

Research Article

A Posteriori Error Analysis for a New Fully Mixed Isotropic Discretization of the Stationary Stokes-Darcy Coupled Problem

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Received 19 June 2020; Accepted 27 September 2020; Published 12 October 2020

Academic Editor: Alberto Fiorenza

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In this paper, we develop an a posteriori error analysis for the stationary Stokes-Darcy coupled problem approximated by conforming the finite element method on isotropic meshes in \mathbb{R}^d , $d \in \{2, 3\}$. The approach utilizes a new robust stabilized fully mixed discretization. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution plus the stabilization terms. It is proven that the a posteriori error estimate provided in this paper is both reliable and efficient.

1. Introduction

There are many serious problems currently facing the world in which the coupling between groundwater and surface water is important. These include questions such as predicting how pollution discharges into streams, lakes, and rivers making its way into the water supply. This coupling is also important in technological applications involving filtration. We refer to the nice overview [1] and the references therein for its physical background, modeling, and standard numerical methods. One important issue in the modeling of the coupled Darcy-Stokes flow is the treatment of the interface condition, where the Stokes fluid meets the porous medium. In this paper, we only consider the so-called Beavers-Joseph-Saffman condition, which was experimentally derived by Beavers and Joseph in [2], modified by Saffman in [3], and later mathematically justified in [4–7].

There are three popular formulations of the coupled Darcy-Stokes flow, namely, the primal formulation, the mixed formulation in the Darcy region, or the fully mixed formulation, see, for example, the works [8–16] for some mathematical analysis. Two different mixed formulations have been studied by Layton et al. in [15]. The first one

enforces the weak continuity of the normal component of the velocity field on the interface, by employing a Lagrange multiplier, while the second one imposes the strong continuity in the functional space. We can call these two mixed formulations the weakly coupled formulation and the strongly coupled formulation, respectively. The weakly coupled formulation gives more freedom in the choice of the discretization in the Stokes side and the Darcy side singly. The authors in [9–11, 15, 17–19] employ the weakly coupled formulation. The works on the strongly coupled formulation have been based on the development of a unified discretization, namely, the Stokes side and the Darcy side are discretized using the same finite element. This approach simplifies the numerical implementation, only if the unified discretization is not significantly more complicated than the commonly used discretizations for the Darcy and Stokes problems. The authors in [8, 20] have proposed a conforming, unified finite element for the strongly coupled mixed formulation. Superconvergence analysis of the finite element methods for the Stokes-Darcy system was studied by Chen et al. in [21]. Other less restrictive discretizations as the nonconforming unified approach [12, 19] or the discontinuous Galerkin (DG) approach have been analyzed in [13, 14, 22]. Due to its

discontinuous nature, some (DG) discretizations for the coupled Darcy-Stokes problem may break the strong coupling in the discrete level [13, 14], as they impose the normal continuity across the interface via interior penalties.

The adaptive techniques have become indispensable tools and unavoidable in the field of study behavior of the error committed during solving partial differential equations (PDE). A posteriori error estimators are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. They are essential to design adaptive mesh refinement algorithms which equidistribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuška and Rheinboldt [23], adaptive finite element methods based on a posteriori error estimates have been extensively investigated.

A posteriori error estimations have been widely studied for both the mixed formulations of the Darcy flow [24–26] and the Stokes flow [27–36]. However, only a few works exist for the coupled Darcy-Stokes problem, see for instance [18, 37–40]. The works in [18, 38] concern the strongly coupled mixed formulation where a $H(\text{div})$ conforming and nonconforming finite element methods have been employed. The papers [37, 39] concern the weakly coupled mixed formulation while [39] uses the primal formulation on the Darcy side. The authors in [40] employ a fully mixed formulation where Raviart-Thomas elements have been used to approximate the velocity in both the Stokes domain and the Darcy domain, and constant piecewise for approximating the pressure.

In [16], a stabilized finite element method for the stationary-mixed Stokes-Darcy problem has been proposed for the fully mixed formulation. The authors have used the well-known MINI elements ($P1b - P1$) to approximate the velocity and pressure in the conduit for the Stokes equation. To capture the fully mixed technique in the porous medium region linear Lagrangian elements, $P1$ has been used for hydraulic (piezometric) head and Brezzi-Douglas-Marini (BDM1) piecewise constant finite elements have been used for Darcy velocity. An a priori error analysis is performed with some numerical tests confirming the convergence rates. However, to our best knowledge, they did not talk about the adaptive method for the fully mixed discretization proposed in [16]. In this case, our main objective is to perform an a posteriori error analysis by constructing reliable and efficient indicator errors. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. We prove that our indicator errors are efficiency and reliability and then are optimal. The difference between our paper and the reference [40] is that our discretization uses MINI elements ($P1b - P1$) to approximate the velocity and pressure in the conduit for Stokes equations; $P1$ -Lagrange elements to approximate hydraulic (piezometric) head and Brezzi-Douglas-Marini (BDM1) piecewise constant finite elements have been used for Darcy velocity. As a result, an additional term is included in the error estimator that measures the stability of the method. In order to treat appropriately this stability term, we further need a special

Helmholtz decomposition ([18], Theorem 3), a regularity result ([18], Theorem 4), and an estimate of the stability error ([18], Theorem 5).

The paper is organized as follows. Some preliminaries and notation are given in Section 2. The efficiency result is derived using the technique of bubble function introduced by Verfürth [41] and used in a similar context by Carstensen [25, 42]. In Section 3, the a posteriori error estimates are derived. We offer our conclusion and the further works in Section 4.

2. Preliminaries and Notations

2.1. Model Problem. We consider the model of a flow in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3), consisting of a porous medium domain Ω_p , where the flow is a Darcy flow, and an open region Ω_f , where the flow is governed by the Stokes equations. The two regions are separated by an interface $\Gamma = \partial\Omega_p \cap \partial\Omega_f$. Let $\Gamma_l = \partial\Omega_l \setminus \Gamma$, $l = f, p$. Each interface and boundary is assumed to be polygonal ($d = 2$) or polyhedral ($d = 3$). We denote by \mathbf{n}_f (resp., \mathbf{n}_p) the unit outward normal vector along $\partial\Omega_f$ (resp., $\partial\Omega_p$). Note that on the interface Γ , we have $\mathbf{n}_f = -\mathbf{n}_p$. Figure 1 shows a sketch of the problem domain, its boundaries, and some other notations.

The fluid velocity and pressure $\mathbf{u}_f(x)$ and $p(x)$ are governed by the Stokes equations in Ω_f :

$$\begin{cases} -2\nu\nabla\cdot\mathbb{D}(\mathbf{u}_f) + \nabla p = \mathbf{f}_f & \text{in } \Omega_f, \\ \nabla\cdot\mathbf{u}_f = 0 & \text{in } \Omega_f, \end{cases} \quad (1)$$

where $\mathbb{T} = -p\mathbb{I} + 2\nu\mathbb{D}(\mathbf{u}_f)$ denotes the stress tensor, and $\mathbb{D}(\mathbf{u}_f) = (1/2)(\nabla\mathbf{u}_f + (\nabla\mathbf{u}_f)^T)$ represents the deformation tensor. The porous media flow is governed by the following Darcy equations on Ω_p through the fluid velocity $\mathbf{u}_p(x)$ and the piezometric head $\phi(x)$:

$$\begin{cases} \mathbf{u}_p = -\mathbf{K}\nabla\phi & \text{in } \Omega_p, \\ \nabla\cdot\mathbf{u}_p = f_p & \text{in } \Omega_p, \end{cases} \quad (2)$$

We impose impermeable boundary conditions, $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on Γ_p , on the exterior boundary of the porous media region, and no-slip conditions, $\mathbf{u}_f = 0$ on Γ_f , in the Stokes region. Both selections of boundary conditions can be modified. On Γ , the interface coupling conditions are conservation of mass, balance of forces, and a tangential condition on the fluid region's velocity on the interface. The correct tangential condition is not completely understood (possibly due to matching a pointwise velocity in the fluid region with an averaged or homogenized velocity in the porous region). In this paper, we take the Beavers-Joseph-Saffman (-Jones), see [2–7], interfacial coupling:

$$\mathbf{u}_f \cdot \mathbf{n}_f + \mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma, \quad (3)$$

$$-\mathbf{n}_f \cdot \mathbb{T} \cdot \mathbf{n}_f = p - 2\nu\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_f) \cdot \mathbf{n}_f = \rho g \phi \quad \text{on } \Gamma, \quad (4)$$

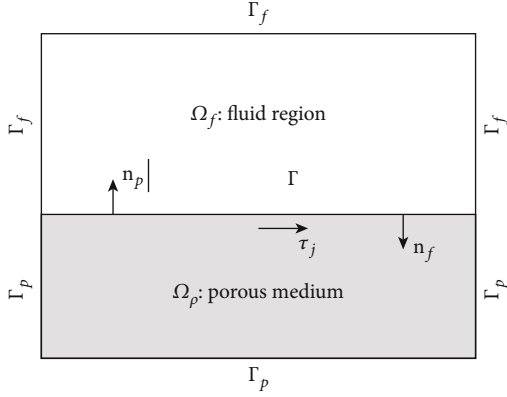


FIGURE 1: Global domain Ω consisting of the fluid region Ω_f and the porous media region Ω_p separated by the interface Γ .

$$\begin{aligned} -\mathbf{n}_f \cdot \mathbb{T} \cdot \boldsymbol{\tau}_j &= -2\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_f) \cdot \boldsymbol{\tau}_j \\ &= \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \boldsymbol{\tau}_j}} \mathbf{u}_f \cdot \boldsymbol{\tau}_j, \quad 1 \leq j \leq (d-1) \quad \text{on } \Gamma. \end{aligned} \quad (5)$$

This is a simplification of the original and more physically realistic Beavers-Joseph conditions (in which $\mathbf{u}_f \cdot \boldsymbol{\tau}_j$ in (2.8) is replaced by $(\mathbf{u}_f - \mathbf{u}_p) \cdot \boldsymbol{\tau}_j$; see [2]). Here, we denote \mathbf{f}_f and f_p are the body forces in the fluid region and source in the porous region, \mathbf{K} is the symmetric positive definite (SPD) hydraulic conductivity tensor, and α is the constant parameter.

We shall also assume that all material and fluid parameters defined above are uniformly positive and bounded, i.e.,

$$0 \leq k_{\min} \leq \lambda(\mathbf{K}) \leq k_{\max} < \infty. \quad (6)$$

2.2. Notations and the Weak Formulation. In this part, we first introduce some Sobolev spaces [43] and norms. If W is a bounded domain of \mathbb{R}^d and m is a nonnegative integer, the Sobolev space $H^m(W) = W^{m,2}(W)$ is defined in the usual way with the usual norm $\|\cdot\|_{m,W}$ and seminorm $|\cdot|_{m,W}$. In particular, $H^0(W) = L^2(W)$ and we write $\|\cdot\|_W$ for $\|\cdot\|_{0,W}$. Similarly, we denote by $(\cdot, \cdot)_W$ the $L^2(W)[L^2(W)]^N$ or $[L^2 W]^{d \times d}$ inner product. For shortness, if W is equal to Ω , we will drop the index Ω , while for any $m \geq 0$, $\|\cdot\|_{m,l} = \|\cdot\|_{m,\Omega_l}$, $|\cdot|_{m,l} = |\cdot|_{m,\Omega_l}$ and $(\cdot, \cdot)_l = (\cdot, \cdot)_{\Omega_l}$, for $l = f, s$. The space $H_0^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Let $[H^m(\Omega)]^d$ be the space of vector-valued functions $\mathbf{v} = (v_1, \dots, v_d)$ with components v_i in $H^m(\Omega)$. The norm and the seminorm on $[H^m(\Omega)]^d$ are given by

$$\begin{aligned} \|\mathbf{v}\|_{m,\Omega} &:= \left(\sum_{i=0}^m \|\mathbf{v}_i\|_{m,\Omega}^2 \right)^{1/2}, \\ |\mathbf{v}|_{m,\Omega} &:= \left(\sum_{i=0}^m |v_i|_{m,\Omega}^2 \right)^{1/2}. \end{aligned} \quad (7)$$

For a connected open subset of the boundary $E \subset \partial\Omega_s \cup \partial\Omega_d$, we write $\langle \cdot, \cdot \rangle_E$ for the $L^2(E)$ inner product (or duality pairing), that is, for scalar-valued functions λ, σ , one defines:

$$\langle \lambda, \sigma \rangle_E := \int_E \lambda \sigma ds. \quad (8)$$

By setting the space

$$H_{\text{div}} := H(\text{div}; \Omega_p) = \left\{ \mathbf{v}_p \in [L^2(\Omega_p)]^d : \nabla \cdot \mathbf{v}_p \in L^2(\Omega_p) \right\}, \quad (9)$$

we introduce the following spaces:

$$\begin{aligned} \mathbf{X}_f &:= \left\{ \mathbf{v}_f \in [H^1(\Omega_f)]^d : \mathbf{v}_f = \mathbf{0} \quad \text{on } \Gamma_f \right\}, \\ Q_f &:= L^2(\Omega_f), \\ \mathbf{X}_p &:= \left\{ \mathbf{v}_p \in H(\text{div}; \Omega_p) : \mathbf{v}_p \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma_p \right\}, \\ Q_p &:= L^2(\Omega_p). \end{aligned} \quad (10)$$

For the spaces \mathbf{X}_f and \mathbf{X}_p , we define the following norms:

$$\begin{aligned} \|\mathbf{v}_f\|_1 &:= \sqrt{\|\mathbf{v}_f\|_{\Omega_f}^2 + |\mathbf{v}_f|_{1,\Omega_f}^2}, \quad \text{with } |\mathbf{v}_f|_{1,\Omega_f} = \|\nabla \mathbf{v}_f\|_{\Omega_f}, \quad \forall \mathbf{v}_f \in \mathbf{X}_f, \\ \|\mathbf{v}_p\|_{\text{div}} &:= \sqrt{\|\mathbf{v}_p\|_{\Omega_p}^2 + \|\nabla \cdot \mathbf{v}_p\|_{\Omega_p}^2}, \quad \forall \mathbf{v}_p \in \mathbf{X}_p. \end{aligned} \quad (11)$$

The variational formulation of the steady-state Stokes-Darcy problem (1)–(5) reads as find $(\mathbf{u}_f, p; \mathbf{u}_p, \phi) \in (\mathbf{X}_f, Q_f; \mathbf{X}_p, Q_p)$ satisfying

$$a_f(\mathbf{u}_f, \mathbf{v}_f) - b_f(\mathbf{v}_f, p) + c_\Gamma(\mathbf{v}_f, \phi) = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f}, \quad \forall \mathbf{v}_f \in \mathbf{X}_f, \quad (12)$$

$$b_f(\mathbf{u}_f, q) = 0, \quad \forall q \in Q_f, \quad (13)$$

$$a_p(\mathbf{u}_p, \mathbf{v}_p) - b_p(\mathbf{v}_p, \phi) - c_\Gamma(\mathbf{v}_p, \phi) = 0, \quad \forall \mathbf{v}_p \in \mathbf{X}_p, \quad (14)$$

$$b_p(\mathbf{u}_p, \psi) = \rho g (f_p, \psi)_{\Omega_p}, \quad \psi \in Q_p, \quad (15)$$

where the bilinear forms are defined as

$$\begin{aligned} a_f(\mathbf{u}_f, \mathbf{v}_f) &:= 2\nu(\mathbb{D}(\mathbf{u}_f), \mathbb{D}(\mathbf{v}_f))_{\Omega_f} \\ &\quad + \sum_{j=1}^{d-1} \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \boldsymbol{\tau}_j}} \langle \mathbf{u}_f \cdot \boldsymbol{\tau}_j, \mathbf{v}_f \cdot \boldsymbol{\tau}_j \rangle_{\Gamma}, \end{aligned}$$

$$a_p(\mathbf{u}_p, \mathbf{v}_p) := \rho g (\mathbf{K}^{-1} \mathbf{u}_p, \mathbf{v}_p)_{\Omega_p},$$

$$b_f(\mathbf{v}_f, p) := (p, \nabla \cdot \mathbf{v}_f)_{\Omega_f},$$

$$\begin{aligned} b_p(\mathbf{v}_p, \phi) &:= \rho g(\phi, \nabla \cdot \mathbf{v}_p)_{\Omega_p}, \\ c_\Gamma(\mathbf{v}_f, \phi) &:= \rho g \langle \phi, \mathbf{v}_f \cdot \mathbf{n}_f \rangle_{\Gamma}. \end{aligned} \quad (16)$$

After introducing, for $\mathbf{U} = (\mathbf{u}_f, p, \mathbf{u}_p, \phi) \in \mathbf{X}_f \times Q_p \times \mathbf{X}_p \times Q_p = \mathbf{H}$ and $\mathbf{V} = (\mathbf{v}_f, q, \mathbf{v}_p, \psi) \in \mathbf{X}_f \times Q_p \times \mathbf{X}_p \times Q_p$,

$$\begin{aligned} \mathcal{L}(\mathbf{U}, \mathbf{V}) &:= a_f(\mathbf{u}_f, \mathbf{v}_f) - b_f(\mathbf{v}_f, p) + b_f(\mathbf{u}_f, q) + a_p(\mathbf{u}_p, \mathbf{v}_p) \\ &\quad - b_p(\mathbf{v}_p, \phi) + b_p(\mathbf{u}_p, \psi) + c_\Gamma(\mathbf{v}_f - \mathbf{v}_p, \phi), \\ \mathcal{F}(\mathbf{V}) &:= (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f} + \rho g(f_p, \psi)_{\Omega_p}, \end{aligned} \quad (17)$$

the weak formulation (12)–(15) can be equivalently rewritten as follows: find $\mathbf{U} \in \mathbf{H}$ satisfying

$$\mathcal{L}(\mathbf{U}, \mathbf{V}) = \mathcal{F}(\mathbf{V}), \quad \forall \mathbf{V} \in \mathbf{H}. \quad (18)$$

It is easy to verify that this variational formulation is well-posedness [16].

To end this section, we recall the following Poincaré, Korn's, and the trace inequalities, which will be used in the later analysis; there exist constant C_p , C_K , and C_v , only depending on ω_f such that for all $\mathbf{v}_f \in \mathbf{X}_f$,

$$\begin{aligned} \|\mathbf{v}_f\| &\leq C_p \|\mathbf{v}_f\|_1, \\ \|\mathbf{v}_f\| &\leq C_K \|\mathbb{D}(\mathbf{v}_f)\|_{\Omega_f}, \\ \|\mathbf{v}_f\|_\Gamma &\leq C_v \|\mathbf{v}_f\|_{\Omega_f}^{1/2} \|\mathbf{v}_f\|_{1, \Omega_f}^{1/2}. \end{aligned} \quad (19)$$

Besides, there exists a constant \tilde{C}_v that only depends on Ω_p such that for all $\psi \in Q_p$,

$$\|\psi\|_{L^2(\Gamma)} \leq \tilde{C}_v \|\psi\|_{\Omega_p}^{1/2} \|\psi\|_{1, \Omega_p}^{1/2}. \quad (20)$$

2.3. Fully Mixed Isotropic Discretization. First, we consider the family of triangulations \mathcal{T}_h of Ω , consisting of \mathcal{T}_h^f and \mathcal{T}_h^p , which are regular triangulations of Ω_f and Ω_p , respectively, where $h > 0$ is a positive parameter. We also assume that on the interface Γ the two meshes of \mathcal{T}_h^f and \mathcal{T}_h^p , which form the regular triangulation $\mathcal{T}_h := \mathcal{T}_h^f \cup \mathcal{T}_h^p$, coincide.

The domain of the uniformly regular triangulation $\bar{\Omega}_f \cup \bar{\Omega}_p$ is such that $\bar{\Omega} = \{UK : K \in \mathcal{T}_h\}$ and $h = \max_{K \in \mathcal{T}_h} h_K$. There exist positive constants c_1 and c_2 satisfying $c_1 h \leq h_K \leq c_2 \rho_K$. To approximate the diameter h_K of the triangle (or tetrahedral) K , ρ_K is the diameter of the greatest ball included in K . Based on the subdivisions \mathcal{T}_h^f and \mathcal{T}_h^p , we can define finite element spaces $\mathbf{X}_{fh} \subset \mathbf{X}_h$, $Q_{fh} \subset Q_f$, $\mathbf{X}_{ph} \subset \mathbf{X}_p$, and $Q_{ph} \subset Q_p$. We consider the well-known MINI elements (P1b–P1) to approximate the velocity and the pressure in the conduit for Stokes equations [44]. To capture the fully mixed technique in the porous medium region linear Lagrangian

elements, P1 are used for hydraulic (piezometric) head and Brezzi-Douglas-Marini (BDM1) piecewise constant finite elements are used for Darcy velocity [45].

In the fluid region, we select for the Stokes problem the finite element spaces $(\mathbf{X}_{fh}, Q_{fh})$ that satisfy the velocity-pressure inf-sup condition: there exists a constant $C_f > 0$, independent of h , such that,

$$\inf_{0 \neq q^h \in Q_{fh}} \sup_{0 \neq \mathbf{v}_f^h \in \mathbf{X}_{fh}} \frac{b_f(\mathbf{v}_f^h, q^h)}{\|\mathbf{v}_f^h\|_{1, \Omega_f} \|q_f^h\|_{\Omega_f}} \geq C_f. \quad (21)$$

In the porous region, we use the finite element spaces $(\mathbf{X}_{ph}, Q_{ph})$ that also satisfy a standard inf-sup condition: there exist a constant $C_p > 0$ such that for all $\phi^h \in Q_{ph}$,

$$\inf_{0 \neq \phi^h \in Q_{ph}} \sup_{0 \neq \mathbf{v}_p^h \in \mathbf{X}_{ph}} \frac{b_p(\mathbf{v}_p^h, \phi^h)}{\|\mathbf{v}_p^h\|_{\text{div}} \|\phi^h\|_{\Omega_p}} \geq C_p. \quad (22)$$

Then, the finite element discretization of (18) is to find $\mathbf{U}_h \in \mathbf{H}_h = \mathbf{X}_{fh} \times Q_{fh} \times \mathbf{X}_{ph} \times Q_{ph}$ such that

$$\mathcal{L}(\mathbf{U}_h, \mathbf{V}_h) + \mathbf{J}_\Gamma(\mathbf{U}_h, \mathbf{V}_h) = \mathcal{F}(\mathbf{V}_h), \quad \forall \mathbf{V}_h \in \mathbf{H}_h. \quad (23)$$

This is the natural discretization of the weak formulation (18) except that the stabilized term $\mathbf{J}_\Gamma(\mathbf{U}_h, \mathbf{V}_h)$ is added. This bilinear form $\mathbf{J}_\Gamma(\cdot, \cdot)$ is defined by

$$\mathbf{J}_\Gamma(\mathbf{U}_h, \mathbf{V}_h) := \frac{\delta}{h} \left\langle (\mathbf{u}_f^h - \mathbf{u}_p^h) \cdot \mathbf{n}_f, (\mathbf{v}_f^h - \mathbf{v}_p^h) \cdot \mathbf{n}_f \right\rangle_\Gamma, \quad 0 < h < 1. \quad (24)$$

We are now able to define the norm on \mathbf{H}_h :

$$\|\mathbf{V}\|_h := \sqrt{\|\mathbf{v}_f^h\|_{1, \Omega_f}^2 + \|q_f^h\|_{\Omega_f}^2 + \|\mathbf{v}_p^h\|_{\text{div}}^2 + \|\psi^h\|_{\Omega_p}^2 + h^{-1} \|(\mathbf{v}_f^h - \mathbf{v}_p^h)\|_\Gamma^2}. \quad (25)$$

We have the following results (see [16], Theorem 2 and Theorem 3):

Theorem 1. *There exists a unique solution $\mathbf{U}_h \in \mathbf{H}_h$ to problem (23) and if the solution $\mathbf{U} \in \mathbf{H}$ of the continuous problem (18) is smooth enough, then we have*

$$\|\mathbf{U} - \mathbf{U}_h\|_h \leq C(\mathbf{U})h. \quad (26)$$

Below, in order to avoid excessive use of constants, the abbreviation $x \leq y$ stand for $x \leq cy$, with c a positive constant independent of x , y , and \mathcal{T}_h .

Remark 2. (Galerkin orthogonality relation). Let $\mathbf{U} = (\mathbf{u}_f, p, \mathbf{u}_p, \phi) \in \mathbf{H}$ be the exact solution and $\mathbf{U}_h = (\mathbf{u}_{fh}, p_h, \mathbf{u}_{ph}, \phi_h) \in \mathbf{H}_h$ be the finite element solution. Then, for any $\mathbf{V}_h = (\mathbf{v}_{fh},$

$p_h, \mathbf{v}_{ph}, \psi_h) \in \mathbf{H}_h$, and using technical regularity result Theorem 4 below, we can subtract (18) to (23) to obtain the Galerkin orthogonality relation:

$$\begin{aligned} \mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{V}_h) &= \mathcal{L}_h(\mathbf{U}, \mathbf{V}_h) - \mathcal{L}_h(\mathbf{U}_h, \mathbf{V}_h) \\ &= \mathcal{L}(\mathbf{U}, \mathbf{V}_h) - \mathcal{L}_h(\mathbf{U}_h, \mathbf{V}_h) \\ &= \mathcal{F}(\mathbf{V}_h) - \mathcal{F}(\mathbf{V}_h) = 0. \end{aligned} \quad (27)$$

Thus, we have the relation:

$$\begin{aligned} 2\nu(\mathbb{D}(\mathbf{e}_f), \mathbb{D}(\mathbf{v}_{fh}))_{\Omega_f} &+ \sum_{j=1}^{d-1} \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j}} \langle \mathbf{e}_f \cdot \boldsymbol{\tau}_j, \mathbf{v}_{fh} \cdot \boldsymbol{\tau}_j \rangle_{\Gamma} \\ &- (\varepsilon_p \nabla \cdot \mathbf{v}_{fh})_{\Omega_f} + (q_h \nabla \cdot \mathbf{e}_f)_{\Omega_f} + \rho g \left[(\mathbf{K}^{-1} \mathbf{e}_p, \mathbf{v}_{ph})_{\Omega_p} \right. \\ &\left. - (\lambda_\phi \nabla \cdot \mathbf{v}_{ph})_{\Omega_p} + (\psi_h \nabla \cdot \mathbf{e}_p)_{\Omega_p} + \langle \lambda_\phi, [\mathbf{v}] \rangle_{\Gamma} \right] = 0, \end{aligned} \quad (28)$$

where here and below, the errors in the velocity and in the pressure of Stokes equations and errors in the hydraulic and Darcy velocity equations are, respectively, defined by

$$\begin{aligned} \mathbf{e}_f &:= \mathbf{u}_f - \mathbf{u}_{fh}, \\ \varepsilon_p &:= p - p_h, \\ \mathbf{e}_p &:= \mathbf{u}_p - \mathbf{u}_{ph}, \end{aligned} \quad (29)$$

$$\lambda_\phi = \phi - \phi_h.$$

3. A Posteriori Error Analysis

3.1. *Some Technical Results.* Our a posteriori analysis requires some analytical results that are recalled. We define the space

$$\mathcal{H} = \left\{ \mathbf{v} \in \mathbf{H}(\text{div}, \Omega) : \mathbf{v}_{|\Omega_f} \in \mathbf{X}_f \text{ and } \mathbf{v}_{|\Omega_p} \in \mathbf{X}_p \right\} \quad (30)$$

with the norm

$$\|\mathbf{v}\|_{\mathcal{H}} := \sqrt{|\mathbf{v}_f|_{1,\Omega_f}^2 + \|\mathbf{v}_p\|_{\Omega_p}^2 + \|\nabla \cdot \mathbf{v}_p\|_{\Omega_p}}. \quad (31)$$

The first one concerns a sort of Helmholtz decomposition of elements of \mathcal{H} . Recall first that if $d = 3$,

$$H_0(\text{curl}, \Omega_p) = \left\{ \psi \in L^2(\Omega_p)^3 : \text{curl } \psi \in L^2(\Omega_p)^3 \text{ and } \psi \times n = 0 \text{ on } \partial\Omega_p \right\}. \quad (32)$$

Theorem 3. (see Ref. [18], page 708). *Any $\mathbf{v} \in \mathcal{H}$ admits the Helmholtz-type decomposition*

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \quad (33)$$

where $\mathbf{v}_0, \mathbf{v}_1 \in \mathcal{H}$ but satisfying $\mathbf{v}_0 \in H^1(\Omega)^d$,

$$\mathbf{v}_1 = \begin{cases} 0 & \text{in } \Omega_f, \\ \text{curl } \beta_p & \text{in } \Omega_p, \end{cases} \quad (34)$$

where $\beta_p \in H_0^1(\Omega_p)$ if $d = 2$, while $\beta_p \in H^1(\Omega_p)^3 \cap H_0(\text{curl}, \Omega_p)$ if $d = 3$, with the estimate

$$\|\mathbf{v}_0\|_{1,\Omega} + \|\beta_p\|_{1,\Omega_p} \lesssim \|\mathbf{v}\|_{\mathcal{H}}. \quad (35)$$

The second result that we need is a regularity result for the solution $\mathbf{U} = (\mathbf{u}_f, p, \mathbf{u}_p, \phi) \in \mathbf{H}$ of (18) is the following theorem:

Theorem 4. (see [18], page 710). *Let $\mathbf{U} \in \mathbf{H}$ be the unique solution of (18). If $\mathbf{K} \in [C^{0,1}(\bar{\Omega}_p)]^{d \times d}$, then there exists $\varepsilon > 0$ such that*

$$\mathbf{u}_{|\Omega_p} \in [H^{1/2+\varepsilon}(\Omega_p)]^d. \quad (36)$$

Let us finish this section by an estimation of the stability error (see [18], Theorem 5):

Theorem 5. *For any $\mathbf{U}_h = (\mathbf{u}_{fh}, p_h, \mathbf{u}_{ph}, \phi_h) \in \mathbf{H}_h$, we have*

$$\inf_{\mathbf{W}_h \in \mathbf{H}_h \cap \mathcal{H}} \|\mathbf{U}_h - \mathbf{W}_h\|_h^2 \leq \mathbf{J}_\Gamma(\mathbf{U}_h, \mathbf{U}_h). \quad (37)$$

3.2. *Error Estimator.* In order to solve the Stokes-Darcy coupled problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriated indicators. In this section, we first define the local and global indicators and then the lower and upper error bounds are derived in Section 3.3.

3.2.1. *Error Equations.* The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier and that involve the data at hand. Thus, we define the error equations: let $\mathbf{U} = (\mathbf{u}_f, p, \mathbf{u}_p, \phi) \in \mathbf{H}$ be the exact solution and $\mathbf{U}_h = (\mathbf{u}_{fh}, p_h, \mathbf{u}_{ph}, \phi_h) \in \mathbf{H}_h$ be the finite element solution. Then, for any $\mathbf{V}_h = (\mathbf{v}_{fh}, q_h, \mathbf{v}_{ph}, \psi_h) \in \mathbf{H}_h$ and $\mathbf{V} = (\mathbf{v}_f, p, \mathbf{v}_p, \psi) \in \mathbf{H}$, using the Helmholtz decomposition (Theorem 3), we have

$$\mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{V}) = \mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{V} - \mathbf{V}_h) \quad (38)$$

$$= \left[\sum_{K \in \mathcal{T}_h^f} \left(\mathbf{R}_K^f(\mathbf{U}_h), \mathbf{V} - \mathbf{V}_h \right)_K + \sum_{K \in \mathcal{T}_h^p} \left(\mathbf{R}_K^p(\mathbf{U}_h), \mathbf{V} - \mathbf{V}_h \right)_K \right], \quad (39)$$

where

$$\begin{aligned}
& \left(\mathbf{R}_K^f(\mathbf{U}_h), \mathbf{V} - \mathbf{V}_h \right)_K \\
&= \left(\mathbf{f}_f + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h, \mathbf{v}_f - \mathbf{v}_{fh} \right)_K \\
&\quad - \left(q - q_h, \nabla \cdot \mathbf{u}_{fh} \right)_K - \left[\sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} \sum_{j=1}^{d-1} \right. \\
&\quad \cdot \left. \left(2\nu \mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \boldsymbol{\tau}_j + \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j}} \mathbf{u}_{fh} \cdot \boldsymbol{\tau}_j, (\mathbf{v}_f - \mathbf{v}_{fh}) \cdot \boldsymbol{\tau}_j \right)_E \right] \\
&\quad + \left[\sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} (p_h - 2\nu \mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \mathbf{n}_f - \rho g \phi_h, (\mathbf{v}_f - \mathbf{v}_{fh}) \cdot \mathbf{n}_f)_E \right], \tag{40}
\end{aligned}$$

$$\begin{aligned}
& \left(\mathbf{R}_K^p(\mathbf{U}_h), \mathbf{V} - \mathbf{V}_h \right)_K \\
&= \left(\text{curl}(\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h), \boldsymbol{\beta}_p - \boldsymbol{\beta}_{ph} \right)_K \\
&\quad + \left(\rho g (f_p - \nabla \cdot \mathbf{u}_{ph}), \psi - \psi_h \right)_K \\
&\quad - \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_p)} (\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h) \times \mathbf{n}_E, \psi - \psi_h)_E \tag{41} \\
&\quad + \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_p)} \left([\rho g \phi_h \mathbf{n}_E]_E, \boldsymbol{\beta}_p - \boldsymbol{\beta}_{ph} \right)_E \\
&\quad - \sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} ([\rho g \phi_h \mathbf{n}_E]_E, [\mathbf{v} - \mathbf{v}_h]_E)_E.
\end{aligned}$$

3.2.2. Residual Error Estimators

Definition 6 (a posteriori error indicators). The residual error estimator is locally defined by

$$\Theta_K = \left[\Theta_{K,f}^2 + \Theta_{K,p}^2 \right]^{1/2} \text{ for each } K \in \mathcal{T}_h, \tag{42}$$

where

$$\begin{aligned}
\Theta_{K,f}^2 &= h_K^2 \left\| \mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h \right\|_K^2 + \left\| \nabla \cdot \mathbf{u}_{fh} \right\|_K^2 + \sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} \\
&\quad \cdot h_E \left\{ \sum_{j=1}^{d-1} \left\| 2\nu \mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \boldsymbol{\tau}_j + \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j}} \mathbf{u}_{fh} \cdot \boldsymbol{\tau}_j \right\|_E^2 \right\} \\
&\quad + \sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} h_E \left\| p_h - 2\nu \mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \mathbf{n}_f - \rho g \phi_h \right\|_E^2, \tag{43}
\end{aligned}$$

$$\begin{aligned}
\Theta_{K,p}^2 &= h_K^2 \left\| \text{curl}(\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h) \right\|_K^2 + \left\| \rho g (f_p - \nabla \cdot \mathbf{u}_{ph}) \right\|_K^2 \\
&\quad + \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_p)} \left\| [\rho g (\mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h) \times \mathbf{n}_p]_E \right\|_E^2 \\
&\quad + \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_p)} h_E \left\| [\rho g \phi_h \mathbf{n}_p]_E \right\|_E^2 \\
&\quad + \sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} \frac{\delta h_E}{h} \left\| [(\mathbf{u}_{fh} - \mathbf{u}_{ph}) \cdot \mathbf{n}_f]_E \right\|_E^2. \tag{44}
\end{aligned}$$

The global residual error estimator is given by

$$\Theta := \left[\sum_{K \in \mathcal{T}_h} \Theta_K^2 \right]^{1/2}. \tag{45}$$

Furthermore, denote the local and global approximation terms by

$$\begin{aligned}
\zeta_K &:= \begin{cases} h_K \|\mathbf{f}_f - \mathbf{f}_{fh}\|_K, & \forall K \in \mathcal{T}_h^f, \\ \rho g \|f_p - f_{ph}\|_K, & \forall K \in \mathcal{T}_h^p, \end{cases} \tag{46} \\
\zeta &:= \left[\sum_{K \in \mathcal{T}_h} \zeta_K^2 \right]^{1/2},
\end{aligned}$$

where the global function $\mathbf{f}_{fh} : \Omega_f \rightarrow \mathbb{R}^d$ is defined by

$$\mathbf{f}_{fh|K} = \mathbf{f}_K = \frac{1}{|K|} \int_K \mathbf{f}_f(x) dx, \quad \forall K \in \mathcal{T}_h^f, \tag{47}$$

while in Ω_p , we take $f_{h|K} = f_K$ for all $K \in \mathcal{T}_h^p$, as the unique element of $\mathbb{P}^1(K)$ such that

$$\int_K f_K(x) q(x) dx = \int_K f(x) q(x) dx \quad \forall q \in \mathbb{P}^1(K). \tag{48}$$

Remark 7. The residual character of each term on the right-hand sides of (43) and (44) is quite clear since if $(\mathbf{u}_{fh}, p_h, \mathbf{u}_{ph}, \phi_h) \in \mathbf{H}_h$ would be the exact solution of (18), then they would vanish.

3.2.3. Analytical Tools

(1) *Inverse Inequalities.* In order to derive the lower error bounds, we proceed similarly as in [25, 42] (see also [46]), by applying inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions. To this end, we recall some notation and introduce further preliminary results. Given $K \in \mathcal{T}_h$, and $E \in \mathcal{E}(K)$, we let b_K and b_E be the usual simplex-bubble and face-bubble functions, respectively (see (1.5) and (1.6) in [41]). In particular, b_K satisfies $b_K \in \mathbb{P}^3(K)$, $\text{supp}(b_K) \subseteq K$, $b_K = 0$ on ∂K , and $0 \leq b_K \leq 1$ on K . Similarly, $b_E \in \mathbb{P}^2(K)$, $\text{supp}(b_E) \subseteq \omega_E := \{K' \in \mathcal{T}_h : E \in \mathcal{E}(K')\}$, $b_E = 0$ on ∂KE , and $0 \leq b_E \leq 1$ in ω_E . We also recall from [47] that, given $k \in \mathbb{N}$, there exists an extension operator $L : C(E) \rightarrow C(K)$ that satisfies $L(p) \in \mathbb{P}^k(K)$ and $L(p)|_E = p$, $\forall p \in \mathbb{P}^k(E)$. A corresponding vectorial version of L , that is, the componentwise application of L , is denoted by L . Additional properties of b_K , b_E , and L are collected in the following lemma (see [47]).

Lemma 8. *Given $k \in \mathbb{N}^*$, there exist positive constants depending only on k and shape-regularity of the triangulations (minimum angle condition), such that for each simplex K and*

$E \in \mathcal{E}(K)$, there hold

$$\|q\|_K \lesssim \|qb_K^{1/2}\|_K \lesssim \|q\|_K, \quad \forall q \in \mathbb{P}^k(K), \quad (49)$$

$$|qb_K|_{1,K} \lesssim h_K^{-1} \|q\|_K, \quad \forall q \in \mathbb{P}^k(K), \quad (50)$$

$$\|p\|_E \lesssim \|b_E^{1/2} p\|_E \lesssim \|p\|_E, \quad \forall p \in \mathbb{P}^k(E), \quad (51)$$

$$\|L(p)\|_K + h_E |L(p)|_{1,K} \lesssim h_E^{1/2} \|p\|_E, \quad \forall p \in \mathbb{P}^k(E). \quad (52)$$

(2) *Continuous Trace Inequality.*

Lemma 9 (continuous trace inequality). *There exists a positive constant $\beta_1 > 0$ depending only on σ_0 such that*

$$\|\mathbf{v}\|_{\partial K}^2 \leq \beta_1 \|\mathbf{v}\|_K \|\mathbf{v}\|_{1,K}, \quad \forall K \in \mathcal{T}_h, \forall \mathbf{v} \in [H^1(K)]^d. \quad (53)$$

(3) *Clément Interpolation Operator.* In order to derive the upper error bounds, we introduce the Clément interpolation operator $I_{\text{Cl}}^0 : H_0^1(\Omega) \rightarrow \mathcal{P}_c^b(\mathcal{T}_h)$ that approximates optimally nonsmooth functions by continuous piecewise linear functions:

$$\mathcal{P}_c^b(\mathcal{T}_h) := \{v \in C^0(\bar{\Omega}) : v|_K \in \mathbb{P}^1(K), \forall K \in \mathcal{T}_h \text{ and } v = 0 \text{ on } \partial\Omega\}. \quad (54)$$

In addition, we will make use of a vector-valued version of I_{Cl}^0 , that is, $I_{\text{Cl}}^0 : [H_0^1(\Omega)]^d \rightarrow [\mathcal{P}_c^b(\mathcal{T}_h)]^d$, which is defined componentwise by I_{Cl}^0 . The following lemma establishes the local approximation properties of I_{Cl}^0 (and hence of I_{Cl}^0), for a proof see [48], Section 3.

Lemma 10. *There exist constants $C_1, C_2 > 0$, independent of h , such that for all $v \in H_0^1(\Omega)$, there hold*

$$\|v - I_{\text{Cl}}^0(v)\|_K \leq C_1 h_K \|v\|_{1,\Delta(K)}, \quad \forall K \in \mathcal{T}_h, \quad (55)$$

$$\|v - I_{\text{Cl}}^0(v)\|_E \leq C_2 h_E^{1/2} \|v\|_{1,\Delta(E)}, \quad \forall E \in \mathcal{E}_h,$$

where $\Delta(K) := \cup\{K' \in \mathcal{T}_h : K' \cap K \neq \emptyset\}$ and $\Delta(E) := \cup\{K' \in \mathcal{T}_h : K' \cap E \neq \emptyset\}$.

3.3. Optimality of $\{\Theta_K\}_{K \in \mathcal{T}_h}$

3.3.1. Reliability Result

Theorem 11 (reliability of Θ). *Let $\mathbf{U} = (\mathbf{u}_f, p, \mathbf{u}_p, \phi) \in \mathbf{H}$ be the exact solution of (18) and $\mathbf{U}_h = (\mathbf{u}_{fh}, p_h, \mathbf{u}_{ph}, \phi_h) \in \mathbf{H}_h$ be the finite element solution of (23). There exist a constant $C_{\text{rel}} > 0$ such that the following estimate holds:*

$$\|\mathbf{U} - \mathbf{U}_h\|_h \leq C_{\text{rel}}(\Theta + \zeta). \quad (56)$$

Proof. We take $\psi_h = 0 = q_h$ in error equation (38)–(41) and

we obtain

$$\begin{aligned} \mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{V}) &= \mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{V} - \mathbf{V}_h) \\ &= \left[\sum_{K \in \mathcal{T}_h^f} (\mathbf{R}_K^f(\mathbf{U}_h), \mathbf{V} - \mathbf{V}_h)_K + \sum_{K \in \mathcal{T}_h^p} (\mathbf{R}_K^p(\mathbf{U}_h), \mathbf{V} - \mathbf{V}_h)_K \right], \end{aligned} \quad (57)$$

where

$$\begin{aligned} (\mathbf{R}_K^f(\mathbf{U}_h), \mathbf{V} - \mathbf{V}_h)_K &= (\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h, \mathbf{v}_f - \mathbf{v}_{fh})_K \\ &\quad - (q, \nabla \cdot \mathbf{u}_{fh})_K + (\mathbf{f}_f - \mathbf{f}_{fh}, \mathbf{v}_f - \mathbf{v}_{fh})_K - \left[\sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} \sum_{j=1}^{d-1} \right. \\ &\quad \cdot \left(2\nu \mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \boldsymbol{\tau}_j + \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j}} \mathbf{u}_{fh} \cdot \boldsymbol{\tau}_j, (\mathbf{v}_f - \mathbf{v}_{fh}) \cdot \boldsymbol{\tau}_j \right)_E \left. \right] \\ &\quad + \left[\sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} (p_h - 2\nu \mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \mathbf{n}_f - \rho g \phi_h, (\mathbf{v}_f - \mathbf{v}_{fh}) \cdot \mathbf{n}_f)_E \right], \end{aligned} \quad (58)$$

$$\begin{aligned} (\mathbf{R}_K^p(\mathbf{U}_h), \mathbf{V} - \mathbf{V}_h)_K &= \left(\text{curl}(\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h), \beta_p - \beta_{ph} \right)_K \\ &\quad + \left(\rho g (f_{ph} - \nabla \cdot \mathbf{u}_{ph}), \psi \right)_K + \rho g (f_p - f_{ph}, \psi)_K \\ &\quad - \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_p)} (\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h) \times \mathbf{n}_E, \psi)_E \\ &\quad + \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_p)} ([\rho g \phi_h \mathbf{n}_E]_E, \beta_p - \beta_{ph})_E \\ &\quad - \sum_{E \in \mathcal{E}_h(\partial K \cap \Gamma)} ([\rho g \phi_h \mathbf{n}_E]_E, [\mathbf{v} - \mathbf{v}_h]_E)_E. \end{aligned} \quad (59)$$

The inf-sup condition of \mathcal{L}_h leads to

$$\|\mathbf{U} - \mathbf{U}_h\|_h \leq C \sup_{\mathbf{V} \in \mathbf{H} \setminus \{0\}} \frac{|\mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{V})|}{\|\mathbf{V}\|_h}. \quad (60)$$

Now, using the error equation (57)–(59), Cauchy-Schwarz inequality, and the Clément operator of Lemma 10, we deduce the estimate (56). The proof is complete.

3.3.2. *Efficiency Result.* To prove local efficiency for $w \subset \Omega$, let us denote by

$$\begin{aligned} \|(\mathbf{v}_f, \mathbf{v}_p)\|_{h,w}^2 &:= \sum_{K \subset \bar{w} \cap \bar{\Omega}_f} |\mathbf{v}_f|_{1,K}^2 + \sum_{K \subset \bar{w} \cap \bar{\Omega}_p} \left(\|\mathbf{v}_p\|_K^2 + \|\text{div}_h \mathbf{v}_p\|_K^2 \right) \\ &\quad + \|\mathbf{v}_f \times \mathbf{n}\|_{\Gamma \cap \bar{w}}^2 + \sum_{K \subset \bar{\Omega}} \mathbf{J}_K(\mathbf{v}_f, \mathbf{v}_f), \end{aligned} \quad (61)$$

where

$$\mathbf{J}_K(\mathbf{v}_f, \mathbf{v}_f) := \sum_{E \in \mathcal{E}_h(\Omega_f) \cap \mathcal{E}_h(K)} \delta h_E^{-1} \left\| [\mathbf{v}_f]_E \right\|_E^2; \quad (62)$$

$$\|(\boldsymbol{\varepsilon}, \boldsymbol{\lambda})\|_w := \|\boldsymbol{\varepsilon}\|_w + \|\boldsymbol{\lambda}\|_w.$$

The main result of this subsection can be stated as follows.

Theorem 12 (efficiency of Θ). *Under the assumptions of Theorem 4, the following lower error bound holds:*

$$\Theta_K \lesssim \|(\mathbf{e}_f, \mathbf{e}_p)\|_{h, \tilde{w}_K} + \|(\boldsymbol{\varepsilon}_p, \boldsymbol{\lambda}_\phi)\|_{\tilde{w}_K} + \sum_{K' \subset \tilde{w}_K} \zeta_{K'}, \quad (63)$$

where \tilde{w}_K is a finite union of neighboring elements of K .

Proof. We begin by bounding each term of the residuals separately.

- (i) Element residual in Ω_f : to estimate $h_K^2 \|\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h\|_K^2$, we choose in error equation (57)–(59) for each $K \in \mathcal{T}_h^f$, $\mathbf{V} = (\mathbf{v}_s^K, 0, \mathbf{v}_p^K, 0)$ and $\mathbf{V}_h = (0, 0, 0, 0)$ with $\mathbf{v}_p^K = 0$ on Ω_p ,

$$\mathbf{v}_f^K = \begin{cases} [\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h] b_K & \text{on } K \in \mathcal{T}_h^f, \\ 0 & \text{on } \Omega_f \setminus K, \end{cases} \quad (64)$$

for obtained, $\mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{V}) = \|[\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h] b_K^{1/2}\|_K^2 + (\mathbf{f}_f - \mathbf{f}_{fh}, \mathbf{v}_f^K)_K$. Noted that $[\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h] b_K \in \mathbb{P}^k(K)$ and vanish on ∂K . Because $\mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{V}) = (\mathbb{D}(\mathbf{e}_f), \mathbf{v}_f^K)_K$ in this case, then we have $(\mathbb{D}(\mathbf{e}_f), \mathbf{v}_f^K)_K = \|[\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h] b_K^{1/2}\|_K^2 + (\mathbf{f}_f - \mathbf{f}_{fh}, \mathbf{v}_f^K)_K$. The first inverse inequality (49) and the Cauchy-Schwarz inequality lead to

$$\begin{aligned} \|\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h\|_K^2 &\sim \|[\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h] b_K^{1/2}\|_K^2 \\ &= \int_K \mathbb{D}(\mathbf{e}_f) [\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h] b_K - (\mathbf{f}_f - \mathbf{f}_{fh}, \mathbf{v}_f^K)_K \\ &\leq |\mathbf{e}_f|_K |(\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h) b_K|_{1,K} + \|\mathbf{f}_f - \mathbf{f}_{fh}\|_K \|\mathbf{v}_f^K\|_K. \end{aligned} \quad (65)$$

Using inverse inequality (50), we deduce the estimate:

$$h_K \|\mathbf{f}_{fh} + 2\nu \nabla \cdot \mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h\|_K \lesssim \|(\mathbf{e}_f, \mathbf{e}_p)\|_{h, \tilde{w}_K} + \zeta_K. \quad (66)$$

- (ii) Divergence element residual in Ω_f (estimation of $\|\nabla \cdot \mathbf{u}_{fh}\|_K^2$): for each $K \in \mathcal{T}_h^f$, we have,

$$\|\nabla \mathbf{u}_{fh}\|_K = \|\nabla(\mathbf{u}_f - \mathbf{u}_{fh})\|_K \lesssim |\mathbf{u}_f - \mathbf{u}_{fh}|_{1,K}. \quad (67)$$

- (iii) Element residual in Ω_p : we have for each $K \in \mathcal{T}_h^p$,

$$\begin{aligned} \|\rho g(f_{ph} - \nabla \cdot \mathbf{u}_{ph})\|_K &= \|\rho g(f_{ph} - \nabla \cdot \mathbf{u}_{ph}) - \rho g(f_p - \nabla \cdot \mathbf{u}_p)\|_K \\ &= \|\rho g \operatorname{div}_h(\mathbf{u}_p - \mathbf{u}_{ph}) - \rho g(f_p - f_{ph})\|_K \lesssim \|(\mathbf{e}_f, \mathbf{e}_p)\|_{h, \tilde{w}_K} + \zeta_K. \end{aligned} \quad (68)$$

- (iv) Curl element residual in Ω_p : for $K \in \mathcal{T}_h^p$, we set $C_K = \operatorname{curl}(\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h)$ and $\mathbf{W}_K = C_K b_K$. Hence, we notice that $\operatorname{curl} \mathbf{W}_K$ belongs to \mathbf{H} and is divergence free; therefore, by Equations (57)–(59), we obtain with $\mathbf{V}_h = \mathbf{0}$ and $\beta_p = \mathbf{W}_K$, $\mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{W}_K) = (\mathbf{R}_K^p(\mathbf{U}_h), \mathbf{W}_K)_K = \|\operatorname{curl}(\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h) b_K^{1/2}\|_K^2$. The first inverse inequality (49) and the Cauchy-Schwarz inequality lead to

$$\begin{aligned} \|\operatorname{curl}(\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h)\|_K^2 &\sim \|\operatorname{curl}(\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h) b_K^{1/2}\|_K^2 \\ &= \int_K [\rho g \mathbf{K}^{-1}(\mathbf{u}_p - \mathbf{u}_{ph}) + \nabla(\phi - \phi_h)] \cdot \operatorname{curl} \mathbf{W}_K \\ &\leq \left(\|\rho g \mathbf{K}^{-1}(\mathbf{u}_p - \mathbf{u}_{ph})\|_K + \|\nabla(\phi - \phi_h)\|_K \right) \cdot \|\operatorname{curl} \mathbf{W}_K\|_K \\ &\leq \left(\|\rho g \mathbf{K}^{-1}(\mathbf{e}_p)\|_K + \|\nabla(\boldsymbol{\lambda}_\phi)\|_K \right) \cdot \|\mathbf{W}_K\|_K. \end{aligned} \quad (69)$$

Again, the inverse inequality (49) allows to get

$$\|\operatorname{curl}(\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h)\|_K \lesssim \|(\mathbf{e}_f, \mathbf{e}_p)\|_{h, \tilde{w}_K} + \|(\boldsymbol{\varepsilon}_p, \boldsymbol{\lambda}_\phi)\|_{\tilde{w}_K}. \quad (70)$$

- (v) Interface elements on Γ : we fix an edge E included in Γ and for a constant r_E fixed later on a unit vector N , we consider

$$\mathbf{W}_E = r_E b_E N, \quad (71)$$

that clearly belongs to \mathbf{H} . Hence, by residual equation (57)–(59), we obtain with $\mathbf{V}_h = \mathbf{0}$,

$$\mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{W}_E) = \left(\mathbf{R}_{K_f}^f(\mathbf{U}_h), \mathbf{W}_E \right)_{K_f} + \left(\mathbf{R}_{K_p}^p(\mathbf{U}_h), \mathbf{W}_E \right)_{K_p}, \quad (72)$$

where K_f (resp., K_p) is the unique triangle/tetrahedron

included in $\bar{\Omega}_f$ (resp., $\bar{\Omega}_p$) having E as edge/face, and

$$\begin{aligned} (\mathbf{R}_{K_f}^f(\mathbf{U}_h), \mathbf{W}_E)_{K_f} &= (\mathbf{f}_f + 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h, \mathbf{W}_E)_{K_f} - (q, \nabla\cdot\mathbf{u}_{fh})_{K_f} \\ &- \left[\sum_{E \in \mathcal{E}_h(\partial K_f \cap \Omega)} \sum_{j=1}^{d-1} \left(2\nu\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \boldsymbol{\tau}_j + \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j}} \mathbf{u}_{fh} \cdot \boldsymbol{\tau}_j, \mathbf{W}_E \cdot \boldsymbol{\tau}_j \right)_E \right] \\ &+ \left[\sum_{E \in \mathcal{E}_h(\partial K_f \cap \Omega)} (p_h - 2\nu\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \mathbf{n}_f - \rho g \phi_h, \mathbf{W}_E \cdot \mathbf{n}_f)_E \right], \\ (\mathbf{R}_K^p(\mathbf{U}_h), \mathbf{W}_E)_{K_p} &= \left(\text{curl}(\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h), \beta_p \right)_{K_p} \\ &+ \left(\rho g (f_p - \nabla \cdot \mathbf{u}_{ph}), \psi \right)_{K_p} \\ &- \sum_{E \in \mathcal{E}_h(\partial K_p \cap \Omega)} (\rho g \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h) \times \mathbf{n}_E, \psi)_E \\ &+ \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega_p)} ([\rho g \phi_h \mathbf{n}_E]_E, \beta_p)_E \\ &- \sum_{E \in \mathcal{E}_h(\partial K \cap \Omega)} ([\rho g \phi_h \mathbf{n}_E]_E, [\mathbf{W}]_E)_E. \end{aligned} \quad (73)$$

Taken $\mathbf{W}_E = \mathbf{0}$ in K_p , $q = 0$ in Ω and for each $j = 1, \dots, d - 1$, $\mathbf{r}_E = 2\nu\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \boldsymbol{\tau}_j + (\alpha/\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j})\mathbf{u}_{fh} \cdot \boldsymbol{\tau}_j$ with $\mathbf{N} = \boldsymbol{\tau}_j$. We have

$$\begin{aligned} \mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{W}_E) &= (\mathbf{f}_f + 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h, \mathbf{W}_E)_{K_f} \\ &- \left(2\nu\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \boldsymbol{\tau}_j + \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j}} \mathbf{u}_{fh} \cdot \boldsymbol{\tau}_j, \mathbf{W}_E \cdot \boldsymbol{\tau}_j \right)_E \\ &= (\mathbf{f}_f + 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h, \mathbf{W}_E)_{K_f} - \|\mathbf{r}_E b_E^{1/2}\|_E^2. \end{aligned} \quad (74)$$

Hence,

$$\begin{aligned} \|\mathbf{r}_E\|_E^2 \sim \|\mathbf{r}_E b_E^{1/2}\|_E^2 &= (\mathbf{f}_f + 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h, \mathbf{W}_E)_{K_f} - \mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{W}_E) \\ &= (\mathbf{f}_{fh} + 2\nu\nabla\cdot\mathbb{D}(\mathbf{u}_{fh}) - \nabla p_h, \mathbf{W}_E)_{K_f} - \int_{K_f} 2\nu\mathbb{D}(\mathbf{e}_f) : \mathbb{D}(\mathbf{W}_E) \\ &+ \int_{K_f} \varepsilon_p \text{div } \mathbf{W}_E + (\mathbf{f}_f - \mathbf{f}_{fh}, \mathbf{W}_E)_{K_f} \\ &- \sum_{j=1}^{d-1} \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j}} (\mathbf{e}_f \cdot \boldsymbol{\tau}_j, \mathbf{W}_E \cdot \boldsymbol{\tau}_j)_E. \end{aligned} \quad (75)$$

Inverse inequalities (50) and (51) and Cauchy-Schwarz inequality lead to

$$\begin{aligned} h_E^{1/2} \left\| 2\nu\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \boldsymbol{\tau}_j + \frac{\alpha}{\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j}} \mathbf{u}_{fh} \cdot \boldsymbol{\tau}_j \right\|_E &\leq \|(\mathbf{e}_f, \mathbf{e}_p)\|_{h, w_E} \\ &+ \|(\varepsilon_p, \lambda_\phi)\|_{w_E} + \sum_{K \subset w_E} \zeta_K, \end{aligned} \quad (76)$$

with $w_E = K_f \cup K_p$.

Taken $\mathbf{W}_E = \mathbf{0}$ in K_f , $q = 0$ in Ω and for each $j = 1, \dots, d - 1$, $\mathbf{r}_E = 2\nu\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \boldsymbol{\tau}_j + (\alpha/\sqrt{\boldsymbol{\tau}_j \cdot \mathbf{K} \cdot \boldsymbol{\tau}_j})\mathbf{u}_{fh} \cdot \boldsymbol{\tau}_j$ with $\mathbf{N} = \mathbf{n}_f$. As before the identities (72)–(74) and the inverse inequalities (50) and (51) lead to

$$\begin{aligned} h_E^{1/2} \|p_h - 2\nu\mathbf{n}_f \cdot \mathbb{D}(\mathbf{u}_{fh}) \cdot \mathbf{n}_f - \rho g \phi_h\|_E &\leq \|(\mathbf{e}_f, \mathbf{e}_p)\|_{h, w_E} \\ &+ \|(\varepsilon_p, \lambda_\phi)\|_{w_E} + \sum_{K \subset w_E} \zeta_K. \end{aligned} \quad (77)$$

(vi) Piezometric head jump in Ω_p : for each edge/face $E \in \mathcal{E}_h(\Omega_p)$, we consider $w_E = T_1 \cup T_2$. As $[\rho g \phi_h \mathbf{n}_p]_E \in [\mathbb{P}^1(E)]^d$, we set

$$\mathbf{W}_E := [\rho g \phi_h \mathbf{n}_p]_E b_E \in [H_0^1(w_E)]^d. \quad (78)$$

Using the residual equation (57)–(59), we obtain with $\mathbf{V}_h = \mathbf{0}$ and $\mathbf{V} = (0, 0, \beta_p, 0)$ where $\beta_p = \mathbf{W}_E$:

$$\mathcal{L}_h(\mathbf{U} - \mathbf{U}_h, \mathbf{V}) = ([\rho g \phi_h \mathbf{n}_E]_E, \mathbf{W}_E)_E. \quad (79)$$

Therefore, Cauchy-Schwarz inequality and inverse inequalities (51) and (52) lead to

$$\begin{aligned} \|[\rho g \phi_h \mathbf{n}_E]_E\|_E^2 &\sim \|[\rho g \phi_h \mathbf{n}_E]_E b_E^{1/2}\|_E^2 = \int_E [\rho g \phi_h \mathbf{n}_E]_E \cdot \mathbf{W}_E \\ &= \sum_{i=1}^2 \rho g (\phi - \phi_h, \nabla \cdot \mathbf{W}_E)_{K_i} \leq \sum_{i=1}^2 \|\rho g \lambda_\phi\|_{K_i} \|\nabla \cdot \mathbf{W}_E\|_{K_i} \\ &\leq \sum_{i=1}^2 \|\lambda_\phi\|_{K_i} h_E^{-1} \|[\rho g \phi_h \mathbf{n}_E]_E\|_E. \end{aligned} \quad (80)$$

In additionally, since by regularity Theorem 4, the jump of \mathbf{u} is zero through all the edges of Ω ; hence, we clearly have $\mathbf{J}_\Gamma(\mathbf{U} - \mathbf{U}_h, \mathbf{U} - \mathbf{U}_h) = \mathbf{J}_\Gamma(\mathbf{U}_h, \mathbf{U}_h)$. Thus,

$$\begin{aligned} h_E^{1/2} \left\| [\rho g \phi_h \mathbf{n}_p]_E \right\|_E + \frac{\delta h_E^{1/2}}{h} \left\| [(\mathbf{u}_{fh} - \mathbf{u}_{ph}) \cdot \mathbf{n}_f]_E \right\|_E &\leq \|(\mathbf{e}_f, \mathbf{e}_p)\|_{h, w_E} + \|(\varepsilon_p, \lambda_\phi)\|_{w_E}. \end{aligned} \quad (81)$$

(vii) Finally, for $E \in \mathcal{E}_h(\partial K \cap \Omega_p)$, we have

$$\begin{aligned} \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h &= \mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h - \mathbf{K}^{-1} \mathbf{u}_p - \nabla \phi \\ &= -[\mathbf{K}^{-1}(\mathbf{u}_p - \mathbf{u}_{ph}) + \nabla(\phi - \phi_h)]. \end{aligned} \quad (82)$$

Thus,

$$\left\| \left[(\mathbf{K}^{-1} \mathbf{u}_{ph} + \nabla \phi_h) \times \mathbf{n}_p \right]_E \right\|_E \lesssim \|(\mathbf{e}_f, \mathbf{e}_p)\|_{h,K} + \|(\varepsilon_p, \lambda_\phi)\|_K. \tag{83}$$

The estimates (66), (67), (70), (76), (77), (81), and (83) provide the desired local lower error bound.

4. Summary

In this paper, we have discussed a posteriori error estimates for a finite element approximation of the Stokes-Darcy system. A residual type a posteriori error estimator is provided that is both reliable and efficient. Many issues remain to be addressed in this area, let us mention other types of a posteriori error estimators or implementation and convergence analysis of adaptive finite element methods.

5. Nomenclature

- (i) $\Omega \subset \mathbb{R}^d, d \in \{2, 3\}$ bounded domain
- (ii) Ω_p : the porous medium domain
- (iii) Ω_f : the fluid region
- (iv) $\Gamma = \partial\Omega_f \cap \partial\Omega_p$
- (v) $\Omega_f = \Omega \setminus \{\Omega_p \cup \Gamma\}$
- (vi) $\Gamma_l = \partial\Omega_l \setminus \Gamma, l = f, p$
- (vii) \mathbf{n}_f (resp., \mathbf{n}_p) the unit outward normal vector along $\partial\Omega_f$ (resp., $\partial\Omega_p$)
- (viii) \mathbf{u} : the fluid velocity
- (ix) p : the fluid pressure
- (x) In 2D, the curl of a scalar function w is given as usual by

$$\text{curl } w := (\partial w / \partial x_2, -\partial w / \partial x_1)^T \tag{84}$$

- (xi) In 3D, the curl of a vector function $\mathbf{w} = (w_1, w_2, w_3)$ is given as usual by $\text{curl } \mathbf{w} := \nabla \times \mathbf{w}$, namely,

$$\text{curl } \mathbf{w} := ((\partial w_3 / \partial x_2) - (\partial w_2 / \partial x_3), (\partial w_1 / \partial x_3) - (\partial w_3 / \partial x_1), (\partial w_2 / \partial x_1) - (\partial w_1 / \partial x_2)) \tag{85}$$

- (xii) \mathbb{P}^k : the space of polynomials of total degree not larger than k
- (xiii) \mathcal{T}_h : triangulation of Ω

- (xiv) \mathcal{T}_h^l : the corresponding induced triangulation of $\Omega_l, l \in \{f, p\}$
- (xv) For any $K \in \mathcal{T}_h, h_K$ is the diameter of K and $\rho_K = 2r_K$ is the diameter of the largest ball inscribed into K
- (xvi) $h := \max_{K \in \mathcal{T}_h} h_K$ and $\sigma_h := \max_{K \in \mathcal{T}_h} (h_K / \rho_K)$
- (xvii) \mathcal{E}_h : the set of all the edges or faces of the triangulation
- (xviii) $\mathcal{E}(K)$: the set of all the edges ($N = 2$) or faces ($N = 3$) of a element K
- (xix) $\mathcal{E}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{E}(K)$
- (xx) $\mathcal{N}(K)$: the set of all the vertices of a element K
- (xxi) $\mathcal{N}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{N}(K)$
- (xxii) For $\mathcal{A} \subset \bar{\Omega}, \mathcal{E}_h(\mathcal{A}) := \{E \in \mathcal{E}_h : E \subset \mathcal{A}\}$
- (xxiii) For $E \in \mathcal{E}_h$, we associate a unit vector \mathbf{n}_E such that \mathbf{n}_E is orthogonal to E and equals to the unit exterior normal vector to $\partial\Omega$
- (xxiv) For $E \in \mathcal{E}_h, [\phi]_E$ is the jump across E in the direction of \mathbf{n}_E
- (xxv) In order to avoid excessive use of constants, the abbreviations $x \lesssim y$ and $x \sim y$ stand for $x \leq cy$ and $c_1x \leq y \leq c_2x$, respectively, with positive constants independent of x, y , or \mathcal{T}_h

Data Availability

There are no data underlying the findings in this paper to be shared.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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