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Partial Menger algebras of terms

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The superposition operation $S^{n,A}$, $n \geq 1$, $n \in \mathbb{N}$, maps to each $(n+1)$-tuple of n-ary operations on a set A an n-ary operation on A and satisfies the so-called superassociative law, a generalization of the associative law. The corresponding algebraic structures are Menger algebras of rank n. A partial algebra of type $(n+1)$ which satisfies the superassociative law as weak identity is said to be a partial Menger algebra of rank n. As a generalization of linear terms we define r-terms as terms where each variable occurs at most r-times. It will be proved that n-ary r-terms form partial Menger algebras of rank n. In this paper, some algebraic properties of partial Menger algebras such as generating systems, homomorphic images and freeness are investigated. As generalization of hypersubstitutions and linear hypersubstitutions we consider r-hypersubstitutions.

Keywords: n-ary operation; n-ary term; superposition of n-ary operations and n-ary terms; linear term; r-term; Menger algebra of rank n; partial Menger algebra of rank n; r-hypersubstitution.

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1. Preliminaries

Let A be a nonempty set. A function $f: A^n \to A, n \in \mathbb{N}^+ := \mathbb{N}\backslash\{0\}$, is said to be an *$n$-ary operation* on A. Let $O^n(A)$ be the set of all n-ary operations defined on A. The *n-ary projections* $e_i^{n,A}$ on A are defined by $e_i^{n,A}(a_1,\ldots,a_n) = a_i$ for all $a_1,\ldots,a_n \in A, n \in \mathbb{N}^+$ and $1 \leq i \leq n$. Let $f \in O^n(A)$ and let $g_1,\ldots,g_n \in O^n(A)$. We define a new n-ary operation $f(g_1,\ldots,g_n)$ by

$$f(g_1,\ldots,g_n)(a_1,\ldots,a_n) := f(g_1(a_1,\ldots,a_n),\ldots,g_n(a_1,\ldots,a_n))$$

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for all $(a_1, \ldots, a_n) \in A^n$. This operation $f(g_1, \ldots, g_n)$ is said to be the *composition* of f with $g_1, \ldots, g_n$. Since $O^n(A)$ is the set of all n-ary operations defined on A, it is closed under composition and contains all projection operations defined on A. This can be described by an $(n+1)$-ary operation

$$S^{n,A}: \ (O^n(A))^{n+1} \to O^n(A),$$
$$(f, g_1, \ldots, g_n) \mapsto S^{n,A}(f, g_1, \ldots, g_n) = f(g_1, \ldots, g_n).$$

Then for any $n \in \mathbb{N}^+$, we obtain an algebra n-clone $A := (O^n(A); S^{n,A})$ of type $(n+1)$ and an algebra $(n\text{-clone } A)^+ := (O^n(A); S^{n,A}, e_1^{n,A}, \ldots, e_n^{n,A})$ of type $(n+1, 0, \ldots, 0)$.

Let us consider the following identities:

(C1) $\tilde{S}^n(\tilde{Z}, \tilde{S}^n(\tilde{Y}_1, \tilde{X}_1, \ldots, \tilde{X}_n), \ldots, \tilde{S}^n(\tilde{Y}_n, \tilde{X}_1, \ldots, \tilde{X}_n)) \approx \tilde{S}^n(\tilde{S}^n(\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_n), \tilde{X}_1, \ldots, \tilde{X}_n)$ $(n \in \mathbb{N}^+)$ (the superassociative law);
(C2) $\tilde{S}^n(\tilde{e_i^n}, \tilde{X}_1, \ldots, \tilde{X}_n) \approx \tilde{X}_i$ $(1 \leq i \leq n, n \in \mathbb{N}^+)$;
(C3) $\tilde{S}^n(\tilde{Y}, \tilde{e_1^n}, \ldots, \tilde{e_n^n}) \approx \tilde{Y}$ $(n \in \mathbb{N}^+)$.

An algebra which satisfies the identity (C1) is said to be a Menger algebra of rank n. It is not difficult to check that n-clone A is a Menger algebra of rank n.

An algebra of type $(n+1, 0, \ldots, 0)$ which satisfies (C1)–(C3) is said to be a unitary Menger algebra of rank n. It is easy to see that $(n\text{-clone } A)^+$ is a unitary Menger algebra of rank n. For $n = 1$, (C1) is the associative law and $(1\text{-clone } A)^+$ is a monoid.

We remark that in (C1)–(C3) $\tilde{Z}, \tilde{Y}_1, \ldots, \tilde{Y}_n, \tilde{X}_1, \ldots, \tilde{X}_n$, are variables for terms, $\tilde{S}^n$ is an operation symbol and $\tilde{e_i^n}$ are symbols for variables. Let M_n be the variety of all Menger algebras of rank n and let $(M_n)^+$ be the variety of all unitary Menger algebras of rank n. The elements of M_n are called abstract Menger algebras of rank n and those of $(M_n)^+$ are called abstract unitary Menger algebras of rank n. Menger algebras of operations, i.e. those of the form n-clone A, are called concrete Menger algebras and unitary Menger algebras of the form $(n\text{-clone } A)^+$ are called concrete unitary Menger algebras of rank n. The following facts were proved first by Dicker in 1963 [6] (see also [7]).

Theorem 1.1. *Every abstract Menger algebra of rank n is isomorphic to a concrete one.*

Theorem 1.2. *Every abstract unitary Menger algebra $(A; S, e_1, \ldots, e_n)$ of rank n is isomorphic to a concrete unitary Menger algebra of rank n such that the set A corresponds to a set of n-ary operations defined on A and the nullary operations $e_1, \ldots, e_n$ correspond to the n-ary projections $e_1^{n,A}, \ldots, e_n^{n,A}$.*

Another class of Menger algebras can be obtained considering n-ary terms of type τ.

Let $(f_i)_{i \in I}$ be an indexed sequence of operation symbols, let $X_n := \{x_1, \ldots, x_n\}$ be a finite alphabet which is disjoint to the set of operation symbols. To every operation symbol f_i there belongs an integer n_i as its *arity*. The type τ is the indexed set $(n_i)_{i\in I}$ of the arities. We define n-ary terms of type τ as follows.

Definition 1.3. Let $n \geq 1$.

(i) Every variable $x_i \in X_n$ is an n-ary term.
(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.
(iii) The set $W_\tau(X_n) = W_\tau(x_1, \ldots, x_n)$ of all n-ary terms is the smallest set which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii).

On the set $W_\tau(X_n)$ we define a superposition operation S^n as follows.

Definition 1.4. Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ. Then the superposition operation S^n (for terms) is inductively defined by the following steps:

(i) If $x_j \in X_n$ is a variable and $t_1, \ldots, t_n \in W_\tau(X_n)$, then $S^n(x_j, t_1, \ldots, t_n) := t_j$ for $1 \leq j \leq n$.
(ii) If $f_i(s_1, \ldots, s_{n_i})$ is a composite term, then $S^n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$.

By simple calculation one shows that the algebra n-clone $\tau := (W_\tau(X_n); S^n)$ of type $(n+1)$ is a Menger algebra of rank n. Similarly, one can check that the algebra $(n\text{-clone } \tau)^+ := (W_\tau(X_n); S^n, x_1, \ldots, x_n)$ of type $(n+1, 0, \ldots, 0)$ is a unitary Menger algebra of rank n.

Now, for any $n \in \mathbb{N}^+$, we consider type $\tau_n := (n, n, \ldots, n)$ (i.e. each operation symbol has arity n) and we define the following set of terms:

$$F^n_{\tau_n} := \{f_i(x_1, \ldots, x_n) \mid i \in I\}.$$

It is not difficult to see that $F^n_{\tau_n} \cup X_n$ generates n-clone τ_n. Moreover, one can show that any mapping $\varphi_n : F^n_{\tau_n} \cup X_n \to W_{\tau_n}(X_n)$ can be extended to an endomorphism of n-clone τ_n. We say that n-clone τ_n is free with respect to itself.

Theorem 1.5. *The Menger algebra* n-clone τ_n *is free with respect to itself, freely generated by* $F^n_{\tau_n} \cup X_n$.

In a similar way one obtains the following.

Theorem 1.6. *The unitary Menger algebra* $(n\text{-clone } \tau_n)^+$ *is free with respect to itself, freely generated by* $F^n_{\tau_n}$.

In any Menger algebra $(G; S)$ of rank n, a binary operation $+$ can be defined by

$$x + y := S(x, y, \ldots, y)$$

The operation $+$ is associative. The semigroup $(G; +)$ is said to be diagonal semigroup (see, e.g. [11]). On the Cartesian power G^n one may define a binary

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operation $*$ by

$$(x_1,\dots,x_n)*(y_1,\dots,y_n) := (S(x_1,y_1,\dots,y_n),\dots,S(x_n,y_1,\dots,y_n)).$$

In the case of Menger algebras of n-ary terms semigroup properties such as idempotent and regular elements and Green's relations for the corresponding semigroups were investigated in [4].

If an algebra of type τ is given, then terms of type τ can be interpreted as term operations of this algebra.

Definition 1.7. Let $\mathcal{A} := (A; (f_i^{\mathcal{A}})_{i\in I})$ be an algebra of type τ and let t be an n-ary term of type τ over X_n. Then t induces an n-ary operation $t^{\mathcal{A}}$ on $\mathcal{A}$, which is called the *term operation induced by the term* t on the algebra $\mathcal{A}$, by the following steps:

(1) If $t = x_j \in X_n$, then $t^{\mathcal{A}} = x_j^{\mathcal{A}} = e_j^{n,A}$; here $e_j^{n,A}$ is the n-ary projection on A defined by $e_j^{n,A}(a_1,\dots,a_n) = a_j$ for all $a_1,\dots,a_n \in A$.
(2) If $t = f_i(t_1,\dots,t_{n_i})$ is an n-ary term of type τ, and $t_1^{\mathcal{A}},\dots,t_{n_i}^{\mathcal{A}}$ are the term operations which are induced by $t_1,\dots,t_{n_i}$, then $t^{\mathcal{A}} = f_i^{A}(t_1^{\mathcal{A}},\dots,t_{n_i}^{\mathcal{A}})$.

Let $W_\tau(X_n)^{\mathcal{A}}$ be the set of all n-ary term operations induced by the terms from $W_\tau(X_n)$ on the algebra $\mathcal{A}$. The set $W_\tau(X_n)^{\mathcal{A}} \subseteq O^n(A)$ is closed under the $(n+1)$-ary operation $S^{n,A}$. Therefore, one obtains an algebra n-clone $\mathcal{A} = (W_\tau(X_n)^{\mathcal{A}}; S^{n,A})$ which turns out to be a Menger algebra of rank n. Since the projections $e_1^{n,A},\dots,e_n^{n,A}$ are the term operations induced by $x_1,\dots,x_n$, respectively, the algebra (n-clone $\mathcal{A})^+ = (W_\tau(X_n)^{\mathcal{A}}; S^{n,A}, e_1^{n,A},\dots,e_n^{n,A})$ is a unitary Menger algebra of rank n. The mapping $v\colon W_\tau(X_n) \to W_\tau(X_n)^{\mathcal{A}}$ defined by $t \mapsto t^{\mathcal{A}}$ satisfies $v(S^n(t,t_1,\dots,t_n)) = S^n(t,t_1,\dots,t_n)^{\mathcal{A}} = t(t_1,\dots,t_n)^{\mathcal{A}} = t^{\mathcal{A}}(t_1^{\mathcal{A}},\dots,t_n^{\mathcal{A}}) = S^{n,A}(t^{\mathcal{A}},t_1^{\mathcal{A}},\dots,t_n^{\mathcal{A}}) = S^{n,A}(v(t),v(t_1),\dots,v(t_n))$. Therefore, v is a homomorphism from n-clone τ onto n-clone $\mathcal{A}$. Similarly, (n-clone $\mathcal{A})^+$ is a homomorphic image of (n-clone $\tau)^+$.

For proofs and more detailed information see, e.g. [4, 6, 7, 10].

2. Partial Menger Algebras of *r*-Terms

For partial algebras there are several possibilities to define homomorphisms and identities. If $\mathcal{A},\mathcal{B}$ are partial algebras of the same type with indexed sets $\{f_i^{\mathcal{A}} \mid i \in I\}$, $\{f_i^{\mathcal{B}} \mid i \in I\}$ of partial fundamental operations on A and on B, respectively, then a mapping $h : A \to B$ is said to be a weak homomorphism if for all fundamental operations the following is satisfied:

If $(a_1,\dots,a_{n_i}) \in \operatorname{dom} f_i^{\mathcal{A}}$, then $(h(a_1),\dots,h(a_{n_i})) \in \operatorname{dom} f_i^{\mathcal{B}}$ and then

$$h(f_i^{\mathcal{A}}(a_1,\dots,a_{n_i})) = f_i^{\mathcal{B}}(h(a_1),\dots,h(a_{n_i})), \quad i \in I,$$

where $\operatorname{dom} f_i^{\mathcal{A}}$ is the domain of the operation $f_i^{\mathcal{A}}$.

An equation $s \approx t$ of terms of type τ is satisfied as an identity in a total algebra $\mathcal{A}$ if the induced term operations $s^{\mathcal{A}}$ and $t^{\mathcal{A}}$ are equal.

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The equation $s \approx t$ is said to be a weak identity of the partial algebra $\mathcal{A}$ if after evaluation there holds: if the right-hand side is defined then the left-hand side is defined and both sides are equal or if the left-hand side is defined then the right-hand side is defined and both sides are equal.

The equation $s \approx t$ is said to be a strong identity of the partial algebra $\mathcal{A}$ if after evaluation there holds: if the right-hand side is defined then the left-hand side is defined and both sides are equal and if the left-hand side is defined then the right-hand side is defined and both sides are equal. That means, the left-hand side is defined if and only if the right-hand side is defined and if one side is defined, the both sides are equal. For more information on partial algebras see, e.g. [1].

There are several possibilities to measure the complexity of a term $t \in W_\tau(X_n)$ (see [5]). $\mathrm{var}(t)$ denotes the set of variables occurring in t. $vb_k(t)$ is the number of occurrences of the variable $x_k, 1 \leq k \leq n$, in t. Here is a formal inductive definition of $vb_k(t)$.

Definition 2.1. (i) If t is a variable from X_n and $k \in \{1, \ldots, n\}$, then $vb_k(t) := 1$ if $t = x_k$ and $vb_k(t) := 0$, otherwise.

(ii) If $t = f_i(t_1, \ldots, t_{n_i})$, then $vb_k(t) := \sum_{j=1}^{n_i} vb_k(t_j)$.

For $S^n(s, t_1, \ldots, t_n)$ we have the following formula.

Lemma 2.2. *For all $k \in \{1, \ldots, n\}$,*

$$vb_k(S^n(s, t_1, \ldots, t_n)) = \sum_{j=1}^{n} vb_j(s) vb_k(t_j).$$

Proof. We prove the lemma by induction on the complexity of s. If $s = x_i$ for $1 \leq i \leq n$, then

$$vb_k(S^n(s, t_1, \ldots, t_n)) = vb_k(t_i) = vb_i(x_i) vb_k(t_i) = \sum_{j=1}^{n} vb_j(x_i) vb_k(t_j).$$

If $s = f_i(s_1, \ldots, s_{n_i})$ is a composite term and if we assume that the formula holds for $s_1, \ldots, s_{n_i} \in W_\tau(X_n)$, then

$$S^n(s, t_1, \ldots, t_{n_i}) = f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$$

and

$$\begin{aligned} &vb_k(f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))) \\ &= \sum_{j=1}^{n_i} vb_k(S^n(s_j, t_1, \ldots, t_n)) \\ &= \sum_{j=1}^{n_i} \left(\sum_{l=1}^{n} vb_l(s_j) vb_k(t_l) \right) \end{aligned}$$

$$= \sum_{j=1}^{n} \left(\sum_{l=1}^{n_i} vb_j(s_l) \right) vb_k(t_j)$$

$$= \sum_{j=1}^{n} vb_j(f_i(s_1, \ldots, s_{n_i}))vb_k(t_j)$$

$$= \sum_{j=1}^{n} vb_j(s)vb_k(t_j).$$

□

Definition 2.3. Let $r \in \mathbb{N}^+$. An n-ary term $t \in W_\tau(X_n)$ is said to be an r-term if each variable from X_n occurs at most r times in t, i.e. if $vb_k(t) \leq r$, for all $1 \leq k \leq n$. Let $W_\tau^r(X_n)$ be the set of all r-terms in $W_\tau(X_n)$.

If $r = 1$, i.e. if each variable occurs at most once in t, then t is said to be linear. Let $W_\tau^{\text{lin}}(X_n)$ be the set of linear terms in $W_\tau(X_n)$. Linear expressions of the form $t = a_1x_1 + a_2x_2 + \cdots + a_lx_l + c$ of a vector space over a field, regarded as two-sorted algebra, are linear terms. In [2], the authors describe algebraic-structural properties of the set of all linear term operations of an algebra $\mathcal{A}$, induced by linear terms. Considering terms over particular algebraic structures, it can happen that each variable occurs only finitely many times, for instance in the case of nilpotent commutative semigroups. The idea to generalize linear terms to r-terms and to consider r-hypersubstitutions comes from [9].

The set of all n-ary r-terms is in general not closed under the superposition operation S^n as the following example shows.

Let $\tau = (2, 1)$ with operation symbols f and g. Let $s = f(g(x_1), x_4)$, $t_1 = f(x_2, x_1)$, $t_2 = f(x_3, g(x_4))$, $t_3 = f(f(x_1, x_2), g(x_3))$, $t_4 = g(f(g(x_1), g(x_2)))$. s, t_1, t_2, t_3, t_4 are linear since $vb_k(s), vb_k(t_1), vb_k(t_2), vb_k(t_3), vb_k(t_4) \leq 1$ for $k = 1, 2, 3, 4$, but

$$vb_1(S^4(s, t_1, t_2, t_3, t_4)) = vb_1(f(g(f(x_2, x_1)), g(f(g(x_1), g(x_2))))) = 2$$

and

$$vb_2(f(g(f(x_2, x_1)), g(f(g(x_1), g(x_2))))) = 2,$$

i.e. $S^4(s, t_1, t_2, t_3, t_4)$ is not linear.

Definition 2.4. Let $r \in \mathbb{N}^+$. The partial superposition operation

$$\bar{S}^n := (W_\tau^r(X_n))^{n+1} \multimap W_\tau^r(X_n)$$

is defined by

$$\bar{S}^n(s, t_1, \ldots, t_n) := \begin{cases} S^n(s, t_1, \ldots, t_n) & \text{if } \displaystyle\sum_{j=1}^{n} vb_j(s)vb_k(t_j) \leq r \ \text{ for all } 1 \leq k \leq n, \\ \text{Not defined} & \text{otherwise.} \end{cases}$$

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Remark 2.5. For $r = 1$ the condition $\sum_{j=1}^{n} vb_j(s)vb_k(t_j) \leq 1$ is equivalent to $\mathrm{var}(t_l) \cap \mathrm{var}(t_s) = \emptyset$ for all $1 \leq l < s \leq n$. Indeed, $\sum_{j=1}^{n} vb_j(s)vb_k(t_j) \leq 1$ implies $vb_j(s) \leq 1$ and $vb_k(t_j) \leq 1$ for all $1 \leq j \leq n, 1 \leq k \leq n$. If there were two terms t_m, t_p and a number $k, 1 \leq k \leq n$, with $vb_k(t_m) = vb_k(t_p) = 1$, then $\sum_{j=1}^{n} vb_j(s)vb_k(t_j) \geq 2$. If conversely, $\mathrm{var}(t_l) \cap \mathrm{var}(t_k) = \emptyset$ for all $1 \leq k < l \leq n$, then $\sum_{j=1}^{n} vb_j(s)vb_k(t_j) \leq 1$.

$(W_\tau^r(X_n); \bar{S}^n)$ is a partial algebra of type $(n+1)$ and $(W_\tau^r(X_n); \bar{S}^n, x_1, \ldots, x_n)$ is a partial algebra of type $(n+1, 0, \ldots, 0)$. We show that the first partial algebra satisfies (C1) as weak identity and that the second one satisfies (C1)–(C3) as weak identities. We define partial Menger algebras of rank n and unitary partial Menger algebras of rank n as follows.

Definition 2.6. Let M be a nonempty set, let $\circ\colon M^{n+1} \to M$ be an $(n+1)$-ary partial operation on M and let $e_1, \ldots, e_n$ be nullary (total) operations on M. If the partial algebra $\mathcal{M} = (M; \circ)$ satisfies (C1) as weak identity, then $\mathcal{M}$ is said to be a partial Menger algebra of rank n. If $\mathcal{M}^+ = (M; \circ, e_1, \ldots, e_n)$ satisfies (C1)–(C3) as weak identities, then $\mathcal{M}^+$ is said to be a unitary partial Menger algebra of rank n.

Theorem 2.7. *$(W_\tau^r(X_n); \bar{S}^n)$ is a partial Menger algebra of rank n.*

Proof. We have to prove that the superassociative identity (C1) is satisfied as weak identity. We show that after evaluation there holds: if the right-hand side is defined then the left-hand side is defined and both sides are equal. If we replace in (C1) the variables by terms $u, s_1, \ldots, s_n, t_1, \ldots, t_n \in W_\tau^r(X_n)$ and the operation symbol by $\bar{S}^n$, we obtain the pair

$$(\bar{S}^n(u, \bar{S}^n(s_1, t_1, \ldots, t_n), \ldots, \bar{S}^n(s_n, t_1, \ldots, t_n)), \bar{S}^n(\bar{S}^n(u, s_1, \ldots, s_n), t_1, \ldots, t_n)).$$

If the right-hand side is defined, then $\bar{S}^n(u, s_1, \ldots, s_n)$ is defined, i.e. $\sum_{j=1}^{n} vb_j(u)vb_k(s_j) \leq r$ for all $1 \leq k \leq n$ and then

$$\bar{S}^n(u, s_1, \ldots, s_n) = S^n(u, s_1, \ldots, s_n).$$

Moreover, $\bar{S}^n(\bar{S}^n(u, s_1, \ldots, s_n), t_1, \ldots, t_n)$ is defined, i.e.

$$\sum_{l=1}^{n} vb_l(S^n(u, s_1, \ldots, s_n))vb_k(t_l) = \sum_{l=1}^{n} \left(\sum_{j=1}^{n} vb_j(u)vb_l(s_j) \right) vb_k(t_l)$$
$$\leq r \quad \text{for all } 1 \leq k \leq n$$

and

$$\bar{S}^n(\bar{S}^n(u, s_1, \ldots, s_n), t_1, \ldots, t_n) = S^n(S^n(u, s_1, \ldots, s_n), t_1, \ldots, t_n).$$

Because of

$$\sum_{l=1}^{n} \left(\sum_{j=1}^{n} vb_j(u)vb_l(s_j) \right) vb_k(t_l) = \sum_{j=1}^{n} vb_j(u) \sum_{l=1}^{n} vb_l(s_j)vb_k(t_l)$$

we have also

$$\sum_{j=1}^{n} vb_j(u) \sum_{l=1}^{n} vb_l(s_j)vb_k(t_l) \leq r.$$

Then it follows that

$$\sum_{l=1}^{n} vb_l(s_j)vb_k(t_l) \leq r \quad \text{for all } 1 \leq k \leq n.$$

This means that $\bar{S}^n(s_j, t_1, \ldots, t_n)$ is defined and

$$\bar{S}^n(s_j, t_1, \ldots, t_n) = S^n(s_j, t_1, \ldots, t_n) \quad \text{for all } 1 \leq j \leq n$$

and that the left-hand side is defined and is equal to

$$S^n(u, S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)).$$

Since S^n satisfies (C1), the superassociative law is satisfied as weak identity.

In a similar way, one can prove that after evaluation there holds: if the left-hand side is defined then the right-hand side is defined and both sides are equal. Therefore, (C1) is even satisfied as a strong identity in the partial algebra $(W_\tau^r(X_n); \bar{S}^n)$. □

Theorem 2.8. *$(W_\tau^r(X_n); \bar{S}^n, x_1, \ldots, x_n)$ is a unitary partial Menger algebra of rank n.*

Proof. We have to prove that (C1)–(C3) are satisfied as weak identities. (C1) is clear. If we replace in (C2), (C3) the variables by terms $s, t_1, \ldots, t_n \in W_\tau^r(X_n)$, symbols for nullary operations by individual variables $x_1, \ldots, x_n, y$ and the operation symbol by $\bar{S}^n$, then we obtain the pairs

$$(\bar{S}^n(x_i, t_1, \ldots, t_n), t_i) \quad \text{and} \quad (\bar{S}^n(s, x_1, \ldots, x_n), s).$$

Since $t_i \in W_\tau^r(X_n)$, we have $vb_k(t_i) \leq r, 1 \leq i \leq n$, and since

$$vb_k(\bar{S}^n(x_i, t_1, \ldots, t_n)) = \sum_{j=1}^{n} vb_j(x_i)vb_k(t_j) = vb_k(t_i) \leq r$$

the left-hand side is defined and $\bar{S}^n(x_i, t_1, \ldots, t_n) = S^n(x_i, t_1, \ldots, t_n) = t_i$ since S^n satisfies (C2).

For the left-hand side of the second pair we have

$$vb_k(\bar{S}^n(s, x_1, \ldots, s_n)) = \sum_{j=1}^{n} vb_j(s)vb_k(x_j) = vb_k(s) \leq r \quad \text{for all } 1 \leq k \leq n$$

since $s \in W_\tau^r(X_n)$. Therefore, the left-hand side is defined and

$$\bar{S}^n(s, x_1, \ldots, x_n) = S^n(s, x_1, \ldots, x_n) = s$$

since S^n satisfies (C3).

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We remark that (C2) and (C3) are even satisfied as strong identities in $(W_\tau^r(X_n); \bar{S}^n, x_1, \ldots, x_n)$. □

3. Properties of $(W_{\tau_n}^r(X_n); \bar{S}^n, x_1, \ldots, x_n)$

In this section, we consider the type τ_n.

Lemma 3.1. *The unitary partial Menger algebra $(W_{\tau_n}^r(X_n); \bar{S}^n, x_1, \ldots, x_n)$ of rank n is generated by $F_{\tau_n}^n$.*

Proof. We show by induction on the complexity of a term $t \in W_\tau^r(X_n)$ that t can be generated from $F_{\tau_n}^n$. Since variables belong to the type, they can be generated. Let $t = f_i(t_1, \ldots, t_n)$ and assume that $t_1, \ldots, t_n$ can be generated. Then

$$\bar{S}^n(t, t_1, \ldots, t_n) = S^n(f_i(x_1, \ldots, x_n), t_1, \ldots, t_n) = f_i(t_1, \ldots, t_n) = t$$

since $\sum_{j=1}^n vb_j(f_i(x_1, \ldots, x_n))vb_k(t_j) \leq r$ because of $vb_j(f_i(x_1, \ldots, x_n)) = 1$ and $\sum_{j=1}^n vb_k(t_j) \leq r$ for all $1 \leq k \leq n$. □

Similarly, one obtains that $F_{\tau_n}^n \cup X_n$ is a generating system of $(W_{\tau_n}^r(X_n); \bar{S}^n)$.

We ask whether Theorem 1.6 is valid for unitary partial Menger algebras of r-terms. We call a unitary partial algebra free with respect to itself if there is a generating system such that each mapping from this generating system to the universe can be extended to a weak endomorphism. We will use that by Theorem 1.6 any mapping $\psi : F_{\tau_n}^n \to W_{\tau_n}(X_n)$ can be extended to an endomorphism of $(n\text{-clone } \tau_n)^+$.

Theorem 3.2. *The unitary partial Menger algebra $(W_{\tau_n}^r(X_n); \bar{S}^n; x_1, \ldots, x_n)$ of rank n is free with respect to itself if and only if $r = 1$,, i.e. if $W_{\tau_n}^r(X_n) = W_{\tau_n}^{\text{lin}}(X_n)$.*

Proof. We prove at first that $(W_{\tau_n}^{\text{lin}}(X_n); \bar{S}^n, x_1, \ldots, x_n)$ is free with respect to itself. By Lemma 3.1, $F_{\tau_n}^n$ is a generating system of $(W_{\tau_n}^r(X_n); \bar{S}^n; x_1, \ldots, x_n)$. Let $\varphi : F_{\tau_n}^n \to W_{\tau_n}^{\text{lin}}(X_n)$ be a mapping. We define a mapping $\bar{\varphi} : W_{\tau_n}^{\text{lin}}(X_n) \to W_{\tau_n}^{\text{lin}}(X_n)$ in the following way:

(i) $\bar{\varphi}(x_i) := x_i, 1 \leq i \leq n$.

(ii) $\bar{\varphi}(f_i(t_1, \ldots, t_n)) := \bar{S}^n(\varphi(f_i(x_1, \ldots, x_n)), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n))$ provided that $\bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n)$ are defined.

We show that $\bar{S}^n(\varphi(f_i(x_1, \ldots, x_n)), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n))$ is defined, provided that $\bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n)$ are defined. By definition of φ and by assumption, $\varphi(f_i(x_1, \ldots, x_n)), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n) \in W_{\tau_n}^{\text{lin}}(X_n)$. At first we see that for any linear term s we have $\text{var}(\bar{\varphi}(s)) \subseteq \text{var}(s)$. Indeed, if $s = x_i$ is a variable from X_n, then

$$\text{var}(\bar{\varphi}(x_i)) = \text{var}(x_i) = x_i.$$

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Now, we consider $s = f_i(s_1, \ldots, s_n)$ and suppose inductively that $\mathrm{var}(\bar{\varphi}(s_i)) \subseteq \mathrm{var}(s_i)$ for $1 \leq i \leq n$. Then

$$\begin{aligned}
\mathrm{var}(\bar{\varphi}(s)) &= \mathrm{var}(\bar{\varphi}(f_i(s_1, \ldots, s_n))) \\
&= \mathrm{var}(\bar{S}^n(\varphi(x_1, \ldots, x_n), \bar{\varphi}(s_1), \ldots, \bar{\varphi}(s_n))) \\
&\subseteq \bigcup_{l=1}^{n} \mathrm{var}(\bar{\varphi}(s_l)) \\
&\subseteq \bigcup_{l=1}^{n} \mathrm{var}(s_l) \\
&= \mathrm{var}(f_i(s_1, \ldots, s_n)) = \mathrm{var}(s).
\end{aligned}$$

As a consequence, we have

$$\mathrm{var}(t_l) \cap \mathrm{var}(t_k) = \emptyset \Rightarrow \mathrm{var}(\bar{\varphi}(t_l)) \cap \mathrm{var}(\bar{\varphi}(t_k)) = \emptyset \quad \text{for all } 1 \leq l < k \leq n.$$

Therefore, the right-hand side of (ii) is defined and is equal to $S^n(\varphi(f_i(x_1, \ldots, x_n)), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n))$. To show that $\bar{\varphi}$ is a weak endomorphism, for arbitrary terms $s, t_1, \ldots, t_n$ we consider the equation

$$\bar{\varphi}(\bar{S}^n(s, t_1, \ldots, t_n)) = \bar{S}^n(\bar{\varphi}(s), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n)).$$

Assume that $(s, t_1, \ldots, t_n) \in \mathrm{dom}\, \bar{S}^n$. Then $s, t_1, \ldots, t_n \in W^{\mathrm{lin}}_{\tau_n}(X_n)$ and $\mathrm{var}\,(t_j) \cap \mathrm{var}(t_k) = \emptyset$ for all $1 \leq j < k \leq n$. It follows that $\bar{\varphi}(s), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n) \in W^{\mathrm{lin}}_{\tau_n}(X_n)$ and $\mathrm{var}(\bar{\varphi}(t_j)) \cap \mathrm{var}(\bar{\varphi}(t_k)) = \emptyset, 1 \leq j < k \leq n$. Thus, $(\bar{\varphi}(s), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n)) \in \mathrm{dom}\, \bar{S}^n$, and by definition of $\bar{S}^n$ we have

$$\bar{\varphi}(\bar{S}^n(s, t_1, \ldots, t_n)) = \bar{\varphi}(S^n(s, t_1, \ldots, t_n))$$

and

$$\bar{S}^n(\bar{\varphi}(s), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n)) = S^n(\bar{\varphi}(s), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n)).$$

Since a mapping $\psi : F^n_{\tau_n} \to W_{\tau_n}(X_n)$ defined by (i) and (ii) can be extended to an endomorphism $\bar{\psi}$ of (n-clone $\tau_n)^+$ and since $\bar{\varphi}$ is the restriction of $\bar{\psi}$ onto $W^{\mathrm{lin}}_{\tau_n}(X_n)$, we obtain

$$\bar{\psi}(S^n(s, t_1, \ldots, t_n)) = S^n(\bar{\psi}(s), \bar{\psi}(t_1), \ldots, \bar{\psi}(t_n))$$

and then also

$$\bar{\varphi}(\bar{S}^n(s, t_1, \ldots, t_n)) = \bar{S}^n(\bar{\varphi}(s), \bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n))$$

and $\bar{\varphi}$ is a weak endomorphism.

If every operation symbol is unary, i.e. $\tau_n = (1, \ldots, 1)$, then $W_{\tau_n}(X_n) = W^{\mathrm{lin}}_{\tau_n}(X_n)$.

Therefore, for $r > 1$ we may assume that there is an index $j \in I$ such that $n_j > 1$ and then for the type τ_n, $n_j > 1$ for all $j \in I$. Following an idea of Lekkoksung

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(see [9]), we consider the cases $n_j \leq r$ and $n_j > r$ for all $j \in I$. In the first case, we consider a mapping $\varphi\colon F_{\tau_n} \to W^r_{\tau_n}(X_n)$ with $\varphi(f_j(x_1,\ldots,x_n)) = f_j(x_1,\ldots,x_1)$ for $j \in I$. Then $f_j(x_1,\ldots,x_1) \in W^r_{\tau_n}(X_n)$ since $vb_1(f_j(x_1,\ldots,x_1)) = n_j \leq r$. Consider a term t constructed from f_j and x_1 such that f_j occurs at least once in t with $vb_1(t) \leq r$, but $vb_1(t) > \frac{r}{n_j}$. Since $n_j \geq 2$, such terms exist. Since

$$\begin{aligned}\bar{\varphi}(f_j(x_1,\ldots,x_1)) &= \bar{\varphi}(\bar{S}^n(f_j(x_1,\ldots,x_n),x_1,\ldots,x_1))\\ &= \bar{\varphi}(S^n(f_j(x_1,\ldots,x_n),x_1,\ldots,x_1))\\ &= S^n(\bar{\varphi}(f_j(x_1,\ldots,x_n),\bar{\varphi}(x_1),\ldots,\bar{\varphi}(x_1))\\ &= S^n(f_j(x_1,\ldots,x_1),x_1,\ldots,x_1))\\ &= f_j(x_1,\ldots,x_1),\end{aligned}$$

for any such term t we have $\bar{\varphi}(t) = t$. Because of $vb_1(f_j(x_1,\ldots,x_n)) = 1, vb_1(t) \leq r$ and

$$vb_1(S^n(f_j(x_1,\ldots,x_n),t,x_2,\ldots,x_n)) = vb_1(f_j(x_1,\ldots,x_n)) \cdot vb_1(t) \leq 1 \cdot r = r,$$

we get $(f_j(x_1,\ldots,x_n),t,x_2,\ldots,x_n) \in \operatorname{dom}\bar{S}^n$.

On the other hand,

$$\begin{aligned}&(\bar{\varphi}(f_j(x_1,\ldots,x_n)),\bar{\varphi}(t),\bar{\varphi}(x_2),\ldots,\bar{\varphi}(x_n))\\ &\quad = (f_j(x_1,\ldots,x_1),t,x_2,\ldots,x_n) \notin \operatorname{dom}\bar{S}^n,\end{aligned}$$

since $vb_1(f_j(x_1,\ldots,x_1)) \cdot vb_1(t) > n_j \cdot \frac{r}{n_j} = r$. Thus, $\bar{\varphi}$ is not a weak endomorphism.

In the second case, we choose a mapping $\varphi : F_{\tau_n} \to W^r_{\tau_n}(X_n)$ with

$$\varphi(f_j(x_1,\ldots,x_n)) = f_j(x_1,\ldots,x_1,x_2,x_3,\ldots,x_{n_j-r+1}) \in W^r_{\tau_n}(X_n).$$

Then

$$(f_j(x_1,\ldots,x_n),f_j(x_1,\ldots,x_n),x_2,\ldots,x_n) \in \operatorname{dom}\bar{S}^n$$

since $vb_1(f_j(x_1,\ldots,x_n)) \cdot vb_1(f_j(x_1,\ldots,x_n)) = 1 \cdot 1 = 1 \leq r$, but $(\bar{\varphi}(f_j(x_1,\ldots, x_n)),\bar{\varphi}(f_j(x_1,\ldots,x_n)),\bar{\varphi}(x_2),\ldots,\bar{\varphi}(x_n)) = (f_j(x_1,\ldots,x_1,x_2,x_3,\ldots,x_{n_j-r+1}), f_j(x_1,\ldots,x_1,x_2,x_3,\ldots,x_{n_j-r+1}),\bar{\varphi}(x_2),\ldots,\bar{\varphi}(x_n)) \notin \operatorname{dom}\bar{S}^n$, since

$$\begin{aligned}&vb_1(f_j(x_1,\ldots,x_1,x_2,x_3,\ldots,x_{n_j-r+1}))\\ &\qquad \cdot vb_1(f_j(x_1,\ldots,x_1,x_2,x_3,\ldots,x_{n_j-r+1})) = r \cdot r > r,\end{aligned}$$

because of $r \geq 2$. □

4. r-Hypersubstitutions

Hypersubstitutions were introduced in [8] and are mappings sending n_i-ary operation symbols to n_i-ary terms. They can be extended to mappings defined on the set of terms. Using these extensions hypersubstitutions can be multiplied and together

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with an identity hypersubstitution one obtains a monoid. These monoids are useful to describe lattices of varieties and other classes of algebraic structures.

We mentioned already that the unitary Menger algebra (n-clone $\tau_n)^+ = (W_{\tau_n}(X_n); S^n, x_1, \ldots, x_n)$ of type $(n+1, 0, \ldots, 0)$ is generated by the set $F^n_{\tau_n} = \{f_i(x_1, \ldots, x_n) \mid i \in I\}$. In [3], hypersubstitutions σ of type τ_n and their extensions $\hat{\sigma}$ defined by

(i) $\hat{\sigma}[x_j] := x_j, 1 \leq j \leq n$;
(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_n)] := S^n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$

were related with endomorphisms of n-clone τ_n.

In Sec. 3, we introduced mappings $\varphi\colon F^n_{\tau_n} \to W_{\tau_n}(X_n)$. Such mappings are called substitutions. For total unitary Menger algebras of rank n substitutions can be extended to endomorphisms [3]. Also for partial unitary Menger algebras $(W^r_{\tau_n}(X_n); \bar{S}^n, x_1, \ldots, x_n)$ by Theorem 3.2 substitutions can be extended by

(1) $\bar{\varphi}(x_i) := x_i, 1 \leq i \leq n$;
(2) $\bar{\varphi}(f_i(t_1, \ldots, t_n)) := \bar{S}^n(\varphi(f_i(x_1, \ldots, x_n)), \bar{\varphi}(t_1), \ldots \bar{\varphi}(t_n))$ provided that $\bar{\varphi}(t_1), \ldots, \bar{\varphi}(t_n)$ are defined,

to weak endomorphisms of these partial Menger algebras if and only if $r = 1$. If the extension of a substitution exists, then a multiplication on the set $\mathrm{Subst}(\tau_n)$ of all substitutions can be defined by $\varphi_1 \otimes \varphi_2 := \bar{\varphi}_1 \circ \varphi_2$. Together with the identity substitution $\mathrm{id}\colon F^n_{\tau_n} \to F^n_{\tau_n}$ defined by $f_i(x_1, \ldots, x_n) \mapsto f_i(x_1, \ldots, x_n)$ for all $i \in I$ the set $\mathrm{Subst}(\tau_n)$ becomes the universe of a monoid $(\mathrm{Subst}(\tau_n); \otimes, \mathrm{id})$. There is a one-to-one correspondence between the set $\{f_i \mid i \in I\}$ and the set $F^n_{\tau_n}$ given by the identity hypersubstitution $\sigma_{\mathrm{id}} : \{f_i \mid i \in I\} \to F^n_{\tau_n}$ defined by $f_i \mapsto f_i(x_1, \ldots, x_n)$ for all $i \in I$. The relationship between hypersubstitutions and substitutions is given by $\sigma = \varphi \circ \sigma_{\mathrm{id}}$ and $\varphi = \sigma \circ \sigma_{\mathrm{id}}^{-1}$. If on $\mathrm{Hyp}(\tau_n)$ a product will be defined by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, then together with the identity hypersubstitution one obtains the monoid $(\mathrm{Hyp}(\tau_n); \circ_h, \sigma_{\mathrm{id}})$ which is isomorphic to the monoid $(\mathrm{Subst}(\tau_n); \otimes, \mathrm{id})$ of all substitutions of type τ_n. For the case $r = 1$, i.e. for the set $W^{\mathrm{lin}}_{\tau_n}(X_n)$ of all linear n-ary terms, the extensions $\bar{\varphi}$ of substitutions are weak endomorphisms of the partial Menger algebra $(W^{\mathrm{lin}}_{\tau_n}(X_n); \bar{S}^n, x_1, \ldots, x_n)$. The monoids of substitutions and of linear hypersubstitutions are total.

Now, we want to apply this approach to unitary partial Menger algebras $(W^r_{\tau_n}(X_n); \bar{S}^n, x_1, \ldots, x_n)$ of rank n if $r \geq 2$.

Definition 4.1. A hypersubstitution σ of type τ_n is said to be an r-hypersubstitution of type τ_n if σ maps every operation symbol to an n-ary r-term: $\sigma\colon \{f_i \mid i \in I\} \to W^r_{\tau_n}(X_n)$. Let $\mathrm{Hyp}^r(\tau_n)$ be the set of all r-hypersubstitutions of type τ_n.

The idea to generalize linear hypersubstitutions to r-hypersubstitutions comes from [9].

Using the partial superposition operations $\bar{S}^n$, we define the extension $\hat{\sigma} : W^r_{\tau_n}(X_n) \to W^r_{\tau_n}(X_n)$ of an r-hypersubstitution σ by

(i) $\hat{\sigma}[x_i] := x_i$ for $1 \le i \le n$.
(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_n)] := \bar{S}^n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$ if $f_i(t_1, \ldots, t_n)$ is a composed r-term and provided that $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$ are already defined.

The right-hand side of (ii) is defined if and only if $\sum_{l=1}^{n} vb_l(\sigma(f_i))vb_k(\hat{\sigma}[t_l]) \le r$. For $r = 1$ the right-hand side is always defined, since $\mathrm{var}(t_l) \cap \mathrm{var}(t_k) = \emptyset$ implies $\mathrm{var}(\hat{\sigma}[t_k]) \cap \mathrm{var}(\hat{\sigma}[t_l]) = \emptyset$.

Since $\sigma(f_i) = \hat{\sigma}(f_i(x_1, \ldots, x_n))$, (ii) corresponds to the equation (2) defining the extension $\bar{\varphi}$ of a substitution φ.

Then application of Theorem 3.2 gives the following corollary.

Corollary 4.2. *The extension $\hat{\sigma}$ of an r-hypersubstitution σ is a weak endomorphism of $(W^r_{\tau_n}(X_n); \bar{S}^n, x_1, \ldots, x_n)$ if and only if $r = 1$.*

If $r \ge 2$ the extension $\hat{\sigma}$ of an r-hypersubstitution is a partial mapping defined by (i) and

(ii′)

$$\bar{\sigma}[f_i(t_1, \ldots, t_n)] := \begin{cases} S^n(\sigma(f_i), \bar{\sigma}[t_1], \ldots, \bar{\sigma}[t_n]) & \text{if } \displaystyle\sum_{l=1}^{n} vb_l(\sigma(f_i))vb_k(\hat{\sigma}[t_l]) \le r \\ & \qquad \text{for all } 1 \le k \le n, \\ \text{Not defined} & \text{otherwise.} \end{cases}$$

For $r = 1$ we define a binary operation $\circ_h$ on $\mathrm{Hyp}^{\mathrm{lin}}(\tau_n)$ by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and obtain the monoid $(\mathrm{Hyp}^{\mathrm{lin}}(\tau_n); \circ_h, \sigma_{\mathrm{id}})$. In the general case, we define a partial binary operation $\bar{\circ}$ by $\sigma_1 \bar{\circ} \sigma_2 := \bar{\sigma}_1 \circ \sigma_2$ which is defined if $\sigma_2(f_i) \in \mathrm{dom}\, \bar{\sigma}_1$ for all $i \in I$ and obtain a partial monoid $(\mathrm{Hyp}(\tau_n)^r; \bar{\circ}, \sigma_{\mathrm{id}})$.

Corollary 4.3. *The partial monoid $(\mathrm{Hyp}^r(\tau_n); \bar{\circ}, \sigma_{\mathrm{id}})$ is total and equal to $(\mathrm{Hyp}^{\mathrm{lin}}(\tau_n); \circ_h, \sigma_{\mathrm{id}})$ if and only if $r = 1$.*

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