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A posteriori error estimation for the Stokes–Darcy coupled problem on anisotropic discretization

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This paper presents an a posteriori error analysis for the stationary Stokes–Darcy coupled problem approximated by finite element methods on anisotropic meshes in $\mathbb{R}^N$, $N = 2$ or 3. Korn's inequality for piecewise linear vector fields on anisotropic meshes is established and is applied to non-conforming finite element method. Then the existence and uniqueness of the approximation solution are deduced for non-conforming case. With the obtained finite element solutions, the error estimators are constructed and based on the residual of model equations plus the stabilization terms. *The lower error bound* is proved by means of bubble functions and the corresponding anisotropic inverse inequalities. In order to prove the *upper error bound*, it is vital that an anisotropic mesh corresponds to the anisotropic function under consideration. To measure this correspondence, a so-called *matching function* is defined, and its discussion shows it to be useful tool. With its help, the *upper error bound* is shown by means of the corresponding anisotropic interpolation estimates and a special Helmholtz decomposition in both media. Copyright © 2016 John Wiley & Sons, Ltd.

Keywords: error estimator; anisotropic solution; stretched elements; Stokes–Darcy coupled problem

1. Introduction

The transport of substances back and forth between surface water and groundwater is a very important problem. We study herein the mathematical model of this setting consisting of the Stokes equations in the fluid region coupled with the Darcy equations in the porous medium, coupled across the interface by the Beavers–Joseph–Saffman conditions (cf. Section 2.1). The system is difficult to approximate because the Darcy and Stokes solutions have very different regularity properties, and, more importantly, the tangential velocity may be discontinuous on the Stokes–Darcy interface.

In certain situations, the solution of the Stokes–Darcy coupled problem exhibits strong directional features. Indeed, it is well known that an internal layer appears at the interface Γ_I (Figure 1) as the permeability tensor degenerates [1, 2]. In that case, anisotropic meshes have to be used in this layer [3].

A posteriori error estimations have been well established for the coupled Stokes–Darcy problem on isotropic meshes, mainly for 2*D* domains [4–8]. To our best knowledge, there is no a posteriori error estimation for the coupled Stokes–Darcy problem valid for anisotropic discretization with finite element methods. Here, we develop such a posteriori error analysis for anisotropic finite elements satisfying minimal assumptions. These assumptions may be summarized as follows: the scheme is stable (not essential but recommended in numerical applications), the velocity space is large enough to contain the conforming $\mathbb{P}^1$ piecewise space and satisfies a Crouzeix–Raviart property (see below for the details). These three properties are satisfied by Crouzeix–Raviart non-conforming finite element on anisotropic meshes. Note that the second property facilitates the process of obtaining reliability result. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. We further prove that our a posteriori error estimator is both reliable and efficient. These main results are summarized in Theorems 6.3 and 6.4.

The outline of the paper is as follows. Some preliminaries and notation are given in Section 2. The discretization (as a mixed formulation) and the general framework with minimal conditions on the mesh and on the element pairs are given in Section 3. Section 4 is devoted to analytical tools. The specific anisotropic interpolation estimates are particularly important. It turns out that an anisotropic mesh should be well aligned with the anisotropic solution. This demand seems to be an inherent feature of anisotropic discretizations

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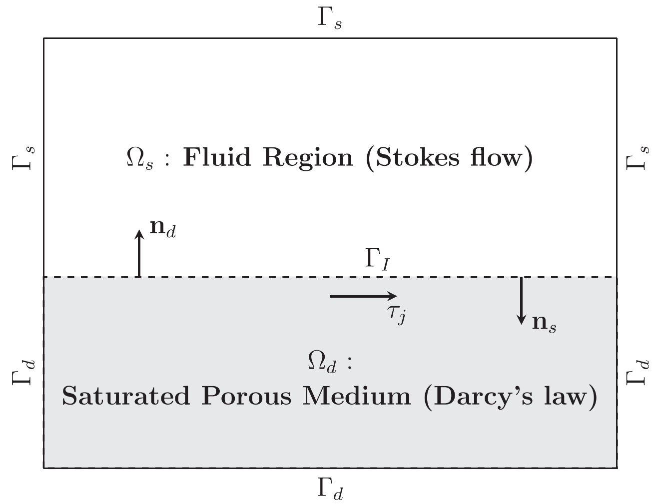


Figure 1. A sketch of the geometry of the problem.

and error estimates. In Section 5, we derive Korn's inequalities for piecewise linear vector fields with respect to simplicial anisotropic triangulations of Ω. Indeed, for the choice of problem (16) defined in the succeeding text, elements satisfying the discrete Korn's inequality on isotropic meshes (studied by Suzanne Brenner in [9]) have to be used [8]. To our best knowledge, the non-conforming elements for the stability of (16) on anisotropic meshes are not yet proved [3, Page 11]. We present in this same section an example of element pair that is covered by our analysis. The actual error bounds are given in Section 6. For the upper error bound, we additionally distinguish between conforming and non-conforming discretization. While all these considerations are made for anisotropic meshes, we simplify the results for the case of an isotropic discretization in Section 6.4 because even in that case, we obtain new results for conforming case. Our exposition treats the two-dimensional case ($N=2$) as well as the three-dimensional case ($N=3$). We then offer our conclusion and the further works in Section 7.

2. Preliminaries and notation

2.1. Model problem

We consider the model of a flow in a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3), consisting of a porous medium domain Ω_d, where the flow is a Darcy flow and an open region $\Omega_s = \Omega \setminus \bar{\Omega}_d$, where the flow is governed by the Stokes equations. The two regions are separated by an interface $\Gamma_I = \partial\Omega_d \cap \partial\Omega_s$. Let $\Gamma_l = \partial\Omega_l \setminus \Gamma_I$, $l = s, d$. Each interface and boundary is assumed to be polygonal ($N=2$) or polyhedral ($N=3$). We denote by $\mathbf{n}_s$ (resp. $\mathbf{n}_d$) the unit outward normal vector along $\partial\Omega_s$ (resp. $\partial\Omega_d$). Note that on the interface Γ_I, we have $\mathbf{n}_s = -\mathbf{n}_d$. Figure 1 gives a schematic representation of the geometry.

For any function v defined in Ω, because its restriction to Ω_s or to Ω_d could play a different mathematical roles (for instance their traces on Γ_I), we will set $v_s = v_{|\Omega_s}$ and $v_d = v_{|\Omega_d}$.

In Ω, we denote by $\mathbf{u}$ the fluid velocity and by p the pressure. The motion of the fluid in Ω_s is described by the Stokes equations

$$\begin{cases} -2\mu \operatorname{div} \mathbf{D}(\mathbf{u}) + \nabla p = \mathbf{f} & \text{in } \Omega_s, \\ \operatorname{div} \mathbf{u} = g & \text{in } \Omega_s, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_s, \end{cases} \tag{1}$$

while in the porous medium Ω_d, by Darcy's law

$$\begin{cases} \mu \mathbf{K}^{-1}\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega_d, \\ \operatorname{div} \mathbf{u} = g & \text{in } \Omega_d, \\ \mathbf{u} \cdot \mathbf{n}_d = 0 & \text{on } \Gamma_d. \end{cases} \tag{2}$$

Here, $\mu > 0$ is the fluid viscosity, $\mathbf{D}$ the deformation rate tensor defined by

$$\mathbf{D}(\psi)_{ij} := \frac{1}{2}\left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right), 1 \leqslant i, j \leqslant N,$$

and $\mathbf{K}$ a symmetric and uniformly positive definite tensor representing the rock permeability and satisfying, for some constants $0 < K_* \leqslant K^* < +\infty$,

$$K_* \xi^T \xi \leqslant \xi^T \mathbf{K}(x) \xi \leqslant K^* \xi^T \xi, \forall x \in \Omega_d, \xi \in \mathbb{R}^N. \tag{3}$$

$\mathbf{f} \in [L^2(\Omega)]^N$ is a term related to body forces and $g \in L^2(\Omega)$ a source or sink term satisfying the compatibility condition

$$\int_\Omega g(x)dx = 0.$$

Finally, we consider the following interface conditions on Γ_I:

$$\mathbf{u}_s \cdot \mathbf{n}_s + \mathbf{u}_d \cdot \mathbf{n}_d = 0, \tag{4}$$

$$p_s - 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_s) \cdot \mathbf{n}_s = p_d, \tag{5}$$

$$\frac{\sqrt{\kappa_j}}{\alpha_1} 2\mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_s) \cdot \tau_j = -\mathbf{u}_s \cdot \tau_j, j = 1, \dots, N-1. \tag{6}$$

Here, Eq. (4) represents mass conservation, Eq. (5) the balance of normal forces and Eq. (6) the Beavers–Joseph–Saffman conditions. Moreover, $\{\tau_j\}_{j=1,\dots,N-1}$ denotes an orthonormal system of tangent vectors on Γ_I, $\kappa_j = \tau_j \cdot \mathbf{K} \cdot \tau_j$, and α_1 is a parameter determined by experimental evidence.

Equations (1)–(6) consist of the model of the coupled Stokes and Darcy flows problem that we will study in the succeeding texts.

2.2. Weak formulation

We begin this subsection by introducing some useful notations. If W is a bounded domain of $\mathbb{R}^N$ and m is a nonnegative integer, the Sobolev space $H^m(W) = W^{m,2}(W)$ is defined in the usual way with the usual norm $\| \cdot \|_{m,W}$ and semi-norm $|\cdot|_{m,W}$. In particular, $H^0(W) = L^2(W)$ and we write $\| \cdot \|_W$ for $\| \cdot \|_{0,W}$. Similarly, we denote by $(\cdot,\cdot)_W$ the $L^2(W)$ $[L^2(W)]^N$ or $[L^2(W)]^{N\times N}$ inner product. For shortness, if W is equal to Ω, we will drop the index Ω, while for any $m \geq 0$, $\| \cdot \|_{m,l} = \| \cdot \|_{m,\Omega_l}$, $| \cdot |_{m,l} = | \cdot |_{m,\Omega_l}$ and $(.,.)_l = (\cdot,\cdot)_{\Omega_l}$, for $l = s, d$. The space $H_0^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Let $[H^m(\Omega)]^N$ be the space of vector-valued functions $\mathbf{v} = (v_1, \dots, v_N)$ with components v_i in $H^m(\Omega)$. The norm and the seminorm on $[H^m(\Omega)]^N$ are given by

$$\| \mathbf{v} \|_{m,\Omega} := \left(\sum_{i=0}^N \| v_i \|_{m,\Omega}^2 \right)^{1/2} \text{ and } |\mathbf{v}|_{m,\Omega} := \left(\sum_{i=0}^N |v_i|_{m,\Omega}^2 \right)^{1/2}. \tag{7}$$

For a connected open subset of the boundary $\Gamma \subset \partial\Omega_s \cup \partial\Omega_d$, we write $\langle .,. \rangle_\Gamma$ for the $L^2(\Gamma)$ inner product (or duality pairing), that is, for scalar valued functions $\lambda, \eta \in L^2(\Gamma)$, one defines

$$\langle \lambda, \eta \rangle_\Gamma := \int_\Gamma \lambda(s)\eta(s)ds \tag{8}$$

We also define the special vector-valued functions space

$$\mathbf{H}(\mathrm{div}\,, \Omega) := \left\{ \mathbf{v} \in [L^2(\Omega)]^N : \mathrm{div}\, \mathbf{v} \in L^2(\Omega) \right\} \tag{9}$$

To give the variational formulation of our coupled problem, we define the following two spaces for the velocity and the pressure:

$$\mathbf{H} := \left\{ \mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega) : \mathbf{v}_s \in [H^1(\Omega_s)]^N, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_s \text{ and } \mathbf{v} \cdot \mathbf{n}_d = 0 \text{ on } \Gamma_d \right\}$$

equipped with the norm

$$\| \mathbf{v} \|_{\mathbf{H}} := \left(|\mathbf{v}|_{1,s}^2 + \| \mathbf{v} \|_d^2 + \| \mathrm{div}\, \mathbf{v} \|_d^2 \right)^{1/2}, \tag{10}$$

and

$$Q = L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q(x)dx = 0 \right\}. \tag{11}$$

Note that the vector-valued functions in $\mathbf{H}$ have (weakly) continuous normal components on Γ_I (consequence of theorem I.2.5 of [10, p. 27]).

Let us further introduce two bilinear forms

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) := 2\mu(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_s + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} \left\langle \mathbf{u}_s \cdot \tau_j, \mathbf{v}_s \cdot \tau_j \right\rangle_{\Gamma_I} + \mu \left(\mathbf{K}^{-1}\mathbf{u}, \mathbf{v} \right)_d,$$

$$\mathbf{b}(\mathbf{v}, q) := -\int_\Omega q \,\mathrm{div}\, \mathbf{v}$$

and two linear forms

$$L(\mathbf{v}) := (\mathbf{f}, \mathbf{v})_\Omega \quad \text{and} \quad G(q) := -(g, q)_\Omega.$$

The weak formulation of the coupled problems (1)–(6) can be stated as follows [11]: find $(\mathbf{u}, p) \in \mathbf{H} \times Q$ such that

$$\begin{cases} \mathbf{a}(\mathbf{u},\mathbf{v}) + \mathbf{b}(\mathbf{v},p) &= L(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}, \\ \mathbf{b}(\mathbf{u},q) &= G(q), \quad \forall q \in Q. \end{cases} \tag{12}$$

Note that if g is of mean zero, (12) directly implies that (1), (2) and (4) hold (the differential equations being understood in the distributional sense), while the interface conditions (5) and (6) are imposed in a weak sense.

This problem has a unique solution as proved in [11, section 3].

Theorem 2.1
If $\mathbf{f} \in [L^2(\Omega)]^N$ and $g \in L^2_0(\Omega)$, there exists a unique solution $(\mathbf{u}, p) \in \mathbf{H} \times Q$ to the problem (12).

We end this section with some notation. In 2*D*, the curl of a scalar function w is given as usual by $\text{curl}w := (\frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1})^\top$ while in 3*D*, the curl of a vector function $\mathbf{w}$ is given as usual by $\text{curl}\mathbf{w} := \nabla \times \mathbf{w}$. Finally, let $\mathbb{P}^k$ be the space of polynomials of total degree not larger than k. In order to avoid excessive use of constants, the abbreviations $x \lesssim y$ and $x \sim y$ stand for $x \leqslant cy$ and $c_1 x \leqslant y \leqslant c_2 x$, respectively, with positive constants independent of x, y or $\mathcal{T}_h$.

3. Anisotropic discretization

The first two sections introduce general aspects of the discretization, for example, the mixed finite element formulation. Section 3.3 is then devoted to the introduction of anisotropic quantities. The general framework (mesh and general assumptions) will be discussed in Section 3.5. As it turns out, the assumptions on the mesh that are introduced for anisotropic elements are quite weak are standard in anisotropic a posteriori error analysis and are similar to the one for isotropic elements. The general assumptions are minimal in order to make a unified analysis.

3.1. Discretization of the domain Ω

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of triangulations of Ω with nondegenerate elements (i.e. triangles for $N = 2$ and tetrahedrons for $N = 3$). We assume that the triangulation is conform with respect to the partition of Ω into Ω_s and Ω_d, namely each $T \in \mathcal{T}_h$ is either in Ω_s or in Ω_d. Let $\mathcal{T}_h^s$ and $\mathcal{T}_h^d$ be the corresponding induced triangulations of Ω_s and Ω_d. For any $T \in \mathcal{T}_h$, we denote by $\mathcal{E}(T)$ (resp. $\mathcal{N}(T)$) the set of its edges ($N = 2$) or faces ($N = 3$) (resp. vertices) and set $\mathcal{E}_h = \bigcup_{T\in\mathcal{T}_h} \mathcal{E}(T)$, $\mathcal{N}_h = \bigcup_{T\in\mathcal{T}_h} \mathcal{N}(T)$. For $\mathcal{A} \subset \bar{\Omega}$, we define

$$\mathcal{E}_h(\mathcal{A}) := \{E \in \mathcal{E}_h : E \subset \mathcal{A}\} \text{ and } \mathcal{N}_h(\mathcal{A}) := \{\mathbf{x} \in \mathcal{N}_h : \mathbf{x} \in \mathcal{A}\}.$$

Notice that $\mathcal{E}_h$ can be split up in the form

$$\mathcal{E}_h = \mathcal{E}_h(\Omega_s^+) \cup \mathcal{E}_h(\Omega_d) \cup \mathcal{E}_h(\partial\Omega_d), \tag{13}$$

where $\Omega_s^+ = \Omega_s \cup \Gamma_s$. Note that $\mathcal{E}_h(\Gamma_I)$ is included in $\mathcal{E}_h(\partial\Omega_d)$.

The measure of an element or edge/face is denoted by $|T| := \text{meas}_N(T)$ and $|E| := \text{meas}_{N-1}(E)$, respectively.

For an edge E of a 2*D* element T, introduce the outer normal vector by $\mathbf{n} = (n_x, n_y)^\top$. Similarly, for a face E of a 3*D* element T, set $\mathbf{n} = (n_x, n_y, n_z)^\top$. Furthermore, for each face E, we fix one of the two normal vectors and denote it by $\mathbf{n}_E$. In the 2*D* case, introduce additionally the tangent vector $\mathbf{t} = \mathbf{n}^\top := (-n_y, n_x)^\top$ such that it is oriented positively (with respect to T). Similarly, set $\mathbf{t}_E := \mathbf{n}_E^\top$.

For any $E \in \mathcal{E}_h$ and any piecewise continuous function φ, we denote by $[\varphi]_E$ its jump across E in the direction of $\mathbf{n}_E$:

$$[\varphi]_E(x) := \begin{cases} \lim\limits_{t\to 0+} \varphi(x + t\mathbf{n}_E) - \lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for an interior edge/face } E, \\ -\lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for a boundary edge/face } E. \end{cases}$$

Note that the sign of $[\varphi]_E$ depends on the orientation of $\mathbf{n}_E$. However, terms such as a gradient jump $[\nabla\varphi\mathbf{n}_E]_E$ are independent of this orientation.

Furthermore, one requires local subdomains (also known as patches). As usual, let w_T be the union of all elements having a common face with T. Similarly, let w_E be the union of both elements having E as face (with appropriate modifications for a boundary face). By $w_{\mathbf{x}}$, we denote the union of all elements having $\mathbf{x}$ as node.

If we have $\psi \in [H^1(T)]^N$ for all T in $\mathcal{T}_h$, then we can define a broken deformation norm on a subset w of Ω by

$$\| \mathbf{D}_{\mathcal{T}_h}(\psi)\|_w^2 := \sum_{T\subset w} \| \mathbf{D}(\psi)\|_T^2. \tag{14}$$

Later on, we specify additional, mild mesh assumptions that are partially due to the anisotropic discretization.

3.2. Discrete mixed formulation

We assume a given velocity (resp. pressure) approximation space $\mathbf{H}_h$ (resp. Q_h) made of polynomials on each element T of the triangulation $\mathcal{T}_h$ and such that $Q_h \subset Q$ (but not necessary $\mathbf{H}_h \subset \mathbf{H}$). A precise description of the properties that these approximation spaces $\mathbf{H}_h$ and Q_h have to satisfy is given in Section 3.5.

Let us introduce the discrete divergence operator $\operatorname{div}_h \in \mathcal{L}(\mathbf{H}_h; Q_h) \cap \mathcal{L}(\mathbf{H}; Q)$ by

$$(\operatorname{div}_h \mathbf{v}_h)_{|T} = \operatorname{div}(\mathbf{v}_{h|T}), \forall T \in \mathcal{T}_h. \tag{15}$$

Because the velocity approximation space $\mathbf{H}_h$ may not be included in the velocity space $\mathbf{H}$, we define the approximate solution by using the weaker bilinear forms $\mathbf{a}_h(.,.)$ and $\mathbf{b}_h(.,.)$:

$$\begin{aligned}\mathbf{a}_h(\mathbf{u}, \mathbf{v}) := 2\mu \sum_{T \in \mathcal{T}_h^s} (\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_T + \sum_{j=1}^{N-1} \frac{\mu \alpha_1}{\sqrt{\kappa_j}} \langle \mathbf{u}_s \cdot \tau_j, \mathbf{v}_s \cdot \tau_j \rangle_{\Gamma_I} \\ + \mu(\mathbf{K}^{-1}\mathbf{u}, \mathbf{v})_{\Omega_d}, \forall \mathbf{u}, \mathbf{v} \in \mathbf{H} + \mathbf{H}_h\end{aligned}$$

and

$$\mathbf{b}_h(\mathbf{v}, q) := -(q, \operatorname{div}_h \mathbf{v})_\Omega, \forall \mathbf{v} \in \mathbf{H} + \mathbf{H}_h, \forall q \in Q_h.$$

Then the finite element discretization of (12) is to find $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times Q_h$ such that

$$\begin{cases} \mathbf{a}_h(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{b}_h(\mathbf{v}_h, p_h) + \mathbf{J}(\mathbf{u}_h, \mathbf{v}_h) &= L(\mathbf{v}_h), \forall \mathbf{v}_h \in \mathbf{H}_h, \\ \mathbf{b}_h(\mathbf{u}_h, q_h) &= G(q_h), \forall q_h \in Q_h. \end{cases} \tag{16}$$

This is the natural discretization of the weak formulation (12) except that the penalizing term $\mathbf{J}(\mathbf{u}_h, \mathbf{v}_h)$ is added (only non-conforming case). These penalizing term will be specified later in Section 5. The space Q_h is equipped with the norm $\| \cdot \|$ while the norm $\| \cdot \|_h$ on $\mathbf{H}_h$ will be also specified later in Section 5 for non-conforming case.

For $\mathbf{v} \in \mathbf{H} \cap \mathbf{H}_h$ and for $q \in Q_h$, we can subtract (16) to (12) to obtain the Galerkin orthogonality relation:

$$\begin{aligned}2\mu\, (\mathbf{D}_{\mathcal{T}_h}(\mathbf{e}), \mathbf{D}(\mathbf{v}))_{\Omega_s} + \sum_{j=1}^{N-1} \frac{\mu \alpha_1}{\sqrt{\kappa_j}} \langle \mathbf{e}_s \cdot \tau_j, \mathbf{v}_s \cdot \tau_j \rangle_{\Gamma_I} + \mu(\mathbf{K}^{-1}\mathbf{e}, \mathbf{v})_d - (\varepsilon, \operatorname{div}_h \mathbf{v})_\Omega \\ - (q, \operatorname{div}_h \mathbf{e})_\Omega - \mathbf{J}(\mathbf{u}_h, \mathbf{v}) = 0, \forall \mathbf{v} \in \mathbf{H} \cap \mathbf{H}_h, \forall q \in Q_h,\end{aligned} \tag{17}$$

where here and in the succeeding text, the errors in the velocity and in the pressure are respectively defined by

$$\mathbf{e} = \mathbf{u} - \mathbf{u}_h \text{ and } \varepsilon = p - p_h,$$

and the broken deformation $\mathbf{D}_{\mathcal{T}_h}(\mathbf{e})$ of the velocity error means its elementwise deformation, namely

$$(\mathbf{D}_{\mathcal{T}_h}(\mathbf{e}))_{|T} = \mathbf{D}(\mathbf{e}_{|T}). \tag{18}$$

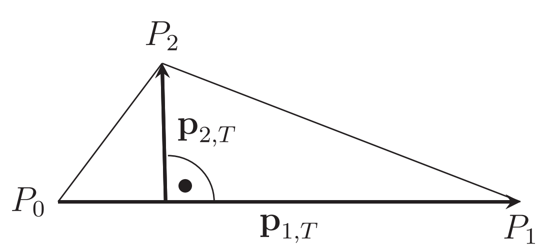


Figure 2. Notation of triangle T.

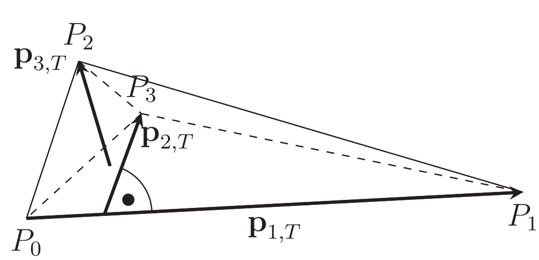


Figure 3. Notation of tetrahedron T.

3.3. Anisotropic quantities

For an element T, we define 2 or 3 anisotropy vectors $\mathbf{p}_{i,T}$, $i = 1, \ldots, N$, that reflect the main anisotropy directions of that element. These anisotropy vectors are defined and visualized below as well (Figures 2 and 3).

The anisotropy vectors $\mathbf{p}_{j,T}$ are enumerated such that lengths are decreasing, that is, $|\mathbf{p}_{1,T}| \geq |\mathbf{p}_{2,T}| \geq |\mathbf{p}_{3,T}|$ in the $3D$ case and analogously in $2D$. The anisotropic lengths of an element T are now defined by $h_{j,T} := |\mathbf{p}_{j,T}|$, ($j = 1, \ldots, N$) which implies $h_{j,T} \geq h_{j+1,T}$, $j = 1, \ldots, N-1$. The smallest of these lengths is particularly important; thus, we introduce $h_{\min,T} := h_{N,T} \equiv \min_{i \in \{1,\ldots,N\}} h_{i,T}$. Finally, the anisotropy vectors $\mathbf{p}_{j,T}$ are arranged columnwise to define a matrix

$$\mathbb{C}_T := [\mathbf{p}_{1,T}, \ldots, \mathbf{p}_{N,T}] \in \mathbb{R}^{N,N} \tag{19}$$

Note that $\mathbb{C}_T$ is orthogonal because anisotropy vectors $\mathbf{p}_{j,T}$ are also orthogonal and

$$\mathbb{C}_T^\top \cdot \mathbb{C}_T = \operatorname{diag}\{h_{1,T}^2, \ldots, h_{N,T}^2\}. \tag{20}$$

Furthermore, introduce the height $h_{E,T}$ over an edge/face E of an element T by

$$h_{E,T} := \frac{N|T|}{|E|}. \tag{21}$$

3.4. Relation between anisotropic mesh and anisotropic function

When investigating a residual error estimator for anisotropic meshes, we want to employ the same basic principles as for isotropic meshes. More precisely, a certain kind of interpolation error estimates is to be derived first. With its help, the finite element error is then bounded globally from above.

Proceeding this way, we naturally use different and more technical methods than for isotropic meshes. But even more important, the results of isotropic meshes can not be transferred identically to anisotropic meshes. A certain factor appears now both at the interpolation error estimates (see Lemma 4.3 of Section 4.2) and the finite element error estimate (cf. Theorem 6.4 of Section 6). This factor is related to how good the chosen anisotropic mesh corresponds to the anisotropic function under consideration. Basically, the better this correspondence, the smaller the factor (but always ≥ 1) and the better the estimate (in a meaning that is to be specified later on). The importance of an anisotropic mesh that corresponds to an anisotropic function can be described and interpreted in different ways (Ref. [12, page 33]).

We present now the definition of an alignment measure, which measures the alignment of mesh and function. Recall first that if $N = 3$,

$$H_0(\operatorname{curl}, \Omega_d) = \{\psi \in L^2(\Omega_d)^3 : \operatorname{curl} \psi \in L^2(\Omega_d)^3 \text{ and } \psi \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega_d\}.$$

Definition 3.1 (**Alignment measure** m) (1) Let $v \in H^1(\Omega_l)$ ($l = s$ or d) be an arbitrary non-constant function and $\mathcal{F}^l$ be a family of triangulations of Ω_l. Define the matching function $m_1(.,.) : H^1(\Omega_l) \times \mathcal{F}^l \longrightarrow \mathbb{R}$ by [3, 12, 13]

$$m_1(v, \mathcal{T}_h^l) := \frac{\sqrt{\sum_{T \in \mathcal{T}_h^l} h_{\min,T}^{-2} \parallel \mathbb{C}_T^\top \nabla v \parallel_T^2}}{\parallel \nabla v \parallel_{\Omega_l}} \tag{22}$$

(2) Let $\psi \in [H^1(\Omega_d)]^3 \cap H_0(\operatorname{curl}, \Omega_d)$ be an arbitrary non-constant function and $\mathcal{F}^d$ be a family of triangulations of Ω_d. Define the matching function $m_2(.,.) : [H^1(\Omega_d)]^3 \cap H_0(\operatorname{curl}, \Omega_d) \times \mathcal{F}^d \longrightarrow \mathbb{R}$ by [13]

$$m_2(\psi, \mathcal{T}_h^d) := \frac{\sqrt{\sum_{T \in \mathcal{T}_h^d} h_{\min,T}^{-4} \parallel \nabla(\mathbb{C}_T^\top \psi)\mathbb{C}_T \parallel_T^2}}{\parallel \nabla \psi \parallel_{\Omega_d}} \tag{23}$$

(3) We set now for $\mathbf{v} \in [H^1(\Omega_l)]^N$, ($l = s$ or d):

$$m(\mathbf{v}, \psi, \mathcal{T}_h) := \begin{cases} m_1(\mathbf{v}, \mathcal{T}_h^s) + m_1(\mathbf{v}, \mathcal{T}_h^d) + m_1(\psi, \mathcal{T}_h^d) & \text{if} \quad \psi \in H_0^1(\Omega_d), \\ m_1(\mathbf{v}, \mathcal{T}_h^s) + m_1(\mathbf{v}, \mathcal{T}_h^d) + m_2(\psi, \mathcal{T}_h^d) & \text{if} \quad \psi \in \mathcal{H}, \end{cases} \tag{24}$$

where

$$\mathcal{H} := [H^1(\Omega_d)]^3 \cap H_0(\operatorname{curl}, \Omega_d).$$

Commentary 3.1 (**Alignment measure**)
For a better understanding, we discuss here the behaviour of the alignment measure. The structure of the matrix $\mathbb{C}_T$ from (19) readily gives the crude bounds for each $i = 1$ or 2

$$1 \leqslant m_i(v, \mathcal{T}_h^l) \leqslant \max_{T \in \mathcal{T}_h^l} \frac{h_{\max,T}}{h_{\min,T}}, \; l = s \text{ or } d, \tag{25}$$

where $h_{\max,T} \equiv h_{1,T}$ temporarily denotes the largest element dimension. Although this bound is pratically useless, it implies an interesting by-product for isotropic meshes. There, one concludes $m_i(v, \mathcal{T}_h^l) \sim 1$ and the alignment measure merges with other constants and thus 'vanishes'.

For anisotropic meshes, the term $\mathbb{C}_T^\top \nabla v$ of (22) and the term $\nabla(\mathbb{C}_T^\top \psi)\mathbb{C}_T$ of (23) contain directional derivatives along the main anisotropic directions $\mathbf{p}_{i,T}$ of the element T [because $\mathbb{C}_T = [\mathbf{p}_{1,T}, \ldots, \mathbf{p}_{N,T}]$, see (19)]. Consider first anisotropic elements that are aligned with an anisotropic function v or ψ. Then the long anisotropic element direction $\mathbf{p}_{1,T}$ is associated with a small directional derivative $\mathbf{p}_{1,T}^\top \cdot \nabla F$ where $F \in \{v, \psi\}$. Conversely, the short direction $\mathbf{p}_{3,T}$ has a comparatively large directional derivative $\mathbf{p}_{3,T}^\top \cdot \nabla F$ with $F \in \{v, \psi\}$. Consequently, the numerator and denominator of $m_i(.,.)$ will be balanced, and $m_i(.,.) \sim 1$. Supplementary details are given in [14].

If the anisotropic mesh is not aligned with an anisotropic function v or ψ, then similar considerations imply that the numerator and denominator of $m_i(.,.)$ are no longer balanced, and thus, $m_i(.,.) >> 1$.

Summarizing, the better the anisotropic mesh $\mathcal{T}_h$ is aligned with an anisotropic function v or ψ, the smaller $m_i(.,.)$ will be . This results in sharper error bounds.

3.5. *Requirements on the mesh and the elements*

Mesh assumptions. Let $a_1, \ldots, a_n$ be the nodes of the triangulation $\mathcal{T}_h$. In addition to the usual conformity conditions of the mesh (see [15, chapter 2]) we demand the following assumptions.

- The number of element that contain the node a_j is bounded uniformly.
- The dimensions of adjacent element must not change rapidly, that is,

$$h_{i,T' \sim h_{i,T}} \; \forall T, T' \text{ with } T \cap T' \neq \emptyset, i = 1 \ldots N. \tag{26}$$

Sometimes, it is more convenient to have face-related data instead of element-related data. Hence, for an interior face $E = T_1 \cap T_2$, we introduce

$$h_{\min,E} := \frac{h_{\min,T_1} + h_{\min,T_2}}{2} \text{ and } h_E := \frac{h_{E,T_1} + h_{E,T_2}}{2}.$$

For boundary faces $E \subset \partial T$, simply set $h_{\min,E} := h_{\min,T}$, $h_E := h_{E,T}$. The last assumption from above readily implies

$$h_E \sim h_{E,T_1} \sim h_{E,T_2} \text{ and } h_{\min,E} \sim h_{\min,T_1} \sim h_{\min,T_2}. \tag{27}$$

General assumptions. In our analysis, a Clément type operator $\mathrm{I}_{\mathrm{Cl}}^0$ plays a vital role. Although the precise definition will be postponed until Section 4.2, we briefly describe the image space of this operator. Roughly speaking, its functions are continuous and piecewise linear for triangle and tetrahedra T. From now on, we use the notation

$$V_{\mathrm{Cl}}^0 := \left[\mathrm{Im}\left(\mathrm{I}_{\mathrm{Cl}}^0\right)\right]^N, \tag{28}$$

for the Clément interpolation space. The general conditions are now as follows.

(G1) The velocity space $\mathbf{H}_h$ is large enough such that it contains the Clément interpolation space, that is, $V_{\mathrm{Cl}}^0 \subset [H_0^1(\Omega)]^N \cap \mathbf{H}_h$;
(G2) In order to obtain robust discrete solutions, the element pairs have to be stable (i.e. the coercivity of the bilinear form $\mathbf{a}_h + \mathbf{J}$ in $\mathbf{Z}_h := \{\mathbf{v}_h \in \mathbf{H}_h : \mathbf{b}_h(\mathbf{v}_h, q_h) = 0 \forall q_h \in Q_h\}$ and the discrete inf-sup condition of $\mathbf{b}_h$ on $\mathbf{H}_h \times Q_h$ are satisfied). Note that this assumption is not necessary to prove error bounds, but in particular, it guarantees existence and uniqueness of the discrete solution of (16).

Crouzeix–Raviart property for non-conforming approximation. For non-conforming approximation, we require a variant of the 'Crouzeix–Raviart' property:

$$(\mathbf{CR}) : \begin{cases} ([\mathbf{u}_h]_E, \mathbf{1})_E &= 0 \quad \forall E \in \mathcal{E}_h(\Omega_s^+), \\ ([\mathbf{u}_h \cdot \mathbf{n}_E]_E, 1)_E &= 0 \quad \forall E \in \mathcal{E}_h(\Omega_d) \cup \mathcal{E}_h(\partial\Omega_d). \end{cases} \tag{29}$$

Remark 3.1
The choice of property (29) is natural because the space $\mathbf{H}_h$ approximates only $H(\mathrm{div}, \Omega_d)$ and not $[H^1(\Omega_d)]^N$, while our a posteriori analysis is only valid in this space for non-conforming case.

4. Analytical tools

4.1. Reference element, affine linear transformation and their properties

Let T be an arbitrary but fixed element. Mainly, we will employ an affine linear mapping F_T that will be defined as follows.

Let $\vec{P}_0$ be the (column) vector from the origin of the coordinate system to P_0 and let $\overrightarrow{P_0P_i}$ be the (column) vectors from P_0 to P_i, $i = 1, \ldots, N$. We define the matrices $\mathbb{A}_T, \mathbb{H}_T \in \mathbb{R}^{N\times N}$ by

$$\mathbb{A}_T := \left(\overrightarrow{P_0P_1}, \ldots, \overrightarrow{P_0P_N}\right) \text{ and } \mathbb{H}_T := \operatorname{diag}\left(h_{1,T}, \ldots, h_{N,T}\right).$$

With the help of the matrix $\mathbb{A}_T$, we now define the affine linear transformation F_T by

$$F_T(\mathbf{x}) := \mathbb{A}_T \cdot \mathbf{x} + \vec{P}_0, \tag{30}$$

with $\mathbf{x} = (x_1, \ldots, x_N)^\top$.

The reference or unitary element $\bar{T}$ is defined by its vertices $\bar{P}_0 = (0,0)^\top$ if $N = 2$ and $\bar{P}_0 = (0,0,0)^\top$ if $N = 3$; and $\bar{P}_i = \mathbf{e}_i^\top$, $i = 1, \ldots, N$, where $\mathbf{e}_i = \delta_{i,j}$ for $j = 1, \ldots, N$. Enumerate the faces $\bar{E}_i := F_T^{-1}(E)$ of $\bar{T}$ such that

$$\bar{E}_i := \bar{T} \cap \{x_i = 0\}, i = 1, \ldots, N \text{ and } \bar{E}_0 := \bar{T} \cap \{|\mathbf{x}|_1 = 1\}, \text{ with } |\mathbf{x}|_1 := \sum_{i=1}^{N} |x_i|.$$

The face $\bar{E}_i$ is opposite the vertex $\bar{P}_i$ (Figure 4).

Variables that are related to the reference element $\bar{T}$ are referred to with a bar (e.g. $\bar{\nabla}$). Note that $\bar{v} := v \circ F_T$. We have the following lemmas (Ref. [12]):

Lemma 4.1 ([12, page 18] **Norms of some matrices**)
Let $T \in \mathcal{T}_h$. Then the following relations hold:

$$\| \mathbb{A}_T^\top \mathbb{C}_T^{-\top} \|_{\mathbb{R}^{N\times N}} = \| \mathbb{C}_T^{-1} \mathbb{A}_T \|_{\mathbb{R}^{N\times N}} \sim 1 \tag{31}$$

$$\| \mathbb{C}_T^\top \mathbb{A}_T^{-\top} \|_{\mathbb{R}^{N\times N}} = \| \mathbb{A}_T^{-1} \mathbb{C}_T \|_{\mathbb{R}^{N\times N}} \sim 1 \tag{32}$$

$$\| \mathbb{C}_T \mathbb{H}_T^{-1} \|_{\mathbb{R}^{N\times N}} = \| \mathbb{H}_T \mathbb{C}_T^{-1} \|_{\mathbb{R}^{N\times N}} = 1 \tag{33}$$

$$\| \mathbb{H}_T^{-1} \|_{\mathbb{R}^{N\times N}} = \| \mathbb{C}_T^{-1} \|_{\mathbb{R}^{N\times N}} = h_{\min,T}^{-1} \tag{34}$$

$$\| \mathbb{A}_T \|_{\mathbb{R}^{N\times N}} \sim \| \mathbb{C}_T \|_{\mathbb{R}^{N\times N}} \sim h_{\max,T} \tag{35}$$

$$\| \mathbb{A}_T^{-1} \|_{\mathbb{R}^{N\times N}} \sim h_{\min,T}^{-1}. \tag{36}$$

Lemma 4.2
[12, page 19] Let $T \in \mathcal{T}_h$ and F_T be the transformation such that $F_T(\bar{\mathbf{x}}) = \mathbb{A}_T \cdot \bar{\mathbf{x}} + \vec{P}_0$. Then we have for each $v \in L^2(T)$ the relations,

$$\| v \|_T = \sqrt{2N|T|} \, \| \bar{v} \|_{\bar{T}}, \tag{37}$$

$$\| v \|_E = \sqrt{\frac{|E|}{|\bar{E}|}} \, \| \bar{v} \|_{\bar{E}}, E \in \mathcal{E}(T). \tag{38}$$

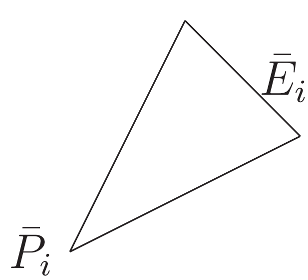


Figure 4. Vertex $\bar{P}_i$ and corresponding face $\bar{E}_i$.

4.2. Clément interpolation

For the analysis, we require some interpolation operator that maps a function from $H_0^1(\Omega)$ to some continuous, piecewise polynomial function V_{Cl}^0. Hence, Lagrange interpolation is unsuitable, but Clément like interpolation techniques have proven to be useful. The image space V_{Cl}^0 will be given by means of its basis functions. To this end, denote by F_T temporarily that affine linear transformation that maps the reference element $\bar{T}$ into the actual element T. For simplicity, we describe the interpolation for scalar functions; for vector-valued functions, the interpolation acts componentwise.

The basis function ϕ_j associated with a node $\mathbf{x}_j$ is now uniquely determined by the condition

$$\phi_j(\mathbf{x}_i) = \delta_i^j \; \forall \mathbf{x}_i \in \mathcal{N}_h(\Omega). \tag{39}$$

Then V_{Cl}^0 is defined as the space spanned by the functions ϕ_j, for all interior nodes $\mathbf{x}_j \in \mathcal{N}_h(\Omega)$. Equivalently, it can be expressed as

$$V_{\text{Cl}}^0 := \left\{ v_h \in C^0(\Omega) : \; v_{h|T} \circ F_T \in \mathbb{P}^1(\bar{T}), \forall T \in \mathcal{T}_h \right\} \cap H_0^1(\Omega), \tag{40}$$

with F_T as above.

Next, the Clément interpolation operator will be defined via the basis functions $\phi_j \in V_{\text{Cl}}^0$.

Definition 4.1 ([3, section 4] **Clément interpolation operator**)
Consider an interior node $\mathbf{x}_j \in \mathcal{N}_h(\Omega)$ and the patch $w_{\mathbf{x}_j \equiv \text{supp}(\phi_j)}$ (cf. Section 3.1). Define the local L^2 projection operator $P_j : L^2(w_{\mathbf{x}_j}) \longrightarrow \mathbb{P}^0(w_{\mathbf{x}_j})$ by

$$\int_{w_{\mathbf{x}_j}} (v - P_j v) w = 0 \; \forall w \in \mathbb{P}^0(w_{\mathbf{x}_j}). \tag{41}$$

For vector-valued functions $\mathbf{v} \in [L^2(w_{\mathbf{x}_j})]^N$, define the projection componentwise.

Then define the Clément interpolation operator $I_{\text{Cl}}^0 : H_0^1(\Omega) \longrightarrow V_{\text{Cl}}^0 \subset H_0^1(\Omega)$ by

$$\mathrm{I}_{\text{Cl}}^0 v := \sum_{\mathbf{x}_j \in \mathcal{N}_h(\Omega)} P_j(v)(\mathbf{x}_j)\phi_j. \tag{42}$$

This operator I_{Cl}^0 acts on functions from $H_0^1(\Omega)$ and preserves zero boundary values. Occasionally, we also require an interpolation operator for functions from $H^1(\Omega)$, that is, without specified boundary values. To this end, denote temporarily the set of boundary nodes by $\mathcal{N}_h(\partial\Omega) = \mathcal{N}_h(\Gamma_s \cup \Gamma_d)$ and define

$$\mathrm{I}_{\text{Cl}} v := \sum_{\mathbf{x}_j \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\partial\Omega)} P_j(v)(\mathbf{x}_j)\phi_j. \tag{43}$$

For vector-valued functions, act componentwise again. From now on, V_{Cl}^0 will always be the space of vector-valued functions.

We need also a Clément-type interpolant mapping a (vector) function in $H_0(\text{curl}, \Omega_d)$ to

$$V_h := \left\{ \mathbf{v}_h \in H_0(\text{curl}, \Omega_d) : \mathbf{v}_{h|T} \in \mathbb{N}_0(T), \forall T \in \mathcal{T}_h^d \right\},$$

where the set $\mathbb{N}_0(T)$ is defined by $\mathbb{N}_0(T) = \left\{ \mathbf{p}(\mathbf{x}) = \mathbf{q} + \mathbf{s} \times \mathbf{x} : \; \mathbf{q}, \mathbf{s} \in \mathbb{R}^3 \right\}$. Such an operator was introduced in [16] where interpolation error estimates were given for isotropic elements. The extension to the anisotropic case has been studied in [13]. The Clément-type interpolation operator is defined with the help of the basis functions $\psi_E \in V_h$, $E \in \mathcal{E}_h(\Omega_d)$, defined by the condition

$$\int_E \psi_E \cdot \mathbf{t}_{E'} = \delta_{E,E'} \; \forall E' \in \mathcal{E}_h(\Omega_d), \tag{44}$$

where $\mathbf{t}_{E'}$ means the unit vector directed along E'. Now following, we use the next definition [13]

Definition 4.2 (**Clément-type interpolation operator**)
For any edge $E \in \mathcal{E}_h(\Omega_d)$, fix one of its adjacent faces F_E. Then define the Clément-type interpolation operator $\mathcal{P}_{\text{Cl}} : [H^1(\Omega_d)]^3 \cap H_0(\text{curl}, \Omega_d) \longrightarrow V_h$ by

$$\mathcal{P}_{\text{Cl}} \psi := \sum_{E' \in \mathcal{E}_h(\Omega_d)} \left(\int_{F_E} (\psi \times \mathbf{n}_{F_E}) \cdot \mathbf{f}_{E'}^{F_E} \right) \mathbf{w}_{E'}, \tag{45}$$

where the (vector) function $\mathbf{f}^{F_E}_{E'}$ are determined by the condition (Ref. [13])

$$\int_{F_E} (\mathbf{w}_{E'} \times \mathbf{n}_{F_E}) \cdot \mathbf{f}^{F_E}_{E''} = \delta_{E',E''}, \forall E', E'' \in \mathcal{E}_h(\bar{\Omega}_d) \cap \partial F_E. \tag{46}$$

We can prove the following interpolation estimates (Ref. [3, 13]):

Lemma 4.3
For all $\mathbf{v} \in [H^1_0(\Omega)]^N$ and for all $\psi \in [H^1(\Omega_d)]^3 \cap H_0(\mathrm{curl}, \Omega_d)$, we have

$$\sum_{T\in\mathcal{T}_h^l} h^{-2}_{\min,T} \parallel \mathbf{v} - I^0_{Cl}\mathbf{v}\|^2_T \lesssim m_1^2(\mathbf{v},\mathcal{T}_h^l) \parallel \nabla \mathbf{v} \parallel^2_{\Omega_l}, \, l = s \text{ or } d \tag{47}$$

$$\sum_{E\in\mathcal{E}_h(\bar{\Omega}_l)} \frac{h_E}{h^2_{\min,E}} \parallel \mathbf{v} - I^0_{Cl}\mathbf{v}\|^2_E \lesssim m_1^2(\mathbf{v},\mathcal{T}_h^l) \parallel \nabla \mathbf{v} \parallel^2_{\Omega_l}, l = s \text{ or } d \tag{48}$$

$$\sum_{T\in\mathcal{T}_h^d} h^{-2}_{\min,T} \parallel \psi - \mathcal{P}_{Cl}\psi \parallel^2_T \lesssim m_2^2(\psi,\mathcal{T}_h^d) \parallel \nabla \psi \parallel^2_{\Omega_d}, \tag{49}$$

$$\sum_{E\in\mathcal{E}_h(\bar{\Omega}_d)} \frac{h_E}{h^2_{\min,E}} \parallel \psi - \mathcal{P}_{Cl}\psi \parallel^2_E \lesssim m_2^2(\psi,\mathcal{T}_h^d) \parallel \nabla \psi \parallel^2_{\Omega_d}. \tag{50}$$

4.3. *Bubble functions, extension operator, inverse inequalities*

For the analysis, we require bubble functions and extension operators that satisfy certain properties. We start with the reference element $\bar{T}$ and define an element bubble function $b_{\bar{T}} \in C(\bar{T})$. We also require an edge bubble function $b_{\bar{E},\bar{T}} \in C(\bar{T})$ for a face $\bar{E} \subset \partial T$ (2*D* case) and a face bubble function $b_{\bar{E},\bar{T}} \in C(\bar{T})$ for a face $\bar{E} \subset \partial T$ (3*D* case). Without loss of generality, assume that $\bar{E}$ is on the $\bar{x}$ axis (2*D* case) or in the $(\bar{x},\bar{y})-$plane (tetrahedral case).
T1 Furthermore, an extension operator $F_{\mathrm{ext}} : C(\bar{E}) \longrightarrow C(\bar{T})$ will be necessary that acts on some function $v_{\bar{E}} \in C(\bar{E})$. Table I gives the definitions in each case. For vector-valued functions, apply the extension operator componentwise.

The element bubble function b_T for the actual element T is obtained simply by the corresponding affine linear transformation. Similarly, the edge/face bubble function $b_{E,T}$ is defined. Later on, an edge/face bubble function b_E is needed on the domain $w_E = T_1 \cup T_2$. This is achieved by an elementwise definition, that is,

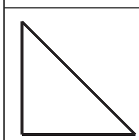
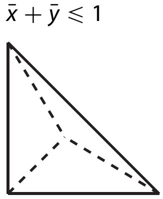
$$b_{E|T_i} := b_{E,T_i}, i = 1, 2.$$

Analogously, the extension operator is defined for functions $v_E \in C(E)$. By the same elementwise definition, we obtain $F_{\mathrm{ext}}(v_E) \in C(w_E)$. With these definitions, one easily checks

$$b_T = 0 \text{ on } \partial T, \; b_E = 0 \text{ on } \partial w_E, \parallel b_T \parallel_\infty = \parallel b_E \parallel_\infty = 1.$$

Next, one requires the so-called inverse inequalities. They can only be expected to hold in some finite-dimensional space. The choice $\mathbb{P}^k$ covers all relevant case of our analysis.

Table I. Bubble functions and extension operator on $\bar{T}$.

Ref. element $\bar{T}$	Bubble functions	Extention operator
 $0 \leqslant \bar{x}, \bar{y}$ $\bar{x} + \bar{y} \leqslant 1$	$b_{\bar{T}} := 3^3\bar{x}\bar{y}(1 - \bar{x} - \bar{y})$ $b_{\bar{E},\bar{T}} := 2^2\bar{x}(1 - \bar{x} - \bar{y})$	$F_{\mathrm{ext}}(v_{\bar{E}})(\bar{x}, \bar{y}) := v_{\bar{E}}(\bar{x})$
 $0 \leqslant \bar{x}, \bar{y}, \bar{z}$ $\bar{x} + \bar{y} + \bar{z} \leqslant 1$	$b_{\bar{T}} := 4^4\bar{x}\bar{y}\bar{z}(1 - \bar{x} - \bar{y} - \bar{z})$ $b_{\bar{E},\bar{T}} := 3^3\bar{x}\bar{y}(1 - \bar{x} - \bar{y} - \bar{z})$	$F_{\mathrm{ext}}(v_{\bar{E}})(\bar{x}, \bar{y}, \bar{z}) := v_{\bar{E}}(\bar{x}, \bar{y})$

Lemma 4.4 (**Inverse inequalities**)
Let $E \in \mathcal{E}(T)$ be an edge/face of an element T. Consider $\mathbf{v}_T \in [\mathbb{P}^{k_0}(T)]^N$ and $\mathbf{v}_E \in [\mathbb{P}^{k_1}(E)]^N$. Then the following equivalences/inequalities hold. The inequality constants depend on the polynomial degree k_0 or k_1 but not on T, E or $\mathbf{v}_T$, $\mathbf{v}_E$.

$$\| \mathbf{v}_T b_T^{1/2} \|_T \sim \| \mathbf{v}_T \|_T \tag{51}$$

$$\| \nabla(\mathbf{v}_T b_T^{1/2}) \|_T \lesssim h_{\min,T}^{-1} \| \mathbf{v}_T \|_T \tag{52}$$

$$\| \mathbf{v}_E b_E^{1/2} \|_E \sim \| \mathbf{v}_E \|_E \tag{53}$$

$$\| F_{\text{ext}}(\mathbf{v}_E) b_E \|_T \lesssim h_{E,T}^{1/2} \| \mathbf{v}_E \|_E \tag{54}$$

$$\| \nabla(F_{\text{ext}}(\mathbf{v}_E) b_E) \|_T \lesssim h_{E,T}^{1/2} h_{\min,T}^{-1} \| \mathbf{v}_E \|_E \tag{55}$$

Proof
Reference [17]. □

4.4. *Anisotropic trace inequalities*

Here, we collect the trace estimates used in the succeeding text.

The first trace inequality is readily obtained by standard scaling techniques [12]. But before, we introduce the following definition [12].

Definition 4.3 (**Directional derivative**)
Let v be a function in $H^1(T)$. The directional derivative $\tilde{D}_{i,T}$ $(i = 1, \dots, N)$ is defined by

$$\left(\tilde{D}_{1,T} v, \dots, \tilde{D}_{1,T} v\right)^\top := \mathbb{H}_T^{-1} \mathbb{C}_T^\top \cdot \nabla v, v \in H^1(T).$$

Here, this derivative $\tilde{D}_{i,T}$ is defined for a fixed element T. Hence, we introduce a derivative $\tilde{D}_i$ which is defined globally for almost all $\mathbf{x} \in \Omega$, and which coincides with $\tilde{D}_{i,T}$ on a element T:

$$\tilde{D}_i v(\mathbf{x}) := \tilde{D}_{i,Tv}(\mathbf{x}) \text{ for } \mathbf{x} \in T.$$

Lemma 4.5 ([12, page 22, lemma 2.3] **First trace inequality**)
Let T be an arbitrary element and E be anedge/face of it. For $\psi \in H^1(T)$, the trace inequality

$$\| \psi \|_E^2 \lesssim h_E^{-1} \left(\| \psi \|_T^2 + \| \mathbb{C}_T^\top \nabla \psi \|_T^2 \right) \tag{56}$$

holds. The componentwise form is

$$\| \psi \|_E^2 \lesssim h_E^{-1} \left(\| \psi \|_T^2 + \sum_{i=1}^{N} h_{i,T}^2 \| \tilde{D}_i \psi \|_T^2 \right). \tag{57}$$

The second, improved trace inequality in the isotropic version (i.e. on the standard element).

Lemma 4.6 ([12, page 23, lemma 2.4] **Second trace inequality**)
Let T be an arbitrary element and E be an edge/face of it. For $\psi \in H^1(T)$, the trace inequality

$$\| \psi \|_E^2 \lesssim h_E^{-1} \| \psi \|_T \left(\| \psi \|_T^2 + \| \mathbb{C}_T^\top \nabla \psi \|_T^2 \right) \tag{58}$$

holds. The componentwise form is

$$\| \psi \|_E^2 \lesssim h_E^{-1} \| \psi \|_T \left(\| \psi \|_T^2 + \sum_{i=1}^{N} h_{i,T}^2 \| \tilde{D}_i \psi \|_T^2 \right). \tag{59}$$

5. Discrete Korn's inequality and example of finite element

5.1. *Discrete Korn's inequality on anisotropic meshes*

In this section, we establish the Discrete Korn's inequality on anisotropic meshes. Indeed, for choice (16), elements satisfying discrete Korn's inequality on isotropic meshes as in [9] have to be used ([8]). To our best knowledge, the elements for the stability of (16) on anisotropic meshes are not yet proved [3, page 11].

Let K be a bounded connected open domain in $\mathbb{R}^N$ ($N = 2$ or $N = 3$) and $\mathcal{T}_h$ anisotropic triangulation of K. The classical Korn inequality (cf. [18–20] and the references therein) states that there exists a (generic) positive constant C_K such that

$$|\mathbf{v}|_{1,K} \leqslant C_K (\| \mathbf{D}(\mathbf{v}) \|_{0,K} + \| \mathbf{v} \|_{0,K}) \ \forall \mathbf{v} \in [H^1(K)]^N. \tag{60}$$

Let $\mathbf{RM}(K)$ be the space of (infinitesimal) rigid motions on K defined by

$$\mathbf{RM}(K) := \left\{ \mathbf{x} \in K \longmapsto \mathbf{a} + \eta \cdot \mathbf{x} : \mathbf{a} \in \mathbb{R}^N \text{ and } \eta \in \mathbf{so}(N) \right\}, \tag{61}$$

where $\mathbf{x} = (x_1, \ldots, x_N)^\top$ is the position vector function on K and $\mathbf{so}(N)$ is the Lie algebra of anti-symmetric $N \times N$ matrices. The space $\mathbf{RM}(K)$ is precisely the kernel of the strain tensor, that is,

$$\forall \mathbf{v} \in [H^1(K)]^N, \mathbf{D}(\mathbf{v}) = \mathbf{0} \Longleftrightarrow \mathbf{v} \in \mathbf{RM}(K). \tag{62}$$

Let Φ be a seminorm on $[H^1(K)]^N$ with the following properties:

$$\exists C_\Phi > 0 / \ \Phi(\mathbf{v}) \leqslant C_\Phi \parallel \mathbf{v} \parallel_{1,K}, \forall \mathbf{v} \in [H^1(K)]^N, \tag{63}$$

where C_Φ is a generic positive constant depending on Φ and

$$\Phi(\mathbf{m}) = 0 \text{ and } \mathbf{m} \in \mathbf{RM}(K) \Longleftrightarrow \mathbf{m} = \text{ a constant vector.} \tag{64}$$

Note that such a seminorm is invariant under the addition of a constant vector $\mathbf{c}$, that is,

$$\Phi(\mathbf{v} + \mathbf{c}) = \Phi(\mathbf{v}), \forall \mathbf{v} \in [H^1(K)]^N, \forall \mathbf{c} \in \mathbb{R}^N. \tag{65}$$

Then, (60), (62), (65) and the compactness of the embedding of $H^1(K)$ into $L^2(K)$ imply that

$$|\mathbf{v}|_{1,K} \leqslant C_\Phi \left(\parallel \mathbf{D}(\mathbf{v}) \parallel_{0,K} + \Phi(\mathbf{v})\right) \forall \mathbf{v} \in [H^1(K)]^N. \tag{66}$$

Let $V_{\mathcal{T}_h} := \left\{\mathbf{v} \in [L^2(K)]^N : \mathbf{v}_{|T} = \mathbf{v}_T \in [\mathbb{P}^1(T)]^N, \forall T \in \mathcal{T}_h\right\}$ be the space of piecewise linear vector fields and $W_{\mathcal{T}_h} := \left\{\mathbf{w} \in [H^1(K)]^N : \mathbf{w}_{|T} = \mathbf{w}_T \in [\mathbb{P}^1(T)]^N, \forall T \in \mathcal{T}_h\right\}$ be the space of continuous piecewise linear vector fields. We define a linear map $F : V_{\mathcal{T}_h} \longrightarrow W_{\mathcal{T}_h}$ as follows. Then $F\mathbf{v}$ is defined by

$$F\mathbf{v}(s) := \frac{1}{|\chi_s|} \sum_{T \in \chi_s} \mathbf{v}_T(s), \forall s \in \mathcal{N}_h, \tag{67}$$

where

$$\chi_s = \{T \in \mathcal{T}_h : s \in \mathcal{N}(T)\}, \tag{68}$$

is the set of simplexes sharing s as a common vertex and $|\chi_s|$ is the number of simplexes in χ_s. Note that $|\chi_s| \lesssim 1$ for all $s \in \mathcal{N}_h$. The following lemma contains the basic estimate for the operator F.

Lemma 5.1 (cf. [9])
It holds that

$$|(\mathbf{v}_T - F\mathbf{v})(s)|^2 \lesssim \sum_{E \in \mathcal{E}_s} |[\mathbf{v}]_E(s)|^2, \ \forall \mathbf{v} \in V_{\mathcal{T}_h}, \ \forall T \in \mathcal{T}_h \text{ and } \forall s \in \mathcal{N}(T), \tag{69}$$

where

$$\mathcal{E}_s := \{E \in \mathcal{E}_h : s \in \partial E\} \tag{70}$$

is the set of interior sides sharing s as a common vertex and $[\mathbf{v}]_E$ is the jump of $\mathbf{v}$ across E.

Theorem 5.1 (**Discrete Korn's inequality**)
Let $\Phi : [H^1(K, \mathcal{T}_h)]^N \longrightarrow \mathbb{R}_+$ be a seminorm satisfying conditions (63)–(64) and, in addition, the condition that

$$(\Phi(\mathbf{v} - F\mathbf{v}))^2 \lesssim \sum_{E \in \mathcal{E}_h(\bar{K})} \frac{h_E}{h^2_{\min,E}} \parallel [\mathbf{v}]_E \parallel_E^2 \forall \mathbf{v} \in V_{\mathcal{T}_h}. \tag{71}$$

Then the following estimate holds:

$$|\mathbf{v}|^2_{[H^1(K,\mathcal{T}_h)]^N} \lesssim \left(\parallel \mathbf{D}_{\mathcal{T}_h}(\mathbf{v}) \|_K^2 + |\Phi(\mathbf{v})|^2 + \sum_{E \in \mathcal{E}_h(\bar{K})} \frac{h_E}{h^2_{\min,E}} \parallel [\mathbf{v}]_E \|_E^2 \right) \forall \mathbf{v} \in V_{\mathcal{T}_h}, \tag{72}$$

where

$$[H^1(K, \mathcal{T}_h)]^N := \left\{\mathbf{v} \in [L^2(K)]^N : \mathbf{v}_{|T} \in [H^1(T)]^N, \forall T \in \mathcal{T}_h\right\}$$

and

$$|\mathbf{v}|_{[H^1(K,\mathcal{T}_h)]^N} := \left(\sum_{T \in \mathcal{T}_h} |\mathbf{v}|^2_{1,T}\right)^{1/2}.$$

Proof
Let $\mathbf{v} \in V_{\mathcal{T}_h}$. We use inequality (66) and the definition of operator Φ. We have

$$\begin{aligned}|\mathbf{v}|^2_{H^1(K,\mathcal{T}_h)} &\leqslant |\mathbf{v}-F\mathbf{v}|^2_{H^1(K,\mathcal{T}_h)} + |F\mathbf{v}|^2_{H^1(K)}\\ &\lesssim \| \mathbf{D}(F\mathbf{v})\|^2_K + (\Phi(F\mathbf{v}))^2 + |\mathbf{v}-F\mathbf{v}|^2_{H^1(K,\mathcal{T}_h)}\\ &\lesssim \| \mathbf{D}_{\mathcal{T}_h}(\mathbf{v})\|^2_K + (\Phi(\mathbf{v}))^2 + (\Phi(\mathbf{v}-F\mathbf{v}))^2 + |\mathbf{v}-F\mathbf{v}|^2_{H^1(K,\mathcal{T}_h)}.\end{aligned}$$

The condition (71) leads immediately to

$$|\mathbf{v}|^2_{H^1(K,\mathcal{T}_h)} \leqslant \| \mathbf{D}_{\mathcal{T}_h}(\mathbf{v})\|^2_K + (\Phi(\mathbf{v}))^2 + \sum_{E\in\mathcal{E}_h(\bar{K})} \frac{h_E}{h^2_{\min,E}} \| [\mathbf{v}]_E\|^2_E + |\mathbf{v}-F\mathbf{v}|^2_{H^1(K,\mathcal{T}_h)}$$

We estimate now the term $|\mathbf{v}-F\mathbf{v}|^2_{H^1(K,\mathcal{T}_h)}$. We have, by a standard inverse estimate (cf. [15, 21]),

$$|\mathbf{v}-F\mathbf{v}|^2_{1,T} \lesssim \| \mathbb{A}_T^{-1} \|^2_{\mathbb{R}^{N\times N}} \, |\det(\mathbb{A}_T)| |\overline{\mathbf{v}-F\mathbf{v}}|^2_{1,\bar{T}}, \tag{73}$$

with

$$|\det(\mathbb{A}_T)| = \frac{|T|}{|\bar{T}|},$$

where $\det(\mathbb{A}_T)$ is the determinant of matrix $\mathbb{A}_T$. Thus, by inequality (36) of Lemma 4.1, we obtain

$$|\mathbf{v}-F\mathbf{v}|^2_{1,T} \lesssim \frac{|T|}{h^2_{\min,T}} |\overline{\mathbf{v}-F\mathbf{v}}|^2_{1,\bar{T}}. \tag{74}$$

As $\mathbf{v}_T - F\mathbf{v}_{|T} \in [\mathbb{P}^1(T)]^N$, then we deduce by norm equivalences (over finite-dimensional spaces) that

$$|\mathbf{v}-F\mathbf{v}|^2_{1,T} \lesssim \frac{|T|}{h^2_{\min,T}} \sum_{s\in\mathcal{N}(T)} |(\mathbf{v}_T - F\mathbf{v})(s)|^2. \tag{75}$$

Lemma 5.1 leads to

$$|\mathbf{v}-F\mathbf{v}|^2_{1,T} \lesssim \frac{|T|}{h^2_{\min,T}} \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} |[\mathbf{v}]_E(s)|^2. \tag{76}$$

Let now $\mathbf{z} = [\mathbf{v}]_E \in [\mathbb{P}^1(E)]^N \longmapsto \bar{\mathbf{z}} \in [\mathbb{P}^1(\bar{E})]^N$. We consider $N = 2$ to simplify. Then $\bar{E} = [0, 1]$. For $\bar{s} = 0$ or 1, we have

$$|\mathbf{z}(s)|^2 = |\bar{\mathbf{z}}(\bar{s})|^2 \leqslant |\bar{\mathbf{z}}(0)|^2 + |\bar{\mathbf{z}}(1)|^2 \sim \| \bar{\mathbf{z}} \|^2_{\bar{E}} = \frac{|\bar{E}|}{|E|} \| \mathbf{z}\|^2_E. \tag{77}$$

Thus, we obtain by inequality (76)

$$|\mathbf{v}-F\mathbf{v}|^2_{1,T} \lesssim \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} \frac{|T|}{|E|} \frac{1}{h^2_{\min,T}} \| [\mathbf{v}]_E\|^2_E. \tag{78}$$

In addition, $h_{E,T} = \frac{N|T|}{|E|}$; hence,

$$|\mathbf{v}-F\mathbf{v}|^2_{1,T} \lesssim \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} \frac{h_{E,T}}{h^2_{\min,T}} \| [\mathbf{v}]_E\|^2_E. \tag{79}$$

Consequently, we have

$$|\mathbf{v}-F\mathbf{v}|_{H^1(K,\mathcal{T}_h)} \lesssim \sum_{T\in\mathcal{T}_h} \sum_{E\in\mathcal{E}(T)} \frac{h_{E,T}}{h^2_{\min,T}} \| [\mathbf{v}]_E\|^2_E, \tag{80}$$

that is,

$$|\mathbf{v}-F\mathbf{v}|_{H^1(K,\mathcal{T}_h)} \lesssim \sum_{T\in\mathcal{T}_h} \sum_{E\in\mathcal{E}(T)} \frac{h_E}{h^2_{\min,E}} \| [\mathbf{v}]_E\|^2_E. \tag{81}$$

The theorem is finally proved. □

Examples of seminorm. The following are examples of Φ that satisfy conditions (63)–(64) and (71). The validity of (63) and (64) is obvious in all three examples.

Example 5.1
Let $\Phi_1 : [H^1(K,\mathcal{T}_h)]^N \longrightarrow \mathbb{R}_+$ be defined by

$$\Phi_1(\mathbf{v}) := \sup_{\mathbf{m}\in\mathbf{RM}(K), \|\mathbf{m}\|_{\Gamma_K}=1, \int_{\Gamma_K}\mathbf{m}=0} \int_{\Gamma_K} |\mathbf{v}\cdot\mathbf{m}|, \forall \mathbf{v} \in [H^1(K,\mathcal{T}_h)]^N, \tag{82}$$

where Γ_K is a measurable subset of ∂K with a positive $N-1$-dimensional volume.

Using (26), (63), (64) and a standard finite element estimate for the L^2-norm, condition (71) can be verified as follows:

$$\begin{aligned}(\Phi_1(\mathbf{v}-F\mathbf{v}))^2 &\leqslant \| \mathbf{v}-F\mathbf{v} \|^2_{\partial K} \\ &\leqslant \sum_{E\in\mathcal{E}_h(\bar{K})} \| \mathbf{v}-F\mathbf{v}\|^2_E \\ &\leqslant \sum_{T\in\mathcal{T}_h} \| \mathbf{v}-F\mathbf{v} \|^2_{\partial T} \\ &\leqslant \sum_{T\in\mathcal{T}_h} \| \mathbf{v}-F\mathbf{v} \|^2_{1,T} \text{ (Trace Theorem)} \\ &\lesssim \sum_{T\in\mathcal{T}_h} \left(\| \nabla(\mathbf{v}-F\mathbf{v})\|^2_T + \| \mathbf{v}-F\mathbf{v}\|^2_T\right).\end{aligned}$$

(1) We estimate the term $\| \mathbf{v}-F\mathbf{v} \|^2_T$.

$$\| \mathbf{v}-F\mathbf{v}\|^2_T = 6|T| \, \| \overline{\mathbf{v}-F\mathbf{v}} \|^2_{\bar{T}} \lesssim |T| \sum_{\bar{s}\in\mathcal{N}(\bar{T})} |(\overline{\mathbf{v}-F\mathbf{v}})(\bar{s})|^2.$$

Norm equivalences (over finite-dimensional spaces) lead to

$$\sum_{\bar{s}\in\mathcal{N}(\bar{T})} |(\overline{\mathbf{v}-F\mathbf{v}})(\bar{s})|^2 \sim \sum_{s\in\mathcal{N}(T)} |(\mathbf{v}-F\mathbf{v})(s)|^2. \tag{83}$$

Hence,

$$\| \mathbf{v}-F\mathbf{v}\|^2_T \lesssim |T| \sum_{s\in\mathcal{N}(T)} |(\mathbf{v}-F\mathbf{v})(s)|^2,$$

and by Lemma 4.1, we deduce

$$\| \mathbf{v}-F\mathbf{v}\|^2_T \lesssim |T| \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} |[\mathbf{v}]_E(s)|^2. \tag{84}$$

We consider the element reference $\bar{T}$ and we take $N = 2$ to simplify. Now, let $\mathbf{z} = [\mathbf{v}]_E \in [\mathbb{P}^1(E)]^N \longmapsto \bar{\mathbf{z}} \in [\mathbb{P}^1(\bar{E})]^N$, with $\bar{E} = [0,1]$ (for $N = 2$).

$$|\mathbf{z}(s)| = |\bar{\mathbf{z}}(\bar{s})| \leqslant |\bar{\mathbf{z}}(0)| + |\bar{\mathbf{z}}(1)| \sim \| \bar{\mathbf{z}} \|_{\bar{E}} . \tag{85}$$

We use (38) and we obtain

$$|\mathbf{z}(s)| \lesssim |E|^{1/2} \| \mathbf{z} \|_E \text{ i.e. } |[\mathbf{v}]_E(s)| \lesssim |E|^{-1/2} \| [\mathbf{v}]_E \|_E .$$

Thus,

$$\| \mathbf{v}-F\mathbf{v}\|^2_T \lesssim \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} \frac{|T|}{|E|} \| [\mathbf{v}]_E\|^2_E \tag{86}$$

$$h_{E,T} = \frac{N|T|}{|E|} \text{ and } 1 \lesssim \frac{1}{h_{\min,T}} \tag{87}$$

Furthermore, inequality (86) implies

$$\| \mathbf{v}-F\mathbf{v}\|^2_T \lesssim \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} \frac{h_{E,T}}{h^2_{\min,T}} \| [\mathbf{v}]_E\|^2_E. \tag{88}$$

Also,

$$h_{E,T} \leqslant h_E \quad \text{and} \quad h_{\min,T} \sim h_{\min,E}. \tag{89}$$

Hence,

$$\| \mathbf{v}-F\mathbf{v}\|^2_T \lesssim \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} \frac{h_E}{h^2_{\min,E}} \| [\mathbf{v}]_E\|^2_E \tag{90}$$

Thus,

$$\sum_{T\in\mathcal{T}_h} \| \mathbf{v}-F\mathbf{v}\|^2_T \lesssim \sum_{E\in\mathcal{E}_h(\bar{K})} \frac{h_E}{h^2_{\min,E}} \| [\mathbf{v}]_E\|^2_E. \tag{91}$$

(2) We estimate the second term $\| \nabla(\mathbf{v}-F\mathbf{v}) \|^2_T$. We have the equalities

$$\nabla(\mathbf{v}-F\mathbf{v}) = \mathbb{A}_T^{-T} \cdot \bar{\nabla}(\overline{\mathbf{v}-F\mathbf{v}}) \text{ and}$$

$$F_T(\bar{x}) = \mathbb{A}_T \cdot \bar{x} + \overrightarrow{P_0} = x \text{ and } dx = J_{F_T} \cdot d\bar{x},$$

where J_{F_T} is Jacobian of transformation. Hence,

$$\| \nabla(\mathbf{v}-F\mathbf{v}) \|_T = |J_{F_T}| \int_{\bar{T}} |\mathbb{A}_T^{-T} \cdot \bar{\nabla}(\overline{\mathbf{v}-F\mathbf{v}})|^2 d\bar{x}$$
$$\leqslant |J_{F_T}| \, \| \mathbb{A}_T^{-T} \|^2_{\mathbb{R}^{N\times N}} \, |\overline{\mathbf{v}-F\mathbf{v}}|^2_{1,\bar{T}}.$$

$$\leqslant \frac{|T|}{h^2_{\min,T}} |\overline{\mathbf{v}-F\mathbf{v}}|^2_{1,\bar{T}} \leqslant \frac{|T|}{h^2_{\min,T}} \| \overline{\mathbf{v}-F\mathbf{v}} \|^2_{1,\bar{T}} .$$

As before, norm equivalences over finite-dimensional spaces lead to

$$\sum_{s\in\mathcal{N}(T)} |(\mathbf{v}-F\mathbf{v})(s)|_{\mathbb{R}^N} \sim \| \overline{\mathbf{v}-F\mathbf{v}} \|_{1,\bar{T}} . \tag{92}$$

That is,

$$\| \nabla(\mathbf{v}-F\mathbf{v}) \|^2_T \lesssim \frac{|T|}{h^2_{\min,T}} \sum_{s\in\mathcal{N}(T)} |(\mathbf{v}-F\mathbf{v})(s)|^2_{\mathbb{R}^N} \tag{93}$$

By Lemma 4.1, we obtain

$$\| \nabla(\mathbf{v}-F\mathbf{v}) \|^2_T \lesssim \frac{|T|}{h^2_{\min,T}} \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} |[\mathbf{v}]_E(s)|^2_{\mathbb{R}^N} \tag{94}$$

Using the same technical as before, we have

$$\| \nabla(\mathbf{v}-F\mathbf{v}) \|^2_T \lesssim \frac{|T|}{h^2_{\min,T}} \times \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} \frac{|T|}{|E|} \frac{1}{h^2_{\min,T}} \| [\mathbf{v}]_E \|^2_E;$$

let

$$\| \nabla(\mathbf{v}-F\mathbf{v}) \|^2_T \lesssim \frac{|T|}{h^2_{\min,T}} \sum_{s\in\mathcal{N}(T)} \sum_{E\in\mathcal{E}_s} \frac{h_{E,T}}{h^2_{\min,T}} \| [\mathbf{v}]_E \|^2_E$$

Finally, we have

$$\sum_{T\in\mathcal{T}_h} |\mathbf{v}-F\mathbf{v}|^2_{1,T} \lesssim \sum_{E\in\mathcal{E}_h(\bar{K})} \frac{h_E}{h^2_{\min,E}} \| [\mathbf{v}]_E \|^2_E \tag{95}$$

Equations (91) and (95) lead to estimate (71).

Example 5.2
Let $\Phi_2 : [H^1(K,\mathcal{T}_h)]^N \longrightarrow \mathbb{R}_+$ be defined by

$$\Phi_2(\mathbf{v}) := \| \mathcal{Q}\mathbf{v} \|_K, \forall \mathbf{v} \in [H^1(K,\mathcal{T}_h)]^N, \tag{96}$$

where $\mathcal{Q}$ is the orthogonal projection from $[L^2(K)]^N$ onto the orthogonal complement of the constant vector fields, that is,

$$\mathcal{Q}\mathbf{v} := \mathbf{v} - \frac{1}{|K|}\int_K \mathbf{v}. \tag{97}$$

Condition (71) can be verified as follows:

$$(\Phi_2(\mathbf{v}-F\mathbf{v}))^2 \lesssim \| \mathbf{v}-F\mathbf{v} \|^2_K = \sum_{T\in\mathcal{T}_h} \| \mathbf{v}-F\mathbf{v} \|^2_T \tag{98}$$

and by (91), we deduce (71).

Example 5.3
Let $\Phi_3 : [H^1(K,\mathcal{T}_h)]^N \longrightarrow \mathbb{R}_+$ be defined by

$$\Phi_3(\mathbf{v}) := \left| \sum_{T\in\mathcal{T}_h} \int_T \operatorname{curl}(\mathbf{v}) \right|, \forall \mathbf{v} \in [H^1(K,\mathcal{T}_h)]^N. \tag{99}$$

Condition (71) can be verified as follows:

$$(\Phi_3(\mathbf{v}-F\mathbf{v}))^2 \lesssim |\mathbf{v}-F\mathbf{v}|^2_{H^1(K,\mathcal{T}_h)} = \sum_{T\in\mathcal{T}_h} |\mathbf{v}-F\mathbf{v}|^2_{1,T} \tag{100}$$

and by (95), condition (71) holds.

5.2. *Example of finite element*

In this section, we present some examples of finite element pairs fulfilling the theoretical assumptions of the previous sections. Their stability is proved on some anisotropic meshes. The anisotropy of the elements is crucial for assumption (G2) that is related to the stability of the (anisotropic) element pairs.

5.2.1. Crouzeix–Raviart elements. In this subsection, for a triangulation of Ω consisting of triangles in $2D$ or the tetrahedra in $3D$, we will use a variant of the non-conforming Crouzeix–Raviart piecewise linear finite element approximation for the velocity and piecewise constant approximation for the pressure, namely

$$\mathbf{H}_h := \{\mathbf{v}_h : \mathbf{v}_{h|T} \in [\mathbb{P}^1(T)]^N\ \forall T \in \mathcal{T}_h, ([\mathbf{v}_h]_E, \mathbf{1})_E = 0\ \forall E \in \mathcal{E}_h(\Omega_s^+), \\ ([\mathbf{v}_h \cdot \mathbf{n}_E]_E, 1)_E = 0 \forall E \in \mathcal{E}_h(\Omega_d) \cup \mathcal{E}_h(\partial\Omega_d)\}$$

and

$$Q_h := \{q_h \in L_0^2(\Omega) : q_{h|T} \in \mathbb{P}^0(T) \forall T \in \mathcal{T}_h\}.$$

The bilinear form $\mathbf{J}(.,.)$ is defined here by following the decomposition (13) of $\mathcal{E}_h$:

$$\mathbf{J}(\mathbf{u},\mathbf{v}) = \mathbf{J}_{\Omega_s^+}(\mathbf{u},\mathbf{v}) + \mathbf{J}_{\Omega_d}(\mathbf{u},\mathbf{v}) + \mathbf{J}_{\partial\Omega_d}(\mathbf{u},\mathbf{v}) \tag{101}$$

where

$$\mathbf{J}_{\Omega_s^+}(\mathbf{u},\mathbf{v}) := (1+2\mu) \sum_{E \in \mathcal{E}_h(\Omega_s^+)} \frac{h_E}{h_{\min,E}^2} \int_E [\mathbf{u}]_E \cdot [\mathbf{v}]_E ds,$$
$$\mathbf{J}_{\Omega_d}(\mathbf{u},\mathbf{v}) := \sum_{E \in \mathcal{E}_h(\Omega_d)} \frac{h_E}{h_{\min,E}^2} \int_E [\mathbf{u}]_E \cdot [\mathbf{v}]_E ds \qquad \text{and,}$$
$$\mathbf{J}_{\partial\Omega_d}(\mathbf{u},\mathbf{v}) := \sum_{E \in \mathcal{E}_h(\partial\Omega_d)} \frac{h_E}{h_{\min,E}^2} \int_E [\mathbf{u}\cdot\mathbf{n}_E]_E [\mathbf{v}\cdot\mathbf{n}_E]_E ds.$$

Note that each element of $\mathcal{E}_h$ only contributes with one jump term in $\mathbf{J}(\mathbf{u},\mathbf{v})$.

We are now able to define the norm on $\mathbf{H}_h$

$$\| \mathbf{v} \|_h := \left(\sum_{T \in \mathcal{T}_h^s} |\mathbf{v}|_{1,T}^2 + \sum_{j=1}^{N-1} \langle \mathbf{v}_s \cdot \tau_j, \mathbf{v}_s \cdot \tau_j \rangle_{\Gamma_I} + \| \mathbf{v} \|_d^2 + \| \operatorname{div}_h \mathbf{v} \|_d^2 + \mathbf{J}(\mathbf{v},\mathbf{v}) \right)^{1/2}.$$

We define the subspace of $\mathbf{H}_h$

$$\mathbf{Z}_h := \{\mathbf{v}_h \in \mathbf{H}_h : \mathbf{b}_h(\mathbf{v}_h, q_h) = 0, \forall q_h \in Q_h\} \tag{102}$$

and the bilinear form on $\mathbf{H}_h \times \mathbf{H}_h$

$$C_h(\mathbf{u}_h, \mathbf{v}_h) := \mathbf{a}_h(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{J}(\mathbf{u}_h, \mathbf{v}_h). \tag{103}$$

Thus, problem (16) leads to

$$\begin{cases} C_h(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{b}_h(\mathbf{v}_h, p_h) &= L(\mathbf{v}_h), \forall \mathbf{v}_h \in \mathbf{H}_h, \\ \mathbf{b}_h(\mathbf{u}_h, q_h) &= G(q_h), \forall q_h \in Q_h \end{cases} \tag{104}$$

From Hölder's inequality and the trace theorem, we derive the boundedness of C_h, L, G and $\mathbf{b}_h$.

Lemma 5.2
There holds

$$|C_h(\mathbf{u}_h, \mathbf{v}_h)| \lesssim \| \mathbf{u}_h \|_h \| \mathbf{v}_h \|_h\ \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{H}_h + \mathbf{H}, \tag{105}$$

$$|\mathbf{b}_h(\mathbf{v}_h, q_h)| \lesssim \| \mathbf{v}_h \|_h \| q_h \|\ \forall \mathbf{v}_h \in \mathbf{H}_h + \mathbf{H}, \forall q_h \in Q_h, \tag{106}$$

$$|L(\mathbf{v}_h)| \lesssim \| \mathbf{v}_h \|_h\ \forall \mathbf{v}_h \in \mathbf{H}_h + \mathbf{H}, \tag{107}$$

$$|G(q_h)| \lesssim \| q_h \|\ \forall q_h \in Q_h \tag{108}$$

Lemma 5.3
If $\mathbf{v}_h \in \mathbf{Z}_h$, then $\operatorname{div}_h \mathbf{v}_h = 0$

Proof
Because $\operatorname{div}_h \mathbf{v}_h \in Q_h$ for $\mathbf{v}_h \in \mathbf{H}_h$, we take $q_h = \operatorname{div}_h \mathbf{v}_h$, leading to $(\operatorname{div}_h \mathbf{v}_h, \operatorname{div}_h \mathbf{v}_h)_\Omega = 0$, and the lemma follows. □

To apply the abstract theory of mixed problems in for example, Girault and Raviart [10] and Brezzi and Fortin [22], we must show that $C_h(.,.)$ is coercive on the constraint set $\mathbf{Z}_h$. This is accomplished in the next lemma.

Lemma 5.4 (**Coercivity**)
$C_h(.,.)$ is coercive on $\mathbf{Z}_h$: there is an $\alpha_h > 0$ such that

$$C_h(\mathbf{v}_h, \mathbf{v}_h) \geq \alpha_h \parallel \mathbf{v}_h \|_h^2, \forall \mathbf{v}_h \in \mathbf{Z}_h \tag{109}$$

Proof
Let $\mathbf{v}_h \in \mathbf{Z}_h$. We have

$$C_h(\mathbf{v}_h, \mathbf{v}_h) = 2\mu \sum_{T \in \mathcal{T}_h^s} \parallel \mathbf{D}(\mathbf{v}_h) \|_T^2 + \mu(\mathbf{K}^{-1}\mathbf{v}_h, \mathbf{v}_h)_{\Omega_d} + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\kappa_j} \parallel \mathbf{v}_h \cdot \tau_j \|_{\Gamma_I}^2$$
$$+ \mathbf{J}_{\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h) + \mathbf{J}_{\Omega_d}(\mathbf{v}_h, \mathbf{v}_h) + \mathbf{J}_{\partial\Omega_d}(\mathbf{v}_h, \mathbf{v}_h).$$

We consider now the seminorm Φ defined by

$$\Phi(\mathbf{v}_h) = \left| \sum_{T \in \mathcal{T}_h^s} \int_T \text{curl } \mathbf{v}_h \right|. \tag{110}$$

We recall that $\text{curl}\mathbf{v}_h \in L^2(T)$ if $N = 2$, while $\text{curl}\mathbf{v}_h \in [L^2(T)]^3$ if $N = 3$ for all $T \in \mathcal{T}_h$. Further,

$$\begin{aligned}
\Phi(\mathbf{v}_h) &= \left| \int_{\Omega_s} \text{curl } \mathbf{v}_h \right| \\
&= \left| \int_{\partial\Omega_s} \gamma_\tau(\mathbf{v}_h) \right| \\
&\leqslant \int_{\Gamma_s} |\gamma_\tau(\mathbf{v}_h)| + \int_{\Gamma_I} |\gamma_\tau(\mathbf{v}_h)| \\
&\lesssim \sum_{E \in \mathcal{E}_h(\Gamma_s)} \int_E |\gamma_\tau(\mathbf{v}_h)| + \sum_{E \in \mathcal{E}_h(\Gamma_I)} \int_E |\gamma_\tau(\mathbf{v}_h)|,
\end{aligned}$$

where for $E \in \mathcal{E}_h(\Omega_s^+)$,

$$\gamma_\tau \mathbf{v}_{h|E} := \begin{cases} \mathbf{v}_h \cdot \tau_{|E} & \text{if } N = 2, \\ \mathbf{v}_h \times \mathbf{n}_{|E} & \text{if } N = 3, \ (\tau \cdot \mathbf{n} = 0 \text{ on } E) \end{cases}$$

- Estimation of $\sum_{E \in \mathcal{E}_h(\Gamma_s)} \int_E |\gamma_\tau(\mathbf{v}_h)|$. We have by Cauchy–Schwarz inequality

$$\begin{aligned}
\sum_{E \in \mathcal{E}_h(\Gamma_s)} \int_E |\gamma_\tau(\mathbf{v}_h)| &\leqslant \sum_{E \in \mathcal{E}_h(\Gamma_s)} \left(\int_E |\gamma_\tau(\mathbf{v}_h)|^2 \right)^{1/2} |\text{diam}(E)|^{1/2} \\
&\leqslant \sum_{E \in \mathcal{E}_h(\Gamma_s)} \left(\int_E |\gamma_\tau(\mathbf{v}_h)|^2 \right)^{1/2} (\text{because diam}(E) \leqslant 1) \\
&\leqslant \sum_{E \in \mathcal{E}_h(\Gamma_s)} \left(\frac{h_E}{h_{\min,E}^2} \right)^{1/2} \left(\int_E |\gamma_\tau(\mathbf{v}_h)|^2 \right)^{1/2} \left(\frac{h_E}{h_{\min,E}^2} \right)^{-1/2} \\
&\lesssim \left\{ \sum_{E \in \mathcal{E}_h(\Gamma_s)} \left(\frac{h_E}{h_{\min,E}^2} \right) \left(\int_E |\gamma_\tau(\mathbf{v}_h)|^2 \right) \right\}^{1/2} \times \\
&\times \left\{ \sum_{E \in \mathcal{E}_h(\Gamma_s)} \left(\frac{h_{\min,E}^2}{h_E} \right) \right\}^{1/2}.
\end{aligned}$$

Now, we set

$$m_h = \left\{ \sum_{E \in \mathcal{E}_h(\Gamma_s)} \left(\frac{h_{\min,E}^2}{h_E} \right) \right\}^{1/2} \in \mathbb{R}, \tag{111}$$

and we have

$$\sum_{E \in \mathcal{E}_h(\Gamma_s)} \int_T |\gamma_\tau(\mathbf{v}_h)| \leqslant m_h \left\{ \sum_{E \in \mathcal{E}_h(\Gamma_s)} \left(\frac{h_E}{h_{\min,E}^2} \right) \left(\int_E |[\mathbf{v}_h]_E|^2 \right) \right\}^{1/2} \tag{112}$$

Hence,

$$\sum_{E \in \mathcal{E}_h(\Gamma_s)} \int_E |\gamma_\tau(\mathbf{v}_h)| \lesssim m_h \left(\mathbf{J}_{\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h) \right)^{1/2}. \tag{113}$$

- We estimate now $\sum_{E\in\mathcal{E}_h(\Gamma_I)}\int_E|\gamma_\tau(\mathbf{v}_h)|$. As before, we apply Cauchy–Schwarz inequality

$$\sum_{E\in\mathcal{E}_h(\Gamma_I)}\int_E|\gamma_\tau(\mathbf{v}_h)| \leqslant \operatorname{diam}(\Gamma_I)^{1/2}\left(\int_{\Gamma_I}|\gamma_\tau(\mathbf{v}_h)|^2\right)^{1/2} \lesssim \mathbf{a}_h(\mathbf{v}_h,\mathbf{v}_h)^{1/2}. \tag{114}$$

By estimates (113) and (114) and Young's inequality, we obtain the estimate

$$(\Phi(\mathbf{v}_h))^2 \lesssim m_h^2\mathbf{J}_{\Omega_s^+}(\mathbf{v}_h,\mathbf{v}_h)+\mathbf{a}_h(\mathbf{v}_h,\mathbf{v}_h) \tag{115}$$

Thus,

$$m_h^2\mathbf{J}_{\Omega_s^+}(\mathbf{v}_h,\mathbf{v}_h)+\mathbf{a}_h(\mathbf{v}_h,\mathbf{v}_h)+\parallel \mathbf{D}_{\mathcal{T}_h^s}(\mathbf{v}_h)\parallel_{\Omega_s}^2 \gtrsim \parallel \mathbf{D}_{\mathcal{T}_h^s}(\mathbf{v}_h)\parallel_{\Omega_s}^2 +(\Phi(\mathbf{v}_h))^2 + \mathbf{J}_{\Omega_s^+}(\mathbf{v}_h,\mathbf{v}_h).$$

Using discrete Korn's inequality of Theorem 5.1, we obtain

$$m_h^2\mathbf{J}_{\Omega_s^+}(\mathbf{v}_h,\mathbf{v}_h)+\mathbf{a}_h(\mathbf{v}_h,\mathbf{v}_h)+\parallel \mathbf{D}_{\mathcal{T}_h^s}(\mathbf{v}_h)\parallel_{\Omega_s}^2 \gtrsim \sum_{T\in\mathcal{T}_h^s}\parallel \nabla(\mathbf{v}_h)\parallel_T^2. \tag{116}$$

Thus, there exists a constant $c_h>0$ such that

$$\mathbf{J}(\mathbf{v}_h,\mathbf{v}_h)+\mathbf{a}_h(\mathbf{v}_h,\mathbf{v}_h)+\sum_{T\in\mathcal{T}_h^s}\parallel \mathbf{D}(\mathbf{v}_h)\parallel_T^2 \geq c_h\left\{\sum_{T\in\mathcal{T}_h^s}\parallel \nabla(\mathbf{v}_h)\parallel_T^2+\mathbf{J}(\mathbf{v}_h,\mathbf{v}_h)\right\},$$

namely

$$C_h(\mathbf{v}_h,\mathbf{v}_h)\geq c_h\left\{\sum_{T\in\mathcal{T}_h^s}\parallel \nabla(\mathbf{v}_h)\parallel_T^2+\mathbf{J}(\mathbf{v}_h,\mathbf{v}_h)\right\}. \tag{117}$$

In addition,

$$C_h(\mathbf{v}_h,\mathbf{v}_h)\geq\sum_{j=1}^{N-1}\parallel \mathbf{v}_h\cdot\tau_j\parallel_{\Gamma_I}^2, \tag{118}$$

$$C_h(\mathbf{v}_h,\mathbf{v}_h)\geq\parallel \mathbf{v}_h\parallel_{\Omega_d}^2. \tag{119}$$

Finally, estimations (117), (118) and (119) lead to estimate (109) of Lemma 5.4 with $\alpha_h=\min\{1/2,c_h\}$. □

In order to verify the discrete inf-sup condition, we define the space

$$\mathbf{X}:=\left\{\mathbf{v}\in\mathbf{H}:\mathbf{v}_{|\Omega_d}\in[H^1(\Omega_d)]^N\right\},$$

and the Crouzeix–Raviart interpolation operator $\mathbf{I}_h:\mathbf{X}\longrightarrow\mathbf{H}_h$ by

$$\int_E\mathbf{v}=\int_E\mathbf{I}_h\mathbf{v},\ \forall T\in\mathcal{T}_h,\forall E\in\mathcal{E}(T). \tag{120}$$

Then we have the following lemma.

Lemma 5.5
$\mathbf{I}_h$ is stable in $[H^1(\Omega)]^N$, namely we have

$$|\mathbf{I}_h\mathbf{v}|_{1,T}\lesssim|\mathbf{v}|_{1,T},\forall\mathbf{v}\in\mathbf{X},\forall T\in\mathcal{T}_h. \tag{121}$$

Hence, we can prove the inf-sup condition by a standard argument.

Lemma 5.6 (**Inf-sup condition**)
There is a constant $\beta_h>0$ (dependent of h) such that

$$\inf_{q_h\in Q_h}\sup_{\mathbf{v}_h\in\mathbf{H}_h}\frac{\mathbf{b}_h(\mathbf{v}_h,q_h)}{\parallel q_h\parallel\times\parallel\mathbf{v}_h\parallel_h}\geq\beta_h. \tag{122}$$

Proof
Consider an arbitrary but fixed $q_h\in Q_h$. Then by [10, corollaire 2.4, page 24], there exists $\mathbf{v}\in[H_0^1(\Omega)]^N\subset\mathbf{X}$, satisfying

$$\begin{cases}\operatorname{div}\mathbf{v} &= -q_h,\ \text{in }\Omega\\ \parallel\mathbf{v}\parallel_{1,\Omega} &\lesssim \parallel q_h\parallel_\Omega.\end{cases} \tag{123}$$

We take $\mathbf{v}_h = \mathbf{I}_h\mathbf{v} \in \mathbf{H}_h$. We have by (121) and Green's formula

$$\mathbf{b}_h(\mathbf{v}-\mathbf{v}_h, q_h) = -\sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}(T)}\int_E q_h\mathbf{n}\cdot(\mathbf{v}-\mathbf{v}_h) = -\sum_{T\in\mathcal{T}_h}\sum_{E\in\mathcal{E}(T)}\int_E \mathbf{n}\cdot(\mathbf{v}-\mathbf{v}_h) = 0.$$

Hence, there exists a constant $C > 0$ such that

$$\mathbf{b}_h(\mathbf{v}_h, q_h) = \mathbf{b}_h(\mathbf{v}, q_h) = \| q_h \|^2 \geq C \| q_h \| \| \mathbf{v} \|_{1,\Omega} . \tag{124}$$

Thus, to obtain inf-sup condition (122), we will show the estimation

$$\| \mathbf{v} \|_{1,\Omega} \gtrsim \| \mathbf{v}_h \|_h . \tag{125}$$

(1) Estimate the term $\sum_{T\in\mathcal{T}_h^s} |\mathbf{v}_h|^2_{1,T}$. We have by (121)

$$\| \mathbf{v} \|^2_{1,\Omega} \gtrsim |\mathbf{v}|^2_{1,\Omega_s} \gtrsim \sum_{T\in\mathcal{T}_h^s} |\mathbf{v}_h|^2_{1,T}. \tag{126}$$

(2) Estimation of term $\sum_{T\in\mathcal{T}_h^d} \| \operatorname{div}\mathbf{v}_h \|^2_T$. We obtain by the same inequality (121),

$$\| \mathbf{v} \|^2_{1,\Omega} \sim |\mathbf{v}|_{1,\Omega} = \sum_{T\in\mathcal{T}_h} |\mathbf{v}|^2_{1,T} \gtrsim \sum_{T\in\mathcal{T}_h^d} |\mathbf{v}|^2_{1,T} \gtrsim \sum_{T\in\mathcal{T}_h^d} \| \operatorname{div}\mathbf{v}_h \|^2_T. \tag{127}$$

(3) To estimate the term $\| \mathbf{v}_h \|^2_{\Omega_d}$, we consider the scalar function $\bar{v} \in H^1(\bar{T})$ and an operator $\bar{S}_h : H^1(\bar{T}) \longrightarrow \mathbb{P}^1(\bar{T})$ defined by

$$\bar{S}_h\bar{v} := \overline{S_h v} := \sum_{i=0}^{N}\left(\frac{1}{|\bar{E}_i|}\int_{\bar{E}_i}\bar{v}\right)\bar{\psi}_i,$$

with $\bar{\psi}_i := 1 - N\bar{\lambda}_i, i = 1, \dots, N$ and $\bar{\lambda}_i(\bar{e}_j) = \delta_{i,j}, j = 1, \dots, N$. Thus, we have the estimations

$$\| \bar{S}_h\bar{v} \|^2_{\bar{T}} \lesssim \sum_{i=0}^{N}\left|\left(\frac{1}{|\bar{E}_i|}\int_{\bar{E}_i}\bar{v}\right)\right|^2 \| \bar{\psi} \|^2_{\bar{T}}, \lesssim \sum_{i=0}^{N}\left(\frac{1}{|\bar{E}_i|}\right) \| \bar{v} \|^2_{\bar{E}_i} \| \bar{\psi} \|^2_{\bar{T}} \lesssim \sum_{i=0}^{N} \| \bar{v} \|^2_{\bar{E}_i}$$

In addition,

$$\| \bar{v} \|^2_{\bar{E}_i} = \frac{|\bar{E}_i|}{|E_i|} \| v \|^2_{E_i}; \tag{128}$$

hence,

$$\| \bar{S}_h\bar{v} \|^2_{\bar{T}} \lesssim \sum_{i=0}^{N}\frac{|\bar{E}_i|}{|E_i|} \| v \|^2_{E_i} \lesssim \sum_{i=0}^{N}\frac{1}{|E_i|} \| v \|^2_{E_i} \lesssim \sum_{i=0}^{N}\frac{h_{E_i,T}}{|T|} \| v \|^2_{E_i} \lesssim \sum_{i=0}^{N}\frac{h_{\max,T}}{|T|} \| v\|^2_T$$

As by (37),

$$\| \overline{S_h v} \|^2_{\bar{T}} = \frac{\| S_h v\|^2_T}{2N|T|}; \tag{129}$$

hence,

$$\| S_h v\|^2_T \lesssim h_{\max,T} \| v\|^2_T. \tag{130}$$

Thus, for a function $\mathbf{v} \in [H^1(T)]^N$, we obtain

$$\| \mathbf{S}_h\mathbf{v}\|_T^2 \lesssim h_{\max,T} \| \mathbf{v}\|_T^2. \tag{131}$$

Further, we conclude by (131), there exists a constant $\beta_{1,h} > 0$ such that

$$\| \mathbf{v} \|_{\Omega_d}^2 \geq \beta_{1,h} \| \mathbf{v}_h \|_{\Omega_d}^2 . \tag{132}$$

(4) To estimate the term $\sum_{j=1}^{N-1} \| \mathbf{v}_{s,h} \cdot \tau_j \|_{\Gamma_I}^2$, we consider arete/face $E \in \mathcal{E}_h(\Gamma_I)$ and we have

$$\begin{aligned}\| \mathbf{v}_{s,h} \cdot \tau_j\|_E^2 &\lesssim \| \mathbf{v}_{s,h}\|_E^2, \text{ (Cauchy-Schwarz inequality)},\\ &\lesssim \| \mathbf{v}_{s,h} - \mathbf{v}\|_E^2 + \| \mathbf{v}\|_E^2, \text{ (triangular inequality)}.\end{aligned}$$

Using first trace inequality of Lemma 4.5 [(i.e. estimate (56)), we obtain

$$\| \mathbf{v}_{s,h} \cdot \tau_j\|_E^2 \lesssim \left[h_E^{-1} \left(\| \mathbf{v}_h - \mathbf{v}\|_T^2 + \| \mathbb{C}_T^\top \nabla(\mathbf{v} - \mathbf{v}_h)\|_T^2\right) + \| \mathbf{v}\|_E^2\right].$$

Cauchy–Schwarz inequality and definition of matrix $\mathbb{C}_T$ lead to

$$\begin{aligned}\| \mathbf{v}_{s,h} \cdot \tau_j\|_E^2 &\lesssim [h_E^{-1} (\| \mathbf{v}_h - \mathbf{v}\|_T^2 + h_{\max,T}^2 \| \nabla(\mathbf{v} - \mathbf{v}_h)\|_T^2) + \| \mathbf{v}\|_E^2]\\ &\lesssim h_E^{-1} \| \mathbf{v}_h - \mathbf{v}\|_T^2 + h_E^{-1}h_{\max,T}^2 \| \nabla(\mathbf{v} - \mathbf{v}_h)\|_T^2 + \| \mathbf{v} \|_{1,\Omega_s}^2 .\end{aligned}$$

Thus, we have the estimate

$$\| \mathbf{v}_{s,h} \cdot \tau_j\|_E^2 \lesssim h_E^{-1} (1 + h_{\max,T}) \| \mathbf{v}\|_T^2 + 2h_E^{-1}h_{\max,T}^2|\mathbf{v}|_{1,T}^2 + \| \mathbf{v} \|_{1,\Omega_s}^2, \tag{133}$$

and hence, there exists a constant $\beta_{2,h} > 0$ such that

$$\| \mathbf{v} \|_{1,\Omega}^2 \gtrsim \beta_{2,h} \sum_{j=1}^{N-1} \| \mathbf{v}_{s,h} \cdot \tau_j \|_{\Gamma_I}^2 . \tag{134}$$

(5) We estimate the term $\mathbf{J}(\mathbf{v}_h, \mathbf{v}_h)$. We remark that $\mathbf{J}(\mathbf{v}, \mathbf{v}) = 0$, because $\mathbf{v} \in [H_0^1(\Omega)]^N$.

(a) Estimate $\mathbf{J}_{\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h)$.

$$\mathbf{J}_{\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h) \lesssim \sum_{T\in\mathcal{T}_h} \mathbf{J}_{T\cap\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h), \text{ with}$$

$$\mathbf{J}_{T\cap\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h) = (1 + 2\mu) \sum_{E\in\mathcal{E}_h(T\cap\Omega_s^+)} \frac{h_E}{h_{\min,E}^2} \| [\mathbf{v}_h]_E\|_E^2.$$

$$\mathbf{J}_{T\cap\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h) = \mathbf{J}_{T\cap\Omega_s^+}(\mathbf{v}_h - \mathbf{v}, \mathbf{v}_h - \mathbf{v}) \tag{135}$$

$$= (1 + 2\mu) \sum_{E\in\mathcal{E}_h(T\cap\Omega_s^+)} \frac{h_E}{h_{\min,E}^2} \| [\mathbf{v}_h - \mathbf{v}]_E\|_E^2. \tag{136}$$

Using theorem trace and first trace inequality of Lemma 4.5, we obtain

$$\begin{aligned}\mathbf{J}_{T\cap\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h) &\lesssim \sum_{E\in\mathcal{E}_h(T\cap\Omega_s^+)} \frac{1}{h_{\min,E}^2} (\| \mathbf{v}_h - \mathbf{v}\|_T^2 + h_{\max,T}^2|\mathbf{v}_h - \mathbf{v}|_{1,T}^2)\\ &\lesssim \sum_{E\in\mathcal{E}_h(T\cap\Omega_s^+)} \frac{1}{h_{\min,T}^2} (\| \mathbf{v}_h - \mathbf{v}\|_T^2 + h_{\max,T}^2|\mathbf{v}_h - \mathbf{v}|_{1,T}^2) .\end{aligned}$$

Applying estimates (121) and (131), we obtain

$$\mathbf{J}_{T\cap\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h) \lesssim \sum_{E\in\mathcal{E}_h(T\cap\Omega_s^+)} \frac{1}{h_{\min,T}^2} (h_{\max,T} \| \mathbf{v} \|_{1,\Omega_s}^2 + h_{\max,T}^2 \| \mathbf{v} \|_{1,\Omega_s}^2) ,$$

namely,

$$\mathbf{J}_{T\cap\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h) \lesssim \sum_{E\in\mathcal{E}_h(T\cap\Omega_s^+)} 2h_{\max,T}h_{\min,T}^{-2} \| \mathbf{v} \|_{1,\Omega_s}^2$$

Thus, there exists a constant $\beta_{3,h} > 0$, such that

$$\| \mathbf{v} \|_{1,\Omega}^2 \gtrsim \beta_{3,h} \mathbf{J}_{\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h). \tag{137}$$

(b) For the terms $\mathbf{J}_{\Omega_d}(\mathbf{v}_h, \mathbf{v}_h)$ and $\mathbf{J}_{\partial\Omega_d}(\mathbf{v}_h, \mathbf{v}_h)$, we use the same technical as in the proof on the bound $\mathbf{J}_{\Omega_s^+}(\mathbf{v}_h, \mathbf{v}_h)$. Further, there exist two constants $\beta_{4,h} > 0$ and $\beta_{5,h} > 0$ such that

$$\| \mathbf{v} \|_{1,\Omega}^2 \gtrsim \beta_{4,h} \mathbf{J}_{\Omega_d}(\mathbf{v}_h, \mathbf{v}_h) \tag{138}$$

$$\| \mathbf{v} \|_{1,\Omega}^2 \gtrsim \beta_{5,h} \mathbf{J}_{\partial\Omega_d}(\mathbf{v}_h, \mathbf{v}_h). \tag{139}$$

The lemma is finally proved with $\beta_h = C \min_{1 \leq i \leq 5} \beta_{i,h}$.

□

Theorem 5.2
The pair $(\mathbf{H}_h, Q_h)$ is stable on the anisotropic meshes.

Proof
Follows directly from Lemmas 5.4 and 5.6. □

6. Error estimators

In order to solve the Stokes–Darcy coupled problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriated indicators. In this section, we first define the local and global indicators and then the lower and upper error bounds are derived.

Our a posteriori analysis requires two technical results that are recalled [8, section 3]. The first one concerns a sort of Helmholtz decomposition of elements of $\mathbf{H}$.

Theorem 6.1 ([8, section 3])
Any $\mathbf{v} \in \mathbf{H}$ admits the Helmholtz type decomposition

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1, \tag{140}$$

where $\mathbf{v}_0, \mathbf{v}_1 \in \mathbf{H}$ but satisfying $\mathbf{v}_0 \in H^1(\Omega)^N$,

$$\mathbf{v}_1 = \begin{cases} \mathbf{0} & \text{in } \Omega_s, \\ \operatorname{curl}\psi & \text{in } \Omega_d, \end{cases} \tag{141}$$

where $\psi \in H_0^1(\Omega_d)$ if $N = 2$, while $\psi \in H^1(\Omega_d)^3 \cap H_0(\operatorname{curl}, \Omega_d)$ if $N = 3$, with the estimate

$$\|\mathbf{v}_0\|_{1,\Omega} + \|\psi\|_{1,\Omega_d} \lesssim \|\mathbf{v}\|_{\mathbf{H}}. \tag{142}$$

The second technical result that we need is a regularity result for the solution $(\mathbf{u}, p)$ of (12).

Theorem 6.2 ([8, section 3])
Let $(\mathbf{u}, p) \in \mathbf{H} \times Q$ be the unique solution of (12). If $\mathbf{f} \in H(\operatorname{curl}, \Omega_d)$ and $\mathbf{K} \in [C^{0,1}(\bar{\Omega}_d)]^{N \times N}$, then there exists $\epsilon > 0$ such that

$$\mathbf{u}_{|\Omega_d} \in [H^{\frac{1}{2}+\epsilon}(\Omega_d)]^N.$$

6.1. Residual error estimators

The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier and that involve the data at hand. To this end, denote the exact element residuals by

$$\mathbf{R}_{s,T} = \mathbf{f} + 2\mu \operatorname{div}\mathbf{D}(\mathbf{u}_h) - \nabla p_h \text{ in } T \in \mathcal{T}_h^s, \tag{143}$$

$$\mathbf{R}_{d,T} = \mathbf{f} - \mu \mathbf{K}^{-1}\mathbf{u}_h - \nabla p_h \text{ in } T \in \mathcal{T}_h^d. \tag{144}$$

As it is common, these exact residuals are replaced by some finite-dimensional approximation called approximate element residual $\mathbf{r}_{l,T}$, $l = s, d$,

$$\mathbf{r}_{l,T} \in [\mathbb{P}^k(T)]^N \text{ on } T \in \mathcal{T}_h^l.$$

This approximation is here achieved by projecting $\mathbf{f}$ on the space of piecewise constant functions in Ω_s and piecewise $\mathbb{P}^1$ functions in Ω_d; more precisely for all $T \in \mathcal{T}_h^s$, we take

$$\mathbf{f}_T = \frac{1}{|T|}\int_T \mathbf{f}(x)\,dx,$$

while for all $T \in \mathcal{T}_h^d$, we take $\mathbf{f}_T$ as the unique element of $[\mathbb{P}^1(T)]^N$ such that

$$\int_T \mathbf{f}_T(x)\cdot\mathbf{q}(x)\,dx = \int_T \mathbf{f}(x)\cdot\mathbf{q}(x)\,dx, \forall \mathbf{q} \in [\mathbb{P}^1(T)]^N.$$

Finally, the global function $\mathbf{f}_h$ is defined by

$$\mathbf{f}_h = \mathbf{f}_T \text{ in } T, \forall T \in \mathcal{T}_h.$$

Hence,

$$\mathbf{r}_{s,T} = \mathbf{f}_T + 2\mu \operatorname{div} \mathbf{D}(\mathbf{u}_h) - \nabla p_h \text{ in } T \in \mathcal{T}_h^s, \tag{145}$$

$$\mathbf{r}_{d,T} = \mathbf{f}_T - \mu\mathbf{K}^{-1}\mathbf{u}_h - \nabla p_h \text{ in } T \in \mathcal{T}_h^d. \tag{146}$$

Next, introduce the gradient jump in normal direction by

$$\mathbf{J}_{E,\mathbf{n}_E} := \begin{cases} [(2\mu\mathbf{D}(\mathbf{u}_h) - p_h\mathbf{I})\cdot\mathbf{n}_E]_E & \text{for an interior edge/face } E,\\ \mathbf{0} & \text{for a boundary edge/face } E,\end{cases}$$

where $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{N\times N}$.

Definition 6.1 (**Residual error estimators**)
For a conforming discretization, the local residual error estimators are defined by

$$\Theta_T := \sqrt{\sum_{i=1}^{6}\Theta_{i,T}^2} \text{ for each } T \in \mathcal{T}_h, \tag{147}$$

with

$$\Theta_{1,T}^2 := \begin{cases} h_{\min,T}^2 \parallel \mathbf{r}_{s,T}\|_T^2 & \text{if} \quad T \in \mathcal{T}_h^s,\\ h_{\min,T}^2 \parallel \mathbf{r}_{d,T}\|_T^2 & \text{if} \quad T \in \mathcal{T}_h^d,\end{cases}$$

$$\Theta_{2,T}^2 := \begin{cases} h_{\min,T}^2 \parallel \operatorname{curl}(\mathbf{f}_h - \mu\mathbf{K}^{-1}\mathbf{u}_h)\|_T^2 & \text{if} \quad T \in \mathcal{T}_h^d,\\ 0 & \text{if} \quad T \in \mathcal{T}_h^s,\end{cases}$$

$$\Theta_{3,T}^2 :=\parallel g - \operatorname{div}\mathbf{u}_h\|_T^2,$$

$$\Theta_{4,T}^2 := \sum_{E\in\mathcal{E}_h(\partial T\cap\bar{\Gamma}_I)} \frac{h_{\min,T}^2}{h_E}\sum_{j=1}^{N-1}\left\| \mathbf{u}_{s,h}\cdot\tau_j + \frac{\sqrt{k_j}}{\alpha_1}2\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\tau_j\right\|_E^2,$$

$$\Theta_{5,T}^2 := \sum_{E\in\partial T\cap\bar{\Gamma}_I}\frac{h_{\min,T}^2}{h_E}\parallel p_{d,h} - p_{s,h} + 2\mu\mathbf{n}_s\cdot\mathbf{D}(\mathbf{u}_{s,h})\cdot\mathbf{n}_s\|_E^2,$$

$$\Theta_{6,T}^2 := \begin{cases} \sum_{E\in\mathcal{E}_h(\partial T\cap\bar{\Omega}_s)} \frac{h_{\min,T}^2}{h_E} \parallel \mathbf{J}_{E,\mathbf{n}_E}\|_E^2 & \text{if} \quad T \in \mathcal{T}_h^s,\\ \sum_{E\in\mathcal{E}_h(\partial T\cap\Omega_d)} \frac{h_{\min,T}^2}{h_E} \parallel [p_h]_E\|_E^2 & \text{if} \quad T \in \mathcal{T}_h^d.\end{cases}$$

For a non-conforming discretization, we set

$$\Theta_T := \sqrt{\sum_{i=1}^{9}\Theta_{i,T}^2} \text{ for each } T \in \mathcal{T}_h, \tag{148}$$

where

$$\Theta_{1,T}^2 := \begin{cases} h_{\min,T}^2 \parallel \mathbf{r}_{s,T} \|_T^2 & \text{if} \quad T \in \mathcal{T}_h^s, \\ h_{\min,T}^2 \parallel \mathbf{r}_{d,T} \|_T^2 & \text{if} \quad T \in \mathcal{T}_h^d, \end{cases}$$

$$\Theta_{2,T}^2 := \begin{cases} h_{\min,T}^2 \parallel \operatorname{curl}(\mathbf{f}_h - \mu \mathbf{K}^{-1}\mathbf{u}_h) \|_T^2 & \text{if} \quad T \in \mathcal{T}_h^d, \\ 0 & \text{if} \quad T \in \mathcal{T}_h^s, \end{cases}$$

$$\Theta_{3,T}^2 := \parallel g - \operatorname{div} \mathbf{u}_h \|_T^2,$$

$$\Theta_{4,T}^2 := \sum_{E \in \mathcal{E}_h(\partial T \cap \bar{\Gamma}_I)} \frac{h_{\min,T}^2}{h_E} \sum_{j=1}^{N-1} \left\| \mathbf{u}_{s,h} \cdot \tau_j + \frac{\sqrt{\kappa_j}}{\alpha_1} 2\mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \tau_j \right\|_E^2,$$

$$\Theta_{5,T}^2 := \sum_{E \in \mathcal{E}_h(\partial T \cap \bar{\Gamma}_I)} \frac{h_{\min,T}^2}{h_E} \parallel p_{d,h} - p_{s,h} + 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \mathbf{n}_s \|_E^2,$$

$$\Theta_{6,T}^2 := \begin{cases} \sum_{E \in \mathcal{E}_h(\partial T \cap \bar{\Omega}_s)} \frac{h_{\min,T}^2}{h_E} \parallel \mathbf{J}_{E,\mathbf{n}_E} \|_E^2 & \text{if } T \in \mathcal{T}_h^s, \\ \sum_{E \in \mathcal{E}_h(\partial T \cap \Omega_d)} \frac{h_{\min,T}^2}{h_E} \parallel [p_h]_E \|_E^2 & \text{if } T \in \mathcal{T}_h^d, \end{cases}$$

$$\Theta_{7,T}^2 := \sum_{E \in \mathcal{E}_h(\partial T \cap \Omega_d)} \frac{h_E}{h_{\min,T}^2} \parallel [\mathbf{u}_h]_E \|_E^2,$$

$$\Theta_{8,T}^2 := \sum_{E \in \mathcal{E}_h(\partial T \cap \partial \Omega_d)} \frac{h_E}{h_{\min,T}^2} \parallel [\mathbf{u}_h \cdot \mathbf{n}_E]_E \|_E^2,$$

$$\Theta_{9,T}^2 := \sum_{E \in \mathcal{E}_h(\partial T \cap \Omega_s^+)} \frac{h_E}{h_{\min,T}^2} (1 + 2\mu) \parallel [\mathbf{u}_h]_E \|_E^2,$$

with $\mathbf{u}_{l,h} := \mathbf{u}_{h|\Omega_l}$ and $p_{l,h} := p_{h|\Omega_l}, l = s, d$.

The global residual error estimator is given by

$$\Theta := \sqrt{\sum_{T \in \mathcal{T}_h} \Theta_T^2}. \tag{149}$$

Furthermore, denote the local and global approximation terms by

$$\zeta_T := \begin{cases} h_{\min,T} \parallel \mathbf{f} - \mathbf{f}_h \parallel_T & \text{if } T \in \mathcal{T}_h^s, \\ h_{\min,T}(\parallel \mathbf{f} - \mathbf{f}_h \parallel_T + \|\operatorname{curl}(\mathbf{f} - \mathbf{f}_h)\|_T) & \text{if } T \in \mathcal{T}_h^d, \end{cases} \tag{150}$$

and

$$\zeta := \sqrt{\sum_{T \in \mathcal{T}_h} \zeta_T^2}. \tag{151}$$

6.2. Proof of the lower error bound

Recall further the notation for the velocity error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and the pressure error $\varepsilon = p - p_h$.

To prove local efficiency for $\omega \subset \Omega$ and $\mathbf{v} \in \mathbf{H} \cup \mathbf{H}_h$, let us denote by

$$\begin{aligned} \parallel \mathbf{v} \parallel_{h,\omega}^2 = & \sum_{T \subset \bar{\omega} \cap \bar{\Omega}_s} |\mathbf{v}|_{1,T}^2 \\ & + \sum_{T \subset \bar{\omega} \cap \bar{\Omega}_d} (\parallel \mathbf{v} \|_T^2 + \parallel \operatorname{div}_h \mathbf{v} \|_T^2) \\ & + \|\mathbf{v}_s \times \mathbf{n}\|_{\Gamma_I \cap \bar{\omega}}^2 + \sum_{T \subset \bar{\omega}} \mathbf{J}_T(\mathbf{v}, \mathbf{v}), \end{aligned}$$

where

$$\begin{aligned} \mathbf{J}_T(\mathbf{v}, \mathbf{v}) = (1 + 2\mu) & \sum_{E \in \mathcal{E}_h(\Omega_s^+) \cap \mathcal{E}(T)} \frac{h_E}{h_{\min,E}^2} \|[\mathbf{v}]_E\|_E^2 \\ + \sum_{E \in \mathcal{E}_h(\Omega_d) \cap \mathcal{E}(T)} \frac{h_E}{h_{\min,E}^2} \|[\mathbf{v}]_E\|_E^2 & + \sum_{E \in \mathcal{E}_h(\partial \Omega_d) \cap \mathcal{E}(T)} \frac{h_E}{h_{\min,E}^2} \|[\mathbf{v} \cdot \mathbf{n}_E]_E\|_E^2, \end{aligned}$$

for non-conforming discretization, and we set

$$\parallel \mathbf{v} \parallel_{h,w}^2 := |\mathbf{v}|_{1,w \cap \bar{\Omega}_s}^2 + \parallel \mathbf{v} \parallel_{w \cap \bar{\Omega}_d}^2 + \parallel \operatorname{div} \mathbf{v} \parallel_{w \cap \bar{\Omega}_d}^2,$$

for conforming discretization.

The main result of this subsection can be stated as follows.

Theorem 6.3 (**Local lower error bound**)
Let $(\mathbf{u}, p) \in \mathbf{H} \times Q$ be the exact solution and $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times Q_h$ be the finite element solution. Then under the assumptions of Theorem 6.2, the following local lower error bound holds:

$$\Theta_T \lesssim \| \mathbf{e} \|_{h,\tilde{\omega}_T} + \| \varepsilon \|_{\tilde{\omega}_T} + \sum_{T' \subset \tilde{\omega}_T} \zeta_{T'}, \tag{152}$$

where $\tilde{\omega}_T$ is a finite union of neighbouring elements of T.

Proof
We begin by bounding each of the residuals separately.

(1) **Element residual in Ω_s.** Set $\mathbf{w}_T := \mathbf{r}_{s,T} b_T \in [H_0^1(T)]^N$ and consider

$$\int_T \mathbf{r}_{s,T} \cdot \mathbf{w}_T = \int_T (\mathbf{f}_h + 2\mu \mathrm{div} \mathbf{D}(\mathbf{u}_h) - \nabla p_h) \cdot \mathbf{w}_T \tag{153}$$

Introduce $\mathbf{f}$ and use the weak formulation (12) to obtain

$$\begin{aligned} \int_T \mathbf{r}_{s,T} \cdot \mathbf{w}_T = & \int_T (\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T \\ & + \int_T (2\mu \mathbf{D}(\mathbf{u}) : \nabla \mathbf{w}_T - p \mathrm{div} \mathbf{w}_T) \\ & + \int_T (2\mu \mathrm{div}\, \mathbf{D}(\mathbf{u}_h) - \nabla p_h) \cdot \mathbf{w}_T . \end{aligned}$$

Integrating by parts in this last term, we obtain

$$\int_T \mathbf{r}_{s,T} \cdot \mathbf{w}_T = \int_T (\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T + 2\mu \int_T \mathbf{D}(\mathbf{e}) : \nabla(\mathbf{w}_T) - \int_T \varepsilon \,\mathrm{div}\, \mathbf{w}_T .$$

Cauchy–Schwarz inequality implies that

$$\int_T \mathbf{r}_{s,T} \cdot \mathbf{w}_T \lesssim \| \mathbf{f} - \mathbf{f}_h \|_T \| \mathbf{w}_T \|_T + (2\mu |\mathbf{e}|_{1,T} + \| \varepsilon \|_T) |\mathbf{w}_T|_{1,T} .$$

The inverse inequalities (51) and (52) and the obvious relation $\| \mathbf{w}_T \|_T \lesssim \| \mathbf{r}_{s,T} \|_T$ imply

$$\| \mathbf{r}_{s,T} \|_T^2 \lesssim (\| \mathbf{f} - \mathbf{f}_h \|_T + h_{\min,T}^{-1} |\mathbf{e}|_{1,T} + h_{\min,T}^{-1} \| \varepsilon \|_T) \| \mathbf{r}_{s,T} \|_T ,$$

or equivalently

$$\Theta_{1,T} \lesssim h_{\min,T} \| \mathbf{f} - \mathbf{f}_h \|_T + |\mathbf{e}|_{1,T} + \| \varepsilon \|_T . \tag{154}$$

(2) **Element residual in** Ω_d. Set $\mathbf{w}_T := \mathbf{r}_{d,T} b_T \in [H_0^1(T)]^N$, use (12) and integrate by parts to obtain

$$\begin{aligned} \int_T \mathbf{r}_{d,T} \cdot \mathbf{w}_T &= \int_T (\mathbf{f}_h - \mu \mathbf{K}^{-1} \mathbf{u}_h - \nabla p_h) \cdot \mathbf{w}_T \\ &= \int_T (\mathbf{f}_h - \mu \mathbf{K}^{-1} \mathbf{u}_h - \nabla p_h) \cdot \mathbf{w}_T \\ &\quad + \int_T (\mu \mathbf{K}^{-1} \mathbf{u} - \mathbf{f}) \cdot \mathbf{w}_T - p \mathrm{div} \mathbf{w}_T) \\ &= \int_T (\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T + \int_T (\mu \mathbf{K}^{-1} \mathbf{e} \cdot \mathbf{w}_T - \varepsilon \mathrm{div} \mathbf{w}_T). \end{aligned}$$

As before, the Cauchy–Schwarz inequality and the inverse inequalities (51) and (52) lead to

$$\Theta_{1,T} \lesssim h_{\min,T} \| \mathbf{f} - \mathbf{f}_h \|_T + \| \mathbf{K}^{-1} \mathbf{e} \|_T + \| \varepsilon \|_T . \tag{155}$$

(3) **Curl element residual in Ω_d.** For $T \in \mathcal{T}_h^d$, we set
$C_T = \mathrm{curl}(\mathbf{f}_h - \mu \mathbf{K}^{-1} \mathbf{u}_h)$ and $\mathbf{w}_T = C_T b_T$. Hence, we notice that $\mathrm{curl} \mathbf{w}_T$ belongs to $\mathbf{H}$ and is divergence free; therefore, by (12), we have

$$\mathbf{a}(\mathbf{u}, \mathrm{curl}\, \mathbf{w}_T) = (\mathbf{f}, \mathrm{curl}\, \mathbf{w}_T),$$

or equivalently

$$\int_T (\mu \mathbf{K}^{-1}\mathbf{u} - \mathbf{f}) \cdot \operatorname{curl} \mathbf{w}_T = 0. \tag{156}$$

But by Green's formula, we may write

$$\int_T C_T \cdot \mathbf{w}_T = \int_T \operatorname{curl}(\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T + \int_T (\mathbf{f} - \mu \mathbf{K}^{-1}\mathbf{u}_h) \cdot \operatorname{curl} \mathbf{w}_T,$$

and by using (156), we deduce that

$$\int_T C_T \mathbf{w}_T = \int_T \operatorname{curl}(\mathbf{f}_h - \mathbf{f}) \cdot \mathbf{w}_T + \int_T \mu \mathbf{K}^{-1}(\mathbf{u} - \mathbf{u}_h) \cdot \operatorname{curl} \mathbf{w}_T.$$

By Cauchy–Schwarz inequality, we obtain

$$\int_T C_T \cdot \mathbf{w}_T \leqslant \|\operatorname{curl}(\mathbf{f}_h - \mathbf{f})\|_T \|\mathbf{w}_T\|_T + \| \mathbf{K}^{-1}\mathbf{e} \|_T \, \|\operatorname{curl} \mathbf{w}_T\|_T.$$

Again, the inverse inequalities (51) and (52) allow to obtain

$$\Theta_{2,T} \lesssim \| \mathbf{K}^{-1}\mathbf{e} \|_T + h_{\min,T} \|\operatorname{curl}(\mathbf{f}_h - \mathbf{f})\|_T. \tag{157}$$

(4) **Divergence element residual in** Ω. We directly see that

$$g - \operatorname{div} \mathbf{u}_h = \operatorname{div} \mathbf{u} - \operatorname{div} \mathbf{u}_h = \operatorname{div} \mathbf{e};$$

hence, by Cauchy–Schwarz inequality, we conclude

$$\Theta_{3,T} = \| g - \operatorname{div} \mathbf{u}_h \|_T \leqslant \| \operatorname{div} \mathbf{e} \|_T . \tag{158}$$

(5) **Interface elements on** Γ_I**.** To estimate $\Theta_{4,T}$ and $\Theta_{5,T}$, we fix an edge E included in Γ_I, and for a constant r_E fixed later on and a unit vector $\mathbf{N}$, we consider

$$\mathbf{w}_E = r_E b_E \mathbf{N}$$

that clearly belongs to $\mathbf{H}$. Hence, the weak formulation (12) yields

$$\mathbf{a}(\mathbf{u}, \mathbf{w}_E) + \mathbf{b}(\mathbf{w}_E, p) = (\mathbf{f}, \mathbf{w}_E)_{\omega_E}$$

that is equivalent to

$$\int_{T_s} (2\mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w}_E) - p \operatorname{div} \mathbf{w}_E) + \int_{T_d} (\mu \mathbf{K}^{-1}\mathbf{u} \cdot \mathbf{w}_E - p \operatorname{div} \mathbf{w}_E) + \sum_{j=1}^{N-1} \frac{\mu \alpha_1}{\sqrt{\kappa_j}} (\mathbf{u}_s \cdot \tau_j, \mathbf{w}_{E,s} \cdot \tau_j)_E = (\mathbf{f}, \mathbf{w}_E)_{\omega_E}, \tag{159}$$

where T_s (resp. T_d) is the unique triangle/tetrahedron included in $\bar{\Omega}_s$ (resp. $\bar{\Omega}_d$) having E as edge/face. On the other hand, integrating by parts in T_s and in T_d yields

$$\begin{aligned} &\int_{T_s} (2\mu \mathbf{D}(\mathbf{u}_h) : \mathbf{D}(\mathbf{w}_E) - p_h \operatorname{div} \mathbf{w}_E) + \int_{T_d} (\mu \mathbf{K}^{-1}\mathbf{u}_h \cdot \mathbf{w}_E - p_h \operatorname{div} \mathbf{w}_E) \\ &+ \sum_{j=1}^{N-1} \frac{\mu \alpha_1}{\sqrt{\kappa_j}} (\mathbf{u}_{s,h} \cdot \tau_j, \mathbf{w}_{E,s} \cdot \tau_j)_E \\ = &-\int_{T_s} (2\mu \operatorname{div} \mathbf{D}(\mathbf{u}_h) - \nabla p_h) \cdot \mathbf{w}_E + \int_{T_d} (\mu \mathbf{K}^{-1}\mathbf{u}_h \cdot \mathbf{w}_E + \nabla p_h) \cdot \mathbf{w}_E \\ &+ \sum_{j=1}^{N-1} \frac{\mu \alpha_1}{\sqrt{\kappa_j}} (\mathbf{u}_{s,h} \cdot \tau_j, \mathbf{w}_{E,s} \cdot \tau_j)_E \\ &- \int_E ([p_h]_E \mathbf{w}_E \cdot \mathbf{n}_E - 2\mu (\mathbf{n}_E \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \mathbf{w}_E). \end{aligned}$$

Subtracting this identity to (159), we find

$$\begin{aligned}
&\int_E ([p_h]_E \mathbf{w}_E \cdot \mathbf{n}_E - 2\mu(\mathbf{n}_E \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \mathbf{w}_E) - \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} (\mathbf{u}_{s,h} \cdot \tau_j, \mathbf{w}_{E,s} \cdot \tau_j)_E \\
&= \int_{T_s} (2\mu \mathbf{D}(\mathbf{e}) : \mathbf{D}(\mathbf{w}_E) - \varepsilon \operatorname{div} \mathbf{w}_E) + \int_{T_d} (\mu \mathbf{K}^{-1} \mathbf{e} \cdot \mathbf{w}_E - \varepsilon \operatorname{div} \mathbf{w}_E) \\
&+ \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} (\mathbf{e}_s \cdot \tau_j, \mathbf{w}_{E,s} \cdot \tau_j)_E \\
&- \int_{T_s} (\mathbf{f} + 2\mu \operatorname{div} \mathbf{D}(\mathbf{u}_h) - \nabla p_h) \cdot \mathbf{w}_E - \int_{T_d} (\mathbf{f} - \mu \mathbf{K}^{-1} \mathbf{u}_h \cdot \mathbf{w}_E - \nabla p_h) \cdot \mathbf{w}_E.
\end{aligned}$$

In that last terms introducing the element residual $\mathbf{r}_{l,T}$, we arrive at

$$\begin{aligned}
&\int_E ([p_h]_E \mathbf{w}_E \cdot \mathbf{n}_E - 2\mu(\mathbf{n}_E \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \mathbf{w}_E) - \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} (\mathbf{u}_{s,h} \cdot \tau_j, \mathbf{w}_{E,s} \cdot \tau_j)_E \\
&= \int_{T_s} (2\mu \mathbf{D}(\mathbf{e}) : \mathbf{D}(\mathbf{w}_E) - \varepsilon \operatorname{div} \mathbf{w}_E) + \int_{T_d} (\mu \mathbf{K}^{-1} \mathbf{e} \cdot \mathbf{w}_E - \varepsilon \operatorname{div} \mathbf{w}_E) \\
&+ \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} (\mathbf{e}_s \cdot \tau_j, \mathbf{w}_{E,s} \cdot \tau_j)_E \\
&- \int_{T_s} (\mathbf{f} - \mathbf{f}_h + \mathbf{r}_{s,T}) \cdot \mathbf{w}_E - \int_{T_d} (\mathbf{f} - \mathbf{f}_h + \mathbf{r}_{d,T}) \cdot \mathbf{w}_E.
\end{aligned} \tag{160}$$

(a) To estimate $\Theta_{4,T}$, for each $j = 1, \ldots, N-1$, we take

$$r_E = \mathbf{u}_h \cdot \tau_j + \frac{\sqrt{\kappa_j}}{\alpha_1} 2\mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_h) \cdot \tau_j \text{ and } \mathbf{N} = \tau_j.$$

Let us assume that $T \equiv T_s$. Recall that $r_E \in \mathbb{P}^k(E)$ for some $k \in \mathbb{N}$ depending on the chosen finite element space. With this choice, identity (160) and inverse inequality (53) yield

$$\begin{aligned}
\|r_E\|_E^2 \lesssim &\int_{T_s} (2\mu \mathbf{D}(\mathbf{e}) : \mathbf{D}(\mathbf{w}_E) - \varepsilon \operatorname{div} \mathbf{w}_E) + \int_{T_d} (\mu \mathbf{K}^{-1} \mathbf{e} \cdot \mathbf{w}_E - \varepsilon \operatorname{div} \mathbf{w}_E) \\
&+ \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} (\mathbf{e}_s \cdot \tau_j, \mathbf{w}_{E,s} \cdot \tau_j)_E \\
&- \int_{T_s} (\mathbf{f} - \mathbf{f}_h + \mathbf{r}_{s,T}) \cdot \mathbf{w}_E - \int_{T_d} (\mathbf{f} - \mathbf{f}_h + \mathbf{r}_{d,T}) \cdot \mathbf{w}_E.
\end{aligned}$$

Hence, the Cauchy–Schwarz inequality, inverse inequalities (54) and (55) and estimates (154) and (157) lead to

$$\frac{h_{\min,T}}{h_{E,T}^{1/2}} \left\| \mathbf{u}_h \cdot \tau_j + \frac{\sqrt{\kappa_j}}{\alpha_1} 2\mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_h) \cdot \tau_j \right\|_E \lesssim |\mathbf{e}|_{h,\omega_E} + \|\varepsilon\|_{\omega_E} + \sum_{T' \subset \omega_E} \zeta_{T'} \tag{161}$$

with $\omega_E = T_s \cup T_d$.

(b) To estimate $\Theta_{5,T}$, we take

$$r_E = p_{d,h} - p_{s,h} + 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \mathbf{n}_s \text{ and } \mathbf{N} = \mathbf{n}_s.$$

As before, identity (160), inverse inequalities (53)–(55) and estimates (154) and (157) lead to

$$\frac{h_{\min,T}}{h_{E,T}^{1/2}} \|p_{d,h} - p_{s,h} + 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \mathbf{n}_s\|_E \lesssim |\mathbf{e}|_{h,\omega_E} + \|\varepsilon\|_{h,\omega_E} + \sum_{T' \subset \omega_E} \zeta_{T'}. \tag{162}$$

(6) **Normal jump in Ω_s.** For each edge/face $E \in \mathcal{E}_h(\Omega_s)$, we consider $\omega_E = T_1 \cup T_2$. Let us assume that $T \equiv T_1$. Recall that $\mathbf{J}_{E,\mathbf{n}_E} \in [\mathbb{P}^k(E)]^N$ for some $k \in \mathbb{N}$ depending on the chosen finite element space. We set

$$\mathbf{w}_E := -F_{\text{ext}}(\mathbf{J}_{E,\mathbf{n}_E}) b_E \in [H_0^1(\omega_E)]^N.$$

First, the weak formulation (12) yields

$$\mathbf{a}(\mathbf{u}, \mathbf{w}_E) + \mathbf{b}(\mathbf{w}_E, p) = (\mathbf{f}, \mathbf{w}_E)_{\omega_E},$$

that is equivalent to

$$\int_{\omega_E} \mathbf{f}\cdot\mathbf{w}_E = \int_{\omega_E} (2\mu\mathbf{D}(\mathbf{u}) - p\mathbf{I}) : \mathbf{D}(\mathbf{w}_E) + \int_{\partial\omega_E} (p\mathbf{I} - 2\mu\mathbf{D}(\mathbf{u}))\mathbf{n}_E\cdot\mathbf{w}_E.$$

By elementwise partial integration, we further have

$$-\int_E \mathbf{J}_{E,\mathbf{n}_E}\cdot\mathbf{w}_E = \int_{\omega_E} (2\mu\mathbf{D}(\mathbf{u}_h) - p_h\mathbf{I}) : \mathbf{D}(\mathbf{w}_E) - \sum_{i=1}^{2}\int_{T_i} (-2\mu\mathrm{div}\,\mathbf{D}(\mathbf{u}_h) + \nabla p_h)\cdot\mathbf{w}_E.$$

Hence, by the previous identity (163), we obtain

$$-\int_E \mathbf{J}_{E,\mathbf{n}_E}\cdot\mathbf{w}_E = \sum_{i=1}^{2}\int_{T_i} (\mathbf{f} - (-2\mu\,\mathrm{div}\,\mathbf{D}(\mathbf{u}_h) + \nabla p_h))\cdot\mathbf{w}_E - \int_{\omega_E} (2\mu\mathbf{D}(\mathbf{e}) - \varepsilon\mathbf{I}) : \mathbf{D}(\mathbf{w}_E)$$

We introduce the approximation $\mathbf{f}_h$ of $\mathbf{f}$ and use the Cauchy–Schwarz inequality and inverse inequalities (53)–(55) to obtain

$$\|\mathbf{J}_{E,\mathbf{n}_E}\|_E \lesssim h_{E,T}^{1/2}\left(\sum_{i=1}^{2}(\|\mathbf{f}-\mathbf{f}_h\|_{T_i} + \|\mathbf{r}_{s,T_i}\|_{T_i})\right) + \frac{h_{E,T}^{1/2}}{h_{\min,T}}(|\mathbf{e}|_{1,\omega_E} + \|\varepsilon\|_{\omega_E})$$

Now, estimates (154) and (157) lead finally to

$$\frac{h_{\min,E}}{h_E^{1/2}}\|\mathbf{J}_{E,\mathbf{n}_E}\|_E \lesssim |\mathbf{e}|_{1,\omega_E} + \|\varepsilon\|_{\omega_E} + \sum_{T'\subset\omega_E} h_{\min,T'}\|\mathbf{f}-\mathbf{f}_h\|_{T'} \tag{163}$$

(7) **Pressure jump in Ω_d.** For each edge/face $E \in \mathcal{E}_h(\Omega_d)$, we consider $\omega_E = T_1 \cup T_2$. As $[p_h]_E \in \mathbb{P}^0(E)$, we set

$$\mathbf{w}_E := [p_h]_E b_E \mathbf{n}_E \in [H_0^1(\omega_E)]^N.$$

First, we notice that as $p \in H^1(\omega_E)$, we have by Green formula

$$\int_{\omega_E} (\nabla p\cdot\mathbf{w}_E + p\,\mathrm{div}\,\mathbf{w}_E) = 0.$$

Again by elementwise partial integration, we further have

$$\int_E [p_h]_E\mathbf{w}_E\cdot\mathbf{n}_E = \sum_{i=1}^{2}\int_{T_i} (\nabla p_h\cdot\mathbf{w}_E + p_h\,\mathrm{div}\,\mathbf{w}_E).$$

Taking the difference of these two identities, we obtain

$$\int_E [p_h]_E\mathbf{w}_E\cdot\mathbf{n}_E = \sum_{i=1}^{2}\int_{T_i} (\nabla(p_h - p)\cdot\mathbf{w}_E + (p_h - p)\,\mathrm{div}\,\mathbf{w}_E).$$

Recalling that $\nabla p = \mathbf{f} - \mu\mathbf{K}^{-1}\mathbf{u}$ and introducing the term $\mathbf{f}_h - \mu\mathbf{K}^{-1}\mathbf{u}_h$, we find

$$\begin{aligned}\int_E [p_h]_E\mathbf{w}_E\cdot\mathbf{n}_E &= \sum_{i=1}^{2}\int_{T_i} (\nabla p_h - \mathbf{f} + \mu\mathbf{K}^{-1}\mathbf{u})\cdot\mathbf{w}_E + (p_h - p)\,\mathrm{div}\,\mathbf{w}_E) \\ &= \sum_{i=1}^{2}\int_{T_i} (\nabla p_h - \mathbf{f}_h + \mu\mathbf{K}^{-1}\mathbf{u}_h)\cdot\mathbf{w}_E + (p_h - p)\,\mathrm{div}\,\mathbf{w}_E) \\ &\quad + \sum_{i=1}^{2}\int_{T_i} (\mathbf{f}_h - \mathbf{f} + \mu\mathbf{K}^{-1}(\mathbf{u} - \mathbf{u}_h))\cdot\mathbf{w}_E.\end{aligned}$$

Now, we fix $T \equiv T_1$ and we apply the Cauchy–Schwarz inequality and inverse inequalities (53)–(55). We obtain

$$\|[p_h]_E\|_E \lesssim \sum_{i=1}^{2} \|\mathbf{r}_{d,T_i}\|_{T_i} h_E^{\frac{1}{2}} + \|p_h - p\|_{T_i} h_E^{-\frac{1}{2}} \tag{164}$$

$$+ h_E^{\frac{1}{2}} \sum_{i=1}^{2} (\|\mathbf{f} - \mathbf{f}_h\|_{T_i} + \|\mathbf{K}^{-1}(\mathbf{u} - \mathbf{u}_h)\|_{T_i}). \tag{165}$$

Finally, by estimates (154) and (157), we deduce that

$$\frac{h_{\min,E}}{h_E^{1/2}} \|[p_h]_E\|_E \lesssim \sum_{T' \subset \omega_E} h_{\min,T'} \|\mathbf{f} - \mathbf{f}_h\|_{T'} + \|\varepsilon\|_{\omega_E} + \|\mathbf{K}^{-1}\mathbf{e}\|_{\omega_E}. \tag{166}$$

(8) **Non-conforming elements.** It remains now to estimate the local indicators $\Theta_{7,T}, \Theta_{8,T}$ and $\Theta_{9,T}$. Because by Theorem 6.2 the jump of $\mathbf{u}$ is zero through all the edges of Ω_d, hence for all $i = 7, 8$ or 9, we clearly have

$$\Theta_{i,T}^2 \lesssim \mathbf{J}_T(\mathbf{u}_h, \mathbf{u}_h) = \mathbf{J}_T(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - \mathbf{u}) \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{h,T}. \tag{167}$$

Summarizing all results provides the desired local lower error bound (152). □

6.3. Proof of the upper error bound

We proceed similarly as [3, section 6]. First, we bound the pressure error (for conforming and non-conforming discretizations). The bound of the velocity error is only derived for conforming case. The bound of the velocity error for non-conforming case will derive as [8] in a forthcoming paper. Section 7 gives the procedure of proving the non-conforming case for the velocity field.

6.3.1. Error in the pressure. We start with an estimate of the pressure error that is valid for conforming and non-conforming elements.

Lemma 6.1 (**Error in the pressure**)
Let $(\mathbf{u}, p) \in \mathbf{H} \times Q$ be the exact solution and $(\mathbf{u}_h, p_h) \in \mathbf{H}_h \times Q_h$ be the finite element solution. Then under the assumptions of Theorem 6.2, there exist functions $\mathbf{v}_{0,\varepsilon} \in [H^1(\Omega)]^N$ and ψ_ε with $\begin{cases} \psi_\varepsilon \in H_0^1(\Omega_d) & \text{if } N = 2, \\ \psi_\varepsilon \in [H^1(\Omega_d)]^3 \cap H_0(\text{curl}, \Omega_d) & \text{if } N = 3, \end{cases}$ both depending on $\varepsilon = p - p_h$ such that the error in the pressure is bounded by

$$\| \varepsilon \| \lesssim m(\mathbf{v}_{0,\varepsilon}, \psi_\varepsilon, \mathcal{T}_h)(\Theta + \zeta) + \| \mathbf{e} \|_h \,. \tag{168}$$

Proof
Because $\varepsilon = p - p_h \in L_0^2(\Omega)$, then [10, corollaire 2.4, page 24] there exists a function $\mathbf{v}_\varepsilon \in [H_0^1(\Omega)]^N \subset \mathbf{H}$ depending the error ε such that

$$\| \varepsilon \| \lesssim \frac{\int_\Omega \varepsilon \operatorname{div} \mathbf{v}_\varepsilon}{\| \nabla \varepsilon \|}. \tag{169}$$

This inequality is equivalent to the continuous inf-sup condition, applied to the pressure error ε. Because the continuous inf-sup condition is not related to the discretization, the inequality constant is independent of any mesh anisotropy. As $\mathbf{v}_\varepsilon \in \mathbf{H}$, then $\mathbf{v}_\varepsilon$ admits the Helmholtz type decomposition (Theorem 6.1) :

$$\mathbf{v}_\varepsilon = \mathbf{v}_{0,\varepsilon} + \mathbf{v}_{1,\varepsilon}, \tag{170}$$

where $\mathbf{v}_{0,\varepsilon} \in [H^1(\Omega)]^N$ and $\mathbf{v}_{1,\varepsilon} = \begin{cases} \mathbf{0} \text{in} \Omega_s \\ \text{curl} \psi_\varepsilon \text{in} \Omega_d, \end{cases}$ where $\psi_\varepsilon \in H_0^1(\Omega_d)$ if $N = 2$ while $\psi_\varepsilon \in [H^1(\Omega_d)]^N \cap H_0(\text{curl}, \Omega_d)$ if $N = 3$ and satisfying estimate (142). For $U = (\mathbf{u}, p) \in \mathbf{H}_h \cup \mathbf{H} \times Q$ and $W = (\mathbf{v}, q) \in \mathbf{H}_h \cup \mathbf{H} \times Q$, we define the continuous bilinear form $\mathbf{L}_h$:

$$\mathbf{L}_h(U, W) := C_h(\mathbf{u}, \mathbf{v}) + \mathbf{b}_h(\mathbf{u}, q) + \mathbf{b}_h(\mathbf{v}, p). \tag{171}$$

Thus,

$$\mathbf{L}_h(U, W) = 2\mu \sum_{T \in \mathcal{T}_h^s} (\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v}))_T + \left(\mu \mathbf{K}^{-1}\mathbf{u}, \mathbf{v}\right)_{\Omega_d} + \sum_{j=1}^{N-1} \frac{\mu \alpha_1}{\sqrt{k_j}} \langle \mathbf{u} \cdot \tau_j, \mathbf{v} \cdot \tau_j \rangle_{\Gamma_I}$$
$$- \sum_{T \in \mathcal{T}_h} (\operatorname{div} \mathbf{u}, q)_T - \sum_{T \in \mathcal{T}_h} (\operatorname{div} \mathbf{v}, p)_T + \mathbf{J}(\mathbf{u}, \mathbf{v}).$$

We set $\mathbf{v}_{\varepsilon,h} = \mathbf{v}_{0,\varepsilon,h} + \mathbf{v}_{1,\varepsilon,h}$ with $\mathbf{v}_{0,\varepsilon,h} = \mathrm{I}_{\mathrm{Cl}}(\mathbf{v}_{0,\varepsilon})$ and

$$\mathbf{v}_{1,\varepsilon,h} = \begin{cases} \text{curl} \psi_{\varepsilon,h} & \text{in} \quad \Omega_d \\ \mathbf{0} & \text{in} \quad \Omega_s, \end{cases}$$

where

$$\psi_{\varepsilon,h} = \begin{cases} \mathrm{I}_{\mathrm{Cl}}^0 \psi_\varepsilon & \text{if} \quad N = 2 \\ \mathcal{P}_{\mathrm{Cl}} \psi_\varepsilon & \text{if} \quad N = 3. \end{cases}$$

Note that $\mathbf{v}_{0,\varepsilon,h}$ belongs to $\mathbf{H}_h \cap [H_0^1(\Omega)]^N$, while $\mathbf{v}_{1,\varepsilon,h}$ simply belongs to $\mathbf{H}_h \cap \mathbf{H}$ ($\psi_{\varepsilon,h}$ being in $H_0^1(\Omega_d)$ if $N = 2$ and $\psi_{\varepsilon,h} \in [H^1(\Omega_d)]^3 \cap H_0(\text{curl}, \Omega_d)$ if $N = 3$, its curl belongs to $H_0(\text{div}, \Omega_d)$ hence $\mathbf{v}_{1,\varepsilon,h}$, its extension by zero in Ω_s, stays in $H_0(\text{div}, \Omega)$). With these definitions and noticing that $\text{div}_h(\mathbf{v}_\varepsilon - \mathbf{v}_h) = \text{div}_h(\mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})$ and that $\mathbf{J}(\mathbf{u}_h, \mathbf{v}_{\varepsilon,h}) = 0$, we may write for $W = (\mathbf{v}_\varepsilon, 0) \in [H_0^1(\Omega)]^N \times Q$, $U_h = (\mathbf{u}_h, p_h)$, $W_h = (\mathbf{v}_{\varepsilon,h}, 0)$ and under the assumptions of Theorem 6.2

$$\begin{aligned}
\mathbf{L}_h(U - U_h, W - W_h) &= \mathbf{L}_h(U - U_h, W) \\
&= 2\mu \sum_{T \in \mathcal{T}_h^s} (\mathbf{D}(\mathbf{e}), \mathbf{D}(\mathbf{v}_\varepsilon))_T + \left(\mu \mathbf{K}^{-1}\mathbf{e}, \mathbf{v}_\varepsilon\right)_{\Omega_d} \\
&\quad + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{k_j}} \langle \mathbf{e} \cdot \tau_j, \mathbf{v}_\varepsilon \cdot \tau_j \rangle_{\Gamma_I} - (\text{div}(\mathbf{v}_\varepsilon), \varepsilon)_\Omega.
\end{aligned}$$

We deduce

$$\begin{aligned}
(\text{div}\, \mathbf{v}_\varepsilon, \varepsilon)_\Omega &= 2\mu \sum_{T \in \mathcal{T}_h^s} (\mathbf{D}(\mathbf{e}), \mathbf{D}(\mathbf{v}_\varepsilon))_T + \left(\mu \mathbf{K}^{-1}\mathbf{e}, \mathbf{v}_\varepsilon\right)_{\Omega_d} \\
&\quad + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{k_j}} \langle \mathbf{e} \cdot \tau_j, (\mathbf{v}_\varepsilon - \mathbf{v}_{\varepsilon,h}) \cdot \tau_j \rangle_{\Gamma_I} \\
&\quad - \mathbf{L}_h(U - U_h, W - W_h).
\end{aligned}$$

Now for a triangle $T \in \mathcal{T}_h^d$, we recall that $\mathbf{v}_{1,\varepsilon} - \mathbf{v}_{1,\varepsilon,h} = \text{curl}(\psi_\varepsilon - \psi_{\varepsilon,h})\text{in}T$ and use Green's formula to obtain

$$\begin{aligned}
\mathbf{L}_h(U - U_h, W - W_h) &= \sum_{T \in \mathcal{T}_h^s} \left\{ (\mathbf{f} - \mathbf{f}_h, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_T + (\mathbf{r}_{s,T}, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_T \right\} \\
&\quad + \sum_{T \in \mathcal{T}_h^d} \Big\{ (\mathbf{f} - \mathbf{f}_h, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_T + (\mathbf{r}_{d,T}, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_T \\
&\quad + \left(\text{curl}(\mathbf{f}_h - \mu \mathbf{K}^{-1}\mathbf{u}_h), \psi_\varepsilon - \psi_{\varepsilon,h}\right)_T + (\text{curl}(\mathbf{f} - \mathbf{f}_h), \psi_\varepsilon - \psi_{\varepsilon,h})_T \Big\} \\
&\quad + \sum_{E \in \mathcal{E}_h(\Omega_d)} ([p_h]_E, (\mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h}) \cdot \mathbf{n}_E)_T \\
&\quad - \sum_{E \in \mathcal{E}_h(\Omega_d)} \left([\mu\, (\mathbf{K}^{-1}\mathbf{u}_h)]_E, \psi_\varepsilon - \psi_{\varepsilon,h}\right)_E - \sum_{E \in \mathcal{E}_h(\Omega_s)} (\mathbf{J}_{E,\mathbf{n}_E}, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_E \\
&\quad + \sum_{E \in \mathcal{E}_h(\bar{\Gamma}_I)} \Big\{ (p_{d,h} - p_{s,h} + 2\mu \mathbf{n}_s \cdot \mathbf{u}_{s,h} \cdot \mathbf{n}_s, (\mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h}) \cdot \mathbf{n}_E)_E \\
&\quad - \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{k_j}} \left(\mathbf{u}_{s,h} \cdot \tau_j + 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \tau_j, (\mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h}) \cdot \tau_j\right)_E \Big\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
(\varepsilon, \text{div}\, \mathbf{v}_\varepsilon)_\Omega &= 2\mu \sum_{T \in \mathcal{T}_h^s} (\mathbf{D}(\mathbf{e}), \mathbf{D}(\mathbf{v}_\varepsilon))_T + \left(\mu \mathbf{K}^{-1}\mathbf{e}, \mathbf{v}_\varepsilon\right)_{\Omega_d} \\
&\quad + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{k_j}} \langle \mathbf{e} \cdot \tau_j, \mathbf{v}_\varepsilon \cdot \tau_j \rangle_{\Gamma_I} \\
&\quad + \sum_{T \in \mathcal{T}_h^s} \left\{ (\mathbf{f} - \mathbf{f}_h, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_T + (\mathbf{r}_{s,T}, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_T \right\} \\
&\quad + \sum_{T \in \mathcal{T}_h^d} \Big\{ (\mathbf{f} - \mathbf{f}_h, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_T + (\mathbf{r}_{d,T}, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_T \\
&\quad + \left(\text{curl}(\mathbf{f}_h - \mu \mathbf{K}^{-1}\mathbf{u}_h), \psi_\varepsilon - \psi_{\varepsilon,h}\right)_T + (\text{curl}(\mathbf{f} - \mathbf{f}_h), \psi_\varepsilon - \psi_{\varepsilon,h})_T \Big\} \\
&\quad + \sum_{E \in \mathcal{E}_h(\Omega_d)} ([p_h]_E, (\mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h}) \cdot \mathbf{n}_E)_T \\
&\quad - \sum_{E \in \mathcal{E}_h(\Omega_d)} \left([\mu\, (\mathbf{K}^{-1}\mathbf{u}_h)]_E, \psi_\varepsilon - \psi_{\varepsilon,h}\right)_E - \sum_{E \in \mathcal{E}_h(\Omega_s)} (\mathbf{J}_{E,\mathbf{n}_E}, \mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h})_E \\
&\quad + \sum_{E \in \mathcal{E}_h(\bar{\Gamma}_I)} \Big\{ (p_{d,h} - p_{s,h} + 2\mu \mathbf{n}_s \cdot \mathbf{u}_{s,h} \cdot \mathbf{n}_s, (\mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h}) \cdot \mathbf{n}_E)_E \\
&\quad - \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{k_j}} \left(\mathbf{u}_{s,h} \cdot \tau_j + 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \tau_j, (\mathbf{v}_{0,\varepsilon} - \mathbf{v}_{0,\varepsilon,h}) \cdot \tau_j\right)_E \Big\}.
\end{aligned}$$

The Cauchy–Schwarz inequality, the trace theorem, the Poincaré–Friedrich inequality and the Clément interpolation results of Lemma 4.3 imply

$$\begin{aligned}(\varepsilon, \operatorname{div} \mathbf{v}_\varepsilon)_\Omega \lesssim\ & 2\mu \parallel \mathbf{D}_{\mathcal{T}_h^s}(\mathbf{e}) \parallel_{\Omega_s} \parallel \mathbf{D}(\mathbf{v}_\varepsilon) \parallel_{\Omega_s} + \mu \parallel \mathbf{K}^{-1}\mathbf{e} \parallel_{\Omega_d} \parallel \mathbf{v}_\varepsilon \parallel_{\Omega_d} \\ & + \left\{ \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{\kappa_j}} \parallel \mathbf{e}\cdot\tau_j \parallel_{\Gamma_l} \right\} \parallel \nabla \mathbf{v}_\varepsilon \parallel_{\Omega_s} \\ & + \{m(\mathbf{v}_{0,\varepsilon}, \psi_\varepsilon, \mathcal{T}_h)\}\,(\Theta + \zeta)(\parallel \nabla \mathbf{v}_{0,\varepsilon} \parallel_\Omega + \parallel \nabla \psi_\varepsilon \parallel_\Omega).\end{aligned}$$

We use property (142) and condition (3) of the tensor **K**, we obtain

$$(\varepsilon, \operatorname{div} \mathbf{v}_\varepsilon)_\Omega \lesssim \left\{ (\Theta + \zeta)\, m(\mathbf{v}_{0,\varepsilon}, \psi_\varepsilon, \mathcal{T}_h) + \parallel \mathbf{e} \parallel_h \right\} \parallel \mathbf{v}_\varepsilon \parallel_{\mathbf{H}}. \tag{172}$$

Let

$$\frac{(\varepsilon, \operatorname{div} \mathbf{v}_\varepsilon)_\Omega}{\parallel \mathbf{v}_\varepsilon \parallel_{1,\Omega}} \lesssim \{m(\mathbf{v}_{0,\varepsilon}, \psi_\varepsilon, \mathcal{T}_h)\}\,(\Theta + \zeta) + \parallel \mathbf{e} \parallel_h. \tag{173}$$

Finally, estimates (169) and (173) finish the proof. □

Remark 6.1
Note that the result of Lemma 6.1 does not depend on regularity result of Theorem 6.2 for conforming case.

6.3.2. Error in the velocity-conforming case.

Lemma 6.2 (**Error in the velocity-conforming case**)
Assume a conform discretization and let $\mathbf{v}_{0,\varepsilon}$, ψ_ε be the functions from Lemma 6.1. Then there exist function $\mathbf{e}_0 \in [H^1(\Omega)]^N$ and $\psi_{\mathbf{e}}$ with $\begin{cases} \psi_{\mathbf{e}} \in H_0^1(\Omega_d) & \text{if } N = 2, \\ \psi_{\mathbf{e}} \in [H^1(\Omega_d)]^3 \cap H_0(\text{curl}, \Omega_d) & \text{if } N = 3, \end{cases}$ both depending on $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ such that the error in the velocity is bounded by

$$\parallel \mathbf{e} \parallel_h \lesssim \left\{ m(\mathbf{e}_0, \psi_{\mathbf{e}}, \mathcal{T}_h) + m(\mathbf{v}_{0,\varepsilon}, \psi_\varepsilon, \mathcal{T}_h)^{1/2} \right\} (\Theta + \zeta). \tag{174}$$

Proof
We define the bilinear form for $U = (\mathbf{u}, p) \in \mathbf{H} \times Q$ and $W = (\mathbf{v}, q) \in \mathbf{H} \times Q$ by $A(U, W) = \mathbf{a}(\mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, p) + \mathbf{b}(\mathbf{u}, q)$. We are setting now $W = (\mathbf{e}, -\operatorname{div} \mathbf{e})$ and we have well $W \in \mathbf{H} \times Q$ because $(\operatorname{div} \mathbf{e}, 1)_\Omega = 0$. Note that for $W_h = (\mathbf{v}_h, q_h) \in \mathbf{H}_h \times Q \subset \mathbf{H} \times Q$, we have $A(U - U_h, W - W_h) = A(U - U_h, W)$ and

$$\begin{aligned} A(U - U_h, W) &= \mathbf{a}(\mathbf{e}, \mathbf{e}) + \mathbf{b}(\mathbf{e}, -\operatorname{div} \mathbf{e}) + \mathbf{b}(\mathbf{e}, \varepsilon) \\ &= 2\mu \parallel \mathbf{D}(\mathbf{e}) \parallel_{\Omega_s}^2 - (\varepsilon, \operatorname{div} \mathbf{e})_\Omega + \parallel \operatorname{div} \mathbf{e} \parallel_\Omega^2 + \\ &\quad + \mu(\mathbf{K}^{-1}\mathbf{e}, \mathbf{e})_{\Omega_d} + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{k_j}} \parallel \mathbf{e}\cdot\tau_j \parallel_{\Gamma_l}^2. \end{aligned}$$

We deduce

$$\begin{aligned} 2\mu \parallel \mathbf{D}(\mathbf{e}) \parallel_{\Omega_s}^2 + \mu(\mathbf{K}^{-1}\mathbf{e}, \mathbf{e})_{\Omega_d} + \sum_{j=1}^{N-1} \frac{\mu\alpha_1}{\sqrt{k_j}} \parallel \mathbf{e}\cdot\tau_j \parallel_{\Gamma_l}^2 + \parallel \operatorname{div} \mathbf{e} \parallel_\Omega^2 \\ = A(U - U_h, W - W_h) + (\varepsilon, \operatorname{div} \mathbf{e})_\Omega \end{aligned}$$

The classical Korn inequality and property (3) of the matrix **K** lead to

$$\parallel \nabla \mathbf{e} \parallel_{\Omega_s}^2 + \parallel \mathbf{e} \parallel_{\Omega_d}^2 + \parallel \operatorname{div} \mathbf{e} \parallel_{\Omega_d}^2 \lesssim A(U - U_h, W - W_h) + (\varepsilon, \operatorname{div} \mathbf{e})_\Omega. \tag{175}$$

As $\mathbf{e} \in \mathbf{H}$, then $\mathbf{e}$ admits the Helmholtz type decomposition (Theorem 6.1) :

$$\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1, \tag{176}$$

where $\mathbf{e}_0 \in [H^1(\Omega)]^N$ and $\mathbf{e}_1 = \begin{cases} \mathbf{0} \text{ in } \Omega_s \\ \text{curl}\psi_{\mathbf{e}} \text{ in } \Omega_d, \end{cases}$ where $\psi_{\mathbf{e}} \in H_0^1(\Omega_d)$ if $N = 2$ while $\psi_{\mathbf{e}} \in [H^1(\Omega_d)]^N \cap H_0(\text{curl}, \Omega_d)$ if $N = 3$ and satisfying estimate (142). Now we define the function $\mathbf{e}_h \in \mathbf{H}_h$ by

$$\mathbf{e}_h = \mathbf{e}_{0,h} + \mathbf{e}_{1,h}, \text{ with } \mathbf{e}_{0,h} = \mathrm{I}_{\mathrm{Cl}}(\mathbf{e}_0),$$

$$\mathbf{e}_{1,h} = \begin{cases} \operatorname{curl} \psi_{\mathbf{e},h} & \text{in} \quad \Omega_d, \\ \mathbf{0} & \text{in} \quad \Omega_s, \end{cases} \tag{177}$$

where

$$\psi_{\mathbf{e},h} = \begin{cases} \mathrm{I}^0_{\mathrm{Cl}} \psi_{\mathbf{e}} & \text{if} \quad N = 2, \\ \mathcal{P}_{\mathrm{Cl}} \psi_{\mathbf{e}} & \text{if} \quad N = 3. \end{cases} \tag{178}$$

Thus, the definition of operator A and integration by parts in each element lead to

$$\begin{aligned}
\| \nabla \mathbf{e} \|^2_{\Omega_s} + \| \mathbf{e} \|^2_{\Omega_d} + \| \operatorname{div} \mathbf{e} \|^2_{\Omega_d} \lesssim \\
&\lesssim \sum_{T\in\mathcal{T}_h^s} \Big\{ (\mathbf{f} - \mathbf{f}_h, \mathbf{e}_0 - \mathbf{e}_{0,h})_T + (\mathbf{r}_{s,T}, \mathbf{e}_0 - \mathbf{e}_{0,h})_T \Big\} \\
&+ \sum_{T\in\mathcal{T}_h^d} \Big\{ (\mathbf{f} - \mathbf{f}_h, \mathbf{e}_0 - \mathbf{e}_{0,h})_T + (\mathbf{r}_{d,T}, \mathbf{e}_0 - \mathbf{e}_{0,h})_T \\
&+ \big(\operatorname{curl}(\mathbf{f}_h - \mu \mathbf{K}^{-1} \mathbf{u}_h), \psi_{\mathbf{e}} - \psi_{\mathbf{e},h}\big)_T + (\operatorname{curl}(\mathbf{f} - \mathbf{f}_h), \psi_{\mathbf{e}} - \psi_{\mathbf{e},h})_T \Big\} \\
&+ \sum_{E\in\mathcal{E}_h(\Omega_d)} ([p_h]_E, (\mathbf{e}_0 - \mathbf{e}_{0,h}) \cdot \mathbf{n}_E)_T \\
&- \sum_{E\in\mathcal{E}_h(\Omega_d)} \big([\mu \gamma_\tau (\mathbf{K}^{-1} \mathbf{u}_h)]_E, \psi_{\mathbf{e}} - \psi_{\mathbf{e},h}\big)_E \\
&- \sum_{E\in\mathcal{E}_h(\Omega_s)} (\mathbf{J}_{E,\mathbf{n}_E}, \mathbf{e}_0 - \mathbf{e}_{0,h})_E \\
&+ \sum_{E\in\mathcal{E}_h(\bar{\Gamma}_I)} \Big\{ (p_{d,h} - p_{s,h} + 2\mu \mathbf{n}_s \cdot \mathbf{u}_{s,h} \cdot \mathbf{n}_s, (\mathbf{e}_0 - \mathbf{e}_{0,h}) \cdot \mathbf{n}_E)_E \\
&- \sum_{j=1}^{N-1} \frac{\mu \alpha_1}{\sqrt{k_j}} \big(\mathbf{u}_{s,h} \cdot \tau_j + 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \tau_j, (\mathbf{e}_0 - \mathbf{e}_{0,h}) \cdot \tau_j\big)_E \Big\} \\
&+ (\varepsilon, \operatorname{div} \mathbf{e})_\Omega + (\operatorname{div} \mathbf{u}_h - g, -\operatorname{div} \mathbf{e})_\Omega.
\end{aligned}$$

Recall that

$$\begin{aligned}
&\| \mathbf{e} \|^2_h = \| \nabla \mathbf{e} \|^2_{\Omega_s} + \| \mathbf{e} \|^2_{\Omega_d} + \| \operatorname{div} \mathbf{e} \|^2_{\Omega_d} \quad \text{and} \\
&\operatorname{div} \mathbf{e} = g - \operatorname{div} \mathbf{u}_h \text{in } \Omega.
\end{aligned}$$

Invoke again the Cauchy–Schwarz inequality, the Clément interpolation inequalities (Section 4.2) and the bound of $\| \varepsilon \|$ from Lemma 6.1. Additionally, employ the triangle inequality for the exact residual $\mathbf{R}_{l,T}$ ($l = s$ or d) as well as the obvious bound $\| \operatorname{div} \mathbf{u}_h \| \leqslant \Theta$ to obtain

$$\begin{aligned}
\| \mathbf{e} \|^2_h \lesssim (\Theta + \zeta) m(\mathbf{e}_0, \psi_{\mathbf{e}}, \mathcal{T}_h) \| \mathbf{e} \|_h + \| \operatorname{div} \mathbf{u}_h - g \|^2_\Omega \\
+ \{ m(\mathbf{v}_{0,\varepsilon}, \psi_\varepsilon, \mathcal{T}_h)(\Theta + \zeta) + \| \mathbf{e} \|_h \} \| g - \operatorname{div} \mathbf{u}_h \|_\Omega .
\end{aligned}$$

Recall also that

$$\| g - \operatorname{div} \mathbf{u}_h \|_\Omega = \sqrt{\sum_{T\in\mathcal{T}_h} \Theta^2_{3,T}} \lesssim \Theta \leqslant (\Theta + \zeta). \tag{179}$$

Hence,

$$\begin{aligned}
\| \mathbf{e} \|^2_h \lesssim (\Theta + \zeta) m(\mathbf{e}_0, \psi_{\mathbf{e}}, \mathcal{T}_h) \| \mathbf{e} \|_h + (\Theta + \zeta) \| \mathbf{e} \|_h \\
+ \{ m(\mathbf{v}_{0,\varepsilon}, \psi_\varepsilon, \mathcal{T}_h)(\Theta + \zeta) + \| \mathbf{e} \|_h \} \Theta.
\end{aligned}$$

Finally, Young's inequality and the trivial relation $1 \leqslant m(.,.,.)$ provide the desired velocity error bound. □

The error bounds for the pressure and velocity immediately yield the following main theorem.

Theorem 6.4 (**Upper error bound-conforming case**)
Assume a conform discretization. Let $\mathbf{v}_{0,\varepsilon}$, ψ_ε be the functions from Lemma 6.1 and $\mathbf{e}_0$, $\psi_{\mathbf{e}}$ be the functions from Lemma 6.2. Then the error is bounded globally from above by

$$\| \mathbf{e} \|_h + \| \varepsilon \| \lesssim \{ m(\mathbf{e}_0, \psi_{\mathbf{e}}, \mathcal{T}_h) + m(\mathbf{v}_{0,\varepsilon}, \psi_\varepsilon, \mathcal{T}_h) \} (\Theta + \zeta). \tag{180}$$

Proof
Follows directly from Lemmas 6.1 and 6.2. □

Commentary 6.1 (**Upper error bound-conforming case**)
The upper error bound (180) contains several alignment measures $m(.,.,.)$. This is in contrast to estimators for isotropic meshes: For anisotropic discretizations, all known estimators are (explicitly or implicitly) based on an anisotropic mesh that is suitably aligned with the anisotropic function.

Compared with the isotropic estimators, our upper error bounds are special in the sense that the alignment measure cannot be evaluated explicitly. However, this should not be considered too much as a disadvantage. For example, the alignment measure $m(\mathbf{e},.,.)$ for the error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ is of size $\mathcal{O}(1)$ for sufficiently good meshes [3]. We expect a similar behaviour for the other alignment measures.

In pratical computations, one may simply use the error estimator without considering the alignment measures [3]. For adaptive algorithms, this is well justified because the lower error bound (152) holds unconditionally.

6.4. Application to isotropic discretization

Because our analysis gives new results for conforming case on isotropic meshes [5–7], we here summarize them. On isotropic discretizations, our analysis holds with $h_{\min,T} \sim h_E \sim h_T$ for $E \in \mathcal{E}(T)$ and the alignment measure $m(.,.,.) \sim 1$. In other words, the given results may be rephrased as follows: the residual error estimator is here given by

$$\Theta_T := \sqrt{\sum_{i=1}^{6} \Theta_{i,T}^2}, \text{ for each } T \in \mathcal{T}_h, \tag{181}$$

where

$$\Theta_{1,T}^2 := h_T^2 \parallel \mathbf{r}_{l,T} \|_T^2, \text{ if } T \in \mathcal{T}_h^l, l = s \text{ or } d,$$
$$\Theta_{2,T}^2 := \begin{cases} h_T^2 \parallel \operatorname{curl}(\mathbf{f}_h - \mu \mathbf{K}^{-1}\mathbf{u}_h)\|_T^2, & \text{if} \quad T \in \mathcal{T}_h^d, \\ 0 & \text{if} \quad T \in \mathcal{T}_h^s, \end{cases}$$
$$\Theta_{3,T}^2 :=\parallel g - \operatorname{div} \mathbf{u}_h \|_T^2,$$
$$\Theta_{4,T}^2 := \sum_{E \in \mathcal{E}_h(\partial T \cap \bar{\Gamma}_I)} h_E \left\{ \sum_{j=1}^{N-1} \parallel \mathbf{u}_{s,h} \cdot \tau_j + \frac{\sqrt{\kappa_j}}{\alpha_1} 2\mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \tau_j \|_E^2 \right\},$$
$$\Theta_{5,T}^2 := \sum_{E \in \mathcal{E}_h(\partial T \cap \bar{\Gamma}_I)} h_E \parallel p_{d,h} - p_{s,h} + 2\mu \mathbf{n}_s \cdot \mathbf{D}(\mathbf{u}_{s,h}) \cdot \mathbf{n}_s \|_E^2,$$
$$\Theta_{6,T}^2 := \begin{cases} \sum_{E \in \mathcal{E}_h(\partial T \cap \bar{\Omega}_s)} h_E \parallel \mathbf{J}_{E,\mathbf{n}_E} \|_E^2, & \text{if} \quad T \in \mathcal{T}_h^s, \\ \sum_{E \in \mathcal{E}_h(\partial T \cap \Omega_d)} h_E \parallel [p_h]_E \|_E^2 & \text{if} \quad T \in \mathcal{T}_h^d, \end{cases}$$

while the approximation term becomes

$$\zeta_T := \begin{cases} h_T \parallel \mathbf{f} - \mathbf{f}_h \parallel_T, & \forall T \in \mathcal{T}_h^s, \\ h_T(\parallel \mathbf{f} - \mathbf{f}_h \parallel_T + \|\operatorname{curl}(\mathbf{f} - \mathbf{f}_h)\|_T), & \forall T \in \mathcal{T}_h^d. \end{cases}$$

We recall that h_T (resp. h_E) is the diameter of T (resp. of E). With these definitions, the lower error bound (152) of Theorem 6.3 holds for any isotropic elements T. On the other hand, the upper bound (180) of Theorem 6.4 reduces to

$$\parallel \mathbf{e} \parallel_h + \parallel \varepsilon \| \lesssim (\Theta + \zeta). \tag{182}$$

7. Conclusion and further works

We have proposed and rigorously analysed a posteriori error estimate for the finite element approximation of Stokes–Darcy equations with anisotropic meshes in $\mathbb{R}^N$, $N \in \{2, 3\}$. The existence and uniqueness are provided for non-conforming finite element method. Different strategies are applied to estimate the lower and upper error bounds. These main results are summarized in Theorems 6.3 and 6.4. For isotropic discretizations, much of the analysis simplifies. The main results are presented in Section 6.4 and the investigations seem to be novel for conforming case.

In the forthcoming works

- Upper error bound (**non-conforming case**). We proved the bound estimation of the pressure using both conforming and non-conforming elements while only show the estimation of the velocity of the conforming case. To obtain the upper bound of the velocity field error for non-conforming case, Clément interpolation operators are not sufficient because additional terms are included in the error estimators (see $\Theta_{i,T}, i \in \{7, 8, 9\}$ in Section 6.1) that measure the non-conformity of the method in the velocity field. In order to treat appropriately this non-conformity, we further need an estimate of the non-conforming error. Thus, we intend to proceed as in [8, proof of theorem 3.3] while building an adapted anisotropic Oswald interpolant that preserves the continuity of the normal trace through the interface. We present also the results of numerical tests with the finite element methods proceeding similarly to [3, 13]. Note that the Oswald interpolation operator that has been built in [8] is only valid for isotropic discretizations.

- Interface conditions. We have adopted in this work Beavers–Joseph–Saffman conditions [23, 24] as interface condition, which is fine here. Note that Beavers–Joseph–Saffman condition is an simplification of Beavers–Joseph [25] by neglecting the tangential velocity in the porous media, but it may lead to an inaccurate accounting of the exchange of fluid between the porous media and the fluid in the conduit. It was once widely used because Beavers–Joseph condition poses some difficulties due to its indefinite contribution to the total energy budget. However, those difficulties were overcome by Cao *et al.* a few years ago [26,27]. That is why we hope in a near further work on this model to adopt the most natural interface condition, namely Beavers–Joseph condition. Furthermore, a comparative study will be made with the current work.
- Analytical tools. In many applications, the fluid region is much narrower than that of the porous medium. Therefore, a widely accepted approximation is to treat the fluid domain as a line (in 2*D* model) or a symmetric tube (in 3*D* model), which is called a pipe flow model [28]. In fact, one previous work [29] dealt with the non-conforming finite element method for such cases. But unfortunately, the toolkits developed in the current manuscript are not valid in the limit that the domain of the flow shrinks to a line or a symmetric tube because, in this case, according to our discretization, the factors $h_{\min,T}$ in continuum pipe flow domain would cancel, and therefore, the quantities $h_{\min,T}^{-1}$ would tend towards the infinity. Thus, we wish in the further considered the appropriate anisotropic discretization used in [30] to constuct the a posteriori error indicators for a coupled continuum pipe-flow/Darcy model in karst aquifers on anisotropic meshes. However, note that a posteriori error estimates for this model with the new non-conforming element used in [29] remain still an open problem on isotropic meshes.
- Boussinesq equations. Finally, we like to extend our results to Boussinesq equations with thermocapillarity effect on the surface and non-homogenous boundary conditions for the velocity and the temperature [31].

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References

1. Discacciati M, Gervasio P, Quarteroni A. Interface control domain decomposition (ICDD) methods for heterogeneous problems. *International Journal for Numerical Methods in Fluids* 2014; **76**(8):471–496.
2. Discacciati M, Quarteroni A. Navier–Stokes/Darcy coupling: modeling, analysis, and numerical approximation. *Revista Matemática Complutense* 2009; **22**:315–426.
3. Creusé E, Kunert G, Nicaise S. A posteriori error estimation for the Stokes problem: anisotropic and isotropic discretizations. *Mathematical Models & Methods in Applied Sciences* 2004; **14**:1297–1341.
4. Chen W, Wang Y. A posteriori error estimate for H(div) conforming mixed finite element for the coupled Darcy–Stokes system. *Journal of Computational and Applied Mathematics* 2014; **255**:502–516.
5. Babuška I, Gatica G. A residual-based a posteriori error estimator for the Stokes–Darcy coupled problem. *SIAM Journal on Numerical Analysis* 2010; **48**:498–523.
6. Cui M, Yan N. A posteriori error estimate for the Stokes–Darcy system. *Mathematical Methods in the Applied Sciences* 2011; **34**:1050–1064.
7. Gatica G, Oyarzùa R, Sayas F-J. A residual-based a posteriori error estimator for a fully-mixed formulation of the Stokes–Darcy coupled problem. *Computer Methods in Applied Mechanics and Engineering* 2011; **200**:1877–1891.
8. Nicaise S, Ahounou B, Houedanou W. A residual-based posteriori error estimates for a nonconforming finite element discretization of the Stokes–Darcy coupled problem: isotropic discretization. *Afrika Matematika, African Mathematical Union and Springer-Verlag Berlin Heidelberg: New York* 2015; **27**(3):701–729 (2016).
9. Brenner S. Korn's inequalities for piecewise H^1 vector fields. *Mathematical and Computer* 2003; **73**:1067–1087.
10. Girault V, Raviart P-A. *Finite Element Methods for Navier–Stokes Equations, Theory and Algorithms*, In Computational Mathematics, vol. 5. Springer: Berlin, 1986.
11. William LJ, Friedhelm S, Ivan Y. Coupling fluid flow with porous media flow. *SIAM Journal on Numerical Analysis* 2002; **40**(6):2195–2218 (2003).
12. Kunert G. A posteriori error estimation for anisotropic tetrahedral and triangular finite element meshes, *Thesis*, Von der Fakultât fûr Mathematik der Technischen Universitât chemnitz genehmigte, 1998.
13. Nicaise S, Creusé E. A posteriori error estimation for the heteregeneous Maxwell equations on isotropic and anisotropic meshes. *Calcolo* 2003; **40**:249–271.
14. Kunert G. Towards anisotropic mesh construction and error estimation in the finite element method. *Numerical Methods for Partial Differential Equations* 2002; **18**:625–648.
15. Ciarlet P. *The Finite Element Method for Elliptic Problems*: Amsterdam: North-Hollande, Society for Industrial and Applied Mathematics, 1978.
16. Kunert G. An a posteriori residual error estimator for the finite element method on anisotropic tetrahedral meshes. *Numerical Mathematics* 2000; **86**:471–490.
17. Kunert Gerd. An a posteriori residual error estimation for the finite element method on anisotropic tetrahedral meshes. *Numerical Mathematics Springer-Verlag* 2000; **86**:471–490.
18. Ciarlet P. *Mathematical Elasticity Volume I: Three-Dimensional Elasticity.* Elsevier Science Publishers B.V.: North-Holland, Amsterdam, 1988. MR 89e:73001.

19. Nitsche J. On Korn's second inequality. *RAIRO Analysis Numérical* 1981; **15**:237–248.
20. Duvaut G, Lions J. *Inequalities in Mechanics and Physics*, Vol. 58. Springer-Verlag: Berlin, 1976,25–91.
21. Nicaise s. *Analyse numérique et équations aux dérivées partielles: Cours et problèmes résolus*, 2000. *Dunod, Paris, ISBN 2100049410,*Posté par Boyaya.
22. Brezzi F, Fortin M. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag: New York, 1991.
23. Jäger W, Mikelić A. On the boundary conditions of the contact interface between a porous medium and a free fluid. *Annali della Scuola Normale Superiore di Pisa Scienze* 1996; **23**:403–465.
24. Jäger W, Mikelić A. On the interface boundary condition of Beavers, Joseph and Saffman. *SIAM Journal on Applied Mathematics* 2000; **60**:1111–1127.
25. Beavers G, Joseph D. Boundary conditions at a naturally permeable wall. *Journal of Fluid Mechanics* 1967; **30**:197–207.
26. Yanzhao C, Gunzburger M, Hua F, Wang X. Coupled Stokes–Darcy model with Beavers–Joseph interface boundary condition. *Communications in Mathematical Sciences* 2010; **8.1**:1–25.
27. Yanzhao C, Gunzburger M, Hu X, Hua F, Wang X. Finite element approximations for Stokes–Darcy flow with Beavers–Joseph interface conditions. *SIAM Journal on Numerical Analysis* 2010; **47.6**:4239–4256.
28. Chen N, Gunzburger M, Huc B, Wang X, Woodruff C. Calibrating the exchange coefficient in the modified coupled continuum pipe-flow model for flows in karst aquifers. *Journal of Hydrology* 2012; **414**:294–301.
29. Wei L, Long X. A new nonconforming finite element with a conforming finite element approximation for a coupled continuum pipe-flow/Darcy model in karst aquifers. *Numerical Methods for Partial Differential Equations* 2016; **32**(3):778–798.
30. Liu W, Wang Z, Li J. Anisotropic finite element approximation for a coupled continuum pipe-flow/Darcy model in karst aquifers. *Computers and Mathematics with Applications* 2014; **68**:86–100.
31. Ahounou B, Paquet L. Existence of a variational solution for the stationary Boussinesq equations with thermocapilarity effect and nonhomogenous boundary. *Afrika Matematika, African Mathematical Union and Springer-Verlag Berlin Heidelberg: New York* 2015; **27**(5):909–921. DOI: 10.1007/s13370-015-0386-6.