International Journal of Geometric Methods in Modern Physics
Vol. 18, No. 8 (2021) 2150125 (23 pages)

DOI: 10.1142/S0219887821501255

Chen-like inequalities on null hypersurfaces with closed rigging of a Lorentzian manifold

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Received 17 February 2021
Accepted 6 April 2021
Published 4 June 2021

We study null hypersurfaces of a Lorentzian manifold with a closed rigging for the hypersurface. We derive inequalities involving Ricci tensors, scalar curvature, squared mean curvatures for a null hypersurface with a closed rigging of a Lorentzian space form and for a screen homothetic null hypersurface of a Lorentzian manifold. We also establish a generalized Chen–Ricci inequality for a screen homothetic null hypersurface of a Lorentzian manifold with a closed rigging for the hypersurface.

Keywords: Lorentzian manifolds; null hypersurfaces; rigging vector field; generalized Chen–Ricci inequality.

Mathematics Subject Classification 2020: 53C50, 53B25, 53B30

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1. Introduction

When the ambient metric restricted to a hypersurface of a semi-Riemannian manifold is degenerate, the hypersurface is called a lightlike (null or degenerate) hypersurface [7]. The study of null hypersurfaces of a Lorentzian manifold is interesting, they produce models of various types of horizons (Kruskal's horizons, Cauchy's horizons, event horizons of a black hole, etc.) in the general theory of relativity. To explore the geometry of null hypersurfaces of semi-Riemannian manifolds and to overcome the anomaly raised because of the existence of degenerate-induced metric on the null hypersurfaces, Duggal and Bejancu [7] adopted a technique of fixing the geometric data formed by a null section and a null transversal section. But this technique has its own disadvantages [8,12]. Recently, Gutierrez and Olea [13] introduced a new rigging technique to study null hypersurfaces of a Lorentzian manifold. This rigging technique is based on the fixing of the geometric data formed by a unique rigging vector field ζ. This technique also allows to construct a Riemannian metric (called the rigged metric) on the null hypersurface and hence the geometry of null hypersurfaces can be explored extensively using a Riemannian metric.

For a submanifold of a Riemannian manifold, several extrinsic invariants are associated beside its intrinsic invariants. Most important intrinsic invariants of a submanifold are sectional curvatures, Ricci curvatures, scalar curvatures, and important extrinsic invariants are the mean curvature function, squared mean curvature, shape operator, etc. In [3], Chen established a sharp relationship between intrinsic invariant (sectional curvature) and the extrinsic invariant (shape operator) for submanifolds in real space forms. Later, for an n-dimensional submanifold M of a Riemannian manifold $\overline{M}$, Chen [4] established a basic inequality involving the main extrinsic invariant (squared mean curvature) and the main intrinsic invariant (Ricci curvature) of submanifolds in a real space form:

$$\operatorname{Ric}(X) \leq \frac{1}{4} n^2 \|H\|^2 + (n-1)c.$$

Let $T_p^1 M$ denote the set of unit vectors in T_pM, then Hong and Tripathi [15], derived a generalized relation between the Ricci curvature and the squared mean curvature for a submanifold M of $\overline{M}$ as

$$\operatorname{Ric}(X) \leq \frac{1}{4} n^2 \mu^2 + \overline{\operatorname{Ric}}_{T_pM}(X),$$

where $\overline{\operatorname{Ric}}_{T_pM}(X)$ is the n-Ricci curvature of T_pM at $X \in T_p^1 M$ with respect to the ambient manifold $\overline{M}$. Later, the above inequality was named as Chen–Ricci inequality by Tripathi in [19]. Hong *et al.* [14] also obtained a generalized relation between the intrinsic scalar curvature of an n-dimensional submanifold M and the extrinsic scalar curvature of a Riemannian manifold $\overline{M}$ as follows:

$$\tau(p) \leq \frac{n(n-1)}{2} \|H\|^2 + \overline{\tau}(T_pM).$$

Later, several authors explored inequalities for non-degenerate submanifolds of different spaces, for example [5,6,17] and many references therein. In [10,11,16] authors obtained Chen-like inequalities on null hypersurfaces of a Lorentzian manifold, particularly, authors established inequalities involving k-Ricci curvature, k-scalar curvature, the screen scalar curvature on a null hypersurface of a Lorentzian manifold.

In this paper, we study null hypersurface M of a Lorentzian manifold $(\overline{M}, \overline{g})$ with a closed rigging ζ for M. We derive inequalities involving Ricci tensors, scalar curvature tensors, squared mean curvatures for a null hypersurface with a closed rigging of a Lorentzian space form and for a screen homothetic null hypersurface of a Lorentzian manifold. We also establish a generalized Chen–Ricci inequality for a screen homothetic null hypersurface of a Lorentzian manifold with a closed rigging for the hypersurface (Theorem 21).

2. Null Hypersurfaces of a Lorentzian Manifold

Let M be a hypersurface of an $(n+2)$-dimensional time-orientable Lorentzian manifold $(\overline{M}, \overline{g})$. The hypersurface M is called a null hypersurface [2] of $\overline{M}$ if the induced metric $g = i^{\star}\overline{g}$ on the hypersurface M is degenerate, where $i : M \hookrightarrow \overline{M}$ is the canonical immersion. We can *choose* a null vector field ξ in TM, denote the complementary distribution to ξ in TM as $S(TM)$, called the screen distribution. Then, there exists a *unique* one-dimensional null distribution $l\mathrm{tr}(TM)$ orthogonal to $S(TM)$ not contained in TM, called the null transversal vector bundle, which is spanned by a null vector field N (as $\overline{M}$ being time-orientable), called the null transversal vector field such that $\overline{g}(N, \xi) = 1$, for more details, see [7].

Let $\overline{\nabla}$ be the Levi-Civita connection of $\overline{M}$ with respect to $\overline{g}$, then local Gauss–Weingarten formulae on TM are

$$\overline{\nabla}_U V = \nabla_U V + B(U, V)N, \quad \overline{\nabla}_U N = -A_N U + \tau(U)N,$$

for any $U, V \in \Gamma(TM)$, where ∇ is the induced linear connection on M, B is the local second fundamental form on TM, A_N is the shape operator on TM and τ is a 1-form on TM, defined by $\tau(U) = \overline{g}(\overline{\nabla}_U N, \xi)$. It should be noted that the local second fundamental form B is symmetric and independent of the choice of the screen distribution and satisfies

$$B(U, \xi) = 0. \tag{1}$$

The induced connection ∇ of M is not a metric connection and satisfies

$$(\nabla_U g)(V, W) = B(U, V)\eta(W) + B(U, W)\eta(V),$$

for any $U, V, W \in \Gamma(TM)$, where η is a 1-form on TM and given by $\eta(U) = \overline{g}(U, N)$, for any $U \in \Gamma(TM)$. Further, the Gauss–Weingarten formulae for $S(TM)$ are

$$\nabla_U X = \nabla^{\star}_U X + C(U, X)\xi, \tag{2}$$

$$\nabla_U \xi = -A^{\star}_{\xi} U - \tau(U)\xi, \tag{3}$$

for any $U \in \Gamma(TM)$, $X \in \Gamma(S(TM))$, where $\nabla^{\star}$ is the induced Levi-Civita connection on $S(TM)$, C is the local second fundamental form on $S(TM)$ and $A^{\star}_{\xi}$ is shape operator of $S(TM)$. Here, the local second fundamental form C is not symmetric and the local second fundamental forms B and C are related to their shape operators as

$$B(X,Y) = g(A^{\star}_{\xi}X, Y), \quad \overline{g}(A^{\star}_{\xi}X, N) = 0, \tag{4}$$

$$C(X,PY) = g(A_N X, PY), \quad \overline{g}(A_N X, N) = 0. \tag{5}$$

Moreover, for $X, Y \in \Gamma(S(TM))$, we also have

$$C(X,Y) - C(Y,X) = \overline{g}(N, [X,Y]), \tag{6}$$

this implies, the screen distribution is integrable if and only if the local second fundamental form C is symmetric.

Let $\overline{R}$ and R be the Riemannian curvature tensor of $\overline{\nabla}$ and ∇ on $\overline{M}$ and M, respectively, and P be the projection morphism of $\Gamma(TM)$ to $\Gamma(S(TM))$. Next, for any $U, V, W \in \Gamma(TM)$ and $X \in \Gamma(S(TM))$, we recall the following Gauss–Codazzi-type equations for M from [7, p. 94]:

$$\begin{aligned}\overline{g}(\overline{R}(U,V)W, X) &= g(R(U,V)W, X) + B(U,W)C(V,X)\\ &\quad - B(V,W)C(U,X),\end{aligned} \tag{7}$$

$$\begin{aligned}\overline{g}(\overline{R}(U,V)W, \xi) &= (\nabla_U B)(V,W) - (\nabla_V B)(U,W)\\ &\quad + B(V,W)\tau(U) - B(U,W)\tau(V),\end{aligned} \tag{8}$$

$$\begin{aligned}\overline{g}(\overline{R}(U,V)PW, N) &= (\nabla_U C)(V,PW) - (\nabla_V C)(U,PW)\\ &\quad + \tau(V)C(U,PW) - \tau(U)C(V,PW).\end{aligned} \tag{9}$$

It is known that the notion of totally umbilical hypersurfaces does not depend on the screen distribution neither on the choice of the null section. Therefore, this notion has sense in degenerate case and null hypersurface M is said to be totally umbilical if $B = \rho g$, for certain $\mathbf{C}^{\infty}$ function ρ on M, moreover, M is totally geodesic if B vanishes identically on M. Since $B(U,\xi) = 0$, then the mean curvature μ of the null hypersurface M with respect to an orthonormal basis $\{e_1, \ldots, e_n\}$ of the screen distribution is

$$\mu = \frac{\text{trac } B}{n} = \frac{1}{n}\sum_{i=1}^{n} B(e_i, e_i). \tag{10}$$

3. Rigging Vector Field for a Null Hypersurface

Let M be a null hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$. Let ζ be a vector field defined in some open set containing M. Denote α be a 1-form $\overline{g}$-metrically equivalent to ζ, that is, $\alpha = \overline{g}(\zeta, \cdot)$. Let $\omega = i^{\star}\alpha$, where $i : M \hookrightarrow \overline{M}$ be the canonical inclusion. Consider the tensors

$$\widehat{g} = \overline{g} + \alpha \otimes \alpha, \quad \widetilde{g} = i^{\star}\widehat{g} = i^{\star}\overline{g} + \omega \otimes \omega. \tag{11}$$

Lemma 1 ([13]). *Given a point $p \in M$, the following statements hold:*

(1) *$\widehat{g}_p$ is degenerate if and only if ζ_p is timelike and unitary for $\overline{g}$.*
(2) *$\widetilde{g}_p$ is Riemannian if and only if $\zeta_p \notin T_pM$.*

Clearly, if ζ_p is timelike and $|\zeta_p| > 1$ (respectively, < 1) then $\widehat{g}_p$ is Riemannian (respectively, Lorentzian).

Definition 2. Let M be a null hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$. A rigging for M is a vector field ζ defined on some open set containing M such that $\zeta_p \notin T_pM$, for any $p \in M$.

Thus, on fixing a rigging ζ for M, we have a Riemannian metric $\widetilde{g}$ on M, called the rigged metric-induced form ζ. The Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ is called the rigged connection on M.

Definition 3. The rigged vector field-induced form the rigging ζ is the $\widetilde{g}$-metrically equivalent vector field to the one-form ω and is denoted by ξ. Therefore $\omega = \widetilde{g}(\xi, \cdot)$.

Lemma 4 ([13]). *The rigged vector field ξ is the unique null vector field in M such that $\overline{g}(\zeta, \xi) = 1$. Moreover, ξ is $\widetilde{g}$-unitary.*

Consider the screen distribution $\mathcal{S}^{\zeta}$ given by $\mathcal{S}^{\zeta} = TM \cap \zeta^{\perp}$, here the screen distribution is denoted by $\mathcal{S}^{\zeta}$ to emphasize that it depends on ζ. It should be noted that $\mathcal{S}^{\zeta}$ is the $\widetilde{g}$-orthogonal subspace to ξ and the null transverse vector field to $\mathcal{S}^{\zeta}$ is given by

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi.$$

Observe that $\overline{g}(N, \xi) = 1$ and $\mathcal{S}^{\zeta}$ is $\overline{g}$-orthogonal to N. For any $U \in \Gamma(TM)$

$$\omega(U) = \widetilde{g}(\xi, U) = \overline{g}(\zeta, U) = \overline{g}(N, U) = \eta(U). \tag{12}$$

A null hypersurface M equipped with a rigging ζ is called normalized and (M, ζ) is called the normalization of M. A normalization (M, ζ) is said to be closed if the rigging ζ is closed, that is, the 1-form α is closed and normalization (M, ζ) is said to be conformal if the rigging ζ is conformal vector field, that is, there exists a function λ on the domain of ζ, such that $L_{\zeta}\overline{g} = 2\lambda\overline{g}$.

Let $\widetilde{\nabla}$ be the Levi-Civita connection (call rigged connection) induced on M by the rigged metric $\widetilde{g}$ and let $\overline{D} = \overline{\nabla} - \widetilde{\nabla}$ and $D = \nabla - \widetilde{\nabla}$. Then Gutierrez and Olea [13] derived the following important relations:

$$\begin{aligned} \overline{g}(\overline{D}(U, V), W) = -\frac{1}{2}\{&\omega(W)(L_{\xi}\widetilde{g})(U, V) + \omega(U)d\omega(V, W) \\ &+ \omega(V)d\omega(U, W)\}, \end{aligned} \tag{13}$$

$$\overline{g}(D(U,V),W) = -\frac{1}{2}\{\omega(W)(L_\xi\widetilde{g})(U,V) + \omega(U)d\omega(V,W) \\ +\omega(V)d\omega(U,W)\} - B(U,V)\omega(W), \qquad (14)$$

for any $U, V, W \in \Gamma(TM)$. Next, let the rigging ζ be closed then its rigged vector field ξ is also closed, using (11) and (14), for any $U \in \Gamma(TM)$ and $X, Y \in \Gamma(S^\zeta)$, we have

$$\widetilde{g}(D(U,X),Y) = g(D(U,X),Y) = -\frac{1}{2}\omega(U)d\omega(X,Y) = 0, \qquad (15)$$

further following the proofs [13, Propositions 3.7 and 3.11], it follows that

$$\widetilde{\nabla}_U Y = \nabla^*_U Y + B(U,Y)\xi, \quad \widetilde{\nabla}_U \xi = -A^*_\xi U. \qquad (16)$$

Hence, we will consider (16) as the Gauss–Weingarten formulae for the screen distribution $\mathcal{S}^\zeta$ of a null hypersurface M with a closed rigging ζ. Now, we recall the following for later uses:

Corollary 5 ([13]). *Given* $U \in \Gamma(TM)$ *and* $X, Y, Z \in \Gamma(S(TM))$, *we have*

- (i) $\widetilde{g}(D(X,U),X) = g(D(X,U),X) = 0$.
- (ii) $\widetilde{g}(D(X,Y),Z) = g(D(X,Y),Z) = 0$.
- (iii) $\widetilde{g}(D(U,\xi),\xi) = -\tau(U) = -\overline{g}(\overline{\nabla}_U\zeta,\xi)$.
- (iv) $-2C(U,X) = d\alpha(U,X) + (L_\zeta\overline{g})(U,X) + \overline{g}(\zeta,\zeta)B(U,X)$.

4. Chen-Like Inequalities

Let P be the projection morphism of $\Gamma(TM)$ onto $\Gamma(\mathcal{S}^\zeta)$, then any $U \in \Gamma(TM)$ can be written as

$$U = PU + \omega(U)\xi. \qquad (17)$$

Assume the rigging ζ to be closed. Let $\widetilde{R}$ be the Riemannian curvature tensor of the rigged connection $\widetilde{\nabla}$ on M, then by straightforward calculations, from (16), we obtain

$$\begin{aligned}
\widetilde{R}(U,V)W &= \widetilde{R}(U,V)PW + \omega(W)\widetilde{R}(U,V)\xi \\
&= R(U,V)W + ((\widetilde{\nabla}_U B)(V,PW) - (\widetilde{\nabla}_V B)(U,PW))\xi \\
&\quad - B(V,PW)A^\star_\xi U + B(U,PW)A^\star_\xi V \\
&\quad - ((\nabla_U C)(V,PW) - (\nabla_V C)(U,PW))\xi \\
&\quad + C(V,PW)A^\star_\xi U - C(U,PW)A^\star_\xi V \\
&\quad + C(V,PW)\tau(U)\xi - C(U,PW)\tau(V)\xi \\
&\quad + \omega(W)\{C(U,A^\star_\xi V)\xi - C(V,A^\star_\xi U)\xi + d\tau(U,V)\xi \\
&\quad - \tau(V)A^\star_\xi U + \tau(U)A^\star_\xi V\},
\end{aligned} \qquad (18)$$

for any $U, V, W \in \Gamma(TM)$, where

$$(\widetilde{\nabla}_U B)(V, W) = U(B(V, W)) - B(\widetilde{\nabla}_U V, W) - B(V, \widetilde{\nabla}_U W),$$

$$(\nabla_U C)(V, PW) = U(C(V, PW)) - C(\nabla_U V, PW) - C(V, \nabla^\star_U PW).$$

Further, using (1), (4), and (11) in the last expression, we derive

$$\begin{aligned}\widetilde{g}(\widetilde{R}(U, V)W, X) &= g(R(U, V)W, X) - B(V, W)B(U, X) + B(U, W)B(V, X)\\ &\quad + C(V, PW)B(U, X) - C(U, PW)B(V, X)\\ &\quad - \omega(W)\tau(V)B(U, X) + \omega(W)\tau(U)B(V, X),\end{aligned} \tag{19}$$

for any $U, V, W \in \Gamma(TM)$ and $X \in \Gamma(S^\zeta)$. Furthermore, using (7) in (19), we have

$$\begin{aligned}\widetilde{g}(\widetilde{R}(U, V)W, X) &= \overline{g}(\overline{R}(U, V)W, X) - B(U, W)C(V, X) + B(V, W)C(U, X)\\ &\quad - B(V, W)B(U, X) + B(U, W)B(V, X)\\ &\quad + C(V, PW)B(U, X) - C(U, PW)B(V, X)\\ &\quad - \omega(W)\tau(V)B(U, X) + \omega(W)\tau(U)B(V, X).\end{aligned} \tag{20}$$

Let $p \in M$ and $\Pi = \operatorname{span}\{e_i, e_j\}$ be a two-dimensional non-degenerate plane of T_pM, then the number

$$K_{ij} = \frac{g(R(e_i, e_j)e_j, e_i)}{g(e_i, e_i)g(e_j, e_j) - g(e_i, e_j)^2},$$

is called the sectional curvature at point $p \in M$. Let $\widetilde{K}_{ij}$ and $\overline{K}_{ij}$ be the sectional curvature of the non-degenerate plane section spanned by e_i and e_j at point p in the hypersurface $(M, \widetilde{g})$ and in the ambient Lorentzian manifold $(\overline{M}, \overline{g})$, respectively. Then $\widetilde{K}_{ij}$ and $\overline{K}_{ij}$ are the intrinsic and extrinsic sectional curvatures of the non-degenerate plane section spanned by e_i and e_j at p. Let $\Pi = \operatorname{span}\{e_i, e_j\}$ be a $\widetilde{g}$-orthogonal non-degenerate plane section to ξ, spanned by unitary and orthogonal vector fields $e_i, e_j \in \Gamma(\mathcal{S}^\zeta)$ then from (20), we obtain

$$\widetilde{K}_{ij} = \overline{K}_{ij} - B_{ij}C_{ji} - B_{ij}C_{ij} + C_{ii}B_{jj} - B_{ii}B_{jj} + B_{ij}^2 + B_{ii}C_{jj}, \tag{21}$$

where $B_{ij} = B(e_i, e_j)$ and $C_{ij} = C(e_i, e_j)$. If the rigging ζ is closed then (iv) of Corollary 5 becomes

$$2C(X, Y) = -\overline{g}(\overline{\nabla}_X \zeta, Y) - \overline{g}(X, \overline{\nabla}_Y \zeta) - \overline{g}(\zeta, \zeta)B(X, Y),$$

implies the local screen second fundamental form C is symmetric and hence from (6), the screen distribution is integrable and (21) can be written as

$$\widetilde{K}_{ij} = \overline{K}_{ij} - 2B_{ij}C_{ij} + C_{ii}B_{jj} - B_{ii}B_{jj} + B_{ij}^2 + B_{ii}C_{jj}. \tag{22}$$

Let $\Pi^{\text{null}}_\xi = \operatorname{span}\{e_i, \xi\}$ be a null plane [1] directed by ξ, where $e_i \in \Gamma(\mathcal{S}^\zeta)$ is a unitary vector field then the null sectional curvature K^{null}_i at p of Π^{null}_ξ with respect to ∇ is a real number, defined as $K^{\text{null}}_i = \frac{g_p(R(e_i,\xi)\xi, e_i)}{g_p(e_i, e_i)} = g_p(R(e_i, \xi)\xi, e_i)$. For the same plane section span$\{e_i, \xi\}$ where $e_i \in \Gamma(\mathcal{S}^\zeta)$ let denote by $\widetilde{K}_{i\xi}$ its $\widetilde{g}$-sectional curvature. Then using (1) in (20), we derive

$$\widetilde{K}_{i\xi} = \overline{K}_i^{\mathrm{null}} - \tau(\xi)B_{ii}. \tag{23}$$

Hence, from (22) and (23), we have the following observation immediately.

Theorem 6. *Let* $(\overline{M}, \overline{g})$ *be a Lorentzian manifold with a closed rigging* ζ *for a null hypersurface* M*. Then if the null hypersurface* M *is totally geodesic then the intrinsic and extrinsic sectional curvatures of* M *coincide.*

Let $\{e_1, \ldots, e_n, e_{n+1} = \xi\}$ be a basis of TM, where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathcal{S}^\zeta$. Then the Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{g}$ on $\overline{M}$ and the Ricci tensor $\widetilde{\mathrm{Ric}}$ of $\widetilde{g}$ are given for vector fields U, V on M by

$$\overline{\mathrm{Ric}}(U,V) = \sum_{j=1}^{n} \overline{g}(\overline{R}(e_j, U)V, e_j) + \overline{g}(\overline{R}(\xi, U)V, N) + \overline{g}(\overline{R}(N, U)V, \xi), \tag{24}$$

$$\widetilde{\mathrm{Ric}}(U,V) = \sum_{j=1}^{n} \widetilde{g}(\widetilde{R}(e_j, U)V, e_j) + \widetilde{g}(\widetilde{R}(\xi, U)V, \xi). \tag{25}$$

By straightforward calculations, using (2) and (3) we get for the induced Riemannian curvature tensor R on the normalized null hypersurface (M, ζ),

$$\begin{aligned} R(U,V)W &= R(U,V)PW + \omega(W)R(U,V)\xi \\ &= R^\star(U,V)W + ((\nabla_U C)(V, PW) - (\nabla_V C)(U, PW))\xi \\ &\quad - C(V, PW)A^\star_\xi(U) + C(U, PW)A^\star_\xi(V) \\ &\quad - \tau(U)C(V, PW)\xi + \tau(V)C(U, PW)\xi \\ &\quad + \omega(W)\{-\nabla^\star_U A^\star_\xi V - \nabla^\star_V A^\star_\xi U - A^\star_\xi[U,V] - C(U, A^\star_\xi V)\xi \\ &\quad + C(V, A^\star_\xi U)\xi - d\tau(U,V)\xi + \tau(V)A^\star_\xi U - \tau(U)A^\star_\xi V\}, \end{aligned} \tag{26}$$

this further implies

$$\begin{aligned} \widetilde{g}(R(\xi,U)V,\xi) &= (\nabla_\xi C)(U, PV) - (\nabla_U C)(\xi, PV) - \tau(\xi)C(U, PV) \\ &\quad + \tau(U)C(\xi, PV) - \omega(V)\{C(\xi, A^\star_\xi U) + d\tau(\xi, U)\}. \end{aligned} \tag{27}$$

Hence, for closed rigging ζ and using [13, Lemma 4.7] we have $\widetilde{\nabla}_U B = \nabla_U B$. This together with (8), (18) and (27) leads to

$$\begin{aligned} \widetilde{g}(\widetilde{R}(\xi,U)V,\xi) &= (\widetilde{\nabla}_\xi B)(U,V) - (\widetilde{\nabla}_U B)(\xi,V) = (\nabla_\xi B)(U,V) - (\nabla_U B)(\xi,V) \\ &= \overline{g}(\overline{R}(\xi,U)V,\xi) - B(U,V)\tau(\xi). \end{aligned} \tag{28}$$

Then, thanks to (20), (24), (25) and (28), we derive

$$\begin{aligned}\widetilde{\mathrm{Ric}}(U,V) = \overline{\mathrm{Ric}}(U,V) &- \overline{g}(\overline{R}(\xi,U)V,N) - \overline{g}(\overline{R}(N,U)V,\xi)\\ &+ \sum_{j=1}^{n}\{-B(e_j,V)C(U,e_j) + B(U,V)C(e_j,e_j)\\ &- B(U,V)B(e_j,e_j) + B(e_j,V)B(U,e_j)\\ &+ C(U,PV)B(e_j,e_j) - C(e_j,PV)B(U,e_j)\\ &- \omega(V)\tau(U)B(e_j,e_j) + \omega(V)\tau(e_j)B(U,e_j)\}\\ &+ (\nabla_\xi B)(U,V) - g(A^*_\xi U, A^*_\xi V), \end{aligned} \tag{29}$$

$$\sum_{j=1}^{n} B(e_j,V)C(U,e_j) = g(A^*_\xi V, A_N U), \tag{30}$$

$$\sum_{j=1}^{n} B(U,e_j)B(V,e_j) = g(A^*_\xi U, A^*_\xi V). \tag{31}$$

Moreover, it is easy to see that for closed rigging ζ,

$$C(e_j,PV) - \tau(e_j)\omega(V) = C(V,e_j). \tag{32}$$

Hence, on using (30)–(32) in (29), we obtain

$$\begin{aligned}\widetilde{\mathrm{Ric}}(U,V) = \overline{\mathrm{Ric}}(U,V) &- \overline{g}(\overline{R}(\xi,U)V,N) - \overline{g}(\overline{R}(\xi,V)U,N)\\ &+ B(U,V)\sum_{j=1}^{n}(C(e_j,e_j) - B(e_j,e_j))\\ &+ (C(U,PV) - \tau(U)\omega(V))\sum_{j=1}^{n} B(e_j,e_j)\\ &- g(A^*_\xi V, A_N U) - g(A^*_\xi U, A_N V) + (\nabla_\xi B)(U,V). \end{aligned} \tag{33}$$

Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of the screen distribution, then from (25), for a fixed $i \in \{1,\ldots,n\}$,

$$\widetilde{\mathrm{Ric}}(e_i) = \sum_{j=1}^{n}\widetilde{g}(\widetilde{R}(e_j,e_i)e_i,e_j) + \widetilde{g}(\widetilde{R}(\xi,e_i)e_i,\xi) = \sum_{j=1}^{n}\widetilde{K}_{ji} + \widetilde{K}_{\xi i}, \tag{34}$$

and

$$\widetilde{\mathrm{Ric}}(\xi) = \sum_{j=1}^{n}\widetilde{g}(\widetilde{R}(e_j,\xi)\xi,e_j) + \widetilde{g}(\widetilde{R}(\xi,\xi)\xi,\xi) = \sum_{j=1}^{n}\widetilde{K}_{j\xi}. \tag{35}$$

On adding (34) and (35), we obtain for the scalar curvature $\widetilde{r}$,

$$\widetilde{r} = \sum_{i=1}^{n}\widetilde{\mathrm{Ric}}(e_i) + \widetilde{\mathrm{Ric}}(\xi) = \sum_{i,j=1}^{n}\widetilde{K}_{ij} + 2\sum_{i=1}^{n}\widetilde{K}_{i\xi}. \tag{36}$$

Similarly, for a quasi-orthonormal frame $\{e_1, \ldots, e_n, \xi, N\}$ for a Lorentzian manifold $\overline{M}$, the scalar curvature $\overline{r}$ of $\overline{M}$ is given as follows:

$$\overline{r}_{TM} = \sum_{i,j=1}^{n} \overline{K}_{ij} + \sum_{i=1}^{n} \overline{K}_{iN} + \sum_{i=1}^{n} \overline{K}_{i\xi} + \sum_{i=1}^{n} \overline{K}_i^{\mathrm{null}},$$

where $\overline{K}_{iN} = \overline{g}(\overline{R}(\xi, e_i)e_i, N)$ and $\overline{K}_{i\xi} = \overline{g}(\overline{R}(N, e_i)e_i, \xi)$. Since

$$\overline{K}_{iN} = \overline{g}(\overline{R}(\xi, e_i)e_i, N) = \overline{g}(\overline{R}(N, e_i)e_i, \xi) = \overline{K}_{i\xi},$$

then

$$\overline{r}_{TM} = \sum_{i,j=1}^{n} \overline{K}_{ij} + 2\sum_{i=1}^{n} \overline{K}_{iN} + \sum_{i=1}^{n} \overline{K}_i^{\mathrm{null}}. \tag{37}$$

From (23), (36) and (37), we derive the following relationship:

$$\widetilde{r} = \overline{r}_{TM} + \sum_{i,j=1}^{n} (\widetilde{K}_{ij} - \overline{K}_{ij}) + \sum_{i=1}^{n} \widetilde{K}_{i\xi} - 2\sum_{i=1}^{n} \overline{K}_{iN} - \tau(\xi)\sum_{i=1}^{n} B_{ii}. \tag{38}$$

Assume that the ambient manifold has constant curvature, that is

$$\overline{R}(U, V)W = \overline{c}\{\overline{g}(V, W)U - \overline{g}(U, W)V\}, \tag{39}$$

for any $U, V, W \in \Gamma(T\overline{M})$.

A normalized null hypersurface (M, ζ) is said to be screen locally conformal [9] if the shape operators A_N and A^*_ξ satisfy

$$A_N = \phi A^*_\xi, \tag{40}$$

where ϕ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ on M. Furthermore, M is said to be screen homothetic if ϕ is a non-zero constant.

It is also easy to check the following relations:

$$\sum_{i,j=1}^{n} B_{ij}C_{ij} = \frac{1}{2}\left\{\sum_{i,j=1}^{n} (B_{ij} + C_{ij})^2 - \sum_{i,j=1}^{n} \{(B_{ij})^2 + (C_{ij})^2\}\right\}, \tag{41}$$

$$\sum_{i,j=1}^{n} B_{ii}C_{jj} = \frac{1}{2}\left\{\left(\sum_{i=1}^{n} B_{ii} + \sum_{j=1}^{n} C_{jj}\right)^2 - \left(\sum_{i=1}^{n} B_{ii}\right)^2 - \left(\sum_{j=1}^{n} C_{jj}\right)^2\right\}. \tag{42}$$

Theorem 7. *Let M be a null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M. Then*

(i)

$$\widetilde{r} \le \overline{r}_{TM} - n(2\overline{c} + n\mu^2 + 2\tau(\xi)\mu) + 2n\mu(\mathrm{trac}\ A_N) + 2\sum_{i,j=1}^{n} B_{ij}^2 + \sum_{i,j=1}^{n} C_{ij}^2.$$

The equality holds if and only if M is a screen homothetic null hypersurface either with $\phi = -1$ such that $A_N\xi = 0$ or M is a totally geodesic null hypersurface.

(ii)

$$\widetilde{r} \geq \overline{r}_{TM} - n(2\overline{c} + n\mu^2 + 2\tau(\xi)\mu) + 2n\mu(\text{trac } A_N) - \sum_{i,j=1}^{n}(B_{ij} + C_{ij})^2.$$

The equality holds if and only if M is a totally geodesic screen homothetic null hypersurface.

(iii) *The equality in above two cases holds if and only if M is a totally geodesic screen homothetic null hypersurface.*

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the screen distribution then using (30), (33) and (39), for a fixed $i \in \{1, \ldots, n\}$, the Ricci curvature of e_i with respect to $\widetilde{g}$ is given by

$$\begin{aligned}\widetilde{\text{Ric}}(e_i) = \overline{\text{Ric}}(e_i) - 2\overline{c} + B_{ii}\sum_{j=1}^{n} C_{jj} - B_{ii}\sum_{j=1}^{n} B_{jj} + C_{ii}\sum_{j=1}^{n} B_{jj}\\ - 2\sum_{j=1}^{n} B_{ij}C_{ij} + (\nabla_\xi B)(e_i, e_i).\end{aligned} \tag{43}$$

From (3), (4), (8), (31) and (39), we have

$$\begin{aligned}(\nabla_\xi B)(e_i, e_i) &= \overline{g}(\overline{R}(\xi, e_i)e_i, \xi) + (\nabla_{e_i} B)(\xi, e_i) - B(e_i, e_i)\tau(\xi)\\ &= g(A^*_\xi e_i, A^*_\xi e_i) - B(e_i, e_i)\tau(\xi) = \sum_{j=1}^{n} B_{ij}^2 - B_{ii}\tau(\xi),\end{aligned}$$

therefore from (43), we derive

$$\begin{aligned}\sum_{i=1}^{n}\widetilde{\text{Ric}}(e_i) = \sum_{i=1}^{n}\overline{\text{Ric}}(e_i) - 2n\overline{c} + 2\sum_{i,j=1}^{n} B_{ii}C_{jj} - \left(\sum_{i=1}^{n} B_{ii}\right)^2\\ - 2\sum_{i,j=1}^{n} B_{ij}C_{ij} + \sum_{i,j=1}^{n} B_{ij}^2 - \tau(\xi)\sum_{i=1}^{n} B_{ii},\end{aligned} \tag{44}$$

and using (33), we also have

$$\widetilde{\text{Ric}}(\xi) = \overline{\text{Ric}}(\xi) - \tau(\xi)\sum_{j=1}^{n} B_{jj}. \tag{45}$$

Hence, from (44) and (45), we obtain

$$\begin{aligned}\widetilde{r} = \overline{r}_{TM} - 2n\overline{c} + 2\sum_{i,j=1}^{n} B_{ii}C_{jj} - \left(\sum_{i=1}^{n} B_{ii}\right)^2 - 2\sum_{i,j=1}^{n} B_{ij}C_{ij}\\ + \sum_{i,j=1}^{n} B_{ij}^2 - 2\tau(\xi)\sum_{i=1}^{n} B_{ii},\end{aligned} \tag{46}$$

and further using (10), we get

$$\widetilde{r}=\overline{r}_{TM}-n(2\overline{c}+n\mu^2+2\tau(\xi)\mu)+2n\mu(\text{trac } A_N)-2\sum_{i,j=1}^{n}B_{ij}C_{ij}+\sum_{i,j=1}^{n}B_{ij}^2. \tag{47}$$

Thus, on using (41) in the last expression, it follows that

$$\begin{aligned}\widetilde{r}&=\overline{r}_{TM}-n(2\overline{c}+n\mu^2+2\tau(\xi)\mu)+2n\mu(\text{trac } A_N)\\&\quad-\sum_{i,j=1}^{n}(B_{ij}+C_{ij})^2+2\sum_{i,j=1}^{n}B_{ij}^2+\sum_{i,j=1}^{n}C_{ij}^2.\end{aligned} \tag{48}$$

□

Now, we recall the following results.

Theorem 8 ([20]). *Let $(M,g,S(TM))$ be an $(n+1)$-dimensional screen conformal lightlike hypersurface of a semi-Riemannian space form $(\overline{M}(\overline{c}),\overline{g})$ with screen conformal function ϕ and $n>2$. Set $\beta=\xi(\phi)-2\phi\tau(\xi)$. Then*

(i) *if $\beta=0$, we have $\overline{c}=0$.*
(ii) *if $\beta\neq 0$, we also have $\overline{c}=0$. Moreover, in this case $S(TM)$ and M are totally geodesic immersed in M and $\overline{M}$, respectively.*

Corollary 9 ([20]). *Let $(M,g,S(TM))$ be an $(n+1)$-dimensional screen homothetic lightlike hypersurface of a semi-Riemannian space form $(\overline{M}(\overline{c}),\overline{g})$ and $n>2$. Then*

(i) *if $\tau(\xi)=0$, we have $\overline{c}=0$.*
(ii) *if $\tau(\xi)\neq 0$, we also have $\overline{c}=0$. Moreover, in this case $S(TM)$ and M are totally geodesic immersed in M and $\overline{M}$, respectively.*

Corollary 10. *Let M be a screen conformal null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M such that $\beta\neq 0$. Then $\widetilde{r}=\overline{r}_{TM}$.*

Corollary 11. *Let M be a null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M. Then*

(i)

$$\widetilde{r}\leq n(n-1)\overline{c}-n\mu(n\mu+2\tau(\xi))+2n\mu(\text{trac } A_N)+2\sum_{i,j=1}^{n}B_{ij}^2+\sum_{i,j=1}^{n}C_{ij}^2.$$

(ii)

$$\begin{aligned}\widetilde{r}&\geq n(n-1)\overline{c}-n\mu(n\mu+2\tau(\xi))+2n\mu(\text{trac } A_N)\\&\quad-\sum_{i,j=1}^{n}(B_{ij}+C_{ij})^2+\sum_{i,j=1}^{n}B_{ij}^2+\sum_{i,j=1}^{n}C_{ij}^2.\end{aligned}$$

For screen homothetic and closed normalization (M, ζ) of a Lorentzian space form $\overline{M}(\overline{c})$ it follows from (47) that

$$\widetilde{r} = n(n-1)\overline{c} - 2\tau(\xi)n\mu + (2\phi - 1)n^2\mu^2 - (2\phi - 1)\sum_{i,j=1}^{n} B_{ij}^2. \tag{49}$$

Hence, from Corollary 9, we have the following assertion.

Corollary 12. *Let M be a screen homothetic null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M.*

- (i) *If $\tau(\xi) \neq 0$ then $\widetilde{r} = 0$.*
- (ii) *If $\tau(\xi) = 0$ then $\widetilde{r} = (2\phi - 1)[n^2\mu^2 - \sum_{i,j=1}^{n} B_{ij}^2]$. Therefore*
  - (a) *if $\phi > \frac{1}{2}$ then $\widetilde{r} \leq (2\phi - 1)n^2\mu^2$, with equality if and only if M is totally geodesic in which case $\widetilde{r} = 0$.*
  - (b) *if $\phi < \frac{1}{2}$ then $\widetilde{r} \leq (1 - 2\phi)\sum_{i,j=1}^{n} B_{ij}^2$, with equality if and only if M is maximal.*
  - (c) *if $\phi = \frac{1}{2}$ then $\widetilde{r} = 0$.*

Next, we recall the following lemma from [9].

Lemma 13. *Let M be a screen homothetic (ϕ is non-zero constant) null hypersurface of a space form $\overline{M}(\overline{c})$. Then $2\phi\tau(\xi)B(U, PV) = -\overline{c}g(U, PV)$.*

Hence, under the hypothesis of above lemma, if $\tau = 0$ then $\overline{c} = 0$ and if $\tau(\xi) \neq 0$ then M is totally umbilical in $\overline{M}$. We also recall the following lemma.

Lemma 14 ([13]). *Let M be a null hypersurface of a Lorentzian manifold $\overline{M}$ and ζ a rigging for it. If ζ is conformal, then $\overline{\nabla}_\xi \xi = 0$, that is, $\tau(\xi) = 0$. Moreover, $\tau(X) = -\frac{1}{2}g(\widetilde{\nabla}_\xi \xi, X)$, for all $X \in \mathcal{S}^\zeta$.*

Hence, on using Lemmas 13 and 14 in (49), we have the following theorem.

Theorem 15. *Let M be a screen homothetic null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a conformal closed rigging ζ for M. If $\phi = \frac{1}{2}$ then $\widetilde{r} = 0$.*

We recall following lemma from [18].

Lemma 16. *If $a_1, \ldots, a_n$ are n-real numbers ($n > 1$), then*

$$\frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)^2 \leq \sum_{i=1}^{n} a_i^2, \tag{50}$$

with equality if and only if $a_1 = \cdots = a_n$.

Theorem 17. *Let M be a screen homothetic null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M such that $\phi > \frac{1}{2}$. Then*

$$\widetilde{r} \leq n(n-1)\big(\overline{c} + (2\phi - 1)\mu^2\big) + \frac{n\overline{c}}{\phi}. \tag{51}$$

Equality of (51) holds at $p \in M$ if and only if p is a totally umbilical point.

Proof. Let M be a screen homothetic null hypersurface of $\overline{M}$ with closed rigging ζ, then from (46), we have

$$\widetilde{r} = \overline{r}_{TM} - 2n\overline{c} + (2\phi - 1)\left(\sum_{i=1}^{n} B_{ii}\right)^2 - (2\phi - 1)\sum_{i,j=1}^{n} B_{ij}^2 - 2\tau(\xi)\sum_{i=1}^{n} B_{ii}. \tag{52}$$

We can write

$$\sum_{i,j=1}^{n} B_{ij}^2 = \sum_{i=1}^{n} B_{ii}^2 + \sum_{i\neq j=1}^{n} B_{ij}^2,$$

then (52) becomes

$$\begin{aligned}\widetilde{r} = \overline{r}_{TM} - 2n\overline{c} + (2\phi - 1)n^2\mu^2 - (2\phi - 1)\sum_{i=1}^{n} B_{ii}^2 - (2\phi - 1)\sum_{i\neq j=1}^{n} B_{ij}^2 \\ - 2\tau(\xi)\sum_{i=1}^{n} B_{ii}.\end{aligned} \tag{53}$$

Using (50), we derive

$$\widetilde{r} \leq \overline{r}_{TM} - 2n\overline{c} + (2\phi - 1)n(n-1)\mu^2 - (2\phi - 1)\sum_{i\neq j=1}^{n} B_{ij}^2 - 2\tau(\xi)\sum_{i=1}^{n} B_{ii}, \tag{54}$$

this further implies

$$\widetilde{r} \leq \overline{r}_{TM} - 2n\overline{c} + (2\phi - 1)n(n-1)\mu^2 - 2\tau(\xi)\sum_{i=1}^{n} B_{ii}.$$

Since $\overline{M}$ is a Lorentzian space form $\overline{M}(\overline{c})$, therefore using Lemma 13, (51) follows. Furthermore, from Lemma 13 and (54), the equality case of (51) holds if and only if $B_{ij} = 0$, for all $i \neq j$ and $B_{11} = B_{22} = \cdots = B_{nn}$. Hence, the equality case of (51) holds if and only if M is totally umbilical. □

Theorem 18. *Let M be an $(n+1)$-dimensional null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M. Then*

(i)

$$\begin{aligned}\widetilde{r} \leq n(n+1)\overline{c} - 2n(\overline{c} + n\mu^2 + \tau(\xi)\mu) + (\text{trac } \overline{A})^2 - (\text{trac } A_N)^2 \\ + 2\sum_{i,j=1}^{n} B_{ij}^2 + \sum_{i,j=1}^{n} C_{ij}^2.\end{aligned}$$

The equality holds if and only if M is a screen homothetic null hypersurface with $\phi = -1$ such that $A_N\xi = 0$.

(ii)

$$\widetilde{r} \geq n(n+1)\overline{c} - 2n(\overline{c} + n\mu^2 + \tau(\xi)\mu) + (\text{trac } \overline{A})^2 - (\text{trac } A_N)^2 - \sum_{i,j=1}^{n} (B_{ij} + C_{ij})^2 + \sum_{i,j=1}^{n} C_{ij}^2.$$

The equality holds if and only if M is a totally geodesic null hypersurface, where

$$\bar{A} = \begin{pmatrix} B_{11} + C_{11} & B_{12} + C_{21} & \dots & B_{1n} + C_{n1} \\ B_{21} + C_{12} & B_{22} + C_{22} & \dots & B_{2n} + C_{n2} \\ \vdots & & & \\ B_{n1} + C_{1n} & B_{n2} + C_{2n} & \dots & B_{nn} + C_{nn} \end{pmatrix}.$$

Proof. On using (41) and (42) in (46), it follows that

$$\widetilde{r} = \overline{r}_{TM} - 2n(\overline{c} + n\mu^2 + \tau(\xi)\mu) + \left(\sum_{i=1}^{n} B_{ii} + \sum_{j=1}^{n} C_{jj}\right)^2 - \left(\sum_{j=1}^{n} C_{jj}\right)^2 - \sum_{i,j=1}^{n} (B_{ij} + C_{ij})^2 + 2\sum_{i,j=1}^{n} B_{ij}^2 + \sum_{i,j=1}^{n} C_{ij}^2, \tag{55}$$

and hence the assertions are complete. □

From (55), we have

$$\widetilde{r} = \overline{r}_{TM} - 2n(\overline{c} + n\mu^2 + \tau(\xi)\mu) + \left(\sum_{i=1}^{n} B_{ii} + \sum_{j=1}^{n} C_{jj}\right)^2 - \left(\sum_{j=1}^{n} C_{jj}\right)^2 - \sum_{i=1}^{n}(B_{ii} + C_{ii})^2 - \sum_{i \neq j=1}^{n} (B_{ij} + C_{ij})^2 + 2\sum_{i,j=1}^{n} B_{ij}^2 + \sum_{i,j=1}^{n} C_{ij}^2. \tag{56}$$

On using (50) in (56), we have following result immediately.

Corollary 19. *Let M be a null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M. Then*

$$\widetilde{r} \leq n(n+1)\overline{c} - 2n(\overline{c} + n\mu^2 + \tau(\xi)\mu) + \frac{(n-1)}{n}(\text{trac } \overline{A})^2 - (\text{trac } A_N)^2 - \sum_{i \neq j=1}^{n} (B_{ij} + C_{ij})^2 + 2\sum_{i,j=1}^{n} B_{ij}^2 + \sum_{i,j=1}^{n} C_{ij}^2.$$

The equality holds if and only if trac $\overline{A} = n(B_{ii} + C_{ii})$, *for* $1 \leq i \leq n$.

In [15] Hong and Tripathi proved the following theorem.

Theorem 20 ([15]). *Let M be an n-dimensional submanifold of a Riemannian manifold $(\overline{M},\overline{g})$. Then, the following statements are true:*

(i) *For $X \in T_p^1M$, it follows that*

$$\mathrm{Ric}(X) \leq \frac{1}{4}n^2\mu^2 + \overline{\mathrm{Ric}}_{T_pM}(X), \tag{57}$$

where $\overline{\mathrm{Ric}}_{T_pM}(X)$ is the n-Ricci curvature of T_pM at $X \in T_p^1M$ with respect to the ambient manifold $\overline{M}$.

(ii) *The equality case of (57) is satisfied by $X \in T_p^1M$ if and only if*

$$\begin{cases} B(X,Y) = 0, & \text{for all } Y \in T_pM \text{ orthogonal to } X,\\ B(X,X) = \dfrac{n}{2}\mu(p). \end{cases}$$

(iii) *The equality case of (57) holds for all $X \in T_p^1M$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Using the Binomial theorem, we recall the following relation between the components of the second fundamental form [11]

$$\begin{aligned}\sum_{i,j=1}^{n}(B_{ij})^2 = \frac{1}{2}n^2\mu^2 + \frac{1}{2}(B_{11} - B_{22} - \cdots - B_{nn})^2\\ + 2\sum_{j=2}^{n}(B_{1j})^2 - 2\sum_{2\leq i<j\leq n} B_{ii}B_{jj} - (B_{ij})^2.\end{aligned} \tag{58}$$

Now, we establish a relation between Ricci curvature and the squared mean curvature for a closed normalization (M,ζ) of a null hypersurface M of a Lorentzian manifold $(\overline{M},\overline{g})$ as follows.

Theorem 21. *Let M be a screen homothetic null hypersurface of a Lorentzian manifold $\overline{M}$ with a closed rigging ζ for M such that $\phi > \frac{1}{2}$. Then, the following assertions are true:*

(i) *For $X \in \mathcal{S}_1^{\zeta} = \{X \in \mathcal{S}^{\zeta} : \widetilde{g}(X,X) = 1\}$*

$$\begin{aligned}\widetilde{\mathrm{Ric}}(X) \leq \frac{(2\phi - 1)}{4}n^2\mu^2 + \overline{\mathrm{Ric}}(X)\\ + 2[(\nabla_X B)(\xi,X) - (\nabla_\xi B)(X,X)] - 4B(X,X)\tau(\xi).\end{aligned} \tag{59}$$

(ii) *The equality case of (59) is satisfied by $X \in \mathcal{S}_1^{\zeta}(p)$ if and only if*

$$\begin{cases} B(X,Y) = 0, & \text{for all } Y \in T_pM \text{ orthogonal to } X,\\ B(X,X) = \dfrac{n}{2}\mu(p). \end{cases} \tag{60}$$

(iii) *The equality case of (59) holds for all $X \in \mathcal{S}_1^{\zeta}(p)$ if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Let M be a screen homothetic null hypersurface of a Lorentzian manifold $\overline{M}$ with a closed rigging ζ for M, then from (22), we have

$$\widetilde{r}_{\mathcal{S}\zeta} = \overline{r}_{\mathcal{S}\zeta} + (2\phi - 1)\left[n^2\mu^2 - \sum_{i,j=1}^{n}(B_{ij})^2\right]. \tag{61}$$

Using (58) in (61), it further follows that

$$\begin{aligned}\frac{(2\phi-1)}{2}n^2\mu^2 &= \widetilde{r}_{\mathcal{S}\zeta} - \overline{r}_{\mathcal{S}\zeta} + \frac{(2\phi-1)}{2}(B_{11} - B_{22} - \cdots - B_{nn})^2\\ &\quad + 2(2\phi-1)\sum_{j=2}^{n}(B_{1j})^2\\ &\quad - 2(2\phi-1)\sum_{2\le i<j\le n}[B_{ii}B_{jj} - (B_{ij})^2].\end{aligned} \tag{62}$$

For a screen homothetic null hypersurface M, from (22), we have

$$(2\phi-1)(B_{ii}B_{jj} - (B_{ij})^2) = \widetilde{K}_{ij} - \overline{K}_{ij},$$

this further can be written as

$$(2\phi-1)\sum_{2\le i<j\le n}[B_{ii}B_{jj} - (B_{ij})^2] = \sum_{2\le i<j\le n}[\widetilde{K}_{ij} - \overline{K}_{ij}]. \tag{63}$$

Now

$$\sum_{2\le i<j\le n}\widetilde{K}_{ij} = \frac{1}{2}\sum_{i,j=2}^{n}\widetilde{K}_{ij} = \frac{1}{2}\sum_{i,j=1}^{n}\widetilde{K}_{ij} - \sum_{j=2}^{n}\widetilde{K}_{1j},$$

further using (34), it follows that

$$\sum_{2\le i<j\le n}\widetilde{K}_{ij} = \frac{1}{2}\widetilde{r}_{\mathcal{S}\zeta} - \widetilde{\mathrm{Ric}}(e_1) + \widetilde{K}_{1\xi}. \tag{64}$$

Similarly

$$\begin{aligned}\sum_{2\le i<j\le n}\overline{K}_{ij} &= \frac{1}{2}\overline{r}_{\mathcal{S}\zeta} - \overline{\mathrm{Ric}}(e_1) + \overline{K}_{1\xi} + \overline{K}_{1N}\\ &= \frac{1}{2}\overline{r}_{\mathcal{S}\zeta} - \overline{\mathrm{Ric}}(e_1) + 2\overline{K}_{1\xi}.\end{aligned} \tag{65}$$

Using (64) and (65) in (63), we derive

$$\begin{aligned}(\widetilde{r}_{\mathcal{S}\zeta} - \overline{r}_{\mathcal{S}\zeta}) - 2(2\phi-1)\sum_{2\le i<j\le n}[B_{ii}B_{jj} - (B_{ij})^2] &= 2[\widetilde{\mathrm{Ric}}(e_1) - \overline{\mathrm{Ric}}(e_1)]\\ &\quad - 2\widetilde{K}_{1\xi} + 4\overline{K}_{1\xi}.\end{aligned} \tag{66}$$

Hence, from (66) and (62), we obtain

$$\begin{aligned}\frac{(2\phi-1)}{2}n^2\mu^2 = 2\big[\widetilde{\mathrm{Ric}}(e_1) - \overline{\mathrm{Ric}}(e_1)\big] - 2\widetilde{K}_{1\xi} + 4\overline{K}_{1\xi} \\ + \frac{(2\phi-1)}{2}[B_{11} - B_{22} - \cdots - B_{nn}]^2 \\ + 2(2\phi-1)\sum_{j=2}^{n}(B_{1j})^2. \qquad (67)\end{aligned}$$

From (28), we have $\widetilde{K}_{1\xi} = \overline{K}_{1\xi} - B_{11}\tau(\xi)$, further using (8), we obtain

$$\begin{aligned}4\overline{K}_{1\xi} - 2\widetilde{K}_{1\xi} &= 2[\overline{K}_{1\xi} + B_{11}\tau(\xi)] \\ &= 2[(\nabla_\xi B)(e_1, e_1) - (\nabla_{e_1} B)(\xi, e_1) + 2B_{11}\tau(\xi)], \qquad (68)\end{aligned}$$

therefore (67) becomes

$$\begin{aligned}\frac{(2\phi-1)}{2}n^2\mu^2 = 2[\widetilde{\mathrm{Ric}}(e_1) - \overline{\mathrm{Ric}}(e_1)] \\ + 2[(\nabla_\xi B)(e_1, e_1) - (\nabla_{e_1} B)(\xi, e_1) + 2B_{11}\tau(\xi)] \\ + \frac{(2\phi-1)}{2}[B_{11} - B_{22} - \cdots - B_{nn}]^2 \\ + 2(2\phi-1)\sum_{j=2}^{n}(B_{1j})^2. \qquad (69)\end{aligned}$$

Thus, the inequality (59) follows from (69). Furthermore, the equality of (59) follows if and only if $B_{12} = B_{13} = \cdots = B_{1n} = 0$, $2B_{11} = B_{11} + B_{22} + \cdots + B_{nn} = n\mu$, and moreover, $B_{i\xi} = 0$, hence (60) follows immediately.

Next, assume that the equality of (59) follows for all $X \in \mathcal{S}_1^\zeta$, then $B_{ij} = 0$, for all $i \neq j$ and $2B_{ii} = n\mu$, for $i \in \{1, 2, \ldots, n\}$, that is, $n(n-2)\mu = 0$. Hence, either $\mu = 0$ or $n = 2$. If $\mu = 0$ and together with $B_{ij} = 0$, for all $i \neq j$ implies p is a totally geodesic point. If $n = 2$ then $2B_{ii} = n\mu$ implies $2B_{11} = 2B_{22} = B_{11} + B_{22}$, hence p is a totally umbilical point and converse is immediate. □

Corollary 22. *Let M be a screen homothetic null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M such that $\phi > \frac{1}{2}$. Then, the following assertions are true*:

(i) *For $X \in \mathcal{S}_1^\zeta = \{X \in \mathcal{S}^\zeta : \widetilde{g}(X, X) = 1\}$*

$$\widetilde{\mathrm{Ric}}_{\mathcal{S}^\zeta}(X) \leq \frac{(2\phi-1)}{4}n^2\mu^2 + \overline{c}(n-1). \qquad (70)$$

(ii) *The equality case of (70) is satisfied by $X \in \mathcal{S}_1^\zeta(p)$ if and only if*

$$\begin{cases} B(X, Y) = 0, & \text{for all } Y \in T_pM \text{ orthogonal to } X, \\ B(X, X) = \dfrac{n}{2}\mu(p). \end{cases}$$

(iii) *The equality case of* (70) *holds for all* $X \in \mathcal{S}_1^\zeta(p)$ *if and only if either* p *is a totally geodesic point or* $n = 2$ *and* p *is a totally umbilical point.*

Next, we recall following important lemma from [5].

Lemma 23. *Let* $n \geq 3$ *be an integer and let* $a_1, \ldots, a_n$ *be* $n-$*real numbers. Then*

$$\sum_{1 \leq i < j \leq n} a_i a_j - a_1 a_2 \leq \frac{(n-2)}{2(n-1)} \left(\sum_{i=1}^{n} a_i \right)^2 . \tag{71}$$

Moreover, the equality holds if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Theorem 24. *Let* M *be a screen homothetic null hypersurface of a Lorentzian manifold* $\overline{M}$ *with a closed rigging* ζ *for* M *such that* $\phi > \frac{1}{2}$ *and* $n > 3$. *Let* Π *be a non-degenerate plane section of* T_pM *spanned by* e_1 *and* e_2. *Then*

$$\begin{aligned} \widetilde{r}_{\mathcal{S}^\zeta} - 2\widetilde{K}(\Pi) \leq \overline{r}_{\mathcal{S}^\zeta} - 2\overline{K}(\Pi) + (2\phi - 1)\frac{n^2(n-2)}{(n-1)}\mu^2 \\ - 2(2\phi - 1)\left(\sum_{1 \leq i < j \leq n} (B_{ij})^2 - B_{12}^2 \right). \end{aligned} \tag{72}$$

The equality of (72) *holds if and only if the mean curvature* μ *of* M *is equal to* $\frac{(n-1)}{n}(B_{11} + B_{22})$.

Proof. Let M be a screen homothetic null hypersurface of a Lorentzian manifold $\overline{M}$ with a closed rigging ζ for M, then from (22), we have

$$\begin{aligned} \widetilde{r}_{\mathcal{S}^\zeta} &= \overline{r}_{\mathcal{S}^\zeta} + (2\phi - 1)\left(\sum_{i=1}^{n} B_{ii} \right)^2 - (2\phi - 1)\sum_{i,j=1}^{n} (B_{ij})^2. \\ &= \overline{r}_{\mathcal{S}^\zeta} + (2\phi - 1)\left(\sum_{i=1}^{n} B_{ii} \right)^2 - (2\phi - 1)\left(\sum_{i=1}^{n} (B_{ii})^2 + \sum_{i \neq j = 1}^{n} (B_{ij})^2 \right), \end{aligned} \tag{73}$$

this implies

$$\widetilde{r}_{\mathcal{S}^\zeta} = \overline{r}_{\mathcal{S}^\zeta} + (2\phi - 1)\left(\left(\sum_{i=1}^{n} B_{ii} \right)^2 - \sum_{i=1}^{n} (B_{ii})^2 \right) - (2\phi - 1)\sum_{i \neq j = 1}^{n} (B_{ij})^2. \tag{74}$$

Since

$$\left(\sum_{i=1}^{n} B_{ii} \right)^2 = \sum_{i=1}^{n} (B_{ii})^2 + 2\sum_{1 \leq i < j \leq n} B_{ii}B_{jj}$$

then (74) becomes

$$\frac{1}{2}\widetilde{r}_{\mathcal{S}^\zeta} = \frac{1}{2}\overline{r}_{\mathcal{S}^\zeta} + (2\phi - 1)\sum_{1 \leq i < j \leq n} B_{ii}B_{jj} - \frac{(2\phi - 1)}{2}\sum_{i \neq j = 1}^{n} (B_{ij})^2. \tag{75}$$

Let Π be a non-degenerate plane section of T_pM spanned by e_1 and e_2, then from (22), we have

$$\widetilde{K}(\Pi) = \overline{K}(\Pi) - (2\phi - 1)B_{12}^2 + (2\phi - 1)B_{11}B_{22}. \tag{76}$$

Subtracting (76) from (75), we derive

$$\begin{aligned}
\frac{1}{2}\widetilde{r}_{\mathcal{S}^{\zeta}} - \widetilde{K}(\Pi) &= \frac{1}{2}\overline{r}_{\mathcal{S}^{\zeta}} - \overline{K}(\Pi) + (2\phi - 1)\left(\sum_{1\leq i<j\leq n} B_{ii}B_{jj} - B_{11}B_{22}\right) \\
&\quad - \frac{(2\phi - 1)}{2}\left(\sum_{i\neq j=1}^{n} (B_{ij})^2 - 2B_{12}^2\right) \\
&= \frac{1}{2}\overline{r}_{\mathcal{S}^{\zeta}} - \overline{K}(\Pi) + (2\phi - 1)\left(\sum_{1\leq i<j\leq n} B_{ii}B_{jj} - B_{11}B_{22}\right) \\
&\quad - (2\phi - 1)\left(\sum_{1\leq i<j\leq n}^{n} (B_{ij})^2 - B_{12}^2\right),
\end{aligned}$$

further, using (71), we obtain

$$\begin{aligned}
\frac{1}{2}\widetilde{r}_{\mathcal{S}^{\zeta}} - \widetilde{K}(\Pi) &\leq \frac{1}{2}\overline{r}_{\mathcal{S}^{\zeta}} - \overline{K}(\Pi) + (2\phi - 1)\frac{(n-2)}{2(n-1)}\left(\sum_{i=n}^{n} B_{ii}\right)^2 \\
&\quad - (2\phi - 1)\left(\sum_{1\leq i<j\leq n}^{n} (B_{ij})^2 - B_{12}^2\right).
\end{aligned} \tag{77}$$

Thus, using (77), proof is complete. □

Corollary 25. *Let M be a screen homothetic null hypersurface of a Lorentzian manifold $\overline{M}$ with a closed rigging ζ for M such that $\phi > \frac{1}{2}$ and $n > 3$. Let Π be a non-degenerate plane section of T_pM spanned by e_1 and e_2. Then*

$$\widetilde{r}_{\mathcal{S}^{\zeta}} - 2\widetilde{K}(\Pi) \leq \overline{r}_{\mathcal{S}^{\zeta}} - 2\overline{K}(\Pi) + (2\phi - 1)\frac{n^2(n-2)}{(n-1)}\mu^2. \tag{78}$$

The equality of (78) holds if and only if $\mu = \frac{(n-1)}{n}(B_{11} + B_{22})$ and the shape operator of M takes the form

$$A_{\xi}^{\star} = \begin{pmatrix} B_{11} & B_{12} & 0 & \dots & 0 \\ B_{21} & B_{22} & 0 & \dots & 0 \\ 0 & 0 & B_{11} + B_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & B_{11} + B_{22} \end{pmatrix}.$$

Furthermore, using Lemma 13, we have the following corollary.

Corollary 26. *Let M be a screen homothetic null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M such that $\phi > \frac{1}{2}$ and $n > 3$. Let Π be a non-degenerate plane section of T_pM spanned by e_1 and e_2. Then*

$$\widetilde{r}_{\mathcal{S}\zeta} - 2\widetilde{K}(\Pi) \leq (n(n-1)-2)\overline{c} + (2\phi - 1)\frac{(n-2)n^2\overline{c}^2}{4(n-1)\phi^2\tau(\xi)^2}. \tag{79}$$

The equality of (79) holds if and only if the mean curvature μ of M is equal to $-\frac{(n-1)}{n}\frac{\overline{c}}{\phi\tau(\xi)}$.

From (75), we can derive

$$\widetilde{r}_{\mathcal{S}\zeta} - \widetilde{r}(\Pi) = \overline{r}_{\mathcal{S}\zeta} - \overline{r}(\Pi) + 2(2\phi - 1)\left(\sum_{1\leq i<j\leq n} B_{ii}B_{jj} - B_{11}B_{22}\right)$$
$$- (2\phi - 1)\sum_{i\neq j=2}^{n}(B_{ij})^2. \tag{80}$$

Hence, on using (71) in (80), we have the following observation immediately.

Corollary 27. *Let M be a screen homothetic null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M such that $\phi > \frac{1}{2}$ and $n > 3$. Let Π be a non-degenerate plane section of T_pM spanned by e_1 and e_2. Then*

$$\widetilde{r}_{\mathcal{S}\zeta} - \widetilde{r}(\Pi) \leq \overline{r}_{\mathcal{S}\zeta} - \overline{r}(\Pi) + (2\phi - 1)\frac{n^2(n-2)}{(n-1)}\mu^2. \tag{81}$$

The equality of (81) holds if and only if the mean curvature μ of M is equal to $\frac{(n-1)}{n}(B_{11} + B_{22})$ and $B_{ij} = 0$, for all $i \neq j$, $2 \leq i, j \leq n$.

For a screen homothetic null hypersurface M of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M, from (73), we derive

$$\widetilde{r}_{\mathcal{S}\zeta} - \widetilde{K}(\Pi) = (n(n-1)-1)\overline{c} + (2\phi - 1)\left(\sum_{i=1}^{n} B_{ii}\right)^2 - (2\phi - 1)\sum_{i,j=1}^{n}(B_{ij})^2$$
$$- (2\phi - 1)B_{11}B_{22} + (2\phi - 1)B_{12}^2,$$

then using Lemma 13, we obtain

$$\widetilde{r}_{\mathcal{S}\zeta} - \widetilde{K}(\Pi) = (n(n-1)-1)\left(\overline{c} + \frac{(2\phi - 1)\overline{c}^2}{4\phi^2\tau^2(\xi)}\right).$$

Hence, we have the following corollary.

Corollary 28. *Let M be a screen homothetic null hypersurface of a Lorentzian space form $\overline{M}(\overline{c})$ with a closed rigging ζ for M such that $n > 2$. Let Π be a* *non-degenerate plane section of T_pM spanned by e_1 and e_2. Then*

$$\widetilde{r}_{\mathcal{S}\varsigma} - \widetilde{K}(\Pi) \leq (n(n-1)-1)\overline{c}, \quad \textit{if } 0 < \phi < \frac{1}{2},$$

$$\widetilde{r}_{\mathcal{S}\varsigma} - \widetilde{K}(\Pi) \geq (n(n-1)-1)\overline{c}, \quad \textit{if } \phi > \frac{1}{2},$$

$$\widetilde{r}_{\mathcal{S}\varsigma} - \widetilde{K}(\Pi) = (n(n-1)-1)\overline{c}, \quad \textit{if } \phi = \frac{1}{2}.$$

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