Advances in Natural and Applied Sciences, 6(6): 841-846, 2012
ISSN 1995-0772

This is a refereed journal and all articles are professionally screened and reviewed

ORIGINAL ARTICLE

Analytical Solution of Blasius problem

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A. Adomou, F.P. Codo, V. Adanhounmè: Analytical Solution of Blasius problem

ABSTRACT

The objective of this paper is to solve the Blasius problem in taking in the case of the boundary-layer flows of pure fluids (non-porous domains) over a flat plate by using the Adomian method. Doing so, we obtained the set of admissible values of the shear-stress on the plate surface.

Key words: Convergent series, Decomposition method, Fluid flow, Shear-stress.

Nomenclature

1. u velocity in the x-direction
2. u_0 velocity of the free stream
3. v velocity in the y-direction
4. x horizontal coordinate
5. y vertical coordinate
6. μ viscosity coefficient
7. ρ density
8. $\nu = \frac{\mu}{\rho}$ kinematic viscosity of the fluid

Introduction

In fluid mechanics, the problem of flow past a flat plate was first introduced by Blasius (1908) by assuming a series solutions. Later, numerical methods were used in L.Howarth, (1938) to obtain the solution of the boundary layer equation. In (M.E.Eglit *et al,* 1996) the first derivative with respect to y of the velocity component in the *x* direction at the point y=0 for the Blasius problem is computed numerically for the estimation of the shear-stress on the plate surface. Later in (P.Vadasz, 1997) one solved the problem above by assuming a finite power series where the objective is to determine the power series coefficients. The purpose of this study is to obtain the solutions for the Blasius problem for two dimensional boundary layer using the Adomian decomposition technique and to compute the admissible values of the shear-stress on the wall, imposing the constraint on the first derivative with respect to y of the velocity component in the *x* direction at the point *y*=0.

Mathematical model:

The physical model considered here consists of a flat plate parallel to the *x*- axis with its leading edge at *x*=0 and infinitely long down stream with constant component u_0 of the velocity.For the mathematical analysis we assume the properties of the fluid such as viscosity and conductivity, to a first approximation, are constant. Under these assumptions the basic equations required for the analysis of the physical phenomenon are the equations of continuity and motion. According to the Boussinesq approximation these equations get the following expressions (M.E.Eglit *et al*, 1996)

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$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (1)$$
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial x}=v\frac{\partial^2 u}{\partial y^2} \quad (2)$$
with the boundary conditions imposed on the flow in (M.E.Eglit *et al*, 1996)
$$\psi=\frac{\partial\psi}{\partial y}=0,\ y=0\ ,\lim_{y\to\infty}\frac{\partial\psi}{\partial y}=u_0 \quad (3)$$
Where ψ is a stream function related to the velocity components as:
$$u=\frac{\partial\psi}{\partial y}, v=\frac{-\partial\psi}{\partial x} \quad (4)$$

3. Analytical solution and convergence results:

In this section we provide the analytical solutions, i.e. the fluid velocity components as sums of convergent series using the Adomian decomposition technique and compute the admissible values of the shear-stress on the plate surface Consider the stream function ψ
$$\psi(x,y)=\sqrt{vu_0x}f(\eta),\qquad \eta=y\sqrt{\frac{u_0}{vx}} \quad (5)$$
Where f is a function three times continuously differentiable on the interval $[0,\eta_0]$ and η_0 a constant positive real. Then the equations (1) and (2) with the boundary conditions (3) are transformed as
$$f'''+\frac{1}{2}ff''=0\ ,\ (0)=f'(0)=0\ ,\ f'(+\infty)=1 \quad (6)$$
where $(.)'$ stands for $\frac{d(.)}{d\eta}$

Definition:

The problem (6) is called the Blasius problem for boundary-layer flows of pure fluids (non-porous domains) over a flat plate.

Let us transform the problem (6) into the nonlinear integral equation. For this purpose, setting $g'(\eta)=f(\eta)$ we can write the equation in (6) as
$$g''''+\frac{1}{2}g'g'''=0 \quad (7)$$
Multiplying by $e^{\frac{1}{2}g}$ and integrating the result from 0 to η we reduce (7) to
$$g'''=Ke^{-\frac{1}{2}g}\text{, where } K=g'''(0)e^{\frac{1}{2}g(0)} \quad (8)$$
Integrating three times the relation (8) from 0 to η,τ,σ and taking into account the boundary conditions in (6) we reduce (8) to the nonlinear integral equation
$$g(\eta)-K\int_0^\eta\int_0^\tau\int_0^\sigma e^{-\frac{1}{2}g(S)}\,dsd\sigma d\tau=a_0 \quad (9)$$
$$a_0=g(0)=const$$
which is a functional equation
$$g-N(g)=a_0\ ,\ \text{where} \quad (10)$$
$$N(g)=K\int_0^\eta\int_0^\tau\int_0^\sigma e^{-\frac{1}{2}g(S)}\,dsd\sigma d\tau\,,$$
$$K=\left[\int_0^{+\infty}e^{-\frac{1}{2}g(S)}ds\right]^{-1}$$
Here $N(g)$is a nonlinear operator from a Hilbert space *H* into *H*. In (G.Adomian, 1990) *G*. Adomian has developed a decomposition technique for solving nonlinear functional equation such as (10). We assume that (10) has a unique solution.The Adomian technique allows us to find the solution of (10) as an infinite series $g=\sum_{n\ge0}g_n$ using the following scheme:
$$g_0=a_0$$
$$g_1=A_0$$
$$\vdots$$
$$g_{n+1}=A_n(g_0,g_{1,\dots,}g_n)$$
$$N(g)=\sum_{n\ge0}A_n\ ,\ \text{where}$$
$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{i\ge0}\lambda^i g_i\right)\right]_{\lambda=0}$$

$n = 0,1,2,....$

The proofs of convergence of the series $\sum_{n\geq 0} g_n$ and $\sum_{n\geq 0} A_n$ are given below.Without loss of generality we set $a_0 = 0$ and we have the following scheme:

$$\frac{d^n}{dg^n}N(g_0) = \left(-\frac{1}{2}\right)^n N(g_0)$$
$$g_0 = 0$$
$$g_1 = \frac{1}{6}K\eta^3 \equiv b_1 K\eta^3$$
$$g_2 = -\frac{1}{72}(K\eta^3)^2 \equiv b_2(K\eta^3)^2$$
$$g_3 = \frac{1}{576}(K\eta^3)^3 \equiv b_3(K\eta^3)^3$$
$$g_4 = -\frac{1}{3888}(K\eta^3)^4 \equiv b_4(K\eta^3)^4$$
$$g_5 = \frac{125}{2985984}(K\eta^3)^5 \equiv b_5(K\eta^3)^5$$
$$g_6 = -\frac{569}{79626240}(K\eta^3)^6 \equiv b_6(K\eta^3)^6.$$

By induction, we have

$$g_{n+1} = \frac{1}{n!}\sum_{k=0}^{n-1} C_{n-1}^k (n-k)!\, g_{n-k}\left[\frac{d^k}{d\lambda^k}\frac{dN}{dh}(h)\right]_{\lambda=0}$$

i.e $g_{n+1} \equiv b_{n+1}(K\eta^3)^{n+1},\ n \geq 1$

$$h = \sum_{i\geq 0} \lambda^i g_i$$

where the b_n are real numbers.Then we obtain

$$g(\eta) = K\frac{\eta^3}{6} - \frac{(K\eta^3)^2}{72} + \frac{(K\eta^3)^3}{576} - \frac{(K\eta^3)^4}{3888} + \frac{125}{2985984}(K\eta^3)^5 - \frac{569}{79626240}(K\eta^3)^6 + \cdots + b_{n+1}(K\eta^3)^{n+1} + \cdots \quad (11)$$

$$f(\eta) = K\frac{\eta^2}{2} - \frac{K^2\eta^5}{12} + \frac{K^3\eta^8}{64} - \frac{K^4\eta^{11}}{324} + \frac{625}{996328}K^5\eta^{14} - \frac{569}{4423680}K^6\eta^{17} + \cdots + 3(n+1)b_{n+1}K^{n+1}\eta^{3n+2} + \cdots \quad (12)$$

$$\psi(x,y) = \sqrt{\nu u_0 x}\left[K\frac{\eta^2}{2} - \frac{K^2\eta^5}{12} + \frac{K^3\eta^8}{64} - \frac{K^4\eta^{11}}{324} + \frac{625}{996328}K^5\eta^{14} - \frac{569}{4423680}K^6\eta^{17} + \cdots + 3(n+1)b_{n+1}K^{n+1}\eta^{3n+2} + \cdots\right] \quad (13)$$

We arrive at the following results:

Lemma 3.1:

The velocity components of the fluid flow u, υ *of the equations of continuity and motion (1) and (2) satisfying the boundary conditions (3) are defined by*

$$u(x,y) = u_0 f'(\eta) = u_0\left[K\eta - \frac{5K^2\eta^4}{12} + \frac{K^3\eta^7}{8} - \frac{11K^4\eta^{10}}{324} + \frac{4375}{498164}K^5\eta^{13} - \frac{9673}{4423680}K^6\eta^{16} + \cdots + 3(n+1)(3n+2)b_{n+1}K^{n+1}\eta^{3n+1} + \cdots\right] \quad (14)$$

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$$v(x,y)=\frac{1}{2}\sqrt{\frac{\nu u_0}{x}}(\eta f'(\eta)-f(n))$$
$$=\frac{1}{2}\sqrt{\frac{\nu u_0}{x}}\Big[K\frac{\eta^2}{2}-\frac{K^2\eta^5}{3}+\frac{7K^3\eta^8}{64}-\frac{5K^4\eta^{11}}{162}+\frac{8125}{996328}K^5\eta^{14}-\frac{9104}{4423680}K^6\eta^{17}+\cdots$$
$$+3(n+1)(3n+1)b_{n+1}K^{n+1}\eta^{3n+2}+\cdots\Big] \quad (15)$$
$$\eta=y\sqrt{\frac{u_0}{\nu x}}$$

Remark 3.1:

The terms approximation of

$$g=\sum_{n=0}^{+\infty}g_n$$

is 18th degree polynomial in $\eta=y\sqrt{\frac{u_0}{\nu x}}$

For the proofs of convergence of the series with general term (11), the following statement holds

Lemma 3.2:

The nonlinear operator $N(g)$ can be developed in entire series with a convergence radius equal to infinity

$$N(g)=K\sum_{n\geq 0}\frac{(-1)^n}{n!2^n}\int_0^\eta\int_0^\tau\int_0^\sigma g^n(S)\,ds\,d\sigma\,d\tau \quad (16)$$

$\|g\|<+\infty$ where $\|g\|=\max_{\eta\in[0,\eta_0]}|g(\eta)|$

Furthermore the following series

$$\sum_{n=0}^{+\infty}A_n \text{ and } \sum_{n=0}^{+\infty}g_n \quad (17)$$

are convergent.

Proof 3.1:

Taking into account the expansion of the function $g\mapsto e^{\frac{-1}{2}g}$ *in entire series and using the uniform convergence of entire series we obtain the formula. By the Cauchy-Hadamard formula for the radius, we prove that the radius is equal to infinity.*

On the other side, taking into account the different expressions of $N(g)$

$$N(g)=\sum_{n\geq 0}\left\{\frac{1}{n!}\sum_{k=0}^{n-1}C_{n-1}^{k}(n-k)!\,g_{n-k}\right.$$
$$\left.\times\left[\frac{d^k}{d\lambda^k}\frac{dN}{dh}\right]_{\lambda=0}\right\}$$
$$N(g)=\sum_{n\geq 0}\frac{(-1)^nK}{n!\,2^n}\int_0^\eta\int_0^\tau\int_0^\sigma g^n(s)\,ds\,d\sigma\,d\tau$$

we obtain that the series $\sum_{i\geq 0}A_i$ is convergent with

$\|g\|<+\infty$ Since

$$\|g\|<+\infty\Leftrightarrow\left\|\sum_{i\geq 0}g_i\right\|<+\infty \quad (18)$$

then the series $\sum_{i\geq 0}g_i$ *is convergent; that completes the proof.*

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4. Error estimate:

In this section we estimate the error by approximating the exact values of the shear-stress Γ by the value $\hat{\Gamma}$ of the shear-stress obtained in (M.E.Eglit *et al* 1996). The shear-stress Γ at the plate surface is defined in (M.E.Eglit *et al.*, 1996) by

$$\Gamma = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{19}$$

Taking into account (14) we obtain

$$\Gamma = K\sqrt{\frac{\mu\rho u_0^3}{x}}, \quad K = \left[\int_0^{+\infty} e^{-\frac{1}{2}g(\eta)} d_\eta\right]^{-1} \tag{20}$$

As K is defined implicitly from (20) we suggest to compute the admissible values of Γ, taking into account $K \approx 0.332$ in (M.E.Eglit *et al* 1996). For this purpose we impose the constraint on the absolute error $|\Gamma - \hat{\Gamma}|$:

$$|\Gamma - \hat{\Gamma}| = |K - 0.332|\sqrt{\frac{\mu\rho u_0^3}{x}} \tag{21}$$

$$\Gamma = K\sqrt{\frac{\mu\rho u_0^3}{x}}; \quad \hat{\Gamma} = 0.332\sqrt{\frac{\mu\rho u_0^3}{x}}$$

Table

x	$\lvert\Gamma - \hat{\Gamma}\rvert$
1	$\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
4	$\frac{1}{2}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
9	$\frac{1}{3}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
16	$\frac{1}{4}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
20	$\frac{1}{2\sqrt{5}}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
25	$\frac{1}{5}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
36	$\frac{1}{6}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
49	$\frac{1}{7}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
60	$\frac{1}{2\sqrt{15}}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
64	$\frac{1}{8}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$
81	$\frac{1}{9}\lvert K - 0.332\rvert\sqrt{\mu\rho u_0^3}$

It is observed from the table, that the absolute error decreases with the increase in the values of x

The approximation may be efficient if the approximation precision is too small,i.e. there exists $n \in \mathbb{N}^*$ such that

$$|K - 0.332|\sqrt{\frac{\mu\rho u_0^3}{x}} < 10^{-n} \Leftrightarrow$$

$$0.332 - \sqrt{\frac{x}{\mu\rho u_0^3}} \times 10^{-n} < K < 0.332 + \sqrt{\frac{x}{\mu\rho u_0^3}} \times 10^{-n}$$

We arrive at the following result

Lemma 4.1:

The admissible values of the shear-stress Γ on the plate surface obtained in (20) belong to the open interval

$$\left]0.332\sqrt{\frac{\mu\rho u_0^3}{x}} - 10^{-n}; 0.332\sqrt{\frac{\mu\rho u_0^3}{x}} + 10^{-n}\right[\quad (22)$$

for each given value of $x > 0$ *and for the given approximation precision depending on* $n \in \mathbb{N}^*$.

Conclusion:

In this paper,we have investigated the analytical solutions for the Blasius problem which are the sums of convergent series, using the Adomian decomposition technique. Then we estimated the error by approximating the exact values of the shear-stress on the plate surface obtained in this paper by the approximate values of the shear-stress obtained in (M.E.Eglit *et al* 1996). Doing so, we constructed the interval of admissible values of the shear-stress on the plate surface.

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