Linear Stability Analysis of Convection of a Maxwell Fluid in a Rotating Anisotropic Porous Layer with Oblique Principal Axes

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Abstract — An analytical method is carried out to investigate linear stability analysis of convection in a rotating anisotropic porous layer heated from below. For the accurate modeling of the anisotropic porous matrix, both mechanical anisotropy about the rotating axis in the vertical direction and hydrodynamical anisotropy prevailing in the horizontal plane whose principal axes oriented in a direction non-coincident with the gravity force are considered. On the basis of the generalized Darcy's law and the modified Darcy-Maxwell-Jeffrey model employed to take into account the properties of the viscoelastic fluid saturating the porous matrix and to include the time derivate and Coriolis terms, the linear stability theory related to the normal mode method has been followed to conduct this analysis. Moreover, the criteria for both stationary and oscillatory convection is derived analytically. The rotation inhibits the onset of convection in both stationary and oscillatory modes. It has been demonstrated that each physical parameter involved in the present analysis has an important effect on the system.

Keywords: Linear Stability Analysis, Convection, Viscoelastic Fluid, Rotating Anisotropic Porous Medium, Oblique Principal Axes

I. INTRODUCTION

Over the past years, interest in fundamental studies of thermal convection in porous media has significantly increased due to its presence in diverse engineering applications in fields such as thermal insulation technology, pebble bed nuclear reactors, cooling of electronic equipment and binary alloy solidification. The study of the effect of external rotation on thermal convection has attracted significant experimental and theoretical interest. Some of the important areas of its pratical applications in engineering include the food processing, chemical process, solidification and centrifugal casting of metals and rotating machinery. Because of its general occurrence in geophysical and oceanic flows, it is important to understand how the Coriolis force influences the structure and transport properties of thermal convection. Rotation thermal convection also provides a system to study hydrodynamic instabilities, pattern formation and spatio-temporal chaos in nonlinear dynamical systems. Further detailed discussions of applications of natural convection in porous media and in particular rotating porous media are presented by Nield and Bejan [1] and Bejan [2]. Previous research of natural convection due to thermal buoyancy resulting from the centrifugal body force (while gravity was neglected) deal with cases of different orientation of the centrifugal force with respect to the temperature gradient. Vadasz [3] presented analytical solutions to the three dimensional natural convection problem in a long rotating porous box (where the centrifugal force was perpendicular to the temperature gradient) whilst Vadasz [4] presented the stability of centrifugally driven natural convection for porous layers adjacent the axis of rotation. The Coriolis effect on the stability of gravity driven convection in porous media was undertaken by Vadasz [5], and it is shown that the Taylor number stabilises the convection. A nonlinear stability analysis was performed for thermal convection in a rotating fluid saturated porous layer using energy stability theory by Straughan [6]. It was reported that the global nonlinear stability Rayleigh number is exactely the same as that for linear instability and for the rotating porous convection problem governed by Darcy equation, subcritical instabilities are not possible. The effect of Coriolis force on the centrifugally driven convection in a rotating porous layer was analyzed by Govender [7]. The marginal stability criterion was established as a characteristic centrifugal Rayleigh number in terms of the wavenumber and the Taylor number.

Moreover, on the one hand, the study of convective flows in a rotating porous medium saturated with viscoelastic fluid has received a very little attention, despite its importance in many fields such as geophysics, petroleum production and centrifugal casting of metals. Further the study of such problems will be very helpful in understanding the effect of rotation on the earth's core, where the molten fluid is viscoelastic, which can become unstable on differential heating, and in the performance of petroleum reservoirs. To the best of authors's knowledge only Laroze et al. [8] have investigated this problem of viscoelastic flow in a rotating porous medium and obtained the convective thresholds for the case of rigid-rigid boundary conditions. Due to various applications in engineering like enhanced oil recovery, ceramic processing and geothermal receivers come in contact with non-Newtonian fluid in porous media. The increasing attention given to this topic, there exit no simple model have been used to study the non-Newtonian fluids in porous media. Rudraiah et al. [9] and Bertal and Cafaro [10], used a Maxwell Jeffrey model to investigate a viscoelastic fluid in a porous layer heated from below. Tan and Masuka [11] proposed a Darcy-Maxwell-Jeffrey model by introducing a flow resistance term to overcome the disavantage encountered in the modified Darcy-Maxwell-Jeffrey model.

On the other hand, in all the above studies the porous media were assumed to be isotropic whereas, in several applications, the porous materials are anisotropic. The inclusion of more physical realism in the matrix properties of the porous medium is important for the accurate modeling of the anisotropic media. Anisotropy, which is generally a consequence of a preferential orientation or asymmetric

geometry of the grain or fibers, is in fact encountered in numerous systems in industry and nature. Despite this fact, natural convection in such anisotropic porous media has received relatively little attention. Only few studies have been reported on convection in an anisotropic porous medium uniformly heated from below. The review of research on convective flow through anisotropic porous media has been well documented by McKibbin [12,13] and Storesletten [14,15]. The effects of an anisotropic permeability on thermal convection in a porous medium began with the investigation of Castinel and Combarnous [16], concerning the onset of motion in a horizontal layer heated from below, and continued with the works of Epherre [17], Kvernvold and Tyvand [18] and Nilsen and Storesletten [19]. Natural convection within enclosures heated from the side was investigated by Kimura *et al.* [20] and Ni and Beckermann [21], for the case when one of the principal axes of anisotropy of permeability is aligned with gravity and by Zhang [22], Degan *et al.* [23] and Degan and Vasseur [24] when the principal axes are inclined with respect to gravity. It was demonstrated by these authors that the effects of the anisotropy considerably modify the convective heat transfer. Recently, Malashetty and Swamy [25] have examined the combined effect of rotation and anisotropy on onset of convection in a horizontal porous layer using linear stability theory and a weak nonlinear theory. It was found that the effect of mechanical anisotropy is to allow the onset of oscillatory convection instead of stationary. Linear and nonlinear thermal instability in a rotating anisotropic medium, saturated with viscoelastic fluid, has been investigated for free-free surfaces by Kumar and Bhadauria [26]. The critera for both stationary and oscillatory convection were derived analytically.

The objective of the present investigation is to study the combined effect of the relaxation time and retardation time of the viscoelastic fluid, the rotation of the system and the anisotropic parameters of the porous matrix on thermal instability of heat transfer in an open-top anisotropic rotating porous layer saturated by a viscoelastic fluid. The porous medium is anisotropic in permeability in the vertical direction about the rotating *z*-axis and in the horizontal plane with its principal axes oriented in a direction that is oblique to the gravity vector.

II. MATHEMATICAL FORMULATION

We consider here the problem of thermal instabilty of viscoelastic fluids saturated horizontal rotating porous cavity caracterized by an anisotropic permeability (see Fig. 1). The porous layer is confined between two parallel horizontal planes at $z = 0$ and $z = H$, the height of the cavity. The planes are extended horizontally in *x* and *y* directions and are rigid-free. A cartesian frame of reference is chosen in such a way that the origin lies on the lower plane and the *z*-axis as vertical upward. The system is rotating about *z*-axis with a constant angular velocity Ω. Adverse temperature gradients is applied across the porous cavity and the lower and upper are kept at temperatures T_H and *Tc*, respectively, with $T_H = T_C + \Delta T$. The anisotropy of the porous medium is characterized firstly, in the horizontal plane (*Oxy*) by the anisotropy ratio $K^* = K_1/K_2$ and the orientation anisotropy angle θ, defined as the angle between the horizontal direction and the principal axis with the permeability *K*2, and secondly, in the vertical direction *Oz* by the anisotropy ratio, $\zeta = K_1/K_3$, being the permeability along *z*-axis.

In order to derive the governing equations of fluid motion and energy transport, the following assumptions are made. It is assumed that the fluid and the porous medium are everywhere in local *thermodynamic* equilibrium. The saturating fuid is viscoelastic, and its motion through the porous medium obeys the model Darcy-Maxwell-Jeffrey and the generalized Darcy's law proposed by Bear [27] to take into account the anisotropy of the porous matrix. The thermophysical properties of the fluid are assumed constant, except for the density in the buoyancy term in the momentum equation (i.e., the Boussinesq approximation is valid).

Under the above approximations the equations governing the conservation of mass, momentum, and energy can be written as follows [11,27]

$$\vec{\nabla}.\vec{V} = 0 \tag{1}$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left\{\frac{\varrho_0}{\delta}\frac{\partial\vec{V}}{\partial t}+\frac{\varrho_0}{\delta}(\vec{\Omega}\Lambda\vec{V})\right\}+\mu(K)^{-1}\left(1+\lambda_2\frac{\partial}{\partial t}\right)\vec{V} = \left(1+\lambda_1\frac{\partial}{\partial t}\right)(-\vec{\nabla}p+\rho\vec{g}) \tag{2}$$

$$\sigma\frac{\partial T}{\partial t}+\vec{\nabla}.\left(\vec{V}T-\alpha\vec{\nabla}T\right)=0 \tag{3}$$

$$\rho = \varrho_0[1-\beta(T-T_0)] \tag{4}$$

In the above equations, V is the superficial flow velocity, *T the* temperature, *t* the time, % the density, g the gravitational acceleration, μ the dynamic viscosity, β the coefficient of thermal expansion of the saturating fluid, σ the heat capacity ratio,$((\rho c)_p/(\rho c)_f)$, λ_1 the relaxation time, λ_2 the retardation time and δ the porosity. The symmetrical three-order inverse permeability tensor $(K)^{-1}$ is defined as

$$(K)^{-1} = \frac{1}{K_3}\begin{bmatrix}\frac{a}{\zeta} & \frac{c}{\zeta} & 0\\ \frac{c}{\zeta} & \frac{b}{\zeta} & 0\\ 0 & 0 & 1\end{bmatrix} \tag{5}$$

where

$$\begin{cases} a = cos^2\varphi + K^* sin^2\varphi \\ a = sin^2\varphi + K^* cos^2\varphi \\ c = \frac{1}{2}(1+K^*)\, sin2\varphi \\ K^* = \frac{K_1}{K_2} \\ \zeta = \frac{K_1}{K_3} \end{cases} \tag{6}$$

At the basic state, the fluid is assumed to be quiescent, therefore the quantities at the basic state are given by

$$V_b = (0,0,0)\,, p = p_b(z), \quad T = T_b(z) \text{ and } \rho = \varrho_b(z) \tag{7}$$

Then at the basic state, governing equations, Eqs. (2) -(4), are

$$\frac{dp_b}{dz} = -\varrho_b g \quad , \qquad \frac{d^2T_b}{dz^2} = 0 \tag{8}$$

where with $T_b = T_H - (\frac{\Delta T}{H})z$

At this basic state, one can impose a small perturbation in the form

$$V = V_b + V', \qquad T = T_b(z) + T', \quad p = p_b(z) + p', \quad \rho = \varrho_b(z) + \rho' \tag{9}$$

where the primes indicate the perturbation state. Substituting Eq. (9) into Eqs. (1)-(4)
using the basic state equations (8) and the following transformations

$$\begin{cases} (x^*, x^*, x^*) = \dfrac{(x',\ y', z')}{H}, \quad t^* = \dfrac{\alpha}{\sigma . H^2} t' \\ (u^*, v^*, w^*) = (u', v', w') \dfrac{H}{\alpha} \quad , \quad T^* = \dfrac{T'}{\Delta T} \\ p^* = \dfrac{K_3}{\mu\alpha} p' \quad , \quad (\lambda_1{}^*, \lambda_2{}^*) = (\lambda_1, \lambda_2) \dfrac{\alpha}{\sigma . H^2} \end{cases} \tag{10}$$

we obtain the dimensionless governing equations, as follows

$$\vec{\nabla} . \vec{V}^* = 0 \tag{11}$$

$$\left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\left\{\frac{1}{PrD} \frac{\partial \vec{V}^*}{\partial t^*} + \sqrt{T\alpha}(\vec{\mathrm{T}} \Lambda \vec{V}^*)\right\} + K_3 (K)^{-1} \left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right) \vec{V}^* = \left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right) (\vec{\nabla} \vec{p}^* - Ra\, \vec{T}^* k) \tag{12}$$

$$\left(\frac{\partial}{\partial t^*} + \vec{V}^* . \vec{\nabla}\right) T^* - \omega^* = \vec{\nabla}^2 T \tag{13}$$

where $P\,rD = \delta P\,r/Da$ is the Vadasz number, $Da = K_3/H^2$ the Darcy number, $T_\alpha = (\frac{2\Omega K_3}{\delta\nu})^2$ the Taylor number, $Ra = K_3 g\beta\rho_0 \Delta T H/(\mu\alpha)$ the Rayleigh number based on the permeability in the vertical direction, $\mathrm{Pr} = \mu/(\varrho_0\alpha)$ the Prandtl number , $\nu = \mu/\varrho_0$ is the kinematic viscosity and k is the unit vector on the z-axis.

The hydrodynamical and thermal boundary conditions considered here are

$$Z^* = 0 : \quad \omega^* = 0, \quad T^* = 1 \tag{14}$$

$$Z^* = 1 : \quad \frac{\partial^2 \omega^*}{\partial^2 Z^*} = 0, \quad T^* = 0 \tag{15}$$

$$Z^* = 0, A : \quad u^* = 0, \quad \frac{\partial T^*}{\partial x^*} = 0 \tag{16}$$

where $A = L/H$ is the aspect ratio of the cavity.
Eliminating the pressure term from Eq. (12) by applying curl on it, one can have

$$\left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\left\{\frac{1}{PrD} \frac{\partial \vec{\omega}}{\partial t^*} + \sqrt{T\alpha}\left(\frac{\partial \vec{V^*}}{\partial z^*}\right)\right\} \left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right) \vec{Q} = \left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right) Ra \left(\frac{\partial T^*}{\partial y^*} \vec{\imath} - \frac{\partial T^*}{\partial x^*} \vec{\jmath}\right) \tag{17}$$

where $\omega = \nabla \Lambda V^*$ denotes the vorticity vector, $\vec{\imath}$ is the unit vector on *x*-axis, $\vec{\jmath}$ the unit vector on *y*-axis, and $\vec{Q} = (Q_1, Q_2, Q_3)$ is such that

$$Q_1 = \frac{\partial \omega^*}{\partial y^*} - \frac{c}{\zeta}\frac{\partial u^*}{\partial z^*} - \frac{b}{\zeta}\frac{\partial v^*}{\partial z^*} \tag{18a}$$

$$Q_2 = -\left(\frac{\partial \omega^*}{\partial x^*} - \frac{a}{\zeta}\frac{\partial u^*}{\partial z^*} + \frac{c}{\zeta}\frac{\partial v^*}{\partial z^*}\right) \tag{18b}$$

$$Q_3 = \frac{1}{\zeta}\left(b\frac{\partial \omega^*}{\partial x^*} - c\frac{\partial u^*}{\partial x^*} - a\frac{\partial u^*}{\partial y^*} + a\frac{\partial v^*}{\partial z^*}\right) \tag{18a}$$

Following Vadasz [5] and Kumar and Bhadauria [26], in most of the traditional porous media applications the value of the Vadasz number is found to be very large. For that reason, the corresponding term has to be neglected in further present analysis.

For the linear stability analysis, let us consider the projection on coordinate axes of the governing equation (17) expressed as

$$\left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\left\{\frac{1}{PrD} \frac{\partial \omega_x{}^*}{\partial t^*} + \sqrt{T\alpha}\left(\frac{\partial u^*}{\partial z^*}\right)\right\} \left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right) Q_1 = \left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right) Ra \frac{\partial T^*}{\partial y^*} \tag{19}$$

$$\left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\left\{\frac{1}{PrD} \frac{\partial \omega_y{}^*}{\partial t^*} + \sqrt{T\alpha}\left(\frac{\partial v^*}{\partial z^*}\right)\right\} \left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right) Q_2 = \left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right) Ra \frac{\partial T^*}{\partial x^*} \tag{20}$$

$$\left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\left\{\frac{1}{PrD} \frac{\partial \omega_z{}^*}{\partial t^*} + \sqrt{T\alpha}\left(\frac{\partial w^*}{\partial z^*}\right)\right\} + \left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right) Q_3 = 0 \tag{21}$$

After derivating Eq. (19) with respect to $y*$, Eq. (20) with respect to $x*$ and Eq. (21) with respect to $z*$, respectively, and by rearranging the terms for $\overrightarrow{V^*}(u^*, 0, z^*)$, one can obtain in the limit of very large Vadasz number which is not applicable for porous media (i.e. $P\,rD \to 0$), the following equations

$$-\left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\sqrt{T\alpha}\frac{\partial^2 u^*}{\partial z^* \partial y^*} + \left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right)\left(\frac{\partial^2 \omega^*}{\partial x^{*2}} + \frac{\partial^2 \omega^*}{\partial y^{*2}} - \frac{c}{\zeta}\frac{\partial^2 u^*}{\partial z^* \partial y^*} - \frac{a}{\zeta}\frac{\partial^2 u^*}{\partial z^* \partial x^*}\right) = \left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right) Ra \left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}}\right) \tag{22}$$

$$\left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\sqrt{T\alpha}\frac{\partial^2 \omega^*}{\partial z^{*2}} = \left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right)\left(\frac{c}{\zeta}\frac{\partial^2 u^*}{\partial z^* \partial x^*} - \frac{a}{\zeta}\frac{\partial^2 u^*}{\partial z^* \partial y^*}\right) \tag{23}$$

From the continuity equation, $\partial u^*/\partial z^* + \partial \omega^*/\partial z^* = 0$, we can deduce that

$$\frac{\partial^2 u^*}{\partial z^* \partial x^*} = -\frac{\partial^2 \omega^*}{\partial z^{*2}} \tag{24}$$

Substituting Eq. (24) into Eqs. (22) and (23), and deducing the expression of $\partial^2 u^*/\partial x^* \partial y^*$ from the resulting equations, one can obtain

$$\left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right)^2 \left(\Delta_1{}^2 + \frac{a}{\zeta}\frac{\partial^2}{\partial z^{*2}}\right)\omega^* + \frac{\zeta}{c}\left\{\frac{c}{\zeta}\left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right) + \left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\sqrt{T\alpha}\right\}^2 \frac{\partial^2 \omega^*}{\partial z^{*2}} - \left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right) Ra \Delta_1^2 T^* = 0 \tag{25}$$

where $\Delta_1^2 = \partial^2/(\partial x^{*2}) + \partial^2/(\partial y^{*2})$ is the horizontal Laplacien

Now we can perform the linear stability analysis in reducing into linear form Eq. (13) by neglecting the non-linear terms. Then, Eq. (13) becomes

$$\left(\frac{\partial}{\partial t^*} - \nabla^2\right) T^* = \omega^* \tag{26}$$

Substituting Eq. (26) into Eq. (25), we obtain

$$\left\{\left(\frac{\partial}{\partial t^*} - \nabla^2\right)\left\{\left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right)\left(\Delta_1{}^2 + \frac{a}{\zeta}\frac{\partial^2}{\partial z^{*2}}\right) + \frac{\zeta}{a}\left\{\frac{\zeta}{c}\left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right) + \left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\sqrt{T\alpha}\right\}^2 \frac{\partial^2}{\partial z^{*2}}\right\} - \left(1 + \lambda_1{}^* \frac{\partial}{\partial t^*}\right)\left(1 + \lambda_2{}^* \frac{\partial}{\partial t^*}\right) Ra\Delta_1^2\right\}\omega^* = 0 \tag{27}$$

The stress rigid-free boundary conditions in terms of *w* are given by

$$z^* = 0\ ; \qquad \omega^* = 0 \tag{28}$$

$$z^* = 1\ ; \qquad \frac{\partial^2 \omega^*}{\partial z^{*2}} = 0 \tag{29}$$

A. *Normal Mode Analysis*

Using the normal mode analysis, we seek the solution of unknown field ω^* in the form

$$\omega^* = W(z^*)\exp[\varepsilon t^* + i(lx^* + i(lx^* + my^*)] \tag{30}$$

where *l* et *m* are horizontal wave numbers and ε is the growth rate, which in general a complex quantity given by $\varepsilon = \varepsilon_r + i\varepsilon_i$. Substituting Eq. (30) into Eq. (27), we obtain a single ordinary differential equation for $W(z*)$ in the form

$$\left\{\left(\varepsilon + \frac{k^2}{A^2} - \mathrm{D}^2\right)\left\{(1 + \lambda_2{}^*\varepsilon)^2\left(-\frac{k^2}{A^2} + \frac{a}{\zeta}\mathrm{D}^2\right) + \frac{\zeta}{a}\left\{(1 + \lambda_1{}^*\varepsilon)\sqrt{T\alpha} + \frac{c}{\zeta}(1 + \lambda_1{}^*\varepsilon)\sqrt{T\alpha}\right\}^2 \mathrm{D}^2\right\} + \frac{k^2}{A^2}(1 + \lambda_1{}^*\varepsilon)(1 + \lambda_2{}^*\varepsilon)\, Ra\right\} W \tag{31}$$

The boundary conditions in terms of *W* are given by

$$z^* = 0\ ; \qquad W = 0 \tag{32}$$

$$z^* = 1\ ; \qquad \frac{\partial^2 W}{\partial z^{*2}} = 0 \tag{33}$$

The boundary value problem considered here can be regarded as an eigenvalue problem. Following many authors, the solution of this problem is assumed to have the forme $W(z^*) = A_n \sin(n\pi z^*)$, where A_n denotes the amplitude which gives the minimum Rayleigh number when $n = 1$, showing that $W(z^*) = A_1 \sin(\pi z^*)$, is the eigenfunction for the marginal stability. Substituting $W(z^*) = A_1 \sin(\pi z^*)$, into Eq. (31), we obtain the expression for the Rayleigh number as follows

$$Ra^{st} = \frac{\left(\varepsilon + \frac{k^2}{A^2} + \pi^2\right)}{\frac{k^2}{A^2}(1 + \lambda_1{}^*\varepsilon)(1 + \lambda_2{}^*\varepsilon)}\left\{(1 + \lambda_1{}^*\varepsilon)^2\left(\frac{k^2}{A^2} + \frac{a}{\zeta}\pi^2\right) + \frac{\zeta\pi^2}{a}\left\{\frac{c}{\zeta}(1 + \lambda_2{}^*\varepsilon) + (1 + \lambda_1{}^*\varepsilon)\sqrt{T\alpha}\right\}^2\right\} \tag{34}$$

B. *Stationary State Analysis*

Stationary convection occurs when $\varepsilon = 0$ and is calculated by the expression of the Rayleigh number Ra^{st}

$$Ra^{st} = \frac{\left(\frac{k^2}{A^2} + \pi^2\right)}{\frac{k^2}{A^2}}\left\{\left(\frac{k^2}{A^2} + \frac{a}{\zeta}\pi^2\right) + \frac{\zeta\pi^2}{a}\left(\frac{c}{\zeta} + \sqrt{T\alpha}\right)^2\right\} \tag{35}$$

Taking $\partial Ra^{st}/(\partial k) = 0$, we obtain the critical wave number k_c^{st} given by

$$k_c^{st} = \pi^2\left\{1 + \left(\frac{a^2 + \zeta^2\left(\frac{c}{\zeta} + \sqrt{T\alpha}\right)^2}{a\zeta}\right)^{1/2}\right\}^2 \tag{37}$$

Similar reults have been obtained by Khumar and Bhadauria [26] while considering the same problem for the case corresponding to an anisotropic porous medium whose the permeabilities K_1 and K_3 are coincident to the horizontal and the vertical principal directions, respectively, i.e., when $a = 1$ and $c = 0$. Therefore, the critical wave and the critical Rayleigh number are given by

$$k_c^{st} = \pi[\zeta(\zeta^{-2} + Ta)]^{1/4} \quad and \quad Ra^{st} = \pi^2\left[1 + \sqrt{\zeta(\zeta^{-2} + Ta)}\right]^2 \tag{38}$$

In the same way, for rotating isotropic porous layer corresponding to take the parameters $a = 1, c = 0\ and\ \zeta = 1$, we obtain after their substitution in Eqs. (37) and (38)

$$k_c^{st} = \pi(1 + Ta)^{1/4} \quad and \quad Ra^{st} = \pi^2\left(1 + \sqrt{1 + Ta}\right)^2 \tag{39}$$

which are nothing but the Vadasz [5] results for the previous identified situation.

Moreover, for the case of non-rotating isotropic porous layer, we put $a = 1$, $c = 0$, $\zeta = 1$ and $Ta = 0$, then Eqs. (36) and (37) give

$$k_c^{st} = \pi \quad and \quad Ra^{st} = 4\pi^2 \tag{40}$$

which are identical with the results of the Newtonian study conducted by Horton and Rogers [28], Lapwood [29], Combarnous and Bories [30].

C. *Oscillatory State Analysis*

For oscillatory marginal state, we must have $\varepsilon_r = 0\ and\ \varepsilon_i \neq 0$ Now putting $\varepsilon = i\varepsilon_i$ into the Eq. (49) and making the denominator as real, we obtai

$$Ra^{osc} = \frac{\left(\frac{k^2}{A^2}+\pi^2\right)(1-{\lambda_2}^*{\lambda_1}^*\varepsilon^2)+({\lambda_1}^*+{\lambda_2}^*)\varepsilon^2}{\frac{k^2}{A^2}\left(1+{\lambda_1}^{*2}\varepsilon^2\right)\left(1+{\lambda_2}^{*2}\varepsilon^2\right)} \times \left\{\left(\frac{k^2}{A^2}+\frac{a}{\zeta}\pi^2\right)\left(1-{\lambda_2}^{*2}\varepsilon^2\right)+\frac{\zeta\pi^2}{a}\left\{\left(\frac{c}{\zeta}+\sqrt{T\alpha}\right)^2-\varepsilon^2\left(\frac{c}{\zeta}{\lambda_2}^*+{\lambda_1}^*\sqrt{T\alpha}\right)^2\right\}\right\} + -\frac{2\varepsilon^2\left\{(1-{\lambda_2}^*{\lambda_1}^*\varepsilon^2)-\left(\frac{k^2}{A^2}+\pi^2\right)({\lambda_1}^*+{\lambda_2}^*)\right\}}{\frac{k^2}{A^2}\left(1+{\lambda_1}^{*2}\varepsilon^2\right)\left(1+{\lambda_2}^{*2}\varepsilon^2\right)}\times \times\left\{{\lambda_2}^*\left(\frac{k^2}{A^2}+\frac{a}{\zeta}\pi^2\right)+2\frac{\zeta\pi^2}{a}\left(\frac{c}{\zeta}{\lambda_2}^*+{\lambda_1}^*\sqrt{T\alpha}\right)\left(\frac{c}{\zeta}+\sqrt{T\alpha}\right)\right\}+i\varepsilon X \quad (41)$$

Where

$$X = \frac{\left(\frac{k^2}{A^2}+\pi^2\right)(1-{\lambda_2}^*{\lambda_1}^*\varepsilon^2)+({\lambda_1}^*+{\lambda_2}^*)\varepsilon^2}{\frac{k^2}{A^2}\left(1+{\lambda_1}^{*2}\varepsilon^2\right)\left(1+{\lambda_2}^{*2}\varepsilon^2\right)}\times\left\{{\lambda_2}^*\left(\frac{k^2}{A^2}+\frac{a}{\zeta}\pi^2\right)+2\frac{\zeta\pi^2}{a}\left(\frac{c}{\zeta}{\lambda_2}^*+{\lambda_1}^*\sqrt{T\alpha}\right)\left(\frac{c}{\zeta}+\sqrt{T\alpha}\right)\right\} + \frac{\left\{(1-{\lambda_2}^*{\lambda_1}^*\varepsilon^2)-\left(\frac{k^2}{A^2}+\pi^2\right)({\lambda_1}^*+{\lambda_2}^*)\right\}}{\frac{k^2}{A^2}\left(1+{\lambda_1}^{*2}\varepsilon^2\right)\left(1+{\lambda_2}^{*2}\varepsilon^2\right)} \times\left\{\left(\frac{k^2}{A^2}+\frac{a}{\zeta}\pi^2\right)\left(1-{\lambda_2}^{*2}\varepsilon^2\right)+\frac{\zeta\pi^2}{a}\left\{\left(\frac{c}{\zeta}+\sqrt{T\alpha}\right)^2-\varepsilon^2\left(\frac{c}{\zeta}{\lambda_2}^*+{\lambda_1}^*\sqrt{T\alpha}\right)^2\right\}\right\} \quad (42)$$

Since oscillatory Rayleigh number *Raosc* should be real, its imaginary part *X* must be zero. Therefore, as $\varepsilon X = 0$ in Eq. (41) and $\varepsilon \neq 0$, we must have $X = 0$. Accordingly, we deduce from $Eq.(42)$ that ε^2 can be calculated by the expression

$$\varepsilon^2 = \frac{2q_2q_4+{\lambda_1}^*{\lambda_2}^*q_1+q_3q_5}{2{\lambda_1}^*{\lambda_2}^*-q_3} + \frac{\sqrt{(2q_2q_4+{\lambda_1}^*{\lambda_2}^*q_1+q_3q_5)^2-4\,{\lambda_1}^*{\lambda_2}^*q_3\left\{2q_2\left(\frac{k^2}{A^2}+\pi^2\right)+q_1q_5\right\}}}{2{\lambda_1}^*{\lambda_2}^*q_3} \quad (43)$$

where

$$q_1 = \left(\frac{k^2}{A^2}+\frac{a}{\zeta}\pi^2\right)+\frac{\zeta\pi^2}{a}\left(\frac{c}{\zeta}+\sqrt{T\alpha}\right)^2 \quad (44a)$$

$$q_2 = {\lambda_2}^*\left(\frac{k^2}{A^2}+\frac{a}{\zeta}\pi^2\right)+\frac{\zeta\pi^2}{a}\left(\frac{c}{\zeta}{\lambda_2}^*+{\lambda_1}^*\sqrt{T\alpha}\right)\left(\frac{c}{\zeta}+\sqrt{T\alpha}\right) \quad (44b)$$

$$q_3 = {\lambda_2}^*\left(\frac{k^2}{A^2}+\frac{a}{\zeta}\pi^2\right)+\frac{\zeta\pi^2}{a}\left(\frac{c}{\zeta}{\lambda_2}^*+{\lambda_1}^*\sqrt{T\alpha}\right)^2 \quad (44c)$$

$$q_4 = \left(\frac{k^2}{A^2}+\pi^2\right){\lambda_1}^*{\lambda_2}^*+({\lambda_1}^*+{\lambda_2}^*) \quad (44d)$$

$$q_5 = 1-\left(\frac{k^2}{A^2}+\pi^2\right)({\lambda_1}^*+{\lambda_2}^*) \quad (44e)$$

III. Results and Discussion

For the discussion of results in this study, we can fundamentally distinguish two types of anisotropy in permeability, namely, the first, the anisotropy in permeability in the vertical direction along *z*-axis, controlled by the parameter ζ, and the second, the anisotropy in permeability in the horizontal plane (Oxy) for which the control parameters are K^* and φ. In the first case of anisotropy, the permeability along the vertical axis, K_3, is more important than others in the porous matrix, and this situation yields $K_3 > K_1$ and $K_3 > K_2$, such that $(\zeta < K^* < 1)$. In the second case of anisotropy, the two permeabilities K_1 and K_2 along the two principal directions in the horizontal plane (Oxy) are predominent in the porous medium, such that $K_1 > K_3$ and $K_2 > K_3$, the resulting condition yields here $(1 < K^* < \zeta)$.

The linear stability analysis gives the conditions for the stationary and oscillatory convection as presented in the $Figs.2-5$ for the case of anisotropic permeability in vertical direction, i.e. when$(\zeta < K^* < 1)$. Figs 2 -5 illustrate the effects of varying the physical parameters on the Rayleigh number *Ra* expressed as a function of *k* the wavenumber when the others are held fixed at $K^* = 0.9$, $\zeta = 0.6$, $Ta = 50$, $\lambda_1^* = 0.5, \lambda_2^* = 0.8,$ and $\varphi = 45^\circ$.

The neutral stability curves for stationary and oscillatory convections are drawn in these figures. It is seen from the latters that the oscillatory convection regime prevails earlier than stationary convection regime for a similar range of the values of the wavenumber *k*, while outside that range of values, the stationary convection regime is noticed. The criteria of linear stability of the convection is predited in terms of the critical Rayleigh number, Ra_c below which the system is stable and becomes unstable above. The necessary condition but not sufficient of stability or overstability is expressed by $\varepsilon^2 \geq 0$ but has to be completed by R^{OSC} less than by R^{st}. So, it is easy to observe from Figs. 2 - 3 that instability prevails initially for a large range of values of *k*. However, it is seen that Ra_{st} becomes less than R^{OSC} with an increase in the wavenumber *k* and therefore the

instability of the convection sets in as stationary regime solution. Moreover, the bifurcation points where the overstable regime solutions branch off the stationary regime solution can be observed easily for different values of the physical parameters. Also, the Taylor number *Ta* from Fig.2 and the permeability ratio ζ in the vertical direction in Fig.3, respectively, increase as the critical Rayleigh number increases and consequently, the onset of convection takes place at the latter point and accordingly the system becomes more stabilized. In Figs. 4 -5, from left to the bifurcation point, it is noticed that the instability of the system sets in as overstability regime solution. And in that interval, upon increasing the relaxation time λ_1^* in Fig. 4, the critical value of *Raosc* increases too and thus the system becomes more stabilized. However, at the right side of the bifurcation point, *Rast* decreases upon further increasing λ_1^*. The inverse situation is observed in Fig.5 from which at the left of the bifurcation point, the retardation time λ_2^* decreases when *Raosc* increases while at the right of that latter point, an increase in λ_2^* corresponds to a decrease in R^{st}. This result has been obtained in the past by Kumar and Bhadauria [26].

In the same line of thought, concerning the case of the horizontal anisotropy in the plane (*Oxy*) i.e. when $(1 < K^* < \zeta)$, Figs. 6-10 depict the effects of Taylor number *Ta*, of permeability ratio K^*, of anisotropic orientation angle, of relaxation time λ_1^*, and of retardation time, λ_2^* , respectively on the Rayleigh number *Ra* expressed as a function of *k* the wavenumber when the others are held fixed at $K^* = 2.5$, $\zeta = 3$, $Ta = 50$, $\lambda_1^* = 0.5$, $\lambda_2^* = 0.8$, and $\varphi = 45°$.
As seen previously, the effect of varying the Tayor number Ta illustrated by Fig. 6 is analogous for this anisotropy case. From Fig.7, the permeability ratio K^* in the horizontal plane increases as the critical Rayleigh number decreases and thus, the convective system becomes more instable further from left to the bifurcation point. However, in Fig. 8, an increase in the anisotropic orientation angle ’ involve an increase in the critical Rayleigh number and consequently, the onset of convection takes place at the latter point and thus the system becomes more stabilized. The behaviour of the relaxation time λ_1^* and the retardation time λ_2^* as described previously reveals the same from Fig. 9 and Fig. 10 respectively, for the case of the anisotropic permeability in the vertical direction.

IV. Conclusion

The problem solved analytically here concerns linear stability analysis of convection of viscoelactic fluid in a rotating anisotropic porous system heated from below. This study performs combined effect of the relaxation time and retardation time of the viscoelastic fluid , the rotation of the system and the anisotropic parameters of the porous matrix on linear stability analyses of convective heat transfer in an open-top anisotropic rotating porous layer. The porous medium is anisotropic in permeability both in the vertical direction rotating about *z*-axis and in the horizontal plane with its principal axes oriented in a direction that is oblique to the gravity vector. The following points are drawn :

1) It is seen that initially instability sets in as overstability for a large value of the wavenumber *k* and however, upon increasing the value of k, *Rast* becomes less than *Raosc* and thus the instability sets in as stationary solution.
2) For the case of anisotropic permeability in vertical direction, i.e. when $(\zeta < K^* < 1)$, both the Taylor number *Ta* and the permeability ratio ζ in the vertical direction increase as the critical Rayleigh number increases and consequently, the onset of convection takes place at the latter point and accordingly the system becomes more stabilized. On increasing the value of relaxation time λ_1^*, and on decreasing the value of retardation time λ_2^* , the critical value of Ra^{osc} increases and as λ_1^* increased and λ_2^* decreased further, we notice a shift in the overstability from the left to the right of bifurcation point and the critical value of *Raosc* decreases.
3) For the case of anisotropic permeability in the horizontal plane (*Oxy*) directions, i.e. when $(1 < K^* < \zeta)$, the effect of varying the Talyor number Ta is similar to that seen previously. The permeability ratio K^* in the horizontal plane increases as the critical Rayleigh number decreases and thus, the convective system becomes more instable further from left to the bifurcation point. However, an increase in the anisotropic orientation angle φ involves an increase in the critical Rayleigh number and consequently, the onset of convection takes place at the ne latter point and thus, the system becomes more stabilized. The behaviour of the relaxation time λ_1^* and the retardation time λ_2^* as described previously reveals the same.

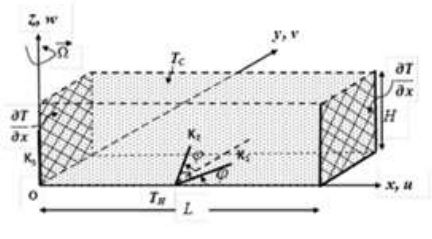

Fig 1

Fig. 1. Physical model and coordinate system.

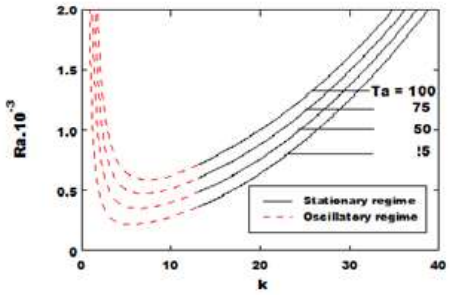

Fig.2 Effect of varying *Ta* on *Ra* for neutral stability for anisotropy in vertical direction case (i.e., for $(\zeta < K^* < 1)$ when $\zeta = 0.6$, $K^* = 0.9$, $\varphi = 45°$, $\lambda_1^* = 0.5$ and $\lambda_2^* = 0.8$.

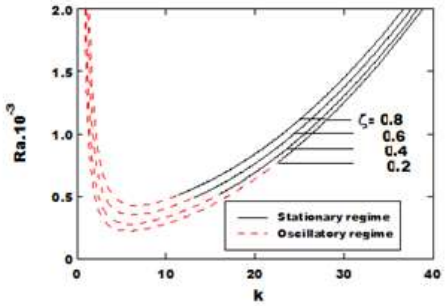

Fig.3 Effect of varying ζ on *Ra* for neutral stability for anisotropy in vertical direction case (i.e., for $(\zeta < K^* < 1)$ when $Ta = 50$, $K^* = 0.9$, $\varphi = 45°$, $\lambda_1^* = 0.5$ and $\lambda_2^* = 0.8$.

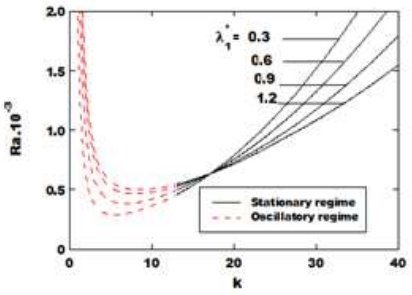

Fig. 4. Effect of varying λ_1^* on *Ra* for neutral stability for anisotropy in vertical direction case (i.e., for $(\zeta < K^* < 1)$ when $\zeta = 0.6$, $K^* = 0.9$, $\varphi = 45°$, and $\lambda_2^* = 0.8$.

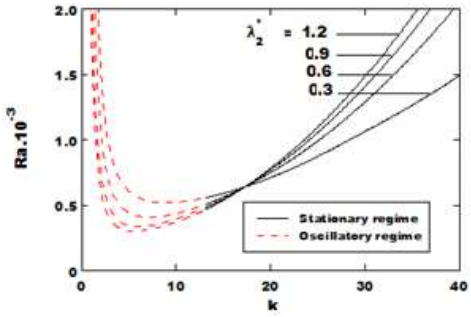

Fig. 5. Effect of varying λ_2^* on Ra for neutral stability for anisotropy in vertical direction case (i.e., for ($\zeta < K^* < 1$) when $\zeta = 0.6$, $K^* = 0.9$, $\varphi = 45°$, and $\lambda_1^* = 0.5$.

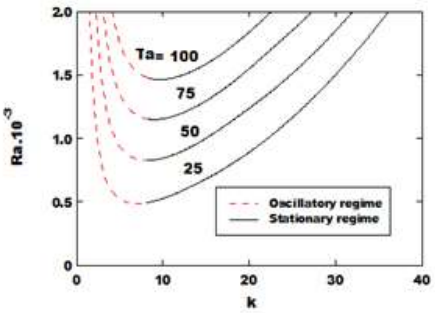

Fig. 6. Effect of varying Ta on Ra for neutral stability for anisotropy in horizontal plane case (i.e., for ($1< K^* < \zeta$) when $\zeta = 3$, $K^* = 2.5$, $\varphi = 45°$, $\lambda_1^* = 0.5$ and $\lambda_2^* = 0.8$.

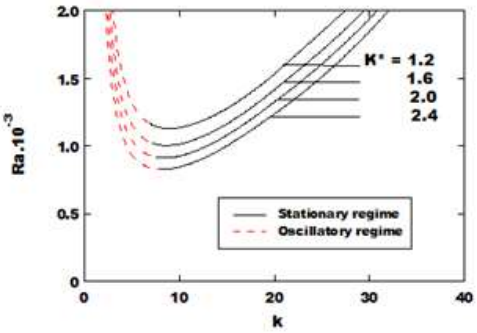

Fig. 7. Effect of varying K^* on Ra for neutral stability for anisotropy in horizontal plane case (i.e., for ($1 < K^* < \zeta$) when $\zeta = 3$, $Ta = 50$, $\varphi = 45$ $\lambda_1^* = 0.5$ and $\lambda_2^* = 0.8$.

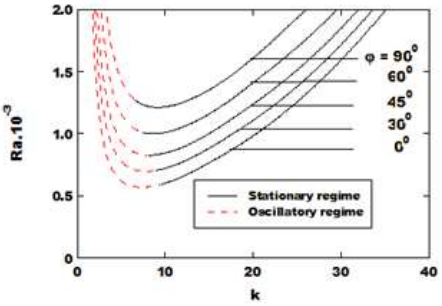

Fig. 8. Effect of varying φ on Ra for neutral stability for anisotropy in horizontal plane case (i.e., for $(1< K^* < \zeta)$) when $\zeta = 3$, $K^* = 2.5$, $Ta = 50$, $\lambda_1^* = 0.5$ and $\lambda_2^* = 0.8$.

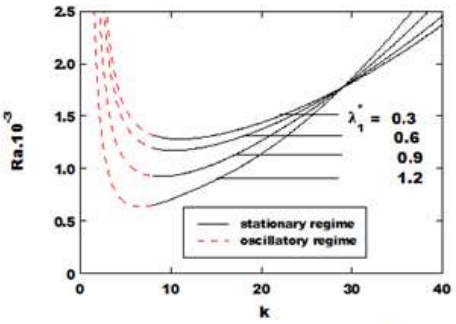

Fig. 9. Effect of varying λ_1^* on Ra for neutral stability for anisotropy in horizontal plane case (i.e., for $(1 < K^* < \zeta)$ when $\zeta = 3$, $K^* = 2.5$, $\varphi = 45°$ and $\lambda_2^* = 0.8$.

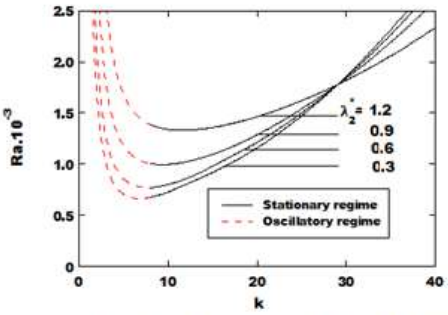

Fig. 10. Effect of varying λ_2^* on Ra for neutral stability for anisotropy in horizontal plane case (i.e., for $(1 < K^* < \zeta)$ when $\zeta = 3$, $K^* = 2.5$, $\varphi = 45°$ and $\lambda_1^* = 0.5$.

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