m.

The space $\mathbb{M}_h$ is equipped with the norm $\|\cdot\|_{\mathbb{M}}$ while the norm on $\mathbb{H}_h$ will be specified later on. The choice of H_h is more natural since the space H_h approximates only $H(\mathrm{div}; \Omega_p)$ and not $[H^1(\Omega_p)]^d$, while our a posteriori error analysis is only valid in this larger space.

Let us introduce the discrete divergence operator $\mathrm{div}_h \in \mathcal{L}(H_h; E_h(\Omega)) \cap \mathcal{L}(H; L_0^2(\Omega))$ by

$$\left(\mathrm{div}_h v_h\right)_{|K} = \mathrm{div}\left(v_{h|K}\right), \forall K \in \mathcal{T}_h, \tag{36}$$

or $\mathrm{div}_h \in \mathcal{L}(X_{ph}; E_h(\Omega_p)) \cap \mathcal{L}(X_p; L_0^2(\Omega_p))$ by

$$\left(\mathrm{div}_h \xi_{ph}\right)_{|K} = \mathrm{div}\left(\xi_{ph|K}\right), \forall K \in \mathcal{T}_h^p. \tag{37}$$

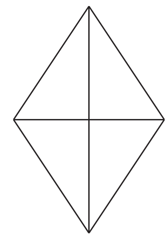

Figure 3: Example of conforming mesh in $\mathbb{R}^2$.

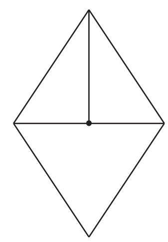

Figure 4: Example of nonconforming mesh in $\mathbb{R}^2$.

Then, for $U_h = (u_h, \eta_{ph}) \in \mathbb{H}_h$, $V_h = (v_h, \xi_{ph}) \in \mathbb{H}_h$ and $Q_h = (Q_{1h}, Q_{2h}) \in \mathbb{M}_h$, we can introduce the following two bilinear forms:

$$\begin{aligned} A_h(U_h, V_h) &:= \sum_{K \in \mathcal{T}_h^f} (2\mu D(u_h), D(v_h))_K + \left(\mu K^{-1} u_h, v_h\right)_{\Omega_p} + \sum_{K \in \mathcal{T}_h^p} \left(2\mu_p D\left(\eta_{ph}\right), D\left(\xi_{ph}\right)\right)_K \\ &\quad + \left(\lambda_p \operatorname{div}_h \eta_{ph}, \operatorname{div}_h \xi_{ph}\right)_{\Omega_p} + \sum_{j=1}^{d-1} \langle \mu \alpha_{BJS} \sqrt{K_j^{-1}}\, u_{fh} \cdot \tau_{f,j}, v_{fh} \cdot \tau_{f,j} \rangle_{\Gamma_{fp}}, \\ B_h(V_h, Q_h) &:= -(Q_{1h}, \operatorname{div}_h v_h)_\Omega - \alpha\left(Q_{2h}, \operatorname{div}_h \xi_{ph}\right)_{\Omega_p}. \end{aligned} \tag{38}$$

Then, we propose the following discrete problem: find $(U_h, P_h) \in \mathbb{H}_h \times Q_h$ with $U_h = (u_h, \eta_{ph}) \in H_h \times X_{ph}$ and $P_h = (p_h, p_{ph}) \in E_h(\Omega) \times E_h(\Omega_p)$ such that

$$\begin{cases} A_h(U_h, V_h) + B_h(V_h, P_h) + J(U_h, V_h) =, & L(V_h) \forall V_h \in \mathbb{H}_h, \\ B_h(U_h, Q_h) =, & G(Q_h) \forall Q_h \in \mathbb{M}_h. \end{cases} \tag{39}$$

This is the natural discretization of the weak formulation (19) only with the penalizing term $J(U_h, V_h)$ added, where $V_h = (v_h, \xi_{ph})$. We define the bilinear form $J(\cdot,\cdot)$ following the decomposition of $\mathcal{E}_h$:

$$J(U_h, V_h) = J_{\Omega_f^+}(u_h, v_h) + J_{\Omega_p}(u_h, v_h) + J_{\partial\Omega_p}(u_h, v_h) + J_{\Omega_p^+}\left(\eta_{ph}, \xi_{ph}\right), \tag{40}$$

where

$$J_{\Omega_f^+}(u_h, v_h) := (1+2\mu) \sum_{E\in E_h(\Omega_f^+)} h_E^{-1} \int_E [u_h]_E \cdot [v_h]_E ds,$$
$$J_{\Omega_p}(u_h, v_h) := \sum_{E\in E_h(\Omega_p)} h_E^{-1} \int_E [u_h]_E \cdot [v_h]_E ds,$$
$$J_{\partial\Omega_p}(u_h, v_h) := \sum_{E\in E_h(\partial\Omega_p)} h_E^{-1} \int_E [u_h \cdot n_E]_E [v_h \cdot n_E]_E ds 0.2\text{cmand}$$
$$J_{\Omega_p^+}(\eta_{ph}, \xi_{ph}) := \sum_{E\in E_h(\Omega_p^+)} h_E^{-1} \int_E (1+2\mu_p)[\eta_{ph}]_E \cdot [\xi_{ph}]_E ds. \quad (41)$$

Here, h_E is the length $(d=2)$ or diameter $(d=3)$ of E. Note that each element of $\mathscr{E}_h$ only contributes with one jump term in $J(U_h, V_h)$.

We are now able to define the norm on $\mathbb{H}_h$:

$$\|V_h\|_{\mathbb{H}_h} := \left[\|v_h\|_{H_h}^2 + \|\xi_{ph}\|_{X_{ph}}^2 + J(V_h, V_h)\right]^{(1/2)}, \quad (42)$$

where

$$\|v_h\|_{H_h} := \left(\sum_{K\in\mathscr{T}_h^f} |v_h|_{1,K}^2 + \sum_{j=1}^{d-1} \langle v_{fh}\cdot\tau_j, v_{fh}\cdot\tau_j\rangle_{\Gamma_{fp}} + \|v_h\|_{\Omega_p}^2 + \|\text{div}_h v_h\|_{\Omega_p}^2\right)^{(1/2)}, \quad (43)$$

$$\|\xi_{ph}\|_{X_{ph}} := \left(\sum_{K\in\mathscr{T}_h^p} |\xi_{ph}|_{1,K}^2\right)^{(1/2)}. \quad (44)$$

The following results hold (Theorems 4.1 and 5.1 in 32).

Theorem 2. *There exists a unique solution* $(U_h, P_h) \in \mathbb{H}_h \times \mathbb{M}_h$ *to a discrete problem (24) and if the solution* $(U, P) \in \mathbb{H} \times \mathbb{M}$ *of the continuous problem (19) is smooth enough, then we have*

$$\|U - U_h\|_{\mathbb{H}_h \cup \mathbb{H}} + \|P - P_h\|_{\mathbb{M}} \lesssim h\left(|u|_{2,\Omega_f} + |u|_{2,\Omega_p} + |\eta_p|_{2,\Omega_p} + |p|_{1,\Omega_f} + |p|_{1,\Omega_p}\right). \quad (45)$$

Here and below, in order to avoid excessive use of constants, the abbreviation $x \lesssim y$ *stands for* $x \le cy$*, with* c *a positive constant independent of* x, y *and* $\mathscr{T}_h$.

3. Error Estimators

In order to solve the Stokes–Biot coupled problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriate indicators. In this section, we first define the local and global indicators, and then the lower and upper error bounds are derived (see Sections 5.2.1 and 5.2.2).

3.1. Residual Error Estimators. The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier, and that involve the data at hand. To this end, the exact element residuals are denoted by

$$\begin{aligned}
R_1 &= f + \nabla\cdot\sigma_f(u_h, p_h) \text{in} K \in \mathscr{T}_h^f,\\
R_2 &= g - \nabla\cdot u_h \text{in} K \in \mathscr{T}_h^f,\\
R_3 &= \mu K^{-1} u_h + \nabla p_h \text{in} K \in \mathscr{T}_h^p,\\
R_4 &= f + \nabla\cdot\sigma_p(\eta_{ph}, p_{ph})) \text{in} K \in \mathscr{T}_h^p,\\
R_5 &= g - \alpha\nabla\cdot\eta_{ph} - \nabla\cdot u_h \text{in} K \in \mathscr{T}_h^p,\\
R_6(j) &= \mu\alpha_{BJF}\sqrt{K_j^{-1}}(u_{fh})\cdot\tau_{f,j} + (\sigma_f(u_h, p_h)n_f)\cdot\tau_{f,j} \text{on} E \in E_h(\partial K \cap \overline{\Gamma}_{fp}),\\
R_7 &= p_{ph} + (\sigma_f(u_{fh}, p_h)n_f)\cdot n_f \text{on} E \in E_h(\partial K \cap \overline{\Gamma}_{fp}).
\end{aligned} \quad (46)$$

As it is common, these exact residuals are replaced by some finite-dimensional approximation called approximate element residual $r_{i,K}$, $i \in \{1, 4\}$:

$$r_{i,K} \in [\mathbb{P}^m(K)]^d \text{on} K \in \mathcal{T}_h^l, l \in \{f, p\}. \quad (47)$$

This approximation is here achieved by projecting f on the space of piecewise constant functions in Ω_l, more precisely for all $K \in \mathcal{T}_h$, we take

$$f_{K,l} = \frac{1}{|K|}\int_K f(x)dx, l \in \{f, p\}, \forall K \in \mathcal{T}_h^l. \quad (48)$$

Finally, the global function f_h is defined by

$$f_{h,l} = f_{K,l} \text{ in } K, \forall K \in \mathcal{T}_h^l. \quad (49)$$

Hence,

$$\begin{aligned} r_{1,K} &:= f_{K,f} + \nabla \cdot \sigma_f(u_h, p_h) \text{in} K \in \mathcal{T}_h^f, \\ r_{4,K} &:= f_{K,p} + \nabla \cdot \sigma_p(\eta_{ph}, p_{ph}) \text{in} K \in \mathcal{T}_h^p, \end{aligned} \quad (50)$$

with $u_{l,h} := u_{h|\Omega_l}$ and $p_{l,h} := p_{h|\Omega_l}, l = f, p$.

Next, the gradient jump in a normal direction is introduced by the following:

$$J_{E,n_E} := \begin{cases} [\sigma_f(u_h, p_h) \cdot n_E]_E, & \text{for an interior (edge/face)} E \in \mathrm{E}_h(\Omega_f), \\ 0, & \text{for a boundary (edge/face)} E \in \mathrm{E}_h(\Gamma_f), \end{cases} \quad (51)$$

$$G_{E,n_E} := \begin{cases} [\sigma_p(\eta_{p,h}, p_{p,h}) \cdot n_E]_E, & \text{for an interior (edge/face)} E \in E_h(\Omega_p), \\ 0, & \text{for a boundary (edge/face)} E \in E_h(\Gamma_p). \end{cases} \quad (52)$$

Definition 1. (residual error estimator). The residual error estimator is locally defined by

$$\Upsilon_K := \left(\sum_{i=1}^{11} \Upsilon_{i,K}^2\right)^{(1/2)} \text{ for each } K \in \mathcal{T}_h, \quad (53)$$

where

$$\Upsilon_{1,K}^2 := \begin{cases} h_K^2 \|r_{1,K}\|_K^2 & \text{if } K \in \mathcal{T}_h^f, \\ h_K^2 \| r_{4,K}\|_K^2 & \text{if } K \in \mathcal{T}_h^p, \end{cases}$$

$$\Upsilon_{2,K}^2 := \begin{cases} \|R_3\|_K^2 & \text{if } K \in \mathcal{T}_h^p, \\ 0 & \text{if } K \in \mathcal{T}_h^f, \end{cases}$$

$$\Upsilon_{3,K}^2 := \begin{cases} \|\text{curl}(R_3)\|_K^2 & \text{if } K \in \mathcal{T}_h^p, \\ 0 & \text{if } K \in \mathcal{T}_h^f, \end{cases}$$

$$\Upsilon_{4,K}^2 := \begin{cases} \|R_2\|_K^2, & \text{if } K \in \mathcal{T}_h^f, \\ \|R_5\|_K^2, & \text{if } K \in \mathcal{T}_h^p, \end{cases}$$

$$\Upsilon_{5,K}^2 := \sum_{E \in \mathrm{E}_h(\partial K \cap \overline{\Gamma}_{fp})} h_E \left\{\sum_{j=1}^{d-1} \|R_6(j)\|_E^2\right\},$$

$$\Upsilon_{6,K}^2 := \sum_{E \in \mathrm{E}_h(\partial K \cap \overline{\Gamma}_{fp})} h_E \|R_7\|_E^2$$

$$\Upsilon_{7,K}^2 := \begin{cases} \sum_{E \in \mathrm{E}_h(\partial K \cap \overline{\Omega}_f)} h_E \left\|J_{E,n_E}\right\|_E^2, & \text{if } K \in \mathcal{T}_h^f, \\ \sum_{E \in \mathrm{E}_h(\partial K \cap \overline{\Omega}_p)} h_E\left(\left\|G_{E,n_E}\right\|_E^2 + \left\|[p_h]_E\right\|_E^2\right), & \text{if } K \in \mathcal{T}_h^p, \end{cases}$$

$$\Upsilon_{8,K}^2 := \sum_{E\in E_h(\partial K\cap\Omega_p)} h_E^{-1}\left\|[u_h]_E\right\|_E^2,$$
$$\Upsilon_{9,K}^2 := \sum_{E\in E_h(\partial K\cap\partial\Omega_p)} h_E^{-1}\left\|[u_h\cdot n_E]_E\right\|_E^2,$$
$$\Upsilon_{10,K}^2 := \sum_{E\in E_h(\partial K\cap\Omega_f^+)} h_E^{-1}(1+2\mu)\left\|[u_h]_E\right\|_E^2$$
$$\Upsilon_{11,K}^2 := \sum_{E\in E_h(\partial K\cap\Omega_p^+)} h_E^{-1}\left\|\left(1+2\mu_p\right)\left[\eta_{ph}\right]_E\right\|_E^2. \tag{54}$$

The global residual error estimator is given by

$$\Upsilon := \left(\sum_{K\in\mathcal{T}_h}\Upsilon_K^2\right)^{(1/2)}. \tag{55}$$

Furthermore, the local and global approximation terms are denoted by

$$\left\{\Psi_K := h_K\left\|f-f_h\right\|_K, \forall K\in\mathcal{T}_h,\right\}, \tag{56}$$

$$\Psi := \left(\sum_{K\in\mathcal{T}_h}\Psi_K^2\right)^{(1/2)}. \tag{57}$$

4. Analytical Tools

4.1. Some Technical Results. Our a posteriori analysis requires some analytical results that are recalled.

The first one concerns a sort of Helmholtz decomposition of elements of H. Recall first that if $d=3$,

$$H_0\left(\mathrm{curl},\Omega_p\right) = \left\{\psi\in L^2\left(\Omega_p\right)^3 : \mathrm{curl}\psi\in L^2\left(\Omega_p\right)^3 \text{and} \psi\times n = 0 \, on\, \partial\Omega_p\right\}. \tag{58}$$

Theorem 3. *(see reference [2], p. 708). Any $v\in H$ admits the Helmholtz-type decomposition*

$$v = v_0 + v_1, \tag{59}$$

where $v_0, v_1 \in H$ but satisfying $v_0\in H^1(\Omega)^d$,

$$v_1 = \begin{cases} 0, & \text{in,} \quad \Omega_f, \\ \mathrm{curl}\beta_p, & \text{in,} \quad \Omega_p, \end{cases} \tag{60}$$

where $\beta_p\in H_0^1(\Omega_p)$ if $d=2$, while $\beta_p\in H^1(\Omega_p)^3\cap H_0(curl,\Omega_p)$, if $d=3$, with the estimate

$$\left\|v_0\right\|_{1,\Omega} + \left\|\beta_p\right\|_{1,\Omega_p} \lesssim \|v\|_H. \tag{61}$$

The second result that we need is a regularity result for the solution $u\in H$ of (28).

Theorem 4. *(see [2], p. 710). Let $(U,P)\in\mathbb{H}\times\mathbb{M}$ be the unique solution of (19) with $U=(u,\eta_p)\in H\times X_p$. If $K\in[C^{0,1}(\overline{\Omega}_p)]^{d\times d}$, then there exists $\delta>0$ such that*

$$u_{|\Omega_p} \in \left[H^{\frac{1}{2}+\delta}\left(\Omega_p\right)\right]^d. \tag{62}$$

Note that the regularity of $u\in[H^{(1/2)+\delta}(\Omega_p)]^d$, with $\delta>0$ allows to give a meaning to $J_{\Omega_p}(u,w)+J_{\partial\Omega_p}(u,w)$ for all $w\in H\cup H_h$ and hence to show that $J(U,W)=0$ for all $W=(w,\xi_p)\in\mathbb{H}\cup\mathbb{H}_h$.

Let us finish this section by an estimation of the nonconformity error (see [2], Theorem 3.3).

Theorem 5. *For any $U_h=(u_h,\eta_{ph})\in\mathbb{H}_h$, we have*

$$\inf_{W_h\in\mathbb{H}_h\cap\mathbb{H}} \left\|U_h-W_h\right\|_{\mathbb{H}_h}^2 \lesssim J\left(U_h,U_h\right). \tag{63}$$

4.2. Clément Interpolation Operator. In order to derive the upper error bounds, we introduce the Clément interpolation operator I_{Cl}^0: $H_0^1(\Omega)\cdot\mathcal{P}_c^b(\mathcal{T}_h)$ that approximates optimally nonsmooth functions by continuous piecewise linear functions:

$$\mathcal{P}_c^b(\mathcal{T}_h) := \left\{v\in C^0(\overline{\Omega}) : v_{|K}\in\mathbb{P}^1(K), \forall K\in\mathcal{T}_h \,\text{and}\, v=0 \,\text{on}\, \partial\Omega\right\}, \tag{64}$$

In addition, we will make use of a vector-valued version of I_{Cl}^0, that is, I_{Cl}^0: $[H_0^1(\Omega)]^d\cdot[\mathcal{P}_c^b(\mathcal{T}_h)]^d$, which is defined componentwise by I_{Cl}^0. The following lemma establishes the local approximation properties of I_{Cl}^0 (and hence of I_{Cl}^0). (for a proof, see [45], Section 3).

Lemma 1. *There exist constants $C_1, C_2 > 0$, independent of h, such that for all $v \in H_0^1(\Omega)$ there hold*

$$\begin{aligned} &\left\| v - I_{Cl}^0(v) \right\|_K \leq C_1 h_K \|v\|_{1,\Delta(K)}, \forall K \in \mathcal{T}_h, 0.2\text{cmand} \\ &\left\| v - I_{Cl}^0(v) \right\|_E \leq C_2 h_E^{1/2} \|v\|_{1,\Delta(E)}, \forall E \in \mathrm{E}_h, \end{aligned} \tag{65}$$

where $\Delta(K) := \cup\{K' \in \mathcal{T}_h \colon K' \cap K \neq \varnothing\}$ *and* $\Delta(E) := \cup\{K' \in \mathcal{T}_h \colon K' \cap E \neq \varnothing\}$.

4.3. Inverse Inequalities. In order to derive the lower error bounds, we proceed similarly as in [40, 41], by applying inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions. To this end, we recall some notation and introduce further preliminary results. Given $K \in \mathcal{T}_h$, and $E \in \mathcal{E}(K)$, we let b_K and b_E be the usual simplex-bubble and face-bubble functions, respectively (see (1.5) and (1.6) in [46]). In particular, b_K satisfies $b_K \in \mathbb{P}^3(K)$, $\operatorname{supp}(b_K) \subseteq K$, $b_K = 0 \text{ on } \partial K$, and $0 \leq b_K \leq 1 \text{ on } K$. Similarly, $b_E \in \mathbb{P}^2(K)$, $\operatorname{supp}(b_E) \subseteq \omega_E \colon = \{K' \in \mathcal{T}_h \colon E \in \mathcal{E}(K')\}$, $b_E = 0 \text{ on } \partial K \setminus E$ and $0 \leq b_E \leq 1 \text{ in } \omega_E$. We also recall from [36] that, given $k \in \mathbb{N}$, there exists an extension operator $L \colon C(E) \cdot C(K)$ that satisfies $L(p) \in \mathbb{P}^k(K)$ and $L(p)_{|E} = p, \forall p \in \mathbb{P}^k(E)$. A corresponding vectorial version of L, that is, the componentwise application of L, is denoted by L. Additional properties of b_K, b_E and L are collected in the following lemma (see [36]).

Lemma 2. *Given $k \in \mathbb{N}^*$, there exist positive constants depending only on k and shape regularity of the triangulations (minimum angle condition), such that for each simplex K and $E \in \mathcal{E}(K)$ there hold*

$$\begin{aligned} \|q\|_K &\lesssim \left\| q b_K^{1/2} \right\|_K \lesssim \|q\|_K, \forall q \in \mathbb{P}^k(K), \\ \left| q b_K \right|_{1,K} &\lesssim h_K^{-1} \|q\|_K, \forall q \in \mathbb{P}^k(K), \\ \|p\|_E &\lesssim \left\| b_E^{1/2} p \right\|_E \lesssim \|p\|_E, \forall p \in \mathbb{P}^k(E), \\ \|L(p)\|_K + h_E |L(p)|_{1,K} &\lesssim h_E^{1/2} \|p\|_E, \forall p \in \mathbb{P}^k(E). \end{aligned} \tag{66}$$

Lemma 3 (continuous trace inequality). *There exists a positive constant $\beta_1 > 0$ depending only on σ_0 such that*

$$\|v\|_{\partial K}^2 \leq \beta_1 \|v\|_K \|v\|_{1,K}, \forall K \in \mathcal{T}_h, \forall v \in \left[H^1(K)\right]^d. \tag{67}$$

5. Main Results

We set $\mathbb{X} \colon = \mathbb{H} \times \mathbb{M}$ and $\mathbb{X}_h \colon = \mathbb{H}_h \times \mathbb{M}_h$ and define on $\mathbb{X}$, the continuous bilinear form $\mathbb{B}$ by

$$\mathbb{B}(\mathbb{U}, \mathbb{W}) := A(U, V) + B(V, P) + B(U, Q) \text{for} \mathbb{U} = (U, P) \text{andfor} \mathbb{W} = (V, Q). \tag{68}$$

We also define on the discrete space $\mathbb{H}_h$, the form,

$$\begin{aligned} \mathbb{B}_h(\mathbb{U}_h, \mathbb{W}_h) := A_h(U_h, V_h) + B_h(V_h, P_h) + B_h(U_h, Q_h) + J(U_h, V_h) \text{ for} \\ \mathbb{U}_h = (U_h, P_h), \mathbb{W}_h = (V_h, Q_h). \end{aligned} \tag{69}$$

The spaces $\mathbb{X}$ and $\mathbb{X}_h$ are equipped with the product norms:

$$\|(U, P)\| = \|U\|_{\mathbb{H}} + \|P\|_{\mathbb{M}} \text{and} \left\|(U_h, P_h)\right\|_h = \left\|U_h\right\|_{\mathbb{H}_h \cup \mathbb{H}} + \left\|P_h\right\|_{\mathbb{M}} \text{respectively.} \tag{70}$$

To prove local efficiency for $\omega \subset \Omega$, let us denote by

$$\begin{aligned} \|(W, Q)\|_{h,\omega}^2 = &\sum_{K \subset \overline{\omega} \cap \overline{\Omega}_f} |v|_{1,K}^2 + \sum_{K \subset \overline{\omega} \cap \overline{\Omega}_p} \left|\xi_p\right|_{1,K}^2 + \sum_{K \subset \overline{\omega} \cap \overline{\Omega}_p} \left(\|v\|_K^2 + \left\|\operatorname{div}_h v\right\|_K^2 \right) + \left\| v_f \times n \right\|_{\Gamma_{fp} \cap \overline{\omega}}^2 \\ &+ \sum_{K \subset \overline{\omega}} J_K(W, W) + \|Q\|_{\omega}^2, \forall (W, Q) = \left(v, \xi_p, Q\right) \in H_h \cup H \times X_p \cup X_{ph} \times \mathbb{M}, \end{aligned} \tag{71}$$

where

$$J_K(W,W) = (1+2\mu) \sum_{E\in E_h\left(\Omega_f^+\right)\cap E(K)} h_E^{-1}\left\|[v]_E\right\|_E^2 + \sum_{E\in E_h\left(\Omega_p\right)\cap E(K)} h_E^{-1}\left\|[v]_E\right\|_E^2 + \sum_{E\in E_h\left(\partial\Omega_p\right)\cap E(K)} h_E^{-1}\left\|[v\cdot n_E]_E\right\|_E^2 + \sum_{E\in E_h\left(\partial\Omega_p\right)\cap E(K)} h_E^{-1}\left\|\left[\left(1+\mu_p\right)\xi_p\right]_E\right\|_E^2. \quad (72)$$

5.1. Optimality Result. The main result of this paper can be stated as follows:

(1) Reliability of $\{\Upsilon_K\}_{K\in\mathcal{T}_h}$: the a posteriori error estimator Υ is consider reliable if it satisfies

$$\left\|\left|(U-U_h,P-P_h)\right|\right\|_h \lesssim \Upsilon + \Psi. \quad (73)$$

(2) Efficiency of $\{\Upsilon_K\}_{K\in\mathcal{T}_h}$:

Under the assumptions of Theorem 4, the following lower error bound holds

$$\Upsilon_K \lesssim \left\|\left|(U-U_h,P-P_h)\right|\right\|_{h,\widetilde{\omega}_K} + \sum_{K'\in\widetilde{\omega}_K} \Psi_{K'}, \quad (74)$$

where $\widetilde{\omega}_K$ is a finite union of neighboring elements of K.

5.2. Proof of the A Posteriori Error Estimate

5.2.1. Proof of the Reliability Estimate. In this section, we shall prove estimate (60).

Let us start with the following result.

Lemma 4. *Let the assumptions of Theorem 4 be satisfied. Then, for all $\mathbb{W}=(V,Q)\in\mathbb{H}\times\mathbb{M}$, we have the estimate*

$$\mathbb{B}_h(\mathbb{U}-\mathbb{U}_h,\mathbb{W}) \lesssim (\widetilde{\Upsilon}+\Psi)\|\|\mathbb{W}\|\|_h, \quad (75)$$

where the estimator $\widetilde{\Upsilon}$ is defined by

$$\widetilde{\Upsilon} := \left\{\sum_{K\in\mathcal{T}_h}\left(\sum_{i=1}^{7}\Upsilon_{i,K}^2\right)\right\}^{\frac{1}{2}}. \quad (76)$$

Proof. Let $\mathbb{W}=(V,Q)\in\mathbb{H}\times\mathbb{M}$ with $V=(v,\xi_p)$ and $Q=(Q_1,Q_2)$. By Theorem 3, **v** admits the decomposition (49) with $v_0,v_1\in H$ and satisfies the properties stated in Theorem 3. Then we take $\mathbb{W}_h=(V_h,0)\in\mathbb{H}_h\times\mathbb{M}_h$ where $V_h=(v_h,\xi_{ph})$ with $v_h=v_{0,h}+v_{1,h}$, where $v_{0,h}=I_{Cl}^0 v_0$ and

$$v_{1,h} = \begin{cases} 0, & \text{in}, \quad \Omega_f, \\ \operatorname{curl} I_{Cl}^0\psi, & \text{in}, \quad \Omega_p, \end{cases} \text{and}\ \xi_{ph} = I_{Cl}^0\xi_p. \quad (77)$$

In $2D$, $I_{Cl}^0\psi$ is the standard Clément interpolant of ψ, while in $3D$, we take the vectorial Clément interpolant from [2] that satisfies the same estimate as the standard one (see [2]). Note that $v_{0,h}$ belongs to $H_h\cap[H_0^1(\Omega)]^d$ while $v_{1,h}$ simply belongs to $H\cap H_h$ ($I_{Cl}^0\psi$ being in $H_0^1(\Omega_p)$ if $d=2$ and $I_{Cl}^0\psi\in H^1(\Omega_p)^3\cap H_0(\text{curl},\Omega_p)$ if $d=3$), its curl belongs to $H_0(\text{div},\Omega_p)$, hence $v_{1,h}$, its extension by zero in Ω_f, stays in $H_0(\text{div},\Omega)$. With these definitions and noticing that $\text{div}(v-v_h)=\text{div}(v_0-v_{0,h})$ and that $J(U_h,V_h)=0$, we may write

$$\begin{aligned}
\mathbb{B}_h(\mathbb{U}-\mathbb{U}_h,\mathbb{W}) &= \mathbb{B}_h(\mathbb{U}-\mathbb{U}_h,\mathbb{W}-\mathbb{W}_h) \\
&= A_h(U-U_h,V-V_h)+B_h(V-V_h,P-P_h)+B_h(U-U_h,Q) \\
&= A_h(U,V-V_h)+B_h(V-V_h,P)+B_h(U,Q) \\
&\quad -[A_h(U_h,V-V_h)+B_h(V-V_h,P_h)+B_h(U_h,Q)] \\
&= L(V-V_h)+G(Q) \\
&\quad -[A_h(U_h,V-V_h)+B_h(V-V_h,P_h)+B_h(U_h,Q)] \\
&= (f,v-v_h)_\Omega-(g,Q_1)_\Omega \\
&\quad -[A_h(U_h,V-V_h)+B_h(V-V_h,P_h)+B_h(U_h,Q)] \\
&= \sum_{K\in T_h}\{(f,v-v_h)_K-(g,Q_1)_K\}-A_h(U_h,V-V_h)-B_h(V-V_h,P_h)-B_h(U_h,Q) \\
&= \sum_{K\in T_h}\{(f,v-v_h)_K\}-\sum_{K\in T_h^f}(g,Q_1)_K \\
&\quad -\sum_{K\in T_h^f}(2\mu D(u_h),D(v-v_h))_K-\sum_{K\in T_h}\left(\mu K^{-1}u_h,v-v_h\right)_K
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{K\in T_h^p}\left(2\mu_p D\left(\eta_{ph}\right), D\left(\xi-\xi_{ph}\right)\right)_K-\sum_{K\in T_h^p}\left(\lambda_p \mathrm{div}_h \eta_{ph}, \mathrm{div}\left(\xi_p-\xi_{ph}\right)\right)_K \\
&-\sum_{j=1}^{d-1}\langle \mu\alpha_{BJS}\sqrt{K_j^{-1}}\, u_{fh}\cdot\tau_{f,j}, \left(v_f-v_{fh}\right)\cdot\tau_{f,j}\rangle_{\Gamma_{fp}}+\sum_{K\in T_h}\left(p_h, \mathrm{div}\left(v_0-v_{0h}\right)\right)_K \\
&+\sum_{K\in T_h^p}\alpha\left(p_{ph}, \mathrm{div}\left(\xi_p-\xi_{ph}\right)\right)_K+\sum_{K\in T_h}\left(Q, \mathrm{div} u_h\right)_K+\sum_{K\in T_h^p}\alpha\left(Q_2, \mathrm{div}\left(\eta_{ph}\right)\right)_K.
\end{aligned} \tag{78}
$$

Integrate by parts element by element and add boundary (resp. internal) terms that appear on the same edge (resp. at the same element) and reminding that $v=v_0$ and $v_h=v_{0,h}$ in Ω_f, we obtain

$$
\begin{aligned}
\mathbb{B}_h\left(\mathbb{U}-\mathbb{U}_h, \mathbb{W}\right)=&\sum_{K\in\mathcal{T}_h^f}\left(R_1, v-v_h\right)_K-\sum_{K\in\mathcal{T}_h^p}\left(R_3, v-v_h\right)_K+\sum_{K\in\mathcal{T}_h^p}\left(R_4, \xi_p-\xi_{ph}\right)_K \\
&-\sum_{K\in\mathcal{T}_h^f}\left(R_2, Q_1\right)_K-\sum_{K\in\mathcal{T}_h^p}\left(R_5, Q_2\right)_K+\sum_{E\in \mathrm{E}_h\left(\Gamma_{fp}\right)}\left(R_7, \left(v_0-v_{0h}\right)\cdot n_f\right)_E \\
&-\sum_{E\in \mathrm{E}_h\left(\Gamma_{fp}\right)}\sum_{j=1}^{d-1}\left(R_6(j), \left(v_0-v_{0h}\right)\cdot\tau_j\right)_E-\sum_{E\in \mathrm{E}_h\left(\Omega_f^+\right)}\left(J_{E,n_E}, v_0-v_{0h}\right)_E \\
&-\sum_{E\in \mathrm{E}_h\left(\Omega_p^+\right)}\left(G_{E,n_E}, \xi_p-\xi_{ph}\right)_E+\sum_{E\in \mathrm{E}_h\left(\Omega_p\right)}\left(\left[p_{ph}\right]_E, \left(v_0-v_{0,h}\right)\cdot n_E\right)_E.
\end{aligned} \tag{79}
$$

We now introduce the approximation f_h of f for appropriated terms and we have

$$
\begin{aligned}
\mathbb{B}_h\left(\mathbb{U}-\mathbb{U}_h, \mathbb{W}\right)=&\sum_{K\in\mathcal{T}_h^f}\left(r_1, v-v_h\right)_K-\sum_{K\in\mathcal{T}_h^p}\left(R_3, v_0-v_{0,h}\right)_K+\sum_{K\in\mathcal{T}_h^p}\left(r_4, \xi_p-\xi_{ph}\right)_K \\
&-\sum_{K\in\mathcal{T}_h^f}\left(R_2, Q_1\right)_K-\sum_{K\in\mathcal{T}_h^p}\left(R_5, Q_2\right)_K+\sum_{E\in \mathrm{E}_h\left(\Gamma_{fp}\right)}\left(R_7, \left(v_0-v_{0h}\right)\cdot n_f\right)_E-\sum_{E\in \mathrm{E}_h\left(\Gamma_{fp}\right)}\sum_{j=1}^{d-1}\left(R_6(j), \left(v_0-v_{0h}\right)\cdot\tau_j\right)_E \\
&-\sum_{E\in \mathrm{E}_h\left(\Omega_f^+\right)}\left(J_{E,n_E}, v_0-v_{0h}\right)_E-\sum_{E\in \mathrm{E}_h\left(\Omega_p^+\right)}\left(G_{E,n_E}, \xi_p-\xi_{ph}\right)_E+\sum_{E\in \mathrm{E}_h\left(\Omega_p\right)}\left(\left[p_{ph}\right]_E, \left(v_0-v_{0,h}\right)\cdot n_E\right)_E \\
&+\sum_{K\in\mathcal{T}_h}\left(f-f_h, v-v_h\right)_K-\sum_{K\in\mathcal{T}_h^p}\left(R_3, v_1-v_{1,h}\right)_K.
\end{aligned} \tag{80}
$$

Now for a triangle $K\in\mathcal{T}_h^p$, we recall that

$$
v_1-v_{1,h}=\mathrm{curl}\left(\psi-I_{Cl}^0\psi\right)\mathrm{in} K, \tag{81}
$$

and use Green's formula to get

$$\sum_{K\in\mathcal{T}_h^p}\left(R_3, v_1 - v_{1,h}\right)_K = \sum_{K\in\mathcal{T}_h^p}\left(R_3, \mathrm{curl}\left(\psi - I_{Cl}^0\psi\right)\right)_K - \sum_{K\in\mathcal{T}_h^p}[\left(\mathrm{curl}R_3, \psi - I_{Cl}^0\psi\right)_K + \left(\mathrm{curl}R_3 \times n, \psi - I_{Cl}^0\psi\right)_{\partial K}]. \tag{82}$$

We deduce the error equation,

$$\begin{aligned}\mathbb{B}_h(\mathbb{U}-\mathbb{U}_h,\mathbb{W}) = &\sum_{K\in\mathcal{T}_h^f}(r_1, v - v_h)_K - \sum_{K\in\mathcal{T}_h^p}\left(R_3, v_0 - v_{0,h}\right)_K + \sum_{K\in\mathcal{T}_h^p}\left(r_4, \xi_p - \xi_{ph}\right)_K \\ &- \sum_{K\in\mathcal{T}_h^f}(R_2, Q_1)_K - \sum_{K\in\mathcal{T}_h^p}(R_5, Q_2)_K + \sum_{E\in\mathrm{E}_h(\Gamma_{fp})}\left(R_7, (v_0 - v_{0h})\cdot n_f\right)_E \\ &- \sum_{E\in\mathrm{E}_h(\Gamma_{fp})}\sum_{j=1}^{d-1}\left(R_6(j), (v_0 - v_{0h})\cdot\tau_j\right)_E - \sum_{E\in\mathrm{E}_h((\Omega_f^+))}\left(J_{E,n_E}, v_0 - v_{0h}\right)_E - \sum_{E\in\mathrm{E}_h(\Omega_p^+)}\left(G_{E,n_E}, \xi_p - \xi_{ph}\right)_E \\ &+ \sum_{E\in\mathrm{E}_h(\Omega_p)}\left(\left[p_{ph}\right]_E, \left(v_0 - v_{0,h}\right)\cdot n_E\right)_E + \sum_{K\in\mathcal{T}_h}(f - f_h, v - v_h)_K + \sum_{K\in\mathcal{T}_h^p}[\left(\mathrm{curl}R_3, \psi - I_{Cl}^0\psi\right)_K \\ &- \left(\mathrm{curl}R_3\times n, \psi - I_{Cl}^0\psi\right)_{\partial K}].\end{aligned} \tag{83}$$

Cauchy–Schwarz inequality and the approximation properties of Lemma 1 imply the required estimate and finish the proof.

The second result of this section is given by the following lemma. □

Lemma 5. *Under the assumptions of Theorem 4, the following estimation holds:*

$$\left\|\left|(U - U_h, P - P_h)\right|\right\|_h \lesssim \widetilde{\Upsilon} + \Psi + \inf_{\mathbb{W}_h\in\mathbb{H}\cap\mathbb{H}_h\times\mathbb{M}_h}\left\|\left|\mathbb{U}_h - \mathbb{W}_h\right|\right\|_h, \tag{84}$$

where $\mathbb{U}_h = (U_h, P_h)$ *and* $\widetilde{\Upsilon}$ *is given by (76).*

Proof. For an arbitrary $\mathbb{W}_h \in \mathbb{H}\cap\mathbb{H}_h\times\mathbb{M}_h$, the inf-sup condition of $\mathbb{B}$ on $\mathbb{H}\times\mathbb{M}$ leads to

$$\left\|\left|\mathbb{U} - \mathbb{W}_h\right|\right\|_h \lesssim \sup_{\mathbb{W}\in\mathbb{H}\times\mathbb{M}}\frac{\mathbb{B}(\mathbb{U} - \mathbb{W}_h, \mathbb{W})}{|||\mathbb{W}|||_h}, \tag{85}$$

hence

$$\left\|\left|\mathbb{U} - \mathbb{W}_h\right|\right\|_h \lesssim \sup_{\mathbb{W}_h\in\mathbb{H}\times\mathbb{M}}\left\{\frac{\mathbb{B}_h(\mathbb{U} - \mathbb{U}_h, \mathbb{W}) + \mathbb{B}_h(\mathbb{U}_h - \mathbb{W}_h, \mathbb{W})}{|||\mathbb{W}|||_h}\right\}. \tag{86}$$

Combining the estimates (62) and (67), it comes

$$\left\|\left|\mathbb{U} - \mathbb{W}_h\right|\right\|_h \lesssim \widetilde{\Upsilon} + \Psi + \sup_{\mathbb{W}\in\mathbb{H}\times\mathbb{M}}\frac{\mathbb{B}_h(\mathbb{U}_h - \mathbb{W}_h, \mathbb{W})}{|||\mathbb{W}|||_h}. \tag{87}$$

The continuity of the operator $\mathbb{B}_h$ implies that

$$\left\|\left|\mathbb{U} - \mathbb{W}_h\right|\right\|_h \lesssim \widetilde{\Upsilon} + \Psi + \left\|\left|\mathbb{U}_h - \mathbb{W}_h\right|\right\|_h. \tag{88}$$

Thus, by the triangular inequality we deduce that

$$\left\|\left|\mathbb{U} - \mathbb{U}_h\right|\right\|_h \lesssim \widetilde{\Upsilon} + \Psi + \left\|\left|\mathbb{U}_h - \mathbb{W}_h\right|\right\|_h, \forall\mathbb{W}_h \in \mathbb{H}\cap\mathbb{H}_h\times\mathbb{M}_h, \tag{89}$$

or equivalently,

$$\left\|\left|(U - U_h, P - P_h)\right|\right\|_h \lesssim \widetilde{\Upsilon} + \Psi + \inf_{\mathbb{W}_h\in\mathbb{H}\cap\mathbb{H}_h\times\mathbb{M}_h}\left\|\left|\mathbb{U}_h - \mathbb{W}_h\right|\right\|_h. \tag{90}$$

Thus, this lemma holds.

Combining Theorem 5 and estimate (65), we have the main result in this section. □

Theorem 6. *Under the assumptions of Theorem 4, the a posteriori error estimator Y satisfies (60).*

5.2.2. Proof of the Efficiency Estimate. In this section, we shall prove the estimate (61). We bound each term of the residual separately. Since by Theorem 4 the jump of $U = (u, \eta_p) \in H\times X_p$ is zero through all the edges of Ω_p; hence, for all $i \in \{8, 9, 10, 11\}$, we clearly have

$$\Upsilon_{i,K}^2 \lesssim J_K(U_h, U_h) = J_K(U_h - U, U_h - U) \lesssim \left\|\left|(U - U_h, P - P_h)\right|\right\|_{h,K}. \tag{91}$$

Hence it remains to estimate the local indicators for $i \le 7$.

(1) Residual element $r_{1,K}$ in Ω_f. Let $K \in \mathcal{T}_h^f$ and set $w_K := r_{1,K}b_K \in [H_0^1(K)]^d$, and consider

$$\int_K r_{1,K} \cdot w_K = \int_K \left[f_{K,f} + \nabla \cdot \sigma_f(u_h, p_h) \right] \cdot w_K. \quad (92)$$

Introduce f and use the weak formulation (19) with $V = (w_K, 0) \in \mathbb{H}$ to obtain

$$\int_K r_{1,K} \cdot w_K = \int_K \left(f_{K,f} - f \right) \cdot w_K + \int_K (2\mu D(u) : D(w_K) - p div w_K) + \int_K \left[2\mu \mathrm{div} D(u_h) - \nabla p_h \right] \cdot w_K. \quad (93)$$

Integrating by parts in this last term, we obtain

$$\int_K r_{1,K} \cdot w_K = \int_K \left(f_{K,f} - f \right) \cdot w_K + 2\mu \int_K D(u - u_h) : D(w_K) - \int_K (p - p_h) \mathrm{div} w_K. \quad (94)$$

Cauchy–Schwarz inequality implies that

$$\int_K r_{1,K} \cdot w_K \lesssim \left\| f - f_{K,f} \right\|_K \| w_K \|_K + \left[2\mu \| \nabla (u - u_h) \|_K + \| p - p_h \|_K \right] \| \nabla w_K \|_K. \quad (95)$$

The inverse inequalities (55) and (56) and the obvious relation $\|w_K\|_K \leq \|r_{1,K}\|_K$ imply

$$h_K \left\| r_{1,K} \right\|_K \lesssim \Psi_K + \left\| \left| (U - U_h, P - P_h) \right| \right\|_{h,K}. \quad (96)$$

(2) Residual Element $r_{4,K}$ in Ω_p. Let $K \in \mathcal{T}_h^p$ and $w_K := r_{4,K} b_K \in [H_0^1(K)]^d$ and we consider

$$\int_K r_{4,K} \cdot w_K = \int_K \left[f_{K,p} + \nabla \cdot \sigma_p \left(\eta_{ph}, p_{ph} \right) \right] \cdot w_K. \quad (97)$$

Introduce f and use the weak formulation (19) with $V = (0, w_K) \in \mathbb{H}$ to obtain

$$\begin{aligned} \int_K r_{4,K} \cdot w_K = {} & \int_K \left(f_{K,p} - f \right) \cdot w_K + \int_K 2\mu_p D\left(\eta_p\right) : D(w_K) + \int_K \lambda_p \nabla \cdot \eta_p \nabla \cdot w_K - \alpha \int_K p_p \nabla \cdot w_K \\ & + \alpha \int_K \nabla p_{ph} \cdot w_K - \int_K \nabla \left[\lambda_p \left(\nabla \cdot \eta_{ph} \right) \right] \cdot w_K - \int_K \nabla \cdot \left[2\mu_p D\left(\eta_{ph}\right) \right] \cdot w_K. \end{aligned} \quad (98)$$

Integrating by parts in these three last terms, we obtain

$$\begin{aligned} \int_K r_{4,K} \cdot w_K = {} & \int_K \left(f_{K,p} - f \right) \cdot w_K + \int_K 2\mu_p D\left(\eta_p - \eta_{ph}\right) : D(w_K) \\ & + \int_K \lambda_p \nabla \cdot \left(\eta_p - \eta_{ph} \right) \nabla \cdot w_K - \alpha \int_K \left(p_p - p_{ph} \right) \nabla \cdot w_K. \end{aligned} \quad (99)$$

Cauchy–Schwarz inequality, the conditions $0 < \lambda_{\min} \leq \lambda_p(x) \leq \lambda_{\max}$ and $0 < \mu_{\min} \leq \mu_p(x) \leq \mu_{\max}$ for all $x \in \Omega_p$, and the inverse inequalities (55) and (56) lead to

$$h_K \left\| r_{4,K} \right\|_K \lesssim \Psi_K + \left\| \left| (U - U_h, P - P_h) \right| \right\|_{h,K}, \quad (100)$$

From (96) and (100), we deduce

$$\Upsilon_{1,K} \lesssim \Psi_K + \left\| \left| (U - U_h, P - P_h) \right| \right\|_{h,K}. \quad (101)$$

(3) Residual Element R_3 in Ω_p. Let $K \in \mathcal{T}_h^p$ and, use the relation $\mu K^{-1}u + \nabla p = 0$ in Ω_p to obtain

$$R_3 = \left[\mu K^{-1}u_h + \nabla p_h\right] = -\left[\mu K^{-1}u + \nabla p\right] + \left[\mu K^{-1}u_h + \nabla (p_h)\right] = -\left[\mu K^{-1}(u - u_h) + \nabla (p - p_h)\right]. \quad (102)$$

As before, Cauchy–Schwarz inequality leads to

$$\Upsilon_{2,K} = \|R_3\|_{0,K} \lesssim \|\|(U - U_h, P - P_h)\|\|_{h,K}. \quad (103)$$

(4) Curl Residual Element $\text{curl}R_3$ in Ω_p: for $K \in \mathcal{T}_h^p$, we use also the relation $\mu K^{-1}u + \nabla p = 0$ in Ω_p to obtain

$$\begin{aligned}\text{curl}R_3 &= \text{curl}\left(\mu K^{-1}u_h + \nabla p_h\right) = -\text{curl}\left[\mu K^{-1}u + \nabla p\right] + \text{curl}\left[\mu K^{-1}u_h + \nabla (p_h)\right] \\ &= -\text{curl}\left[\mu K^{-1}(u - u_h) + \nabla (p - p_h)\right].\end{aligned} \quad (104)$$

Using Cauchy–Schwarz inequality, we obtain

$$\Upsilon_{3,K} = \|\text{curl}R_3\|_{0,K} \lesssim \|\|(U - U_h, P - P_h)\|\|_{h,K}. \quad (105)$$

(5) Residual element R_2 in Ω_f: we directly see that

$$g - \text{div}u_h = \text{div}u - \text{div}u_h = \text{div}(u - u_h). \quad (106)$$

Hence by Cauchy–Schwarz inequality, we conclude

$$\|R_2\|_K \lesssim \|\|(U - U_h, P - P_h)\|\|_{h,K}. \quad (107)$$

(6) Residual element R_5 in Ω_p: as before, we directly, see that

$$g - \alpha\nabla \cdot \eta_{ph} + \nabla \cdot u_h = \alpha\nabla \cdot \left(\eta_p - \eta_{ph}\right) + \nabla \cdot (u - u_h). \quad (108)$$

Hence, by Cauchy–Schwarz inequality, we have

$$\|R_5\|_K \lesssim \|\|(U - U_h, P - P_h)\|\|_{h,K}. \quad (109)$$

Inequalities (80) and (81) lead to

$$\Upsilon_{4,K} \lesssim \|\|(U - U_h, P - P_h)\|\|_{h,K}. \quad (110)$$

(7) Normal jump J_{E,n_E} in Ω_f: for each edge/face $E \in \mathcal{E}_h(\Omega_f)$, we consider $w_E = K_1 \cup K_2$. As $J_{E,n_E} \in [\mathbb{P}^0(E)]^d$, we set

$$w_E := -J_{E,n_E} b_E \in \left[H_0^1(w_E)\right]^d. \quad (111)$$

First, the weak formulation (19) with $V = (w_E, 0) \in \mathbb{H}$ yields

$$A(U, V) + B(V, P) = L(V), \quad (112)$$

that is equivalent to

$$\int_{w_E} f \cdot w_E = \int_{w_E} 2\mu D(u) : D(w_E) - \int_{w_E} p\text{div}w_E - \int_{\partial w_E} \sigma_f(u, p) n_E \cdot w_E. \quad (113)$$

By elementwise partial integration, we further have

$$-\int_E J_{E,n_E} \cdot w_E = \int_{w_E} 2\mu D(u_h) : D(w_E) - \int_{w_E} p_h \text{div}(w_E) - \sum_{i=1}^{2} \int_{K_i} (-2\mu\text{div}D(u_h) + \nabla p_h) \cdot w_E. \quad (114)$$

Hence, by the previous identity (83), we obtain

$$-\int_E J_{E,n_E}\cdot w_E = \sum_{i=1}^{2}\int_{K_i}[f + 2\mu \operatorname{div} D(u_h) + \nabla p_h]\cdot w_E - \int_{\omega_E} D(u-u_h): D(w_E) + \int_{\omega_E}(p-p_h)\operatorname{div}(w_E). \tag{115}$$

We introduce the approximation f_h of f and use the Cauchy–Schwarz inequality and the inverse inequalities (57)–(58) to obtain

$$\|J_{E,n_E}\|_E \lesssim h_E^{(1/2)}\left(\sum_{i=1}^{2}\left(\|f-f_h\|_{K_i} + \|r_{1,K_i}\|_{K_i}\right)\right) + h_E^{(-1/2)}\left(|u-u_h|_{1,\omega_E} + \|p-p_h\|_{\omega_E}\right). \tag{116}$$

The previous bound (74) of r_{1,K_i} and the obvious estimate $h_E \leqslant h_K$ imply that

$$h_E^{(1/2)}\|J_{E,n_E}\|_E \lesssim |u-u_h|_{1,\omega_E} + \|p-p_h\|_{\omega_E} + \sum_{K' \subset \omega_E} h_{K'}\|f-f_h\|_{K'}. \tag{117}$$

(8) Pressure Jump in Ω_p: for each edge/face $E \in \mathscr{E}_h(\Omega_p)$, we consider $\omega_E = K_1 \cup K_2$. As $[p_h]_E \in \mathbb{P}^0(E)$, we set

$$w_E := [p_h]_E b_E n_E \in \left[H_0^1(\omega_E)\right]^d. \tag{118}$$

First, we notice that as $p \in H^1(\omega_E)$, we have by Green formula

$$\int_{\omega_E}(\nabla p\cdot w_E + p \operatorname{div} w_E) = 0. \tag{119}$$

Again by elementwise partial integration, we further have

$$\int_E [p_h]_E w_E.n_E = \sum_{i=1}^{2}\int_{K_i}(\nabla p_h \cdot w_E + p_h div w_E). \tag{120}$$

Taking the difference between these two identities, we obtain

$$\int_E [p_h]_E w_E.n_E = \sum_{i=1}^{2}\int_{T_i}(\nabla(p_h-p)\cdot w_E + (p_h-p)\operatorname{div} w_E). \tag{121}$$

Recalling that $\nabla p = -\mu K^{-1}u$ and introducing the term $\mu K^{-1}u_h$, we find

$$\begin{aligned}\int_E [p_h]_E w_E.n_E &= \sum_{i=1}^{2}\int_{K_i}\left(\nabla p_h + \mu K^{-1}u\right)\cdot w_E + (p_h-p)\operatorname{div} w_E\\ &= \sum_{i=1}^{2}\int_{K_i}\left(\nabla p_h + \mu K^{-1}u_h\right)\cdot w_E + (p_h-p)\operatorname{div} w_E + \sum_{i=1}^{2}\int_{K_i}\left(\mu K^{-1}(u-u_h)\right)\cdot w_E.\end{aligned} \tag{122}$$

Cauchy–Schwarz inequality and inverse inequalities lead to

$$\|[p_h]_E\|_E \lesssim \sum_{i=1}^{2}\|R_3\|_{K_i} h_E^{\frac{1}{2}} + \|p_h - p\|_{K_i} h_E^{-\frac{1}{2}} + h_E^{\frac{1}{2}}\sum_{i=1}^{2}\left\|K^{-1}(u-u_h)_{K_i}\right\|. \tag{123}$$

Since $h_E \leq 1$, then by (103), we deduce that

$$h_E^{\frac{1}{2}}\|[p_h]_E\|_E \lesssim \|p-p_h\|_{\omega_E} + \left\|K^{-1}(u-u_h)\right\|_{\omega_E}. \tag{124}$$

(9) Normal Jump G_{E,n_E} in Ω_p: for each edge/face $E \in \mathscr{E}_h(\Omega_p)$, we consider $w_E = K_1 \cup K_2$. As $G_{E,n_E} \in [\mathbb{P}^0(E)]^d$, we set

$$w_E := -G_{E,n_E} b_E \in \left[H_0^1(w_E)\right]^d. \quad (125)$$

First the weak formulation (19) with $V = (0, w_E) \in H \times X_p$ yields

$$A(U,V) + B(V,P) = L(V), \quad (126)$$

that is equivalent to

$$\begin{aligned}\int_{w_E} f \cdot w_E &= \int_{w_E} \mu K^{-1} u \cdot w_E + \int_{w_E} 2\mu_p D(\eta_p) : D(w_E) + \\ &\int_{w_E} \lambda_p \nabla \cdot \eta_p \nabla \cdot w_E - \alpha \int_{w_E} p_p \nabla \cdot w_E - \int_{\partial w_E} \left[\sigma_p(\eta_p, p_p) n_E\right] \cdot w_E.\end{aligned} \quad (127)$$

By elementwise partial integration, we further have

$$\begin{aligned}-\int_{w_E} G_{E,n_E} \cdot w_E &= \int_{w_E} 2\mu_p D(\eta_{ph}) : D(w_E) + \int_{w_E} \lambda_p \nabla \cdot \eta_{ph} \nabla \cdot w_E - \alpha \int_{w_E} p_{ph} \nabla \cdot w_E \\ &- \sum_{i=1}^{2} \int_{K_i} -\sigma_p(\eta_{ph}, p_{ph}) \cdot w_E.\end{aligned} \quad (128)$$

By the previous identity, we obtain

$$\begin{aligned}-\int_{w_E} G_{E,n_E} \cdot w_E = &-\int_{w_E} 2\mu_p D(\eta_p - \eta_{ph}) : D(w_E) - \int_{w_E} \lambda_p \nabla \cdot (\eta_p - \eta_{ph}) \nabla \cdot w_E + \\ &\alpha \int_{w_E} (p_p - p_{ph}) \nabla \cdot w_E + \sum_{i=1}^{2} \int_{K_i} \left[f_h + \sigma_p(\eta_{ph}, p_{ph})\right] \cdot w_E + \\ &\int_{w_E} \mu K^{-1}(u - u_h) \cdot w_E + \int_{w_E} \left[\mu K^{-1} u_h + \nabla p_{ph}\right] \cdot w_E + \sum_{i=1}^{2} \int_{K_i} (f - f_h) \cdot w_E.\end{aligned} \quad (129)$$

The previous bounds of $r_{4,K}$, R_3, and the obvious estimate $h_E \leq h_K$ imply that

$$h_E^{1/2} \left\| G_{E,n_E} \right\|_E \lesssim \left\| (U - U_h, P - P_h) \right\|_{h,w_E} + \Psi_{K_1} + \Psi_{K_2}. \quad (130)$$

From (84), (85), and (86), we deduce the estimation

$$\Upsilon_{7,K} \lesssim \left\| (U - U_h, P - P_h) \right\|_{h,\widetilde{w}_K} + \sum_{K' \subset \widetilde{w}_K} \Psi_{K'}. \quad (131)$$

(10) Interface elements on Γ_{fp} ($\Upsilon_{5,K}$ and $\Upsilon_{6,K}$): to estimate $\Upsilon_{5,K}$ and $\Upsilon_{6,K}$, we fix an edge E included in Γ_{fp} and, for a constant r_E fixed later on and a unit vector i, we consider

$$w_E := r_E b_E i, \quad (132)$$

that clearly belongs to H. We take $W = (w_E, 0)$ and the weak formulation (19) yields

$$\int_{w_E} f \cdot w_E = A(U,W) + B(W,P), \quad (133)$$

that is equivalent to

$$\int_{w_E} f\cdot w_E = \int_{K_f}\left[2\mu D(u): D(w_E) - p\nabla\cdot w_E\right] + \int_{K_p}\left[\mu K^{-1}u\cdot w_E - p\nabla\cdot w_E\right] + \sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\,u_f\cdot\tau_{f,j}w_E\cdot\tau_{f,j}\right], \tag{134}$$

where K_f (resp. K_p) is the unique triangle/tetrahedron included in $\overline{\Omega}_f$ (resp. $\overline{\Omega}_p$) having E as edge/face. On the other hand, integrating by parts in K_f and in K_p yields

$$\begin{aligned}&\int_{K_f}(2\mu D(u_h): D(w_E) - p_h\mathrm{div}w_E) + \int_{K_p}\left(\mu K^{-1}u_h\cdot w_E - p_h\mathrm{div}w_E\right)\\&+\sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\,u_{f,h}\cdot\tau_{f,j}w_{E,f}\cdot\tau_{f,j}\right]\\&= -\int_{K_f}(2\mu divD(u_h) - \nabla p_h)\cdot w_E + \int_{K_p}\left(\mu K^{-1}u_h\cdot w_E + \nabla p_h\right)\cdot w_E\\&+\sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\,u_{f,h}\cdot\tau_{f,j}w_{E,f}\cdot\tau_{f,j}\right]\\&-\int_E([p_h]_E w_E\cdot n_E - 2\mu\left(D\left(u_{f,h}n_E\right)\cdot w_E\right).\end{aligned}\tag{135}$$

Subtracting this identity to (89), we find

$$\begin{aligned}&\int_E([p_h]_E w_E\cdot n_E - 2\mu\left(D\left(u_{f,h}n_E\right)\cdot w_E\right) - \sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\,u_{fh}\cdot\tau_{f,j}w_{E,f}\cdot\tau_{f,j}\right]\\&= \int_{K_f}(2\mu D(u-u_h): D(w_E) - (p-p_h)divw_E) + \int_{K_p}\left(\mu K^{-1}(u-u_h)\cdot w_E - (p-p_h)\mathrm{div}w_E\right)\\&+\sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\,(u_f-u_{fh})\cdot\tau_{f,j}w_{E,f}\cdot\tau_{f,j}\right]\\&-\int_{K_f}(f + 2\mu divD(u_h) - \nabla p_h)\cdot w_E - \int_{K_p}\left(-\mu K^{-1}u_h\cdot w_E - \nabla p_h\right)\cdot w_E.\end{aligned}\tag{136}$$

In that last terms introducing the element residual $r_{1,K}$ and R_3, we arrive at

$$\begin{aligned}&\int_E([p_h]_E w_E\cdot n_E - 2\mu\left(D\left(u_{f,h}\right)n_E\cdot w_E\right) - \sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\left(u_f-u_{fh}\right)\cdot\tau_{f,j}w_{E,f}\cdot\tau_{f,j}\right]\\&= \int_{K_f}(2\mu D(u-u_h): D(w_E) - (p-p_h)\mathrm{div}w_E)\\&+\int_{K_p}\left(\mu K^{-1}(u-u_h)\cdot w_E - (p-p_h)\mathrm{div}w_E\right) + \sum_{j=1}^{d-1}\int_{\Gamma_{fp}}\left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\left(u_f-u_{fh}\right)\cdot\tau_{f,j}w_{E,f}\cdot\tau_{f,j}\right]\\&-\int_{K_f}\left(f - f_h + r_{f,K}\right)\cdot w_E - \int_{K_p}R_3\cdot w_E.\end{aligned}\tag{137}$$

(a) To estimate $\Upsilon_{5,K}$, for each $j = 1, \ldots, d-1$, we take $r_E = R_6(j)$ and $i = \tau_j$. With this choice, the identity (90) and the inverse inequality (57) yield

$$\begin{aligned}\|r_E\|_E^2 \lesssim & \int_{K_f} \left[(2\mu D(u-u_h) : D(w_E) - (p-p_h)\operatorname{div} w_E) \right] + \int_{K_p} \left[\left(\mu K^{-1}(u-u_h)\cdot w_E - (p-p_h)\operatorname{div} w_E \right) \right] \\ & + \sum_{j=1}^{d-1} \int_{\Gamma_{fp}} \left[\mu\alpha_{BJS}\sqrt{K_j^{-1}}\left(u_f - u_{fh}\right)\cdot \tau_{f,j} w_{E,f}\cdot\tau_{f,j} \right] - \int_{K_f} \left(f - f_h + r_{1,K} \right)\cdot w_E - \int_{K_p} R_3 \cdot w_E. \end{aligned} \tag{138}$$

Hence, Cauchy–Schwarz inequality, the inverse inequalities (58), and the estimates of bounds r_{1,K_l} and R_3 lead to

$$h_E^{\frac{1}{2}} \|R_6(j)\|_E \lesssim |u-u_h|_{h,\omega_E} + \|p-p_h\|_{h,\omega_E} + \Psi_{K_f} + \Psi_{K_p}, \tag{139}$$

with $\omega_E = K_f \cup K_p$.
Thus,

$$\Upsilon_{5,K} \lesssim \left\|\left|(U-U_h, P-P_h)\right|\right\|_{h,\widetilde{\omega}_K} + \sum_{K' \subset \widetilde{\omega}_K} \Psi_{K'}. \tag{140}$$

(b) To estimate $\Upsilon_{6,K}$, we take $r_E = R_7$ and $i = n_f$. As before, the identity (90), the inverse inequalities (57) and (58), and the estimates of bounds $r_{K,l}$ and R_3 lead to

$$\Upsilon_{6,K} \lesssim \left\|\left|(U-U_h, P-P_h)\right|\right\|_{h,\widetilde{\omega}_K} + \sum_{K' \subset \widetilde{\omega}_K} \Psi_{K'}. \tag{141}$$

Combining estimates (72), (77), (78), (79), (82), (85), (87), and (92), we have the main result of this section.

Theorem 7. *Under the assumptions of Theorem 4, the family* $\{\Upsilon_K\}_{K\in\mathcal{T}_h}$ *satisfies (61).*

6. Summary

In this paper, we have discussed a posteriori error estimates for a finite element approximation of the Stokes–Biot system. A residual type a posteriori error estimator is provided, that is both reliable and efficient. Many issues remain to be addressed in this area, let us mention other types of a posteriori error estimators or implementation and convergence analysis of adaptive finite element methods. Further, it is well known that an internal layer appears at the interface Γ_{fp} as the permeability tensor degenerates, in that case, anisotropic meshes have to be used in this layer (see, for instance [8]). Hence, we intend to extend our results to such anisotropic meshes.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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