

# Principal eigenvalue in an unbounded domain with indefinite potential

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**Abstract.** In this work, we discuss the existence and the non-existence of principal eigenvalue in an unbounded domain of  $\mathbb{R}^N$  for some potentials which change sign. We also give certain properties of this principal eigenvalue.

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## 1. Introduction

This work is mainly concerned with the existence of a positive principal eigenvalue for the following problem

$$-\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \quad (1.1)$$

where  $\Omega$  is an unbounded smooth domain in  $\mathbb{R}^N$ ,  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ ,  $1 < p < +\infty$ , is the  $p$ -Laplacian,  $\lambda$  is the eigenvalue parameter,  $V$  and  $m$  are weight functions possibly indefinite whose properties will be specified later. By principal eigenvalue, we mean  $\lambda > 0$  such that (1.1) admits a nontrivial  $u$ , with  $u \geq 0$ , in a suitable weak sense.

Many works have been devoted to the existence of a principal eigenvalue in the last years due to the importance of the validity of the weighted Poincaré inequality. The case  $V \equiv 0$  has been studied, among others, by [3, 8, 11, 12] under different hypothesis on  $m$ . They prove that there exists a first positive principal eigenvalue, denoted by  $\lambda_1(m)$  and defined by

$$\lambda_1(m) := \inf \left\{ E(u) = \int_{\Omega} |\nabla u|^p, u \in W \text{ and } \int_{\Omega} m|u|^p = 1 \right\},$$

in the case of bounded and unbounded domains, where  $W$  is a suitable Sobolev space. The case  $V \geq 0$  is technically very similar to the case  $V \equiv 0$ , as the

energy associated

$$E_V(u) := \int_{\Omega} (|\nabla u|^p + V(x)|u|^p)$$

has similar properties with the energy  $E$  defined above (see for example [17]). Problems (1.1) with  $V$  changing sign and satisfying certain conditions were recently considered in [18] (with  $m \equiv 1$ ) [13, 10] (with  $m$  indefinite,  $p = 2$  and additional hypothesis on  $V$  and  $m$ ) [5, 6, 9, 11, 16] (with  $m$  indefinite). In [9, 16], the use of the value

$$\alpha(V, m) := \inf \left\{ E_V(u); u \in W_0^{1,p}(\Omega), \int_{\Omega} |u|^p = 1 \text{ and } \int_{\Omega} m|u|^p = 0 \right\}$$

plays an important role in the proof of existence of a principal eigenvalue, where  $\Omega$  is a bounded domain. The authors introduce the notion of eigencurve  $\mu(\lambda)$  and give a necessary and sufficient condition, on the indefinite weights  $V, m$ , for the existence of principal eigenvalues.

Of special interest for our purpose is the work in [11] where the study of problems (1.1) with  $V \equiv 0$ , was carried out. In [11] the existence of a principal eigenvalue was proved by using a certain compact imbedding into suitable weighted Lebesgue spaces. Some properties of this principal eigenvalue was established: regularity of eigenvalue, simplicity and unicity.

Our purpose in this work is to extend the results obtained in [9, 11] to the case of problem (1.1). When trying to adapt the approach in [11], the relevant functional is

$$E_V(u) := \int_{\Omega} (|\nabla u|^p + V(x)|u|^p).$$

We will prove the coerciveness or the existence of converging minimizing sequence of  $E_V$  restricted to the manifold

$$M := \left\{ u \in W_g, B(u) := \int_{\Omega} m_1|u|^p = 1 \right\},$$

where  $m_1$  is related to the weight  $m$  as a “sort of positive part” and  $W_g$  is a suitable weighted Sobolev space in  $\Omega$  which will be defined later.

The existence of a principal eigenvalue for (1.1) is derived in Sects. 3 and 4. In Sect. 3 where the low dimensions are considered, we assume the validity on  $\Omega$  of a weighted Poincaré inequality of the form:

$$\int_{\Omega} g|u|^p \leq K(\Omega, g) \int_{\Omega} |\nabla u|^p, \quad \forall u \in C_c^\infty(\Omega), \tag{H_g}$$

for some function  $g \geq 0, \neq 0$  suitably related to the weight  $m$  and for some positive constant  $K(\Omega, g)$ . Section 4 is devoted to the case of high dimensions. We assume there, that the function  $m_1$  lies in certain Lebesgue space. In Sect. 5 we deal with some non existence result when some of the conditions are dropped. Some properties of the principal eigenvalue are also given: regularity of eigenfunction, unicity of the principal eigenvalue. In the preliminary Sect. 2, we recall some compactity results and various Sobolev imbeddings to be used later.

## 2. Preliminaries and technical results

Throughout this work,  $\Omega$  is an unbounded smooth domain of  $\mathbb{R}^N$ . We will denote by  $\Omega_R = \Omega \cap B(0, R)$  and  $\Omega'_R = \Omega \setminus B(0, R)$  for  $R > 0$ , where  $B(0, R)$  is the ball of radius  $R$  centred at the origin. We will assume that for  $R > 0$ , the open set  $\Omega_R$  is smooth. As in [11], we call “admissible” a nonnegative function  $\rho$  defined on  $\Omega$  and satisfying the following local integrability condition:

$$\rho \in L^s(\Omega_R) \quad \forall R > 0, \tag{H_0}$$

for some  $s > \frac{N}{p}$ , when  $N \geq p$  and  $s = 1$  when  $N < p$ .

Associated with such function  $\rho$ , we define, when  $\rho \not\equiv 0$ , the weighted space  $W_\rho$  as the closure of  $C_c^\infty(\Omega)$  with respect to the norm

$$\|u\|_{W_\rho} := \left[ \int_\Omega (|\nabla u|^p + \rho|u|^p) \right]^{1/p}.$$

The following imbeddings hold (cf. e.g [7, 11]):  $W_\rho \hookrightarrow D^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$  if  $N > p$  and  $W_\rho \hookrightarrow W^{1,p}(\Omega_R)$  for all  $p$ . Here  $D^{1,p}(\Omega)$  denotes when  $N > p$  the closure of  $C_c^\infty(\Omega)$  with respect to the norm  $(\int_\Omega |\nabla u|^p)^{1/p}$  and  $p^* = Np/(N - p)$  is the critical Sobolev exponent. For the proof of the imbedding  $W_\rho \hookrightarrow W^{1,p}(\Omega_R)$ , in low dimensions, we use the following lemma, whose proof can be found in [11].

**Lemma 2.1.** *Suppose  $N \leq p$  and  $\rho$  admissible,  $\rho \not\equiv 0$ . Then for any  $R$  sufficiently large (such that  $\rho \not\equiv 0$  on  $\Omega_R$ ), there exists a constant  $K_R$  such that*

$$\int_{\Omega_R} |u|^p \leq K_R \left[ \int_{\Omega_R} (|\nabla u|^p + \rho|u|^p) \right] \quad \forall u \in C_c^\infty(\Omega). \tag{2.1}$$

We write the weight  $m$  in the form  $m = m_1 - m_2$ , with  $m_1, m_2$  admissible functions. Note that the decomposition  $m = m_1 - m_2$  does not necessarily coincide with the usual decomposition  $m = m^+ - m^-$ , where  $m^\pm = \max(\pm m, 0)$ .

## 3. Existence of a principal eigenvalue in low dimensions

In this section, we will assume that  $N \leq p$  and consider successively a weight  $m$  either nonnegative or indefinite in  $\Omega$ .

Let us assume that the weight  $m_1$  and the potential  $V$  satisfy the following hypothesis

(H<sub>1</sub>): There exists an admissible function  $g$  such that (H<sub>g</sub>) holds and

$$m_1(x) \leq \theta(x)g(x) \quad \text{a.e in } \Omega,$$

where the function  $\theta$  verifies

$$\|\theta\|_{L^\infty(\Omega'_R)} \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

(H<sub>2</sub>): There exists a positive constant  $c$  with,  $0 \leq c < \frac{1}{K(\Omega, g)}$  such that

$$|V(x)| \leq cg(x) \quad \text{a.e in } \Omega,$$

where  $K(\Omega, g)$  and  $g$  are defined in (H<sub>g</sub>) with  $g$  admissible and  $g \not\equiv 0$ .

The inequality connecting  $m_1$  and  $g$  in  $(H_1)$  above means that  $m_1(x) = 0(g(x))$  as  $|x| \rightarrow +\infty, x \in \Omega$ . It implies that  $W_g \hookrightarrow W_{m_1}$ .

By a solution  $u$  of (1.1), we mean a weak solution, i.e.  $u \in W_g$  such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v + \int_{\Omega} V(x) |u|^{p-2} uv = \lambda \int_{\Omega} m(x) |u|^{p-2} uv, \quad \forall v \in W_g. \tag{3.1}$$

Note that every term in (3.1) is well defined.

The following theorem concerns the case  $m$  is a nonnegative function.

**Theorem 3.1.** *Let  $N \leq p$  and  $m = m_1$ , with  $m_1$  admissible and  $m \not\equiv 0$ . Assume that  $m_1$  satisfies  $(H_1)$  and  $V$  satisfies  $(H_2)$ . Then (1.1) admits a principal eigenvalue  $\lambda_1(V, m)$  with corresponding eigenfunction  $\varphi \in W_g$ .*

The proof of Theorem 3.1 relies on the following lemma whose proof can be found in [11]. We give the sketch of this proof for the comprehension of the reader. It is here the assumption about the function  $\theta$  in  $(H_1)$  is used.

**Lemma 3.2.** *Under the hypotheses of Theorem 3.1, one has the following compact imbedding  $W_g \hookrightarrow L^p(m_1, \Omega)$ , where  $L^p(m_1, \Omega)$  denotes the  $L^p$  space on  $\Omega$  with weight  $m_1$ .*

*Proof.* Let  $u_k \rightharpoonup u$  in  $W_g$ . Splitting the integral over  $\Omega$  into integral over  $\Omega_R$  and  $\Omega'_R$ , one has by Hölder inequality and  $(H_1)$

$$\begin{aligned} \int_{\Omega} m_1 |u_k - u_l|^p &= \int_{\Omega_R} m_1 |u_k - u_l|^p + \int_{\Omega'_R} m_1 |u_k - u_l|^p \\ &\leq \|m_1\|_{L^s(\Omega_R)} \|u_k - u_l\|_{L^{ps'}(\Omega_R)}^p + \|\theta\|_{L^\infty(\Omega'_R)} \int_{\Omega'_R} g |u_k - u_l|^p \end{aligned}$$

Take  $\varepsilon > 0$ . By  $(H_1)$  for  $R$  sufficiently large, the second term above is less than  $\frac{\varepsilon}{2}$ , uniformly with respect to  $k, l$ . Fixing such  $R$  and using the compact imbedding  $W^{1,p}(\Omega_R) \hookrightarrow L^{ps'}(\Omega_R)$ , one obtains that the first term is also less than  $\frac{\varepsilon}{2}$  for  $k, l$  sufficiently large. The conclusion then follows.  $\square$

*Proof of Theorem 3.1.* The proof is partly adapted from [11]. We give here some details for the sake of completeness.

Let us consider the following functionals

$$E_V(u) := \int_{\Omega} (|\nabla u|^p + V(x)|u|^p), \quad B(u) := \int_{\Omega} m_1 |u|^p,$$

which are  $C^1$  on  $W_g$ . Set

$$\lambda_1(V, m_1) := \inf \{ E_V(u), u \in W_g \text{ and } B(u) = 1 \}.$$

We aim to prove that this infimum is achieved at some nonnegative function and by Lagrange multipliers rule, we will conclude that  $\lambda_1(V, m_1)$  is a principal eigenvalue for (1.1).

It is easy, using  $(H_2)$ , to prove that

$$E_V(u) \geq \alpha \|u\|_{W_g}^p \quad \forall u \in W_g, \tag{3.2}$$

where  $\alpha \in [0, \frac{1-cK(\Omega,g)}{1+K(\Omega,g)}[$  and then  $E_V$  is coercive on  $W_g$ . On the other hand let  $u_k$  be a minimizing sequence for  $\lambda_1(V, m_1)$ , i.e

$$\begin{cases} E_V(u_k) \rightarrow \lambda_1(V, m_1) & (i) \\ B(u_k) = 1 & (ii) \end{cases}$$

Using (3.2) and (i), one deduces that  $u_k$  is a bounded sequence in  $W_g$ . Thus there exists  $u_0 \in W_g$  such that, up to a subsequence,  $u_k$  converges weakly to  $u_0$  in  $W_g$  and strongly in  $L^p(m_1, \Omega)$  (using Lemma 3.2). Consequently, from (ii), one has  $B(u_0) = 1$  which implies

$$\lambda_1(V, m_1) \leq E_V(u_0) \leq \liminf E_V(u_k) \leq \lambda_1(V, m_1).$$

The infimum is then achieved at  $u_0$ . By Lagrange multipliers rule,  $\lambda_1(V, m_1)$  is an eigenvalue for (1.1). Moreover  $E_V(|u|) = E_V(u)$  for any  $u \in W_g$ , so that  $\lambda_1(V, m_1)$  possesses a nonnegative eigenfunction. Serrin estimates (cf. Theorem 8 in [19]) and Harnack inequality (arguing as [8]) allow us to conclude that the eigenvalue  $\lambda_1(V, m_1)$  is principal.  $\square$

Next we consider an arbitrary weight  $m$  in the form  $m = m_1 - m_2$ , where  $m_1, m_2$  are nonnegative functions. For fixed  $\lambda$ , We consider the following eigenvalue problem with parameter  $\beta(\lambda)$

$$-\Delta_p u + (V + \lambda m_2)|u|^{p-2}u = \beta(\lambda)m_1|u|^{p-2}u. \tag{3.3}$$

It is clear that  $\lambda$  is an eigenvalue of (1.1) if and only if  $\lambda$  is a fixed point of  $\beta(\lambda)$  where  $\beta(\lambda)$  is an eigenvalue of (3.3). Searching such a  $\lambda$  is our goal in the next paragraph.

Let start to prove the following result.

**Theorem 3.3.** *Suppose  $N \leq p$  and let  $m = m_1 - m_2$  with  $m_1, m_2$  admissibles. Assume that  $m_1$  satisfies  $(H_1)$  and  $V$  satisfies  $(H_2)$ . Assume also that*

$$m^+ \not\equiv 0 \tag{H_3}$$

*Then (1.1) admits a principal eigenvalue  $\lambda_1(V, m_1 - m_2)$ , having an eigenfunction which belongs to  $W_{g+m_2}$  (where  $g$  is provided by  $(H_1)$ ).*

*Proof.* We will first prove that problem (3.3) admits a principal eigenvalue  $\lambda_1(V + \lambda m_2, m_1)$ . Indeed let us denote by

$$\begin{aligned} A_\lambda(u) &:= \int_\Omega [|\nabla u|^p + (V(x) + \lambda m_2)|u|^p] \quad \text{and set} \\ \beta(\lambda) &:= \inf\{A_\lambda(u) : u \in W_{g+m_2} \text{ and } B(u) = 1\}. \end{aligned}$$

As in the proof of Theorem 3.1, we will prove that  $\beta(\lambda)$  is achieved at some nonnegative function  $u_0 \in W_{g+m_2}$ , which yields to the existence of a principal eigenvalue for (3.3).

We will first prove that  $A_\lambda$  is coercive on  $W_{g+m_2}$ . Taking  $\alpha_1 = \min(\frac{\lambda}{2}, \frac{1-cK(\Omega,g)}{1+cK(\Omega,g)})$ , and using (3.2) we obtain

$$A_\lambda(u) \geq \alpha_1 \|u\|_{W_{g+m_2}}^p \quad \text{for all } u \in W_{g+m_2}, \tag{3.4}$$

so that  $A_\lambda$  is coercive on  $W_{g+m_2}$  for any  $\lambda > 0$ . Arguing as in the proof of Theorem 3.1, one deduces that for any  $\lambda > 0$ , (3.3) has a principal eigenvalue  $\beta(\lambda) := \lambda_1(V + \lambda m_2, m_1)$ .

Our purpose in the second step of the proof is to show the existence of a fixed point for the function  $\beta(\lambda)$ , which will clearly yields the conclusion of Theorem 3.3.

Since for  $\lambda > 0$  and for all  $u \in W_{g+m_2}$  with  $B(u) = 1$ ,

$$A_\lambda(u) = E_V(u) + \lambda \int_\Omega m_2(x)|u|^p \geq E_V(u) \geq \lambda_1(V, m_1),$$

where  $\lambda_1(V, m_1)$  is defined in Theorem 3.1, one deduces that

$$\beta(\lambda) \geq \lambda_1(V, m_1).$$

Consequently,  $\beta(\lambda) \geq \lambda$  for  $0 < \lambda \leq \lambda_1(V, m_1)$ . On the other hand  $(H_3)$  implies that there exists a nonempty open subset, with positive measure,  $\Omega_0 \subset \Omega$  such that

$$m = m_1 - m_2 > 0 \quad \text{a.e in } \Omega_0. \tag{3.5}$$

Let  $u_0 \in W_{g+m_2}$  with  $u_0 \not\equiv 0$  and  $\text{supp } u_0 \subset \Omega_0$ . Then for  $\lambda$  sufficiently large

$$A_\lambda(u_0) - \lambda \int_\Omega m_1|u_0|^p = \int_\Omega |\nabla u_0|^p - \lambda \int_\Omega m|u_0|^p < 0. \tag{3.6}$$

From (3.6) (taking  $v_0 = u_0/[B(u_0)]^{1/p}$  if necessary), one deduces that  $\beta(\lambda) < \lambda$  for  $\lambda$  sufficiently large. Moreover  $\beta$  is a concave u.s.c function from  $\mathbb{R}_0^+$  to  $\mathbb{R}$  (since it is the infimum of a family of affine functions of  $\lambda$ ), and so in particular  $\beta$  is continuous on  $\mathbb{R}_0^+$ . It follows that  $\beta$  admits a positive fixed point  $\lambda_0$  (ie  $\beta(\lambda_0) = \lambda_0$ ), and the theorem is proved.  $\square$

### 4. Existence of principal eigenvalue in high dimensions

As in the previous section, we will investigate the case when the weight  $m$  is a positive function and when it is indefinite. We assume that  $N > p$ . Let us begin with the first result when the weight is positive.

**Theorem 4.1.** *Suppose  $N > p$  and let  $m = m_1$  with  $m_1$  admissible and  $m_1 \not\equiv 0$ . Assume that*

$$m_1 \in L^{N/p}(\Omega) \tag{H_4}$$

$$V \in L^{N/p}(\Omega) \quad \text{and} \quad \|V\|_{L^{N/p}(\Omega)} \leq \frac{1}{c_1^p}, \tag{H_5}$$

where  $c_1$  is the constant of the imbedding of  $D^{1,p} \hookrightarrow L^{p^*}(\Omega)$ . Then the problem (1.1) admits a principal eigenvalue  $\lambda_1(V, m_1)$ , having an eigenfunction  $\varphi \in D^{1,p}(\Omega)$ .

The proof of Theorem 4.1 relies on the following lemma

**Lemma 4.2.** [4] *Assume that  $N > p$ . The functional*

$$u \mapsto \int_{\Omega} \omega |u|^q$$

*is well defined and weakly continuous in  $D^{1,p}(\Omega)$ , for  $1 \leq q < p^*$  and  $\omega \in L^{(p^*/q)'}(\Omega)$ .*

*Proof.* The proof is partly adapted from the corresponding lemma in [11, 12].

Let  $u_k \rightharpoonup u$  in  $D^{1,p}(\Omega)$ . Then, for any fixed  $R > 0$ ,  $u_k \rightarrow u$  in  $L^{qs'}(\Omega_R)$ . Given  $\varepsilon > 0$ , we pick  $R$  sufficiently large such that

$$\|\omega\|_{L^{(p^*/q)'(\Omega'_R)}} \|u_k - u_l\|_{L^{p^*}(\Omega)}^q \leq \frac{\varepsilon}{2},$$

uniformly with respect to  $k, l$  (which is possible since  $\omega \in L^{(p^*/q)'(\Omega)}$ ). Then splitting the integrals over  $\Omega$  into integrals over  $\Omega_R$  and over  $\Omega'_R$ , we get

$$\begin{aligned} \int_{\Omega} \omega |u_k - u_l|^q &= \int_{\Omega_R} \omega |u_k - u_l|^q + \int_{\Omega'_R} \omega |u_k - u_l|^q \\ &\leq \|\omega\|_{L^s(\Omega_R)} \cdot \|u_k - u_l\|_{L^{qs'}(\Omega_R)}^q + \|\omega\|_{L^{(p^*/q)'(\Omega'_R)}} \cdot \|u_k - u_l\|_{L^{p^*}(\Omega)}^q. \end{aligned}$$

Fixing such a  $R$ , one deduces from the compact imbedding  $W^{1,p}(\Omega_R) \hookrightarrow L^{qs'}(\Omega_R)$  that

$$\|\omega\|_{L^s(\Omega_R)} \cdot \|u_k - u_l\|_{L^{qs'}(\Omega_R)}^q \leq \frac{\varepsilon}{2},$$

for  $k$  and  $l$  sufficiently large and consequently  $u_k$  converges to  $u$  in  $L^q(\omega, \Omega)$ . □

*Proof of Theorem 4.1.* Let  $E_V$  be defined, as in Sect. 3, by:

$$E_V(u) := \int_{\Omega} [|\nabla u|^p + V(x)|u|^p].$$

From  $(H_5)$  and the imbedding  $D^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ , one easily establishes the following inequalities

$$(1 - c_1^p) \|V\|_{L^{N/p}(\Omega)} \|u\|_{D^{1,p}(\Omega)}^p \leq E_V(u) \leq (1 + c_1^p) \|V\|_{L^{N/p}(\Omega)} \|u\|_{D^{1,p}(\Omega)}^p, \tag{4.1}$$

which implies that  $E_V$  is coercive on  $D^{1,p}(\Omega)$ . Then, applying Lemma 4.2 (with  $\omega = m_1$  and  $q = p$ ), the result follows by minimization of  $E_V$  on the manifold  $M := \{u \in D^{1,p}(\Omega) : B(u) = 1\}$ , as in Sect. 3. □

Next, we turn to the case  $m = m_1 - m_2$  with nontrivial positive part  $m^+$  (i.e,  $m^+ \not\equiv 0$  in  $\Omega$ ).

**Theorem 4.3.** *Suppose  $N > p$  and let  $m = m_1 - m_2$  with  $m_1, m_2$  admissible functions. Assume that  $m$  satisfies  $(H_3)$  and  $m_1$  satisfies  $(H_4)$ . Assume further that  $V$  satisfies  $(H_5)$ . Then the problem (1.1) admits a principal eigenvalue  $\lambda_1(V, m_1 - m_2)$ , having an eigenfunction  $\varphi \in W_{m_2}$ .*

*Proof.* Let us write, for  $\lambda > 0$ ,

$$A_\lambda(u) := \int_\Omega (|\nabla u|^p + (V(x) + \lambda m_2(x))|u|^p), \quad B(u) := \int_\Omega m_1(x)|u|^p.$$

From (4.1), one easily proves that

$$A_\lambda(u) \geq \min(\delta, \lambda) \|u\|_{W_{m_2}}^p, \quad \text{for all } u \in W_{m_2},$$

where  $\delta = 1 - c_1^p \|u\|_{L^{N/p}(\Omega)} > 0$ .

Hence  $A_\lambda$  is coercive on  $W_{m_2}$ , then using Lemma 4.2 and Theorem 4.1, one deduces that the problem

$$-\Delta_p u + V(x)|u|^{p-2}u + \lambda m_2|u|^{p-2}u = \beta(\lambda)m_1|u|^{p-2}u \quad \text{in } \Omega \quad (4.2)$$

admits a principal eigenvalue  $\beta(\lambda)$  defined by

$$\beta(\lambda) = \lambda_1(V + \lambda m_2, m_1) := \inf\{A_\lambda(u) : u \in W_{m_2} \text{ and } B(u) = 1\}.$$

Arguing as in the proof of Theorem 3.3, one can prove that  $\beta(\lambda) \geq \lambda$  for  $\lambda > 0$  small and also for  $\lambda$  sufficiently large

$$\int_\Omega [|\nabla u_0|^p + (V - \lambda m)|u_0|^p] < 0,$$

with  $u_0 \in W_{m_2}, u_0 \not\equiv 0$  and  $\text{supp } u_0 \subset \Omega_0$ , where  $\Omega_0$  is provided by  $(H_3)$ . So  $\beta(\lambda) < \lambda$  for  $\lambda$  sufficiently large. Using the facts that  $\beta$  is concave and continuous in  $[0, +\infty[$ , one deduces the existence of a fixed point for  $\beta$  which is, in fact, a principal eigenvalue for (1.1).  $\square$

We give below some properties of the function  $\beta(\lambda)$ .

**Proposition 4.4.** *Let  $\lambda_n \rightarrow \lambda$  and  $(\beta(\lambda_n), \varphi_n), (\beta(\lambda), \varphi_\lambda)$  be the corresponding eigenpairs for problem (4.2). Then  $\beta(\lambda_n) \rightarrow \beta(\lambda)$  and  $\varphi_n \rightarrow \varphi_\lambda$  in  $W_{m_2}$ . Moreover  $\beta(\lambda)$  is differentiable and for any  $\lambda_0 \in \mathbb{R}$ , we have*

$$\beta'(\lambda_0) = \int_\Omega m_2 \varphi_{\lambda_0}^p \quad (4.3)$$

*Proof.* The proof borrows some ideas from [1, 5, 14].

Continuity of the function  $\beta$  follows from the fact that  $\beta$  is concave and concave function is continuous on its domain of definition.

To prove that  $\varphi_n \rightarrow \varphi_\lambda$ , we proceed as follows. Let  $\Lambda \subset \mathbb{R}$  be bounded. For  $\lambda_n \in \Lambda$ , since

$$\beta(\lambda_n) = \int_\Omega [|\nabla \varphi_n|^p + (V(x) + \lambda_n)|\varphi_n|^p] \leq K \quad (\text{constant}),$$

(using continuity of  $\beta$ ), one deduces that  $\int_\Omega |\nabla \varphi_n|^p$  and  $\int_\Omega m_2|\varphi_n|^p$  are bounded. Thus for some  $\tilde{\varphi} \in W_{m_2}$ ,  $\varphi_n$  converges to  $\tilde{\varphi}$  weakly in  $W_{m_2}$  and strongly in  $L^p(\omega, \Omega)$  for  $\omega \in L^{N/p}(\Omega)$  as  $\lambda_n \rightarrow \lambda \in \bar{\Lambda}$ . Next, taking  $\varphi_k - \varphi_l$  as testing function for  $\varphi_k$  in (4.2), one has

$$\begin{aligned} & \int_\Omega |\nabla \varphi_k|^{p-2} \nabla \varphi_k \cdot \nabla (\varphi_k - \varphi_l) + \int_\Omega V(x) \varphi_k^{p-1} (\varphi_k - \varphi_l) + \lambda_k \int_\Omega m_2 \varphi_k^{p-1} (\varphi_k - \varphi_l) \\ & = \beta(\lambda_k) \int_\Omega m_1 \varphi_k^{p-1} (\varphi_k - \varphi_l). \end{aligned} \quad (4.4)$$

Then

$$\begin{aligned}
 0 \leq J(\lambda_k, \lambda_l) &:= \int_{\Omega} (|\nabla \varphi_k|^{p-2} \nabla \varphi_k - |\nabla \varphi_l|^{p-2} \nabla \varphi_l) \cdot \nabla (\varphi_k - \varphi_l) \\
 &= - \int_{\Omega} V(x) (\varphi_k^{p-1} - \varphi_l^{p-1}) (\varphi_k - \varphi_l) \\
 &\quad - \lambda_k \int_{\Omega} m_2 (\varphi_k^{p-1} - \varphi_l^{p-1}) (\varphi_k - \varphi_l) \\
 &\quad - (\lambda_k - \lambda_l) \int_{\Omega} m_2 \varphi_l^{p-1} (\varphi_k - \varphi_l) \\
 &\quad + \beta(\lambda_k) \int_{\Omega} m_1 (\varphi_k^{p-1} - \varphi_l^{p-1}) (\varphi_k - \varphi_l) \\
 &\quad + [\beta(\lambda_k - \beta(\lambda_l))] \int_{\Omega} m_1 \varphi_l^{p-1} (\varphi_k - \varphi_l). \tag{4.5}
 \end{aligned}$$

Hence, from the monotonicity of the function  $x \mapsto |x|^{p-2}x$ , we have

$$\begin{aligned}
 0 \leq J(\lambda_k, \lambda_l) &\leq \int_{\Omega} |V(x)| |\varphi_k^{p-1} - \varphi_l^{p-1}| |\varphi_k - \varphi_l| \\
 &\quad + |\lambda_k - \lambda_l| \int_{\Omega} m_2 \varphi_l^{p-1} |\varphi_k - \varphi_l| \\
 &\quad + \beta(\lambda_k) \int_{\Omega} m_1 |\varphi_k^{p-1} - \varphi_l^{p-1}| |\varphi_k - \varphi_l| \\
 &\quad + |\beta(\lambda_k - \beta(\lambda_l))| \int_{\Omega} m_1 \varphi_l^{p-1} |\varphi_k - \varphi_l|. \tag{4.6}
 \end{aligned}$$

Using Hölder inequality, the hypothesis  $(H_4)$ ,  $(H_5)$  and Lemma 4.2 (for  $q = p$ ), one sees that  $\int_{\Omega} |V(x)| |\varphi_k^{p-1} - \varphi_l^{p-1}| |\varphi_k - \varphi_l| \rightarrow 0$  and  $\int_{\Omega} m_1 |\varphi_k^{p-1} - \varphi_l^{p-1}| |\varphi_k - \varphi_l| \rightarrow 0$ . Furthermore Hölder inequality implies that  $\int_{\Omega} m_2 \varphi_l^{p-1} |\varphi_k - \varphi_l|$  and  $|\int_{\Omega} m_1 \varphi_l^{p-1} |\varphi_k - \varphi_l|$  (using the fact  $D^{1,p}(\Omega) \hookrightarrow L^p(m_1, \Omega)$ ) remain bounded. Since  $(\lambda_k - \lambda_l) \rightarrow 0, (\beta(\lambda_k) - \beta(\lambda_l)) \rightarrow 0$  for  $k, l$  sufficiently large, we conclude that the right hand side of (4.6) goes to 0 as  $k, l \rightarrow +\infty$ .

To go on with the proof, we observe that for some constant  $d = d(p)$  and for any  $a, b \in \mathbb{R}^N$ ,

$$|a - b|^p \leq \{(|a|^{p-2}a - |b|^{p-2}b)(a - b)\}^{r/2} (|a|^p + |b|^p)^{1-r/2}, \tag{4.7}$$

where  $r = p$  if  $p \in ]1, 2[$  and  $r = 2$  if  $p \geq 2$  (cf. [15]). Applying (4.7) and Hölder inequality, one easily derives that  $\nabla \varphi_n \rightarrow \nabla \tilde{\varphi}$  in  $L^p(\Omega)$ . Moreover the calculation leading to (4.5) gives,

$$\begin{aligned}
 0 &\leq \lambda_k \int_{\Omega} m_2(\varphi_k^{p-1} - \varphi_l^{p-1})(\varphi_k - \varphi_l) \\
 &\leq - \int_{\Omega} V(x)(\varphi_k^{p-1} - \varphi_l^{p-1})(\varphi_k - \varphi_l) - \lambda_k \int_{\Omega} m_2(\varphi_k^{p-1} - \varphi_l^{p-1})(\varphi_k - \varphi_l) \\
 &\quad - (\lambda_k - \lambda_l) \int_{\Omega} m_2 \varphi_l^{p-1}(\varphi_k - \varphi_l) + \beta(\lambda_k) \int_{\Omega} m_1(\varphi_k^{p-1} - \varphi_l^{p-1})(\varphi_k - \varphi_l) \\
 &\quad + [\beta(\lambda_k - \beta(\lambda_l))] \int_{\Omega} m_1 \varphi_l^{p-1}(\varphi_k - \varphi_l).
 \end{aligned} \tag{4.8}$$

where the right hand side goes to zero. We then deduce from above that  $\int_{\Omega} m_2|\varphi_k - \varphi_l|^p \rightarrow 0$  by applying successively (4.7) and Hölder inequality. Consequently  $\varphi_n \rightarrow \tilde{\varphi}$  in  $W_{m_2}$ . On the other hand, one has

$$\begin{aligned}
 \beta(\lambda) &= \lim_{n \rightarrow +\infty} \beta(\lambda_n) \\
 &= \lim_{n \rightarrow +\infty} \int_{\Omega} [|\nabla \varphi_n|^p + (V(x) + \lambda_n m_2)\varphi_n^p] = A_{\lambda}(\tilde{\varphi})
 \end{aligned} \tag{4.9}$$

and since  $\int_{\Omega} m_1 \tilde{\varphi}^p = 1$  (because  $\int_{\Omega} m_1 \varphi_n^p = 1$  and  $W_{m_2} \hookrightarrow L^p(m_1, \Omega)$ ), one derives that  $\tilde{\varphi} = \varphi_{\lambda}$  and the first part of Proposition 4.4 is proved.

To prove the differentiability of  $\beta(\lambda)$ , let write

$$\begin{aligned}
 \beta(\lambda_n) &= \int_{\Omega} [|\nabla \varphi_n|^p + (V(x) + \lambda_n m_2)\varphi_n^p] \\
 &= \int_{\Omega} [|\nabla \varphi_n|^p + (V(x) + \lambda m_2)\varphi_n^p] + (\lambda_n - \lambda) \int_{\Omega} m_2 \varphi_n^p \\
 &\geq \beta(\lambda) + (\lambda_n - \lambda) \int_{\Omega} m_2 \varphi_n^p.
 \end{aligned}$$

Replacing  $\lambda$  (resp.  $\varphi_{\lambda}$ ) by  $\lambda_n$  (resp.  $\varphi_n$ ) in the last inequality we have

$$(\lambda_n - \lambda) \int_{\Omega} m_2 \varphi_n^p \leq \beta(\lambda_n) - \beta(\lambda) \leq (\lambda_n - \lambda) \int_{\Omega} m_2 \varphi_{\lambda}^p.$$

Dividing by  $(\lambda_n - \lambda)$  and letting  $\lambda_n \rightarrow \lambda$ , we obtain (4.3) □

*Remark 4.5.* In Sect. 3 (respectively Sect. 4), hypotese  $(H_2)$  in Theorems 3.1 and 3.3 (resp.  $(H_5)$  in Theorems 4.1 and 4.3) guarantees the positivity of the principal eigenvalue. In what follows, we aim to weak these hypotheses, that will allow the principal eigenvalue to be either positive or negative.

For that let us introduce the following number (cf. [9, 16])

$$\alpha(V, m) := \inf \left\{ E_V(u), u \in D^{1,p}(\Omega), \int_{\Omega} m|u|^p = 0 \text{ and } \int_{\Omega} |u|^p = 1 \right\}.$$

One has the following

**Theorem 4.6.** *Suppose  $N > p$  and let  $m = m_1 - m_2$  with  $m_1, m_2$  admissible functions. Assume that  $m$  satisfies  $(H_3)$  and  $(H_4)$ . Assume further that  $V$  changes sign and satisfies*

$$V \in L^{N/p}(\Omega). \tag{H'_5}$$

Then the problem (1.1) admits a principal eigenvalue  $\lambda_1(V, m_1 - m_2)$ , having an eigenfunction  $\varphi \in D^{1,p}(\Omega)$  if  $\alpha(V, m) > 0$ .

*Proof.* We define

$$\lambda_1(V, m) := \inf \left\{ E_V(u), u \in D^{1,p}(\Omega) \text{ and } \int_{\Omega} m|u|^p = 1 \right\}.$$

We have that  $\lambda_1(V, m) > -\infty$ . Indeed assume by contradiction that there exists a sequence  $(u_k)$  such that  $\int_{\Omega} m|u_k|^p = 1$  and  $E_V(u_k) \rightarrow -\infty$ . Then

$$\int_{\Omega} |\nabla u_k|^p + \int_{\Omega} V^+|u_k|^p - \int_{\Omega} V^-|u_k|^p \rightarrow -\infty,$$

which implies that  $\int_{\Omega} V^-|u_k|^p \rightarrow +\infty$ . Set  $w_k = u_k / (\int_{\Omega} V^-|u_k|^p)^{1/p}$ , then  $\int_{\Omega} V^-|w_k|^p = 1$  and

$$E_V(w_k) = \frac{E_V(u_k)}{\int_{\Omega} V^-|u_k|^p} \leq 0, \text{ for } k \text{ sufficiently large.}$$

Moreover we have

$$\|w_k\|_{D^{1,p}(\Omega)}^p = \frac{\|u_k\|_{D^{1,p}(\Omega)}^p}{\int_{\Omega} V^-|u_k|^p} \leq \frac{\int_{\Omega} |\nabla u_k|^p + \int_{\Omega} V^+|u_k|^p}{\int_{\Omega} V^-|u_k|^p} \leq 1.$$

Hence  $w_k$  is bounded in  $D^{1,p}(\Omega)$ . Consequently, up to a subsequence  $w_k \rightharpoonup w_0$  in  $D^{1,p}(\Omega)$ ,  $\int_{\Omega} m|w_k|^p \rightarrow \int_{\Omega} m|w_0|^p$  and  $\int_{\Omega} V^-|w_k|^p \rightarrow \int_{\Omega} V^-|w_0|^p$  (using Lemma 4.2). We deduce that  $w_0 \not\equiv 0$  since  $\int_{\Omega} V^-|w_k|^p = 1$ .

On the other hand, since  $E_V$  is lower semi-continuous, we have

$$E_V(w_0) \leq \liminf E_V(w_k) \leq 0.$$

Furthermore

$$\int_{\Omega} m|w_k|^p = \frac{\int_{\Omega} m|u_k|^p}{\int_{\Omega} V^-|u_k|^p} = \frac{1}{\int_{\Omega} V^-|u_k|^p} \rightarrow 0.$$

Then  $\int_{\Omega} m|w_0|^p = 0$  and the function  $\tilde{w}_0 = \frac{w_0}{(\int_{\Omega} |w_0|^p)^{1/p}}$  is admissible in the definition of  $\alpha(V, m)$  so that  $\alpha(V, m) \leq A(\tilde{w}_0) \leq 0$ , a contradiction.

To go on with the proof, let  $u_k$  be a minimizing sequence for  $\lambda_1(V, m)$ , ie.  $\int_{\Omega} m|u_k|^p = 1$  and  $E_V(u_k) \rightarrow \lambda_1(V, m)$ . It suffices to prove that  $u_k$  is bounded in  $D^{1,p}(\Omega)$ . Indeed, assume by contradiction that  $\|u_k\|_{D^{1,p}(\Omega)} \rightarrow +\infty$  and consider  $t_k = \frac{u_k}{\|u_k\|_{D^{1,p}(\Omega)}}$ . Then, up to subsequence,  $t_k \rightarrow t_0$  weakly in  $D^{1,p}(\Omega)$ ,

$$E_V(t_k) = \frac{E_V(u_k)}{\|u_k\|_{D^{1,p}(\Omega)}^p} \rightarrow 0 \text{ and } \int_{\Omega} m|t_k|^p = \frac{\int_{\Omega} m|u_k|^p}{\|u_k\|_{D^{1,p}(\Omega)}^p} = \frac{1}{\|u_k\|_{D^{1,p}(\Omega)}^p} \rightarrow 0.$$

Moreover

$$\|t_k\|_{D^{1,p}(\Omega)}^p = E_V(t_k) - \int_{\Omega} V|t_k|^p \rightarrow - \int_{\Omega} V|t_0|^p.$$

Hence

$$\|t_0\|_{D^{1,p}(\Omega)}^p \leq \liminf \|t_k\|_{D^{1,p}(\Omega)}^p = - \int_{\Omega} V|t_0|^p,$$

and consequently  $E_V(t_0) \leq 0$ . Moreover  $\int_{\Omega} m|t_0|^p = 0$  and we claim that  $t_0 \not\equiv 0$ . Indeed if  $t_0 \equiv 0$  then  $\|t_k\|_{D^{1,p}(\Omega)}^p \rightarrow -\int_{\Omega} V|t_0|^p = 0$ , a contradiction since  $\|t_k\|_{D^{1,p}(\Omega)}^p = 1$ . Arguing as above, we get a contradiction with the assumption on  $\alpha(V, m)$ .  $\square$

*Remark 4.7.* (i) When  $\Omega$  is an bounded domain and  $V \equiv V_0$  (constant function), one has  $\alpha(V, m) = V_0 + \alpha(0, m)$ . Since  $\alpha(0, m) \geq 0$  one can choose  $V_0$  such that  $\alpha(V, m) > 0$ .

(ii) It is easily seen that  $\alpha(V, m) = \infty$  if and only if  $m > 0$  in  $\Omega$ .

(iii) It is not difficult to construct an example of functions  $V$  and  $m$  such that  $\alpha(V, m) > 0$  when  $\Omega$  is an unbounded domain. The following example is partly adapted from Example 10.1 of [9].

*Example 1.* We will assume that  $N > 2$ . Let  $\Omega_1$  be a bounded domain of  $\mathbb{R}^N$  such that  $\Omega_1 \subset \Omega$  and  $B_0 \subset \Omega_1$  be an open subset such that

$$\|\varphi_1\|_{2,B_0}^2 := \int_{B_0} \varphi_1^2 dx = \frac{1}{2},$$

where  $\varphi_1$  is the first  $L^2(\Omega_1)$ -normed positive eigenfunction of  $(-\Delta, H_0^1(\Omega_1))$  with  $\lambda_1$  as associated positive eigenvalue (i.e  $-\Delta\varphi_1 = \lambda_1\varphi_1$  in  $\Omega_1$  and  $\varphi_1 = 0$  in  $\partial\Omega_1$ ). Set for  $\varepsilon > 0$

$$m = \begin{cases} \varepsilon & \text{in } B_0 \\ -\varepsilon & \text{in } \Omega_1 \setminus B_0 \\ 0 & \text{in } \Omega \setminus \Omega_1 \end{cases}$$

and

$$V = \begin{cases} -c\lambda_1 + \frac{1}{2\varepsilon} & \text{in } B_0 \\ -c\lambda_1 - \frac{1}{2\varepsilon} & \text{in } \Omega_1 \setminus B_0 \\ 0 & \text{in } \Omega \setminus \Omega_1 \end{cases}$$

where  $c$  is a positive constant. We denote by  $\tilde{\varphi}_1$  the extension by 0 to  $\Omega$  of  $\varphi_1$ .

(a) From  $\|\varphi_1\|_{2,B_0}^2 = \frac{1}{2}$ , we deduce that  $\int_{\Omega} m\tilde{\varphi}_1^2 dx = 0$ ,  $\int_{\Omega} \tilde{\varphi}_1^2 dx = 1$  and  $\|\varphi_1\|_{2,B_0}^2 = \|\varphi_1\|_{2,\Omega_1 \setminus B_0}^2 = \frac{1}{2}$ .

(b)  $E_V(\tilde{\varphi}_1) = E_V(\varphi_1)$

$$\begin{aligned} &= \lambda_1 + (-c\lambda_1 + \frac{1}{2\varepsilon}) \|\varphi_1\|_{2,B_0}^2 + (-c\lambda_1 - \frac{1}{2\varepsilon}) \|\varphi_1\|_{2,\Omega_1 \setminus B_0}^2 \\ &= (1 - c)\lambda_1 \end{aligned}$$

so that if  $0 < c < 1$  then  $E_V(\tilde{\varphi}_1) > 0$ .

(c) If  $u \in H_0^1(\Omega)$  is such that  $\int_{\Omega} m|u|^2 dx = 0$  then  $\|u\|_{2,B_0} = \|u\|_{2,\Omega_1 \setminus B_0}$  and

$$\begin{aligned} E_V(u) &= \int_{\Omega} |\nabla u|^2 + \int_{\Omega} V|u|^2 \\ &= \int_{\Omega} |\nabla u|^2 - \left(c\lambda_1 + \frac{1}{2\varepsilon}\right) \|u\|_{2,\Omega_1 \setminus B_0}^2 - \left(c\lambda_1 - \frac{1}{2\varepsilon}\right) \|u\|_{2,B_0}^2 \end{aligned}$$

$$\begin{aligned} &= \int_{\Omega} |\nabla u|^2 - c\lambda_1 \|u\|_{2,\Omega_1}^2 + \frac{1}{2\varepsilon} \left( \|u\|_{2,B_0}^2 - \|u\|_{2,\Omega_1 \setminus B_0}^2 \right) \\ &= \int_{\Omega} |\nabla u|^2 - c\lambda_1 \|u\|_{2,\Omega_1}^2 \geq (1 - c)\lambda_1 \|u\|_{2,\Omega_1}^2 \end{aligned}$$

since, by definition of  $\lambda_1$ , one has

$$\int_{\Omega} |\nabla u|^2 \geq \int_{\Omega_1} |\nabla u|^2 \geq \lambda_1 \int_{\Omega_1} |u|^2 \quad \text{for all } u \in H_0^1(\Omega).$$

Hence one deduces that  $\alpha(V, m) > 0$  if  $0 < c < 1$ .

Next we give examples of functions  $g$  and domains  $\Omega$  for which assumption  $H_g$  holds.

*Example 2.* We borrow some idea from [11].

Let  $N = 1$ ,  $p > 1$  and  $\Omega = ]0, +\infty[$ . If  $g$  is admissible function with  $m \not\equiv 0$  and satisfies

$$\int_0^{+\infty} x^{p-1} g(x) dx < +\infty, \tag{4.10}$$

then  $(H_g)$  holds. Indeed writing  $u(x) = \int_0^x u'(t) dt$  and using Hölder inequality, one has

$$\begin{aligned} |u(x)|^p &\leq \left( \int_0^x |u'(t)| dt \right)^p \\ &\leq \left( \int_0^x dt \right)^{p/p'} \left( \int_0^x |u'(t)|^p dt \right) \\ &\leq x^{p-1} \left( \int_0^{+\infty} |u'(t)|^p dt \right). \end{aligned}$$

Hence

$$\int_0^{+\infty} g(x) |u(x)|^p dx \leq \left( \int_0^{+\infty} x^{p-1} g(x) dx \right) \left( \int_0^{+\infty} |u'(t)|^p dt \right).$$

Consequently if  $m_1$  is admissible,  $m_1 \not\equiv 0$ , satisfies  $(H_1)$  with respect to the function  $m$  such as (4.10) holds, and  $V$  satisfies  $(H_2)$ , then Theorem 3.1 applies and yields a principal eigenvalue for the problem

$$\begin{cases} -(|u'|^{p-2} u')' + V|u|^{p-2} u = \lambda m_1(x) |u|^{p-2} u & \text{in } ]0, +\infty[ \\ u(0) = 0 \end{cases} \tag{4.11}$$

The following proposition describes a situation where (4.11) does not admit a principal eigenvalue.

**Proposition 4.8.** *Let  $m_1$  be admissible function such that for some  $0 < \alpha < p-1$*

$$\int_0^{+\infty} x^\alpha m_1(x) dx = +\infty \tag{4.12}$$

*Then (4.11) does not admit a principal eigenvalue.*

*Proof.* The proof is partly adapted from that of Proposition 3.6 of [11].

Let us define for  $R > 1$  the following function

$$u_R(x) = \begin{cases} x & \text{if } x \in [0, 1] \\ x^{\alpha/p} & \text{if } x \in [1, R] \\ R^{\alpha/p} - R^{\frac{\alpha}{p}-1}(x - R) & \text{if } x \in [R, 2R] \\ 0 & \text{if } x \geq 2R \end{cases}$$

We have

(i)

$$\begin{aligned} \int_0^{+\infty} |u'_R(x)|^p dx &= 1 + \left(\frac{\alpha}{p}\right)^p \int_1^R x^{\alpha-p} dx + R^{\alpha-p} \int_R^{2R} dx \\ &= 1 + R^{\alpha-p+1} + \frac{1}{p-\alpha-1} \left(\frac{\alpha}{p}\right)^p [1 - R^{\alpha-p+1}] \\ &\leq 2 + \left(\frac{\alpha}{p}\right)^p \frac{1}{p-\alpha-1} \quad \text{since } R > 1. \end{aligned}$$

(ii)

$$\int_0^{+\infty} m_1(x)|u_R(x)|^p dx \geq \int_1^R m_1(x)|u_R(x)|^p dx = \int_1^R x^\alpha m_1(x) dx.$$

Then from (4.12),  $\int_0^{+\infty} m_1(x)|u_R(x)|^p dx \rightarrow +\infty$  as  $R \rightarrow +\infty$ .

(iii)

$$\begin{aligned} \int_0^{+\infty} g(x)|u_R(x)|^p dx &\leq \int_0^R x^\alpha g(x) dx + R^\alpha \int_R^{2R} g(x) dx \\ &\leq \int_0^{2R} x^\alpha g(x) dx \\ &\leq \int_0^1 x^\alpha g(x) dx + \int_1^{2R} x^{p-1} g(x) dx. \end{aligned}$$

(4.10) implies that  $\int_0^{+\infty} g(x)|u_R(x)|^p dx < +\infty$  for  $R$  sufficiently large. From  $(H_2)$ , one deduces that  $\int_0^{+\infty} V(x)|u_R(x)|^p dx < +\infty$  for  $R$  sufficiently large.

Now assume that a principal eigenvalue  $\lambda$  exists for the problem (4.11).

Then

$$E_V(u) = \int_0^{+\infty} |u'(x)|^p dx + \int_0^{+\infty} V(x)|u(x)|^p dx \geq \lambda \int_0^{+\infty} m_1(x)|u(x)|^p dx, \quad \forall u \in W_m.$$

In particular

$$0 < \lambda \leq \frac{\int_0^{+\infty} |u'_R(x)|^p dx}{\int_0^{+\infty} m_1(x)|u_R(x)|^p dx} + \frac{\int_0^{+\infty} V(x)|u_R(x)|^p dx}{\int_0^{+\infty} m_1(x)|u_R(x)|^p dx}.$$

Since the right hand side goes to 0 as  $R \rightarrow +\infty$ , a contradiction follows.  $\square$

In particular taking  $m_1(x) = \frac{1}{(1+x)^\beta}$ ,

- If  $\beta < p$ , one can find  $\alpha$  such that  $\beta < 1 + \alpha < p$  and (4.12) holds. Then Proposition 4.8 yields non existence result.

- On contrary, if  $\beta > p$  then with  $\varepsilon > 0$  such that  $\beta - \varepsilon > p$ , one has that  $m(x) = \frac{1}{(1+x)^{\beta-\varepsilon}}$  satisfies (4.10). This yields existence result.

*Remark 4.9.* When  $N \leq p$ , an inequality of the form  $(H_g)$  with  $g$  admissible,  $g \not\equiv 0$ , never holds for  $\Omega = \mathbb{R}^N$ , as notified in [11].

*Example 3.* Take  $N > p$ ,  $\Omega = \mathbb{R}^N$ ;  $m = m_1 \not\equiv 0$ ,  $m_1 \in L^{N/p}(\mathbb{R}^N)$  and  $g(x) = \frac{1}{1+|x|^p}$ . Denoting by  $C_{N,p}$  the (inverse of the) Hardy constant. Then  $(H_g)$  holds since by Hardy inequality, one has

$$\int_{\mathbb{R}^N} g|u|^p \leq \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \leq C_{N,p} \int_{\mathbb{R}^N} |\nabla u|^p \quad \text{for all } u \in C_c^\infty(\mathbb{R}^N).$$

Then choosing a potential  $V$  which satisfies  $(H'_5)$ , one can apply Theorem 4.6 to get an existence result.

### 5. Non-existence of principal eigenvalue

In this section, we give a nonexistence result. It is related to the case when the second condition in hypohese  $(H_5)$  does not hold for the potential  $V$ .

Let consider problem (1.1) with a certain class of potentials  $V$  which change sign, let the following set:

$$\mathcal{V} := \left\{ V / \int_{\Omega} [|\nabla u|^p + V(x)|u|^p] \geq 0, \quad \forall u \in D^{1,p}(\Omega) \right\}.$$

We claim that  $\mathcal{V}$  is not empty. Indeed choose  $V$  which changes sign such that

$$\max[V^-(x)(1 + |x|^p)] \leq K(N, p),$$

where  $K(N, p) = (\frac{N-p}{p})^p$  being the inverse of the constant in Hardy's inequality i.e.,

$$\int_{\Omega} |\nabla u|^p \geq K(N, p) \int_{\Omega} \frac{|u|^p}{|x|^p} \quad \forall u \in C_c^\infty(\Omega).$$

One has

$$\begin{aligned} \int_{\Omega} |\nabla u|^p &\geq K(N, p) \int_{\Omega} \frac{|u|^p}{|x|^p} \geq \max[V^-(x)(1 + |x|^p)] \int_{\Omega} \frac{|u|^p}{|x|^p} \\ &\geq \int_{\Omega} V^-(x)|u|^p \\ &\geq - \int_{\Omega} V(x)|u|^p \quad \forall u \in C_c^\infty(\Omega), \end{aligned}$$

so such  $V$  belong to  $\mathcal{V}$ .

In the following Theorem which generalizes an analogous result in [10], we prove the non existence of principal eigenvalue of problem (1.1) if  $V \notin \mathcal{V}$ .

**Theorem 5.1.** *Suppose  $N > p$ ,  $m_1$  admissible function which satisfies  $(H_4)$ . Assume that  $V$  changes sign in  $\Omega$ ,  $V \in L^{N/p}(\Omega)$  and  $V \notin \mathcal{V}$ . Moreover assume that*

$$(m_1 - \theta)^+ \not\equiv 0 \tag{H_6}$$

$$\max \left[ \frac{(V - t\theta)^-}{\theta} \right] \leq K(N, p) \leq \max \left( \frac{V^-}{\theta} \right), \tag{H7}$$

for all parameter  $t > 0$ , where  $\theta(x) = \frac{1}{(1+|x|)^p}$ . Then problem (1.1) with  $m = m_1$  does not admit a positive principal eigenvalue.

*Proof.* We consider, the following problem with parameter  $t$ :

$$-\Delta_p u + [V(x) - t\theta(x)]|u|^{p-2}u = \lambda[m_1(x) - \theta(x)]|u|^{p-2}u. \tag{5.1}$$

The eigenvalue problem (5.1) satisfies the hypotheses of Theorem 4.1, so it admits a principal eigenvalue

$$\lambda_1(t) := \inf \left\{ \int_{\Omega} [|\nabla u|^p + (V(x) - t\theta(x))]|u|^p, \int_{\Omega} [m_1(x) - \theta(x)]|u|^p = 1 \right\}. \tag{5.2}$$

Assume that there exists  $t_1 > 0$  such that  $\lambda_1(t_1) = t_1$ . Then  $t_1$  is a principal eigenvalue for (1.1) and consequently we have

$$\int_{\Omega} [|\nabla u|^p + V(x)|u|^p] \geq t_1 \int_{\Omega} m_1(x)|u|^p \quad \text{for all } u \in D^{1,p}(\Omega).$$

which implies that  $\int_{\Omega} [|\nabla u|^p + V(x)|u|^p] \geq 0$  for all  $u \in D^{1,p}(\Omega)$ . But this contradicts the fact that  $V \notin \mathcal{V}$ . If there exists a certain rank  $j > 1$  such that  $\lambda_j(t_1) = t_1$ , then  $t_1$  is the first eigenvalue of Problem (1.1) associated with an eigenfunction  $\varphi_1$  which is has not necessary the same sign. Hence  $t_1$  is not a principal eigenvalue. □

Let us conclude this paragraph by giving some regularity and simplicity results.

**Theorem 5.2.** *Assume that  $m = m_1 - m_2$  with  $m_1, m_2$  admissible functions;  $V, m \in L^{N/p}(\Omega)$ . Let  $\lambda_1(V, m)$  be a principal eigenvalue of (1.1) with  $u \in W_{m_2}$ ,  $u \neq 0$ ,  $u \geq 0$  an associated eigenfunction. Then*

- (i)  $u \in C_{loc}^{\alpha}(\Omega)$  for some  $\alpha \in ]0, 1[$  and  $u > 0$  in  $\Omega$ .
- (ii) *There is no other principal eigenvalue of (1.1) and  $\lambda_1(V, m)$  is simple.*

*Proof.* It follows from Lemma 2.1 (for  $N \leq p$ ) or from Sobolev imbeddings (for  $N > p$ ) that for any  $R > 0$ ,  $u \in W^{1,p}(\Omega_R)$ . Then we deduce from Serrin  $L^{\infty}$  estimate and Tolksdorf regularity result (cf. [19,20]) that  $u \in C_{loc}^{\alpha}(\Omega)$  for some  $0 < \alpha < 1$ . Moreover, applying Harnack inequality, yields that  $u$  is  $> 0$  in  $\Omega$ .

To prove uniqueness of  $\lambda_1(V, m)$ , we will use an argument based on Picone’s identity that we will recall. Let  $L, R$  be defined, for any  $\xi \geq 0$  and  $\Phi > 0$  two a.e differentiable function, as

$$L(\xi, \Phi) := |\nabla \xi|^p + (p - 1) \frac{\xi^p}{\Phi^p} |\nabla \Phi|^p - p \frac{\xi^{p-1}}{\Phi^{p-1}} |\nabla \Phi|^{p-2} \nabla \Phi \nabla \xi.$$

$$R(\xi, \Phi) := |\nabla \xi|^p - |\nabla \Phi|^{p-2} \nabla \Phi \nabla \left( \frac{\xi^p}{\Phi^p} \right).$$

By Picone's identity (cf.[2]),  $0 \leq L(\xi, \Phi) = R(\xi, \Phi)$  and if  $\frac{\xi}{\Phi} \in W_{\text{loc}}^{1,1}(\Omega)$  then  $L(\xi, \Phi) = 0$  iff  $\xi = c\Phi$  for some positive constant  $c$ . Now assume that  $(\lambda_1(V, m), u)$  and  $(\lambda, v)$  are two eigenpairs for (1.1) with  $u, v > 0$  in  $\Omega$ . Let  $\varphi_k \in C_c^\infty(\Omega)$  be a sequence which converges to  $v$  in  $W_{m_2}$ . Then, for a subsequence,  $\varphi_k$  and  $\nabla\varphi_k$  converge a.e in  $\Omega$  to  $v$  and  $\nabla v$  respectively. We then take  $|\varphi_k|^p/u^{p-1}$  as test function in the equation satisfying by  $u$ , and using Picone's identity, we have:

$$\begin{aligned} 0 \leq L(\varphi_k, u) &= \int_{\Omega} |\nabla\varphi_k|^p - \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \left( \frac{|\xi|^p}{u^{p-1}} \right) \\ &= \int_{\Omega} |\nabla\varphi_k|^p + \int_{\Omega} V(x)|\varphi_k|^p - \lambda_1(V, m) \int_{\Omega} m|\varphi_k|^p. \end{aligned}$$

Passing to the limit, as  $k$  going to infinity, yields

$$0 \leq L(v, u) \leq \int_{\Omega} |\nabla v|^p + \int_{\Omega} V(x)v^p - \lambda_1(V, m) \int_{\Omega} mv^p,$$

and since  $(\lambda, v)$  is also an eigenpair for (1.1), we have

$$0 \leq L(v, u) \leq (\lambda - \lambda_1(V, m)) \int_{\Omega} mv^p.$$

If  $\int mv^p \geq 0$  (resp.  $\int mv^p \leq 0$ ) and  $\lambda \leq \lambda_1(V, m)$  (resp.  $\lambda \geq \lambda_1(V, m)$ ), then  $L(v, u) = 0$  and we conclude that  $v = cu$  and  $\lambda = \lambda_1(V, m)$ .  $\square$

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