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Closed equations of the two-point functions for tensorial group field theory

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Abstract
In this paper, we provide the closed equations that satisfy two-point correlation functions of rank 3 and 4 tensorial group field theory. The formulation of the current problem extends the method used by Grosse and Wulkenhaar (2009 arXiv:0909.1389) to the tensor case. Ward–Takahashi identities and Schwinger–Dyson equations are combined to establish a nonlinear integral equation for the two-point functions. In the three-dimensional case, the solution of this equation is given perturbatively at the second order of the coupling constant.

Keywords: renormalization, tensorial group field theory, Ward–Takahashi identities, Schwinger–Dyson equation, two-point correlation functions
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1. Introduction

Random tensor models [1–3] extend matrix models [4] as promising candidates to understand quantum gravity (QG) in higher dimensions, $D \geqslant 3$. The formulation of such models is based on a Feynman path integral that randomly generates graphs representing simplicial pseudo-manifolds of dimension D. The equivalent of the t'Hooft large N limit [5, 6] for these tensor models has been recently discovered by Gurau [7–9]. The large N limit behaviour is a powerful tool which allows us to understand the continuous limit of these models through, for instance, the study of critical exponents and phase transitions [10–12].

With the advent of the field theory formulation of random tensor models, henceforth called tensorial group field theory (TGFT) [13–24], one addresses several different questions such as renormalizability (for removing divergences) and the study of the ultraviolet (UV) behaviour of these models. It turns out that renormalization can be consistently defined for

TGFTs and most of them, for the higher rank $D \geqslant 3$, are UV asymptotically free [21]. This is, of course, very encouraging for the geomeogenesis scenario [15, 23, 24].

It becomes more and more convincing that random tensors and TGFTs will take a growing role in providing answers to the QG conundrum. Despite all these results, a lot of conceptual and technical questions arise in this framework for obtaining a final and emergent theory of general relativity [3]. Among other goals, it would be strongly desirable to establish more connections with other studies and important results around gravity. This paper provides the first glimpses of the extension of the recent full resolution of the correlation functions in the Grosse–Wulkenhaar (GW) model [27–29].

One of the main purposes of a field theory is to find the exact value of the Green's functions, which are also called correlation functions. Obviously, this can be a highly nontrivial task. In the rare cases where this is successfully done, one calls the model exactly solvable. In a recent work, the renormalizable noncommutative scalar field theory, called the GW model, was solved [28–43]. This particular noncommutative field theory projects on a matrix model, and then one can see a model for QG in two dimensions. Let us review this model arising in noncommutative geometry. Grosse and Wulkenhaar modified the propagator of the noncommutative field theory by adding a harmonic term, and showed that the resulting functional action is renormalizable at all orders of perturbation. The proof of this claim was given using the matrix basis dual to the Moyal space of functions. In [34–36], a new proof of the renormalizability was given in direct space using multiscale analysis [37]. The GW propagator breaks the $U(N)$ symmetry invariance in the infrared regime, but is asymptotically safe in the UV regime [32, 38, 39]. The model is also noninvariant under the translation and rotation of spacetime. The only known invariance satisfied by the model is the so-called Langman–Szabo duality [40]. At the perturbative level, the associated Feynman graphs are ribbon graphs. In a recent remarkable contribution, Grosse and Wulkenhaar successfully solve all correlators in this model. Using both Ward–Takahashi identities and the Schwinger–Dyson equation, these authors provide, via Hilbert transform, a nonlinear integral equation for the two-point functions [29]. From this result, they were able to generate solutions for all correlators. Thus, the GW model is exactly nonperturbatively solvable. The question is whether or not this method may apply to other models, in particular to TGFTs dealing with higher rank tensors. We give a partial positive answer to this question. Indeed, as we will show in this paper, the resolution method can be applied to find nonlinear equations for the correlations here as well. Due to the highly nontrivial equations and combinatorics, the full resolution of all correlators deserves more work, which should be addressed elsewhere.

This paper is organized as follows: in section 2, we derive the Ward–Takahashi identities of arbitrary rank D TGFT. In section 3 we give the closed equation of the two-point correlation functions for the rank 3 TGFT. We also give the solution of this equation at the second order of perturbation. In section 4 we provide the closed equation of the rank 4 tensor field. We give a summary of our results and a review of the paper in section 5.

2. Ward–Takahashi identities for an arbitrary *D*-tensor field model

TGFTs are generally defined by an action, $S[\bar{\varphi}, \varphi]$, that depends on the field, φ, and its conjugate, $\bar{\varphi}$, defined on the compact Lie group, G (i.e., $\varphi: G^D \longrightarrow \mathbb{C}$; $(g_1, \cdots, g_D) \longmapsto \varphi(g_1, \cdots, g_D)$). For simplicity, we will always consider $G = U(1)$. We are using the Fourier transformation of the field and are defining the momentum variable associated with the group elements $[g] = (g_1, g_2, \cdots, g_D) \in U(1)^D$ as $[p] = (p_1, p_2, \cdots, p_D) \in \mathbb{Z}^D$. Using the parametrization $g_k = e^{i\theta_k}$, we write

$$\varphi(g_1, \cdots, g_D) = \sum_{p_i \in \mathbb{Z}} \varphi(p_1, \cdots, p_D) e^{i\sum_k \theta_k p_k}, \quad \theta_i \in [0, 2\pi). \tag{1}$$

The Fourier transform of the field φ is denoted by $\varphi_{12\ldots D} =: \varphi(p_1, \cdots, p_D) =: \varphi_{[D]}$ for simplicity. The functional action, $S[\bar{\varphi}, \varphi]$, is written in a general case as

$$S[\bar{\varphi}, \varphi] = \sum_{p_i} \bar{\varphi}_{12\ldots D} C^{-1}\big(p_1, p_2, \cdots, p_D; p'_1, p'_2, \cdots p'_D\big) \varphi_{12\ldots D} \prod_{i=1}^{D} \delta_{p_i p'_i} + S^{\text{int}}, \tag{2}$$

where C stands for the propagator and S^{int} collects all vertex contributions of the interaction. Let $d\mu_C$ be the field measure associated with the covariance C, and we have the relation

$$C([p]; [p']) = \int d\mu_C \ \varphi_{[p]} \bar{\varphi}_{[p']}, \quad d\mu_C = \prod_{[p]} d\bar{\varphi}_{[p]} \ d\varphi_{[p]} \, e^{-\bar{\varphi}_{[p]} \ C^{-1}([p],[p]) \, \varphi_{[p]}}. \tag{3}$$

The Green's functions or N-point correlation functions are defined by the relation

$$G\Big([p]_1, [p]_2, \cdots [p]_N\Big) = \frac{1}{\mathcal{Z}} \int d\mu_C \ \ \varphi_{[p]_1} \bar{\varphi}_{[p]_1} \cdots \varphi_{[p]_N} \bar{\varphi}_{[p]_N} \, e^{-S^{\text{int}}}, \tag{4}$$

where $\mathcal{Z}$ is the normalization factor (also called the partition function) given by

$$\mathcal{Z} = \int d\mu_C \ e^{-S^{\text{int}}}. \tag{5}$$

Let us write the interaction term of action (2) as $S^{\text{int}} = \lambda V[\bar{\varphi}, \varphi] =: \sum_k \lambda_k V_k[\bar{\varphi}, \varphi]$. The main idea of the perturbative theory is to expand the Green's functions as

$$\begin{aligned} G\big([p]_1, [p]_2, \cdots [p]_N\big) &= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int d\mu_C \ V^n[\bar{\varphi}, \varphi] \ \varphi_{[p]_1} \bar{\varphi}_{[p]_1} \cdots \varphi_{[p]_N} \bar{\varphi}_{[p]_N} \\ &= \sum_{n=0}^{\infty} \lambda^n \, G_N^{(n)}. \end{aligned} \tag{6}$$

Using this formula, the Green's functions can be computed order-by-order using Dyson's theorem.

We consider the rank D tensor field, φ, and its conjugate, $\bar{\varphi}$, which are transformed under the tensor product of D fundamental representations of the unitary group, $\mathcal{U}_{\otimes}^{N_D} := \otimes_{i=1}^{D} U(N_i)$. Let $U^{(a)} \in U(N_a)$, $a = 1, 2, \cdots, D$. The field, φ, and its conjugate $\bar{\varphi}$ are transformed under $U(N_a)$ as

$$\varphi_{12\ldots D} \to \left[U^{(a)}\varphi\right]_{12\cdots a \cdots D} = \sum_{p'_a \in Z} U^{(a)}_{p_a p'^a} \varphi_{12\cdots a' \cdots D}, \tag{7}$$

$$\bar{\varphi}_{12\ldots D} \to \left[\bar{\varphi} U^{\dagger(a)}\right]_{12\cdots a \cdots D} = \sum_{p'_a \in Z} \bar{U}^{(a)}_{p_a p'^a} \bar{\varphi}_{12\cdots a' \cdots D}. \tag{8}$$

p'_a or simply a' is the momentum index at position a in the expression $\varphi_{12\cdots a' \cdots D}$. The kinetic action in (2) is re-expressed as follows:

$$S^{\text{kin}}[\bar{\varphi}, \varphi] = \sum_{p_1, \cdots, p_D} \varphi_{12\ldots D} M_{12\ldots D} \bar{\varphi}_{12\ldots D}, \quad M_{12\ldots D} = C^{-1}_{12\ldots D}. \tag{9}$$

$M_{12\ldots D}$ is the inverse of the propagator associated with the model. Rank D tensor fields are represented by half lines made with D segments called strands. A propagator is a D-stranded line that, as usual, connects vertices. The variation of the action S^{kin} under infinitesimal

$U(N_a)$ transformation is given by

$$\delta^{(a)}\Big[S^{\rm kin}\Big] = -\,\mathrm{i}\sum_{p_1,\cdots,p_D}\Big[M\Big(\varphi\big[\bar{B}\bar{\varphi}\big]^{(a)} - [B\varphi]^{(a)}\bar{\varphi}\Big)\Big]_{12\cdots D}, \tag{10}$$

where B is the infinitesimal Hermitian operator corresponding to the generator of unitary group $U(N_a)$; that is,

$$U^{(a)}_{pp'} = \delta^{(a)}_{pp'} + \mathrm{i}B^{(a)}_{pp'} + O\big(B^2\big), \quad \bar{U}^{(a)}_{pp'} = \delta^{(a)}_{pp'} - \mathrm{i}\bar{B}^{(a)}_{pp'} + O\big(\bar{B}^2\big), \tag{11}$$

with $\bar{B}^{(a)}_{pp'} = B^{(a)}_{p'p}$. Consider now the theory defined with external source $F[\varphi, \bar{\varphi}; \eta, \bar{\eta}]$ as

$$F[\eta, \bar{\eta}] = \sum_{p_1,\cdots,p_D} \bar{\varphi}_{12\cdots D}\eta_{12\cdots D} + \bar{\eta}_{12\cdots D}\varphi_{12\cdots D}. \tag{12}$$

The partition function of the model is re-expressed as

$$\mathcal{Z}[\eta, \bar{\eta}] = \int \mathrm{d}\varphi\mathrm{d}\bar{\varphi}\ \mathrm{e}^{-S[\varphi,\bar{\varphi}]+F[\varphi,\bar{\varphi};\eta,\bar{\eta}]}. \tag{13}$$

Under $U(N_a)$, infinitesimal transformation

$$\delta^{(a)}[F] = \mathrm{i}\sum_{p_1,\cdots,p_D}\Big[\bar{\eta}\,[B\varphi]^{(a)} - \big[\bar{B}\bar{\varphi}\big]^{(a)}\eta\Big]_{12\cdots D}. \tag{14}$$

Let $\delta^{(\otimes)}$ be the total variation under the action of the group element, $U^{(1)} \otimes U^{(2)} \otimes \cdots \otimes U^{(D)} \in \mathcal{U}^{N_D}_{\otimes}$. Then we get the following proposition.

Proposition 1. *The kinetic terms of action (2), (i.e., $S^{\rm kin}$ and F) are, respectively, transformed linearly as*

$$\delta^{(\otimes)}S^{\rm kin} = \sum_{a=1}^{D}\delta^{(a)}S^{\rm kin}, \quad \delta^{(\otimes)}F = \sum_{a=1}^{D}\delta^{(a)}F. \tag{15}$$

Then, $\delta^{(\otimes)}S = 0$ if and only if $\delta^{(a)}S = 0$ for all functional quantity S, which depends on φ, $\bar{\varphi}$, η, and $\bar{\eta}$.

We assume that $N_i = N,\ \ i = 1, 2\cdots D$, and we take the interaction terms such that they are invariant under the transformation $U^{(a)}$ (i.e., $\delta^{(a)}S^{\rm int} = 0$). This is the new input in TGFTs: the $U(N_a)$ tensor invariance must be the one defining the interaction [1]. Note that the measure $\mathrm{d}\varphi\mathrm{d}\bar{\varphi}$ is also invariant under $U^{(a)}$. The variation of the partition function can be performed for $a = 1$, and the results for all values of $a \in \{1, 2, \cdots, D\}$ may be deduced using proposition 1. We write

$$\begin{aligned}\frac{\delta^{(1)}\ \ln\mathcal{Z}[\eta, \bar{\eta}]}{\delta B_{p_m p_n}} &= \frac{1}{\mathcal{Z}[\eta, \bar{\eta}]}\int \mathrm{d}\varphi\mathrm{d}\bar{\varphi}\Bigg\{\mathrm{i}\sum_{p_2,\cdots,p_D}\Big(M_{n\,2\cdots D}\varphi_{n\,2\cdots D}\bar{\varphi}_{m\,2\cdots D}\\ &\quad - M_{m\,2\cdots D}\bar{\varphi}_{m\,2\cdots D}\varphi_{n\,2\cdots D}\Big)\\ &\quad + \mathrm{i}\sum_{p_2,\cdots,p_D}\Big(\bar{\eta}_{m\,2\cdots D}\varphi_{n\,2\cdots D} - \bar{\varphi}_{m\,2\cdots D}\eta_{n\,2\cdots D}\Big)\Bigg\}\mathrm{e}^{-S[\varphi,\bar{\varphi}]+F[\varphi,\bar{\varphi};\eta,\bar{\eta}]}\\ &= 0.\end{aligned} \tag{16}$$

Now take $\partial_{\bar\eta}\partial_\eta$ of the above expression. We get only the connected components of the correlation functions as

$$\sum_{[p]}\left(M_{m\,2\cdots D}-M_{n\,2\cdots D}\right)\left\langle\left[\frac{\partial(\bar\eta\varphi)}{\partial\bar\eta}\frac{\partial(\bar\varphi\eta)}{\partial\eta}\right]\varphi_{n\,2\cdots D}\bar\varphi_{m\,2\cdots D}\right\rangle_c$$
$$=\sum_{[p]}\left\langle\frac{\partial\left(\bar\eta_{m\,2\cdots D}\varphi_{n\,2\cdots D}\right)}{\partial\bar\eta}\left[\frac{\partial(\bar\varphi\eta)}{\partial\eta}\right]-\frac{\partial\left(\bar\varphi_{m\,2\cdots D}\eta_{n\,2\cdots D}\right)}{\partial\eta}\left[\frac{\partial(\bar\eta\varphi)}{\partial\bar\eta}\right]\right\rangle_c, \tag{17}$$

which can be simply written as

$$\sum_{[p]}\left(M_m-M_n\right)\left\langle\left[\frac{\partial(\bar\eta\varphi)}{\partial\bar\eta}\frac{\partial(\bar\varphi\eta)}{\partial\eta}\right]\varphi_n\bar\varphi_m\right\rangle_c$$
$$=\sum_{[p]}\left\langle\frac{\partial\left(\bar\eta_m\varphi_n\right)}{\partial\bar\eta}\frac{\partial(\bar\varphi\eta)}{\partial\eta}\right\rangle_c-\sum_{[p]}\left\langle\frac{\partial\left(\bar\varphi_m\eta_n\right)}{\partial\eta}\frac{\partial(\bar\eta\varphi)}{\partial\bar\eta}\right\rangle_c. \tag{18}$$

Note that equation (18) is valid for all position indices $a = 1, 2, \cdots, D$. Let us also remark that for $m = n$, the left hand side (lhs) of equation (18) vanishes. In the double derivative $\partial_{\bar\eta}\partial_\eta$, we fix the indices such that $\bar\eta_{[\alpha]}\eta_{[\beta]}$. Then comes the following proposition.

Proposition 2. *For index a = 1 (corresponding to $U^{(1)}$), we get the Ward–Takahashi identity*

$$\sum_{p_2,\cdots,p_D}\left(M_{m2\cdots D}-M_{n2\cdots D}\right)\left\langle\varphi_{[\alpha]}\bar\varphi_{[\beta]}\varphi_{n2\cdots D}\bar\varphi_{m2\cdots D}\right\rangle_c$$
$$=\delta_{m\alpha_1}\left\langle\varphi_{n\alpha_2\cdots\alpha_D}\bar\varphi_{\beta_1\cdots\beta_D}\right\rangle_c-\delta_{n\beta_1}\left\langle\bar\varphi_{m\beta_2\cdots\beta_D}\varphi_{\alpha_1\cdots\alpha_D}\right\rangle_c, \tag{19}$$

which can be re-expressed for an arbitrary position, a, taking any value in $\{1, 2, \cdots, D\}$ *as*

$$\left(M_m-M_n\right)\left\langle\left[\varphi_m\bar\varphi_n\right]\varphi_n\bar\varphi_m\right\rangle_c=\left\langle\varphi_n\bar\varphi_n\right\rangle_c-\left\langle\bar\varphi_m\varphi_m\right\rangle_c,\quad\left[\varphi_m\bar\varphi_n\right]=\sum_{p_2,\cdots,p_D}\varphi_{n2\cdots D}\bar\varphi_{m2\cdots D}. \tag{20}$$

We emphasize that the positions taken by indices m and n in relation (20) are the positions of the momentum index, p_a, used in the transformation $U^{(a)}$. In conclusion, there are exactly D Ward–Takahashi identities for the rank D TGFTs associated with this type of invariance. Note that the Ward–Takahashi identities for the Boulatov model can be found in [46]. The result obtained therein radically differs from the present identities found in (20). Furthermore, we mention that we are not considering the TGFT with a gauge invariance condition on the fields, as is found in [18, 19]. We consider here the simplest TGFT, as treated in [13, 14]. Most of the results of this work could be extended to this different framework with little additional work, since only the propagator will be modified. Thus, one expects similar Ward identities in that gauge invariant framework.

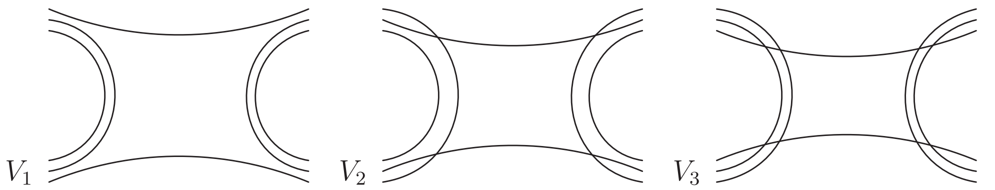

Figure 1. The vertices of the rank 3 tensor model.

3. Two-point functions of rank 3 TGFT

In this section, we consider the just renormalizable rank 3 TGFT on the compact $U(1)$ group, addressed first in [15]. The rank 3 tensor field is defined by φ: $U(1)^3 \longrightarrow \mathbb{C}$, and we expand in Fourier modes as

$$\varphi(g_1, g_2, g_3) = \sum_{p_j \in \mathbb{Z}} \varphi_{123} \mathrm{e}^{\mathrm{i}p_1\theta_1} \mathrm{e}^{\mathrm{i}p_2\theta_2} \mathrm{e}^{\mathrm{i}p_3\theta_3}, \quad \theta_i \in [0, 2\pi). \tag{21}$$

As usual, write $\varphi_{123} := \varphi_{p_1 p_2 p_3}$. The renormalizable $3D$ tensor model is defined by the action S_{3D}, in which the kinetic term takes the form

$$S_{3D}^{\mathrm{kin}} = \sum_{[p]} \bar{\varphi}_{123} \, C_{123}^{-1} \, \varphi_{123}, \tag{22}$$

where C_{123} is the propagator. We write the resulting action for the bare quantities, which involve the bare mass, m, and the three wave function renormalizations, $Z_{\rho=1,2,3}$, each of which is associated with a strand index, $a = 1, 2, 3$. The field strength can be modified as follows:

$$\varphi \longrightarrow \left(Z_1 Z_2 Z_3\right)^{\frac{1}{6}} \varphi = Z^{1/2} \varphi, \quad Z_\rho = 1 + \partial_{b_\rho} \Gamma_{b_1 b_2 b_3} \Big|_{b_{1,2,3}=0}, \tag{23}$$

where $\Gamma_{b_1 b_2 b_3}$ is the self-energy or one particle irreducible (1PI) two-point functions. Then, the renormalized propagator takes the form

$$C_{abc} = Z^{-1} \Big(|a| + |b| + |c| + m^2 \Big)^{-1}, \quad a, b, c \in \mathbb{Z}. \tag{24}$$

m stands for the bare mass. The interaction of the model is defined by the three contributions V_1, V_2, and V_3, expressed in momentum space as

$$\begin{aligned} S_{3D}^{\mathrm{int}} = \lambda_1 Z^2 \sum_{\substack{1,2,3 \\ 1',2',3'}} \varphi_{123} \bar{\varphi}_{321'} \varphi_{1'2'3'} \bar{\varphi}_{3'2'1} + \lambda_2 Z^2 \sum_{\substack{1,2,3 \\ 1',2',3'}} \varphi_{123} \bar{\varphi}_{32'1} \varphi_{1'2'3'} \bar{\varphi}_{3'21'} \\ + \lambda_3 Z^2 \sum_{\substack{1,2,3 \\ 1',2'3'}} \varphi_{123} \bar{\varphi}_{3'21} \varphi_{1'2'3'} \bar{\varphi}_{32'1'} = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3. \end{aligned} \tag{25}$$

The vertices of this model are represented in figure 1.

After reducing some constraints, the Ward–Takahashi identities (20) now find the form

$$\sum_{p_2, p_3} \left(M_{m23} - M_{n23}\right) \left\langle \varphi_{m23} \bar{\varphi}_{n23} \varphi_{nab} \bar{\varphi}_{mab} \right\rangle_c = \left\langle \varphi_{nab} \bar{\varphi}_{nab} \right\rangle_c - \left\langle \bar{\varphi}_{mab} \varphi_{mab} \right\rangle_c \tag{26}$$

$$\sum_{p_1, p_3} \left(M_{1m3} - M_{1n3}\right) \left\langle \varphi_{1m3} \bar{\varphi}_{1n3} \varphi_{anb} \bar{\varphi}_{amb} \right\rangle_c = \left\langle \varphi_{anb} \bar{\varphi}_{anb} \right\rangle_c - \left\langle \bar{\varphi}_{amb} \varphi_{amb} \right\rangle_c \tag{27}$$

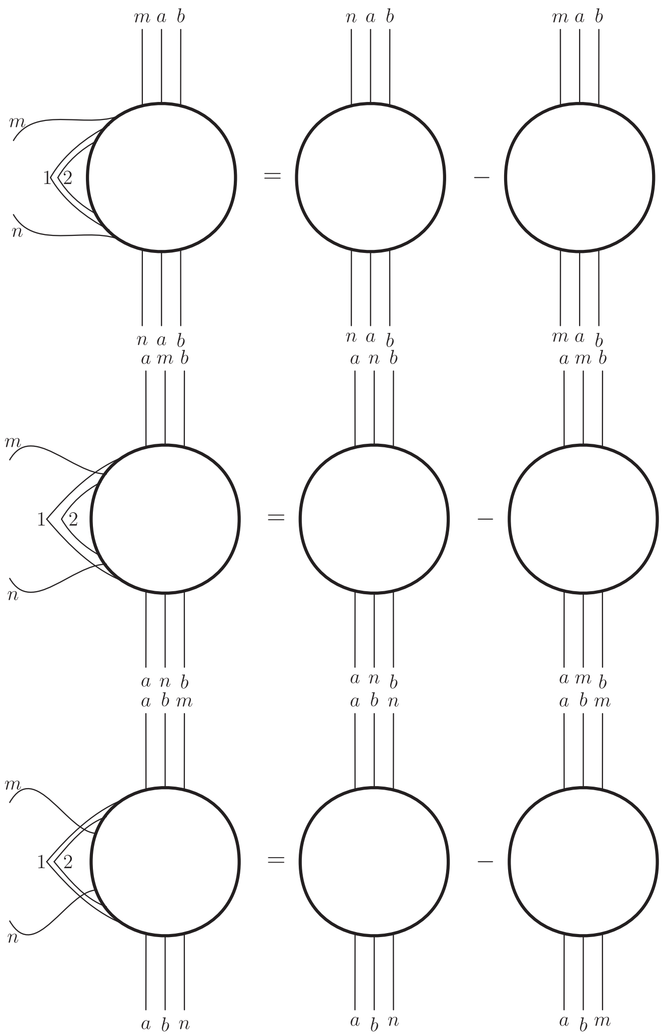

Figure 2. Ward–Takahashi identities.

$$\sum_{p_1,p_2}\left(M_{12\,m}-M_{12n}\right)\left\langle\varphi_{12\,m}\bar{\varphi}_{12n}\varphi_{abn}\bar{\varphi}_{abm}\right\rangle_c=\left\langle\varphi_{abn}\bar{\varphi}_{abn}\right\rangle_c-\left\langle\bar{\varphi}_{abm}\varphi_{abm}\right\rangle_c, \tag{28}$$

with $M_{abc}=C^{-1}_{abc}$. Equations (26), (27), and (28) are given graphically in figure 2. Let $G^{\text{ins}}_{[mn]ab}$ be the two-point functions with insertion (2, 3); that is,

$$G^{\text{ins}}_{[mn]ab}=\sum_{p_2,p_3}\left\langle\varphi_{m23}\bar{\varphi}_{n23}\varphi_{nab}\bar{\varphi}_{mab}\right\rangle_c. \tag{29}$$

The rest of this section is devoted to finding, perturbatively, the exact value of the renormalizable two-and four-point functions. We will use the Schwinger–Dyson equation, and then combine it with Ward–Takahashi identities to yield the closed equation that satisfies the connected two-and four-point functions. The Schwinger–Dyson equation is represented

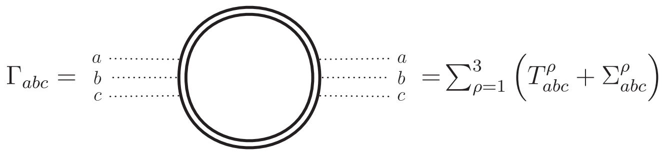

Figure 3. Schwinger–Dyson equation for 1PI two-point functions.

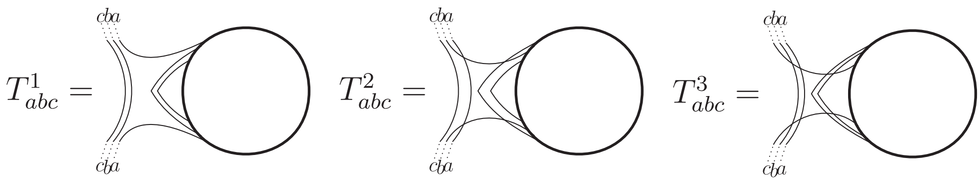

Figure 4. Definition of T^1_{abc}, T^2_{abc} and T^3_{abc}.

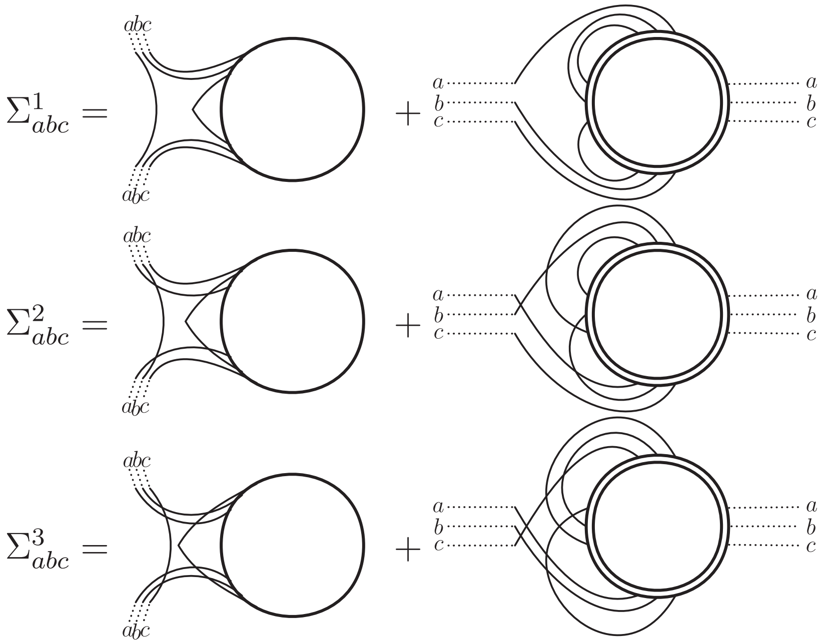

Figure 5. Definition of $\sum^1_{abc}$, $\sum^2_{abc}$ and $\sum^3_{abc}$.

graphically in figure 3. The quantities T^{ρ}_{abc} and Σ^{ρ}_{abc} for $\rho = 1, 2, 3$, are given in figures 4 and 5.

In figure 3, the quantity Γ_{abc} is the self-energy or 1PI two-point function that expresses as

$$\Gamma_{abc} = \sum_{\rho=1}^{3} \Gamma^{\rho}_{abc}, \quad \text{where} \quad \Gamma^{\rho}_{abc} = T^{\rho}_{abc} + \Sigma^{\rho}_{abc}. \tag{30}$$

Also, in figures 3–5 a single circle represents a connected graph and a double circle stands for a 1PI subgraph. Let us consider now the decomposition given in figure 6. The lhs of this

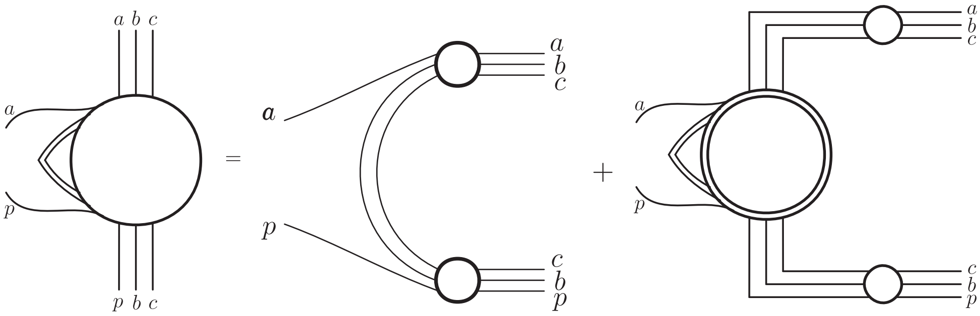

Figure 6. Decomposition of the two-point functions with insertion: case where $\rho = 1$.

equation collects all connected graphs that have the vertex insertion. By cutting this vertex out, one gets a four-point function, but the four-point function can either be disconnected (first graph on the right-hand side (rhs)), or connected (second graph on the rhs). The connected four-point function must have a 1PI four-point function somewhere as its core, and then fully connected two-point functions attached to its four legs. Now, multiplying this equation by G^{-1}_{abc} means that, on the rhs, in the first graph the upper (bc)-branch attached to the insertion vertex is removed, and in the second graph the (abc)-branch attached to the 1PI four-point functions is removed. If one now sums over p and uses the fact that the newly created vertex is $\lambda_1 Z^2$, one gets precisely the function Σ^{ρ}_{abc}. Then equation (30) can be written explicitly using the decomposition of figure 6 as

$$\Sigma^1_{abc} = Z^2\lambda_1 \sum_p G^{-1}_{abc} G^{\rm ins}_{[ap]bc}, \quad T^1_{abc} = Z^2\lambda_1 \sum_{p,q} G_{apq}. \tag{31}$$

In the same manner, we can obtain the decomposition of figure 7, which allows us to obtain the relations

$$\Sigma^2_{abc} = Z^2\lambda_2 \sum_p G^{-1}_{abc} G^{\rm ins}_{[bp]ca}, \quad T^2_{abc} = Z^2\lambda_2 \sum_{p,q} G_{pbq} \tag{32}$$

and

$$\Sigma^3_{abc} = Z^2\lambda_3 \sum_p G^{-1}_{abc} G^{\rm ins}_{[cp]ab}, \quad T^3_{abc} = Z^2\lambda_3 \sum_{p,q} G_{pqc}. \tag{33}$$

Therefore, using the expressions (31), (32), and (33), the 1PI two-point functions take the form

$$\begin{aligned}
\Gamma_{abc} &= Z^2\lambda_1 \sum_{p,q} G_{apq} + Z^2\lambda_2 \sum_{p,q} G_{pbq} + Z^2\lambda_3 \sum_{p,q} G_{pqc} \\
&\quad + Z^2\lambda_1 \sum_p G^{-1}_{abc} G^{\rm ins}_{[ap]bc} + Z^2\lambda_2 \sum_p G^{-1}_{abc} G^{\rm ins}_{[bp]ca} + Z^2\lambda_3 \sum_p G^{-1}_{abc} G^{\rm ins}_{[cp]ab} \\
&= Z^2\lambda_1 \sum_{p,q} G_{apq} + Z^2\lambda_2 \sum_{p,q} G_{pbq} + Z^2\lambda_3 \sum_{p,q} G_{pqc} + Z\lambda_1 \sum_p G^{-1}_{abc} \frac{G_{abc} - G_{pbc}}{|p| - |a|} \\
&\quad + Z\lambda_2 \sum_p G^{-1}_{abc} \frac{G_{bca} - G_{pca}}{|p| - |b|} + Z\lambda_3 \sum_p G^{-1}_{abc} \frac{G_{cab} - G_{pab}}{|p| - |c|}.
\end{aligned} \tag{34}$$

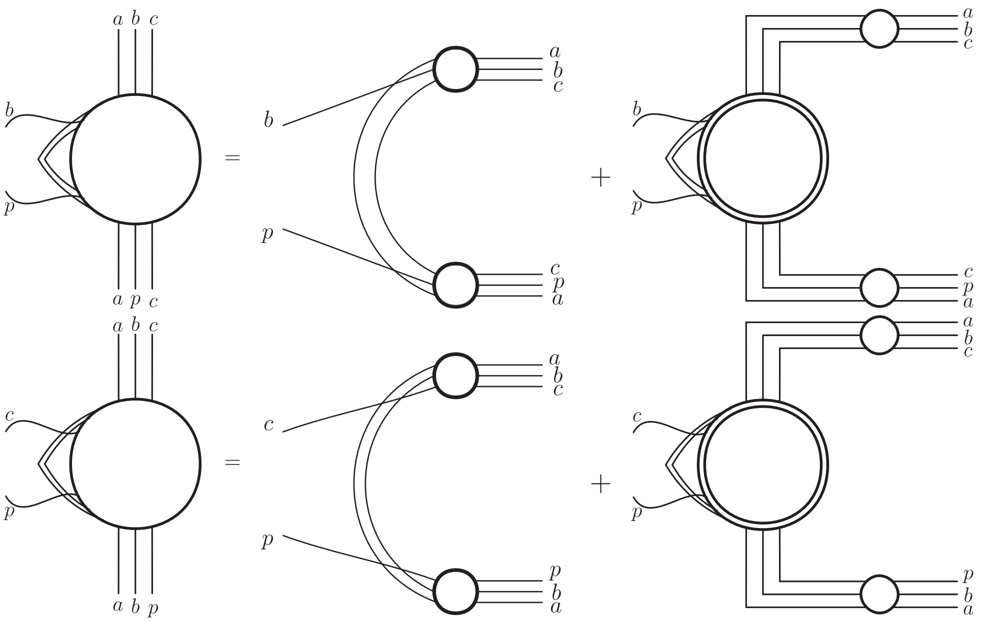

Figure 7. Decomposition of the two-point functions with insertion: case where $\rho = 2$ and $\rho = 3$.

We assume now that the function G_{abc} satisfies the condition

$$G_{abc} = G_{bca} = G_{cab}, \tag{35}$$

and then we get the following proposition.

Proposition 3. *Symmetry properties: the connected two-point functions, Γ^2_{abc}, can be obtained using Γ^1_{abc}, and can replace $a \to b$, $b \to c$, and $c \to a$, respectively. In the same manner, Γ^3_{abc} can be obtained using Γ^1_{abc}, $a \to c$, $b \to a$, and $c \to b$, respectively.*

Now, using the relation $G^{-1}_{abc} = M_{abc} - \Gamma_{abc}$, we get

$$\Gamma^1_{abc} = Z^2\lambda_1\left[\sum_{pq}\frac{1}{M_{apq} - \Gamma^1_{apq}} + \sum_p \frac{1}{M_{pbc} - \Gamma^1_{pbc}} - \sum_p \frac{1}{M_{pbc} - \Gamma^1_{pbc}}\frac{\Gamma^1_{abc} - \Gamma^1_{pbc}}{Z(|a| - |p|)}\right], \tag{36}$$

$$\Gamma^2_{abc} = Z^2\lambda_2\left[\sum_{pq}\frac{1}{M_{pbq} - \Gamma^2_{pbq}} + \sum_p \frac{1}{M_{pca} - \Gamma^2_{pca}} - \sum_p \frac{1}{M_{pca} - \Gamma^2_{pca}}\frac{\Gamma^2_{bca} - \Gamma^2_{pca}}{Z(|b| - |p|)}\right], \tag{37}$$

$$\Gamma^3_{abc} = Z^2\lambda_3\left[\sum_{pq}\frac{1}{M_{pqc} - \Gamma^3_{pqc}} + \sum_p \frac{1}{M_{pab} - \Gamma^3_{pab}} - \sum_p \frac{1}{M_{pab} - \Gamma^3_{pab}}\frac{\Gamma^3_{cab} - \Gamma^3_{pab}}{Z(|c| - |p|)}\right]. \tag{38}$$

For the rest of this section, we consider the connected two-point functions Γ^1_{abc}, and finally Γ^2_{abc} and Γ^3_{abc} will be deduced using proposition (3). Then we pass to renormalized quantities using the Taylor expansion as

$$\Gamma^1_{abc} = ZM^{\text{bar}}_{abc} - M^{\text{phys}}_{abc} + \Gamma^{\text{phys}}_{abc}, \tag{39}$$

with

$$M^{\text{phys}}_{abc} = |a| + |b| + |c| + m^2_{\text{phys}}, \quad M^{\text{bar}}_{abc} = |a| + |b| + |c| + m^2_0. \tag{40}$$

Equation (39) can be obtained after using equation (23) and

$$m^2 = \frac{m^2_{\text{phys}} + \Gamma^1_{000}}{Z}, \tag{41}$$

with the renormalization condition

$$\Gamma^{\text{phys}}_{000} = 0 = \partial\Gamma^{\text{phys}}_{000}, \tag{42}$$

such that after replacing the expression of M_{abc}, we get

$$\Gamma^1_{abc} = (Z-1)\Big(|a| + |b| + |c|\Big) + Zm^2 - m^2_{\text{phys}} + \Gamma^{\text{phys}}_{abc}, \tag{43}$$

which expresses the relation between renormalized and bare quantities. When we set $\lambda_1 = \lambda$, equation (34) takes the form

$$\begin{aligned} & Zm^2 - m^2_{\text{phys}} + (Z-1)(|a| + |b| + |c|) + \Gamma^{\text{phys}}_{abc} \\ & = Z^2\lambda \sum_{p,q} \frac{1}{|p| + |q| + |a| + m^2_{\text{phys}} - \Gamma^{\text{phys}}_{pqa}} \\ & + Z\lambda \Bigg[\sum_p \frac{1}{|p| + |b| + |c| + m^2_{\text{phys}} - \Gamma^{\text{phys}}_{pbc}} \\ & - \frac{1}{|p| + |b| + |c| + m^2_{\text{phys}} - \Gamma^{\text{phys}}_{pbc}} \frac{\Gamma^{\text{phys}}_{abc} - \Gamma^{\text{phys}}_{pbc}}{(|a| - |p|)} \Bigg]. \end{aligned} \tag{44}$$

For $a = b = c = 0$, the relation of mass variation after renormalization is written as

$$\begin{aligned} Zm^2 - m^2_{\text{phys}} = & Z^2\lambda \sum_{p,q} \frac{1}{|p| + |q| + m^2_{\text{phys}} - \Gamma^{\text{phys}}_{pq0}} + Z\lambda \sum_p \frac{1}{|p| + m^2_{\text{phys}} - \Gamma^{\text{phys}}_{p00}} \\ & - Z\lambda \sum_p \frac{1}{|p| + m^2_{\text{phys}} - \Gamma^{\text{phys}}_{p00}} \frac{\Gamma^{\text{phys}}_{p00}}{|p|}. \end{aligned} \tag{45}$$

Without any confusion, in the rest of this paper we replace the renormalized mass m_{phys} by m_0. Inserting equation (45) into (44), we get the closed equation of the two-point functions of renormalizable rank 3 TGFT as

$$
\begin{aligned}
(Z-1)\,(|a|+|b|+|c|)+\Gamma^{\text{phys}}_{abc} = {}& Z^2\lambda \sum_{p,q}\Bigg[\frac{1}{|p|+|q|+|a|+m_0^2-\Gamma^{\text{phys}}_{pqa}} \\
& -\frac{1}{|p|+|q|+m_0^2-\Gamma^{\text{phys}}_{pq0}}\Bigg] \\
& +Z\lambda\sum_p\Bigg[\frac{1}{|p|+|b|+|c|+m_0^2-\Gamma^{\text{phys}}_{pbc}} \\
& -\frac{1}{|p|+|b|+|c|+m_0^2-\Gamma^{\text{phys}}_{pbc}}\frac{\Gamma^{\text{phys}}_{abc}-\Gamma^{\text{phys}}_{pbc}}{(|a|-|p|)} \\
& -\frac{1}{|p|+m_0^2-\Gamma^{\text{phys}}_{p00}}+\frac{1}{|p|+m_0^2-\Gamma^{\text{phys}}_{p00}}\frac{\Gamma^{\text{phys}}_{p00}}{|p|}\Bigg].
\end{aligned}
\tag{46}
$$

Equation (46) is still very complicated compared to an equivalent equation in [27]. To simplify it and get an explicit solution, we pass to the integral transforms. The process is to set

$$
\sum_{p\in\mathbb{Z}} = 2\int_0^\infty \mathrm{d}|p|,\qquad \sum_{p,q\in\mathbb{Z}} = 2\int_0^\infty |p|\mathrm{d}|p|.
\tag{47}
$$

We also assume that $\Gamma_{abc}=\Gamma_{|a||b||c|}$. Then we get the integral equation of (46) as

$$
\begin{aligned}
(Z-1)\big(|a|+|b|+|c|\big)+\Gamma^{\text{phys}}_{abc} = {}& 2Z^2\lambda\int_0^\infty |p|\mathrm{d}|p|\Bigg[\frac{1}{2|p|+|a|+m_0^2-\Gamma^{\text{phys}}_{ppa}} \\
& -\frac{1}{2|p|+m_0^2-\Gamma^{\text{phys}}_{pp0}}\Bigg] \\
& +2Z\lambda\int_0^\infty \mathrm{d}|p|\Bigg[\frac{1}{|p|+|b|+|c|+m_0^2-\Gamma^{\text{phys}}_{pbc}} \\
& -\frac{1}{|p|+m_0^2-\Gamma^{\text{phys}}_{p00}} \\
& -\frac{1}{|p|+|b|+|c|+m_0^2-\Gamma^{\text{phys}}_{pbc}}\frac{\Gamma^{\text{phys}}_{abc}-\Gamma^{\text{phys}}_{pbc}}{(|a|-|p|)} \\
& +\frac{1}{|p|+m_0^2-\Gamma^{\text{phys}}_{p00}}\frac{\Gamma^{\text{phys}}_{p00}}{|p|}\Bigg]
\end{aligned}
\tag{48}
$$

with $p\in\mathbb{R}^+$. We introduce a change of variables,

$$
|a| = m_0^2\frac{\alpha}{1-\alpha},\qquad |b| = m_0^2\frac{\beta}{1-\beta},\qquad |c| = m_0^2\frac{\gamma}{1-\gamma},\qquad |p| = m_0^2\frac{\rho}{1-\rho},
\tag{49}
$$

$$\Gamma^{\rm phys}_{abc} = m_0^2 \frac{\Gamma_{\alpha\beta\gamma}}{(1-\alpha)(1-\beta)(1-\gamma)}. \tag{50}$$

We also take the cutoff, Λ, such that $p_\Lambda = m_0^2 \frac{\Lambda}{1-\Lambda}$. Let us now define the quantity $G_{\alpha\beta\gamma}$ as

$$1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma-\Gamma_{\alpha\beta\gamma} = \frac{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}{G_{\alpha\beta\gamma}}. \tag{51}$$

Let $\mathcal{J}_{\alpha\beta\gamma}$, $\mathcal{L}_{\alpha\beta\gamma}$, and $\mathcal{K}_\alpha$ be three integrals whose relation is given by

$$\mathcal{J}_{\alpha\beta\gamma} = \int_0^1 \frac{{\rm d}\rho}{(\alpha-\rho)} \frac{G_{\rho\beta\gamma}}{(1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta)}, \tag{52}$$

$$\mathcal{L}_{\alpha\beta\gamma} = \int_0^1 \frac{{\rm d}\rho}{(1-\rho)} \frac{G_{\rho\beta\gamma}-1}{(\alpha-\rho)}, \tag{53}$$

$$\mathcal{K}_\alpha = m_0^2 \frac{\int_0^1 \frac{\rho {\rm d}\rho}{(1-\rho)}\left(\frac{(1-\alpha)G_{\rho\rho\alpha}}{1-\rho^2-2\alpha\rho+2\alpha\rho^2} - \frac{G_{\rho\rho 0}}{1-\rho^2}\right)}{1+\frac{2\lambda}{m_0^2}\int_0^1 {\rm d}\rho \left(\frac{G'_{\rho 00}}{\rho} + G_{\rho 00}\right)}. \tag{54}$$

Then we get the following theorem.

Theorem 1. *The connected two-point functions, $G_{\alpha\beta\gamma}$, of the renormalizable rank 3 TGFT on $U(1)$ satisfy the closed integral equation*

$$\begin{aligned} G_{\alpha\beta\gamma} = 1 + \lambda' \Bigg\{ & \mathcal{Y} + \int_0^1 {\rm d}\rho\, G_{\rho 00} + (1-\alpha)(1-\beta)(1-\gamma)\mathcal{J}_{\alpha\beta\gamma} \\ & + \frac{(1-\alpha)(1-\beta)(1-\gamma)G_{\alpha\beta\gamma}}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}\Bigg[-\mathcal{Y} - \int_0^1 {\rm d}\rho\, G_{\rho 00} + \mathcal{K}_\alpha \\ & + \int_0^1 {\rm d}\rho\, \frac{G_{\rho\beta\gamma}-G_{\rho 00}}{1-\rho} - \int_0^1 {\rm d}\rho\, \frac{G_{\rho\beta\gamma}}{\alpha-\rho} + (1-\alpha)\mathcal{L}_{\alpha\beta\gamma} - \mathcal{L}_{000}\Bigg]\Bigg\} \end{aligned} \tag{55}$$

where

$$\mathcal{Y} = \lim_{\epsilon\to 0}\int_0^1 {\rm d}\rho\, \frac{G_{\rho\epsilon 0}-G_{\rho 00}}{\epsilon\rho}, \qquad \lambda' = \frac{2\lambda}{m_0^2}. \tag{56}$$

Proof. Using the transformations given in equations (49) and (50), the expression (48) takes the form

$$
\begin{aligned}
&(Z-1)\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}+\frac{\gamma}{1-\gamma}\right)+\frac{\Gamma_{\alpha\beta\gamma}}{(1-\alpha)(1-\beta)(1-\gamma)}\\
&\quad=2Z^2\lambda\int_0^\Lambda \frac{\rho \mathrm{d}\rho}{(1-\rho)}\left[\frac{(1-\alpha)}{1-\rho^2-2\alpha\rho+2\alpha\rho^2-\Gamma_{\rho\rho\alpha}}-\frac{1}{1-\rho^2-\Gamma_{\rho\rho 0}}\right]\\
&\quad+\frac{2Z\lambda}{m_0^2}\int_0^\Lambda \frac{\mathrm{d}\rho}{(1-\rho)}\left[\frac{(1-\beta)(1-\gamma)}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta-\Gamma_{\rho\beta\gamma}}-\frac{1}{1-\Gamma_{\rho 00}}\right.\\
&\quad-\frac{1}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta-\Gamma_{\rho\beta\gamma}}\frac{(1-\rho)\Gamma_{\alpha\beta\gamma}-(1-\alpha)\Gamma_{\rho\beta\gamma}}{\alpha-\rho}\\
&\quad\left.+\frac{1}{1-\Gamma_{\rho 00}}\frac{\Gamma_{\rho 00}}{\rho}\right].
\end{aligned}
\tag{57}
$$

Note that β and γ are symmetric parameters in equation (57). This implies that $\Gamma_{\alpha\beta\gamma}=\Gamma_{\alpha\gamma\beta}$. Let us now take $\frac{\partial}{\partial\alpha}\big|_{\alpha=\beta=\gamma=0}$ and $\frac{\partial}{\partial\beta}\big|_{\alpha=\beta=\gamma=0}$ of the above equation. We come to the relations that satisfy the renormalized wave function, Z:

$$
\begin{aligned}
Z-1=2Z^2\lambda\int_0^\Lambda \frac{\rho \mathrm{d}\rho}{(1-\rho)}&\frac{\left(-1+2\rho-\rho^2+\Gamma'_{\rho\rho 0}+\Gamma_{\rho\rho 0}\right)}{\left(1-\rho^2-\Gamma_{\rho\rho 0}\right)^2}\\
&-\frac{2Z\lambda}{m_0^2}\int_0^\Lambda \mathrm{d}\rho\frac{\Gamma_{\rho 00}}{\rho^2\left(1-\Gamma_{\rho 00}\right)},
\end{aligned}
\tag{58}
$$

and

$$
\begin{aligned}
Z-1=\frac{2Z\lambda}{m_0^2}\int_0^\Lambda \frac{\mathrm{d}\rho}{(1-\rho)}&\left[\frac{-1+\rho+\Gamma_{\rho 00}+\Gamma'_{\rho 00}}{\left(1-\Gamma_{\rho 00}\right)^2}\right.\\
&\left.-\frac{\left(\rho+\Gamma'_{\rho 00}\right)\Gamma_{\rho 00}}{\rho\left(1-\Gamma_{\rho 00}\right)^2}-\frac{\Gamma'_{\rho 00}}{\rho\left(1-\Gamma_{\rho 00}\right)}\right],
\end{aligned}
\tag{59}
$$

where we take $\Gamma'_{\rho 00}=:\frac{\partial\Gamma_{\rho\beta\gamma}}{\partial\beta}\big|_{\beta=\gamma=0}$ or $\Gamma'_{\rho 00}=:\frac{\partial\Gamma_{\rho\beta\gamma}}{\partial\gamma}\big|_{\beta=\gamma=0}$ and $\Gamma'_{\rho\rho 0}=:\frac{\partial\Gamma_{\rho\rho\alpha}}{\partial\alpha}\big|_{\alpha=0}$. Now let us pass to the new function, $G_{\alpha\beta\gamma}$, given in (51). We find the following relations:

$$
\rho+\Gamma'_{\rho 00}=\frac{\rho}{G_{\rho 00}}+\frac{G'_{\rho 00}}{G^2_{\rho 00}},\quad 2\rho-2\rho^2+\Gamma'_{\rho\rho 0}=\frac{2\rho(1-\rho)}{G_{\rho\rho 0}}+\frac{\left(1-\rho^2\right)G'_{\rho\rho 0}}{G_{\rho\rho 0}}.
\tag{60}
$$

Therefore, equation (59) reduces to

$$
Z^{-1}=1+\frac{2\lambda}{m_0^2}\int_0^\Lambda \mathrm{d}\rho\left[\frac{G'_{\rho 00}}{\rho}+G_{\rho 00}\right],
\tag{61}
$$

and (57) takes the form

$$
\begin{aligned}
&ZG_{\alpha\beta\gamma} - 1 - (Z-1)\frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}G_{\alpha\beta\gamma}\\
&=\frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}G_{\alpha\beta\gamma}\Bigg\{2Z^2\lambda\int_0^{\Lambda}\frac{\rho\mathrm{d}\rho}{(1-\rho)}\Bigg[\frac{(1-\alpha)G_{\rho\rho\alpha}}{1-\rho^2-2\alpha\rho+2\alpha\rho^2}\\
&-\frac{G_{\rho\rho0}}{1-\rho^2}\Bigg]+\frac{2Z\lambda}{m_0^2}\int_0^{\Lambda}\frac{\mathrm{d}\rho}{(1-\rho)}\Bigg[\frac{(1-\beta)(1-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta}-G_{\rho00}\\
&+\frac{(1-\alpha)\big(G_{\rho\beta\gamma}-1\big)}{(\alpha-\rho)}+\frac{G_{\rho00}-1}{\rho}-\frac{G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta}\\
&\times\frac{(1-\rho)(1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma)\big(G_{\alpha\beta\gamma}-1\big)}{(\alpha-\rho)G_{\alpha\beta\gamma}}\Bigg]\Bigg\}. \qquad (62)
\end{aligned}
$$

Inserting (61) into the lhs of (62) and dividing by Z, one gets

$$
\begin{aligned}
G_{\alpha\beta\gamma} &= Z^{-1}-\frac{2\lambda}{m_0^2}\frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}G_{\alpha\beta\gamma}\int_0^{\Lambda}\mathrm{d}\rho\left(\frac{G'_{\rho00}}{\rho}+G_{\rho00}\right)\\
&-\frac{2\lambda}{m_0^2}\int_0^{\Lambda}\mathrm{d}\rho\frac{(1-\alpha)(1-\beta)(1-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta}\cdot\frac{\big(G_{\alpha\beta\gamma}-1\big)}{(\alpha-\rho)}\\
&+\frac{(1-\alpha)(1-\beta)(1-\gamma)G_{\alpha\beta\gamma}}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}\Bigg\{\frac{2\lambda}{Z^{-1}}\int_0^{\Lambda}\frac{\rho\mathrm{d}\rho}{(1-\rho)}\Bigg[\frac{(1-\alpha)G_{\rho\rho\alpha}}{1-\rho^2-2\alpha\rho+2\alpha\rho^2}\\
&-\frac{G_{\rho\rho0}}{1-\rho^2}\Bigg]+\frac{2\lambda}{m_0^2}\int_0^{\Lambda}\frac{\mathrm{d}\rho}{(1-\rho)}\Bigg[\frac{(1-\beta)(1-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta}-G_{\rho00}\\
&+\frac{(1-\alpha)\big(G_{\rho\beta\gamma}-1\big)}{(\alpha-\rho)}+\frac{G_{\rho00}-1}{\rho}\Bigg]\Bigg\}. \qquad (63)
\end{aligned}
$$

Replacing (61) with (63) yields

$$
\begin{aligned}
G_{\alpha\beta\gamma} &= 1+\frac{2\lambda}{m_0^2}\Bigg\{\int_0^{\Lambda}\mathrm{d}\rho\left(\frac{G'_{\rho00}}{\rho}+G_{\rho00}\right)-\frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}G_{\alpha\beta\gamma}\\
&\times\int_0^{\Lambda}\mathrm{d}\rho\left(\frac{G'_{\rho00}}{\rho}+G_{\rho00}\right)-\int_0^{\Lambda}\mathrm{d}\rho\frac{(1-\alpha)(1-\beta)(1-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta}\cdot\\
&\times\frac{\big(G_{\alpha\beta\gamma}-1\big)}{(\alpha-\rho)}\\
&+\frac{(1-\alpha)(1-\beta)(1-\gamma)G_{\alpha\beta\gamma}}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}
\end{aligned}
$$

$$\times\left[m_0^2 \frac{\int_0^{\Lambda} \frac{\rho d\rho}{(1-\rho)} \left(\frac{(1-\alpha) G_{\rho\rho\alpha}}{1-\rho^2-2\alpha\rho+2\alpha\rho^2} - \frac{G_{\rho\rho 0}}{1-\rho^2} \right)}{1+\frac{2\lambda}{m_0^2}\int_0^{\Lambda} d\rho \left(\frac{G'_{\rho 00}}{\rho} + G_{\rho 00} \right)} \right.$$
$$+\int_0^{\Lambda} \frac{d\rho}{(1-\rho)} \left(\frac{(1-\beta)(1-\gamma) G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta} - G_{\rho 00} + \frac{(1-\alpha)\left(G_{\rho\beta\gamma}-1\right)}{(\alpha-\rho)} \right.$$
$$\left.\left.\left. + \frac{G_{\rho 00}-1}{\rho} \right) \right] \right\}. \tag{64}$$

Simplifying identical terms, we get the result of theorem 1. □

Note that $G_{000} = 1$ and $\partial G_{000} = 0$. Equation (64) shows the occurrence of the singular integral kernel, $\int_0^{\Lambda} \frac{d\rho}{\rho-\alpha}$, $\int_0^{\Lambda} \frac{d\rho}{1-\rho}$, and $\int_0^{\Lambda} \frac{d\rho}{\rho}$ for $\Lambda = 1$, which needs to be removed. We will use the Cauchy principal value of the divergent integrals, and also take the limit value at points 0 and 1; that is,

$$\int_0^1 = \lim_{\epsilon\to 0}\left[\int_0^{a-\epsilon} + \int_{a+\epsilon}^1 \right], \quad a \in (0, 1), \quad \int_0^1 = \lim_{\epsilon\to 0, \epsilon'\to 1} \int_{\epsilon}^{\epsilon'}. \tag{65}$$

The nonlinear integral equation (55) is of the form

$$G_{\alpha\beta\gamma} = 1 + \lambda \int_0^1 f\left(G_{\alpha\beta\gamma}, G_{\rho\beta\gamma}, G_{\rho\alpha 0}, G_{\rho 00}, \mathcal{Y}, \alpha, \beta, \gamma\right) d\rho. \tag{66}$$

Now we can easily see that (55) suffers from a lack of symmetry. This inconvenience is due to the position of parameter α. So given that $\alpha = 0$, we get the symmetric solution given in the following proposition.

Proposition 4. *At first order in λ, the solution of equation (55) for $\alpha = 0$ is given by*

$$G_{0\beta\gamma} = 1 + \lambda' \left[1 + \frac{(1-\beta)(1-\gamma)}{1-\beta\gamma} \left(\ln \frac{1+\beta\gamma-\beta-\gamma}{1-\beta\gamma} - 1 \right) \right] = 1 + \lambda' \mathcal{K}_{0\beta\gamma}. \tag{67}$$

Then, using the symmetry properties of proposition 3, we get the symmetric solution $G^{\rm sym}_{\alpha\beta\gamma}$ as

$$G^{\rm sym}_{\alpha\beta\gamma} = 1 + \lambda'_1 \mathcal{K}_{0\beta\gamma} + \lambda'_2 \mathcal{K}_{0\gamma\alpha} + \lambda'_3 \mathcal{K}_{0\alpha\beta}, \tag{68}$$

with $\lambda'_\rho = \frac{2\lambda_\rho}{m_0^2}$; $\rho = 1, 2, 3$ and $\alpha, \beta, \gamma \in [0, 1)$.

We use the symmetry relation (35) and get the following result.

Theorem 2. *The closed equation of the symmetric two-point functions, $G_{\alpha\beta\gamma}$, satisfies the nonlinear integral equation*

$$G_{\alpha\beta\gamma} = 1 + \lambda' \Bigg\{ \mathcal{Y} + \int_0^1 d\rho\, G_{\rho 00} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma} \Bigg[\int_0^1 d\rho \Bigg(\frac{G_{\rho\beta\gamma}}{\alpha-\rho}$$
$$+ \frac{(2\beta\gamma-\beta-\gamma)G_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\gamma\beta+2\rho\gamma\beta} \Bigg) + G_{\alpha\beta\gamma} \Bigg(\int_0^1 d\rho\, \frac{G_{\rho\alpha 0}-G_{\rho 00}}{1-\rho} + \int_0^1 d\rho\, \frac{G_{\rho 00}}{\rho} - \mathcal{Y}$$
$$- \int_0^1 d\rho\, G_{\rho 00} - G_{0\alpha 0}^{-1} \Bigg(\int_0^1 d\rho\, \frac{G_{\rho\alpha 0}}{\rho}$$
$$+ \alpha \int_0^1 d\rho\, \frac{G_{\rho\alpha 0}}{1-\alpha\rho} + \int_0^1 d\rho\, \frac{G_{\rho 00}}{\alpha-\rho} \Bigg) \Bigg) \Bigg] \Bigg\}. \tag{69}$$

Proof. Using the relation (35), we can extract the quantity $\mathcal{K}_\alpha$ after simplification as

$$\mathcal{K}_\alpha = -G_{0\alpha\beta}^{-1} \Bigg[\int_0^1 d\rho\, \frac{G_{\rho\alpha\beta}}{\rho} - (2\alpha\beta-\alpha-\beta) \int_0^1 d\rho\, \frac{G_{\rho\alpha\beta}}{1-\alpha\rho-\beta\rho-\alpha\beta+2\alpha\beta\rho}$$
$$+ \int_0^1 d\rho\, \frac{G_{\rho 0\beta}}{\alpha-\rho} - \beta \int_0^1 d\rho\, \frac{G_{\rho 0\beta}}{1-\beta\rho} \Bigg] + \int_0^1 d\rho\, \frac{G_{\rho\alpha\beta}-G_{\rho 0\beta}}{1-\rho}$$
$$+ \int_0^1 d\rho \left(\frac{1}{\rho} + \frac{1}{\alpha-\rho} \right). \tag{70}$$

Then, note that $\mathcal{K}_\alpha$ is a function only of parameter α. We then take $\beta = 0$ in the last equation, and we get

$$\mathcal{K}_\alpha = -G_{0\alpha 0}^{-1} \left(\int_0^1 d\rho\, \frac{G_{\rho\alpha 0}}{\rho} + \alpha \int_0^1 d\rho\, \frac{G_{\rho\alpha 0}}{1-\alpha\rho} + \int_0^1 d\rho\, \frac{G_{\rho 00}}{\alpha-\rho} \right)$$
$$+ \int_0^1 d\rho\, \frac{G_{\rho\alpha 0}-G_{\rho 00}}{1-\rho} + \int_0^1 d\rho \left(\frac{1}{\rho} + \frac{1}{\alpha-\rho} \right). \tag{71}$$

By replacing the relation (71) in expression (64), we get the desired result. □

Now we are reaching the point where it is possible to give the solution of equation (69). Let us write the solution of this equation as

$$G_{\alpha\beta\gamma} = 1 + \sum_{n=1}^{\infty} (\lambda')^n \mathcal{X}_{\alpha\beta\gamma}^{(n)}, \qquad \mathcal{X}_{000}^{(n)} = 0. \tag{72}$$

The n order terms, $\mathcal{X}_{\alpha\beta\gamma}^{(n)}$, can be deduced by iteration. We give here the quantities $\mathcal{X}_{\alpha\beta\gamma}^{(1)}$ and $\mathcal{X}_{\alpha\beta\gamma}^{(2)}$ in the following statement.

Proposition 5. *Perturbatively, at second order in λ, the symmetry solution of equation (69) using the Cauchy principal value is given by*

$$G_{\alpha\beta\gamma} = 1 + \lambda' \mathcal{X}^{(1)}_{\alpha\beta\gamma} + \lambda'^2 \Bigg\{ \frac{\pi^2}{6} - \frac{3}{2} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}$$
$$\times\Bigg[\mathcal{X}^{(1)}_{\alpha\beta\gamma}\left(\ln\frac{(1-\alpha)^2}{\alpha} - 1\right)$$
$$+\int_0^1 \mathrm{d}\rho \frac{(2\beta\gamma-\beta-\gamma)\mathcal{X}^{(1)}_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\beta\gamma+2\beta\gamma\rho} + \int_0^1 \mathrm{d}\rho \frac{\mathcal{X}^{(1)}_{\rho\alpha 0}-\mathcal{X}^{(1)}_{\rho 00}}{1-\rho} - \alpha\int_0^1 \mathrm{d}\rho\, \frac{\mathcal{X}^{(1)}_{\rho\alpha 0}}{1-\alpha\rho}$$
$$-\frac{\pi^2}{6} + \frac{3}{2} - \int_0^1 \mathrm{d}\rho \frac{\mathcal{X}^{(1)}_{\rho\alpha 0}-\mathcal{X}^{(1)}_{\rho 00}+\mathcal{X}^{(1)}_{0\alpha 0}}{\rho} - \mathcal{X}^{(1)}_{0\alpha 0}\ln\frac{(1-\alpha)^2}{\alpha}\Bigg]\Bigg\} + \mathcal{O}\left(\lambda'^3\right), \quad (73)$$

where $G_{000} = 1$ *and where the first order term* $\mathcal{X}^{(1)}_{\alpha\beta\gamma}$ *is*

$$\mathcal{X}^{(1)}_{\alpha\beta\gamma} = 1 + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}\left(\ln(1-\alpha) - 1 + \ln\frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma}\right). \quad (74)$$

The exact values of the integrals in the rhs of (73) are given using the following relations:

$$\int_0^1 \mathrm{d}\rho\, \frac{\mathcal{X}^{(1)}_{\rho\beta\gamma}}{a-\rho} = \ln\frac{a}{1-a} + \frac{(1-\beta)(1-\gamma)}{1-a\beta-a\gamma-\beta\gamma+2a\beta\gamma}\left(-1+\ln\frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma}\right)$$
$$\times\left((1-a)\ln\frac{a}{1-a} + \frac{\beta\gamma-\beta-\gamma+1}{2\beta\gamma-\beta-\gamma}\ln\frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma}\right)$$
$$+\frac{(1-a)(1-\beta)(1-\gamma)}{1-a\beta-a\gamma-\beta\gamma+2a\beta\gamma}\left(\frac{\pi^2}{6} - Li_2\frac{-a}{1-a}\right.$$
$$\left.+\ln(1-a)\ln\frac{a}{1-a} - \frac{1}{1-a}\right)$$
$$+\frac{(1-\beta\gamma)(1-\beta)(1-\gamma)}{(\beta+\gamma-2\beta\gamma)(1-a\beta-a\gamma-\beta\gamma+2a\beta\gamma)}$$
$$\times\left(\frac{\pi^2}{6} - Li_2\frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma}\right.$$
$$\left.-\ln\frac{\beta+\gamma-2\beta\gamma}{1-\beta\gamma}\ln\frac{\beta\gamma-\beta-\gamma+1}{1-\beta\gamma}\right) \quad (75)$$

and

$$\int_0^1 \mathrm{d}\rho \frac{\mathcal{X}^{(1)}_{\rho\alpha 0}-\mathcal{X}^{(1)}_{\rho 00}+\mathcal{X}^{(1)}_{0\alpha 0}}{\rho} = \frac{(1-\alpha)^2}{\alpha}\ln(1-\alpha)(\ln(1-\alpha)-1) - (1-\alpha)\left(\frac{\pi^2}{6}-1\right)$$
$$+\frac{1-\alpha}{\alpha}\left(\ln\alpha\ln(1-\alpha) + Li_2(1-\alpha) - \frac{\pi^2}{6}\right) + \frac{\pi^2}{6} - 1, \quad (76)$$

where

$$Li_2(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^2}, \quad Li_2(1) = \frac{\pi^2}{6}, \quad Li_2(-1) = -\frac{\pi^2}{12}, \quad Li_2(0) = 0. \quad (77)$$

Let us immediately emphasize that the above solution is related to the coupling constant, λ_1. To establish the full solution of the two-point functions of our model, which takes into account the three coupling constants λ_ρ, $\rho = 1, 2, 3$, we must use the symmetry condition of proposition 3. The end result is given by the sum of three equations (36), (37) and (38). Therefore, the two-point functions, $G^{\rm sym}_{\alpha\beta\gamma}$, of the 3$D$ tensor model are given by the relation

$$
\begin{aligned}
G^{\rm sym}_{\alpha\beta\gamma} =& 1 + \lambda'_1\mathcal{X}^{(1)}_{\alpha\beta\gamma} + \lambda'^2_1\Bigg\{\frac{\pi^2}{6} - \frac{3}{2} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}\\
&\times\Bigg[\mathcal{X}^{(1)}_{\alpha\beta\gamma}\left(\ln\frac{(1-\alpha)^2}{\alpha} - 1\right)\\
&+\int_0^1 {\rm d}\rho\frac{(2\beta\gamma-\beta-\gamma)\mathcal{X}^{(1)}_{\rho\beta\gamma}}{1-\beta\rho-\gamma\rho-\beta\gamma+2\beta\gamma\rho} + \int_0^1 {\rm d}\rho\frac{\mathcal{X}^{(1)}_{\rho\alpha 0}-\mathcal{X}^{(1)}_{\rho 00}}{1-\rho} - \alpha\int_0^1 {\rm d}\rho\,\frac{\mathcal{X}^{(1)}_{\rho\alpha 0}}{1-\alpha\rho}\\
&-\frac{\pi^2}{6} + \frac{3}{2} - \int_0^1 {\rm d}\rho\frac{\mathcal{X}^{(1)}_{\rho\alpha 0}-\mathcal{X}^{(1)}_{\rho 00}+\mathcal{X}^{(1)}_{0\alpha 0}}{\rho} - \mathcal{X}^{(1)}_{0\alpha 0}\ln\frac{(1-\alpha)^2}{\alpha}\Bigg]\Bigg\} + \mathcal{O}\left(\lambda_1'^3\right)\\
&+\lambda'_2\mathcal{X}^{(1)}_{\beta\gamma\alpha} + \lambda'^2_2\Bigg\{\frac{\pi^2}{6} - \frac{3}{2} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}\\
&\times\Bigg[\mathcal{X}^{(1)}_{\beta\gamma\alpha}\left(\ln\frac{(1-\beta)^2}{\beta} - 1\right)\\
&+\int_0^1 {\rm d}\rho\frac{(2\alpha\gamma-\alpha-\gamma)\mathcal{X}^{(1)}_{\rho\gamma\alpha}}{1-\alpha\rho-\gamma\rho-\alpha\gamma+2\alpha\gamma\rho} + \int_0^1 {\rm d}\rho\frac{\mathcal{X}^{(1)}_{\rho\beta 0}-\mathcal{X}^{(1)}_{\rho 00}}{1-\rho} - \beta\int_0^1 {\rm d}\rho\,\frac{\mathcal{X}^{(1)}_{\rho\beta 0}}{1-\beta\rho}\\
&-\frac{\pi^2}{6} + \frac{3}{2} - \int_0^1 {\rm d}\rho\frac{\mathcal{X}^{(1)}_{\rho\beta 0}-\mathcal{X}^{(1)}_{\rho 00}+\mathcal{X}^{(1)}_{0\beta 0}}{\rho} - \mathcal{X}^{(1)}_{0\beta 0}\ln\frac{(1-\beta)^2}{\beta}\Bigg]\Bigg\} + \mathcal{O}\left(\lambda'^3_2\right)\\
&+\lambda'_3\mathcal{X}^{(1)}_{\gamma\beta\alpha} + \lambda'^2_3\Bigg\{\frac{\pi^2}{6} - \frac{3}{2} + \frac{(1-\alpha)(1-\beta)(1-\gamma)}{1-\alpha\beta-\alpha\gamma-\beta\gamma+2\alpha\beta\gamma}\Bigg[\mathcal{X}^{(1)}_{\gamma\beta\alpha}\left(\ln\frac{(1-\gamma)^2}{\gamma} - 1\right)\\
&+\int_0^1 {\rm d}\rho\frac{(2\alpha\beta-\alpha-\beta)\mathcal{X}^{(1)}_{\rho\alpha\beta}}{1-\alpha\rho-\beta\rho-\alpha\beta+2\alpha\beta\rho} + \int_0^1 {\rm d}\rho\frac{\mathcal{X}^{(1)}_{\rho\gamma 0}-\mathcal{X}^{(1)}_{\rho 00}}{1-\rho} - \gamma\int_0^1 {\rm d}\rho\,\frac{\mathcal{X}^{(1)}_{\rho\gamma 0}}{1-\gamma\rho}\\
&-\frac{\pi^2}{6} + \frac{3}{2} - \int_0^1 {\rm d}\rho\frac{\mathcal{X}^{(1)}_{\rho\gamma 0}-\mathcal{X}^{(1)}_{\rho 00}+\mathcal{X}^{(1)}_{0\gamma 0}}{\rho} - \mathcal{X}^{(1)}_{0\gamma 0}\ln\frac{(1-\gamma)^2}{\gamma}\Bigg]\Bigg\} + \mathcal{O}\left(\lambda'^3_3\right) \quad (78)
\end{aligned}
$$

where $\lambda'_\rho = 2\lambda_\rho/m_0^2$; $\rho = 1, 2, 3$, $\alpha, \beta, \gamma \in (0, 1)$; and $G_{000} = 1$. Note that solution (78) satisfies condition (35) if and only if we set $\lambda_1' = \lambda_2' = \lambda_3'$. Let us also emphasize that the higher-order solution can be attained perturbatively by iteration.

4. Closed equation for two-point functions of rank 4 TGFT

The same method used in last section will be implemented here to establish the renormalized two-point functions of the rank 4 tensor field first given in [13]. We provide the master equation of the two-point functions. The action, S_{4D}, of the model is also subdivided into two terms as

$$S_{4D} = S_{4D}^{\mathrm{kin}} + S_{4D}^{\mathrm{int}}. \tag{79}$$

The kinetic term, S_{4D}^{kin}, is given by

$$S_{4D}^{\mathrm{kin}} = \sum_{p_j \in \mathbb{Z}} \varphi_{1234} \left(\sum_{i=1}^{4} p_i^2 + m^2 \right) \bar{\varphi}_{1234}. \tag{80}$$

Note that in a four-dimensional case, the renormalization is guaranteed by the presence of the propagator associated with the heat kernel [26]:

$$C([p]) = \left(\sum_{i=1}^{4} p_i^2 + m^2 \right)^{-1} = M_{1234}^{-1}. \tag{81}$$

S_{4D}^{int} is related to the interaction, which is divided into three fundamental contributions ($V_{6,1}$, $V_{6,2}$, and $V_{4,1}$) given by

$$V_{6;1} = \sum_{p_j \in \mathbb{Z}} \varphi_{1234}\, \bar{\varphi}_{1'234}\, \varphi_{1'2'3'4'}\, \bar{\varphi}_{1''2'3'4'}\, \varphi_{1''2''3''4''}\, \bar{\varphi}_{12''3''4''} + \text{permutations} \tag{82}$$

$$V_{6;2} = \sum_{p_j \in \mathbb{Z}} \varphi_{1234}\, \bar{\varphi}_{1'2'3'4}\, \varphi_{1'2'3'4'}\, \bar{\varphi}_{1''234'}\, \varphi_{1''2''3''4''}\, \bar{\varphi}_{12''3''4''} + \text{permutations} \tag{83}$$

$$V_{4;1} = \sum_{p_j \in \mathbb{Z}} \varphi_{1234}\, \bar{\varphi}_{1'234}\, \varphi_{1'2'3'4'}\, \bar{\varphi}_{12'3'4'} + \text{permutations}, \tag{84}$$

and an anomalous term, $(V_{4,2})$,

$$V_{4;2} = \left(\sum_{p_j \in \mathbb{Z}} \bar{\varphi}_{1234}\, \varphi_{1234} \right) \left(\sum_{p_j \in \mathbb{Z}} \bar{\varphi}_{1'2'3'4'}\, \varphi_{1'2'3'4'} \right). \tag{85}$$

This last vertex is not taken into account in the computation of the correlation functions because it is disconnected and does not contribute to the melonic Feynman graph of the theory. This vertex could be interpreted as the generation of a scalar matter field out of pure gravity [13]. The vertices are represented in figure 8.

Let us immediately emphasize that vertices of the type $V_{6,1}$ and $V_{4,1}$ are parametrized by four indices, $\rho \in \{1, 2, 3, 4\}$; the vertices contributing to $V_{6,2}$ are parametrized by six index values, $\rho\rho' \in \{1.2, 1.3, 1.4, 2.3, 2.4, 3.4\}$. The couple, $\rho\rho'$ will be totally symmetric (i.e., $\rho\rho' = \rho'\rho$). One can check that these interactions are invariant under $U(N_a)$ transformations. Then, the same procedure for finding the Ward–Takahashi identities applies. The Ward–Takahashi identities of equation (20) are re-expressed as

$$\left(M_{m234} - M_{n234}\right) \left\langle \left[\varphi_m \bar{\varphi}_n\right]_{234} \varphi_{n234} \bar{\varphi}_{m234} \right\rangle_c = \left\langle \varphi_{n234} \bar{\varphi}_{n234} \right\rangle_c - \left\langle \bar{\varphi}_{m234} \varphi_{m234} \right\rangle_c \tag{86}$$

$$\left(M_{1m34} - M_{1n34}\right) \left\langle \left[\varphi_m \bar{\varphi}_n\right]_{134} \varphi_{1n34} \bar{\varphi}_{1m34} \right\rangle_c = \left\langle \varphi_{1n34} \bar{\varphi}_{1n34} \right\rangle_c - \left\langle \bar{\varphi}_{1m34} \varphi_{1m34} \right\rangle_c \tag{87}$$

$$\left(M_{12m4} - M_{12n4}\right) \left\langle \left[\varphi_m \bar{\varphi}_n\right]_{124} \varphi_{12n4} \bar{\varphi}_{12m4} \right\rangle_c = \left\langle \varphi_{12n4} \bar{\varphi}_{12n4} \right\rangle_c - \left\langle \bar{\varphi}_{12m4} \varphi_{12m4} \right\rangle_c \tag{88}$$

$$\left(M_{123m} - M_{123n}\right) \left\langle \left[\varphi_m \bar{\varphi}_n\right]_{123} \varphi_{123n} \bar{\varphi}_{123m} \right\rangle_c = \left\langle \varphi_{123n} \bar{\varphi}_{123n} \right\rangle_c - \left\langle \bar{\varphi}_{123m} \varphi_{123m} \right\rangle_c. \tag{89}$$

Figure 9 gives the Schwinger–Dyson equation of the two-point functions. This figure collects the 1PI two-point functions. Let us discuss the contributions in figure 9. The graphs of figure 10 are related to the graphs made with vertex $V_{6,1}$. The first graph of this figure is denoted by $T^{6,\,1}_{abcd}$, and the sum of the other two is $\Sigma^{6,\,1}_{abcd}$. The graphs of figure 11 are related to the graphs built with vertex $V_{6,2}$. The first graph of this figure is called $T^{6,\,2}_{abcd}$ and the sum of the other two is $\Sigma^{6,\,2}_{abcd}$. In the same manner, the graphs of figure 12 take into account the graphs built with vertex $V_{4,1}$. The first graph is called $\Sigma^{4,\,1}_{abcd}$ and the sum of the other two is $T^{4,\,1}_{abcd}$. Then, the relations given in figures 10, 11, and 12 are re-expressed simply as

$$\Gamma^{6,\,1}_{abcd} = \sum_{\rho} \Gamma^{6,\,1;\,\rho}_{abcd}, \quad \Gamma^{6,\,2}_{abcd} = \sum_{\rho\rho'} \Gamma^{6,\,2;\,\rho\rho'}_{abcd} \quad \Gamma^{4,\,1}_{abcd} = \sum_{\rho} \Gamma^{4,\,1;\,\rho}_{abcd} \tag{90}$$

with

$$\Gamma^{6,\,1;\,\rho}_{abcd} = T^{6,\,1;\,\rho}_{abcd} + \Sigma^{6,\,1;\,\rho}_{abcd}, \quad \Gamma^{6,\,2;\,\rho\rho'}_{abcd} = T^{6,\,2;\,\rho\rho'}_{abcd} + \Sigma^{6,\,2;\,\rho\rho'}_{abcd}, \quad \Gamma^{4,\,1;\,\rho}_{abcd} = T^{4,\,1;\,\rho}_{abcd} + \Sigma^{4,\,1;\,\rho}_{abcd}. \tag{91}$$

Therefore, the equation in figure 9 takes the form

$$\Gamma_{abcd} = \Gamma^{6,\,1}_{abcd} + \Gamma^{4,\,1}_{abcd} + \Gamma^{6,\,2}_{abcd}. \tag{92}$$

All of the above quantities are obtained by using the following symmetry properties.

Proposition 6.

- $\Gamma^{6,\,1;\,2}_{abcd}$ *can be obtained using* $\Gamma^{6,\,1;\,1}_{abcd}$ *and replacing* $a \to b$, *and* $b \to a$.
- $\Gamma^{6,\,1;\,3}_{abcd}$ *is obtained using* $\Gamma^{6,\,1;\,1}_{abcd}$ *and replacing* $a \to c$, $b \to a$, *and* $c \to b$.
- $\Gamma^{6,\,1;\,4}_{abcd}$ *is obtained using* $\Gamma^{6,\,1;\,1}_{abcd}$ *and replacing* $a \to d$, $b \to a$, $c \to b$, *and* $d \to c$.
  These above symmetries are well satisfied for $\Gamma^{4,\,1;\,\rho}_{abcd}$. *In the case of* $\Gamma^{6,\,2;\,\rho\rho'}_{abcd}$, *we get:*
- $\Gamma^{6,\,2;\,13}_{abcd}$ *can be obtained using* $\Gamma^{6,\,2;\,14}_{abcd}$ *and by replacing* $a \to b$, *and* $b \to a$.
- $\Gamma^{6,\,2;\,12}_{abcd}$ *is obtained by replacing in* $\Gamma^{6,\,2;\,14}_{abcd}$, $a \to c$, $b \to a$, *and* $c \to b$.
- $\Gamma^{6,\,2;\,23}_{abcd}$ *is obtained by replacing in* $\Gamma^{6,\,2;\,14}_{abcd}$, $a \to b$, $b \to a$, $c \to d$, *and* $d \to c$.
- $\Gamma^{6,\,2;\,24}_{abcd}$ *is obtained by replacing in* $\Gamma^{6,\,2;\,14}_{abcd}$, $c \to d$ *and* $d \to c$.
- $\Gamma^{6,\,2;\,34}_{abcd}$ *is obtained by replacing in* $\Gamma^{6,\,2;\,14}_{abcd}$, $b \to c$, $c \to d$, *and* $d \to b$.

We then focus our attention on $\Gamma^{1}_{abcd} = (\Gamma^{6,\,1;\,1}_{abcd} + \Gamma^{4,\,1;\,1}_{abcd}) + \Gamma^{6,\,2;\,14}_{abcd}$. We also call $G^{\text{ins}}_{[mn]abc}$ the two-point functions with insertion (1,2,3), wherein the momentum indices, p_1, p_2, p_3, are summed; that is,

$$G^{\text{ins}}_{[mn]abc} = \sum_{p_1,p_2,p_3} \left\langle \varphi_{m123}\bar{\varphi}_{n123}\varphi_{nabc}\bar{\varphi}_{mabc} \right\rangle_c. \tag{93}$$

The following relations are satisfied:

$$\Sigma^{6,\,1;\,1}_{abcd} = Z^2\lambda_{6,1;1}C_{abcd}\sum_{p} G^{-1}_{abcd}G^{\text{ins}}_{[ap]bcd}, \quad T^{6,\,1;\,1}_{abcd} = Z^2\lambda_{6,1;1}C_{abcd}\sum_{p,q,r} G_{pqra}, \tag{94}$$

$$\Sigma^{6,\,2;\,14}_{abcd} = Z^2\lambda_{6,2;14}C_{abcd}\sum_{p} G^{-1}_{abcd}G^{\text{ins}}_{[ap]bcd}, \quad T^{6,\,2;\,14}_{abcd} = Z^2\lambda_{6,2;14}C_{abcd}\sum_{p,q,r} G_{pqra}, \tag{95}$$

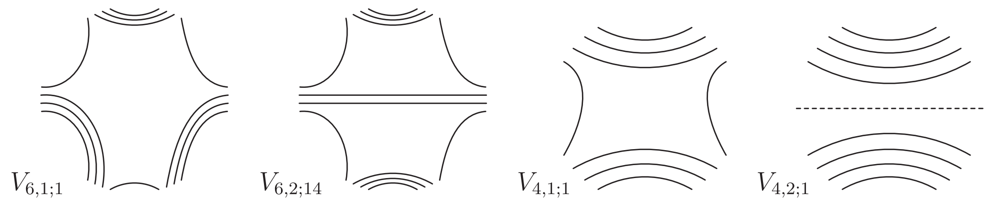

Figure 8. Vertex representation of a $4D$ tensor model.

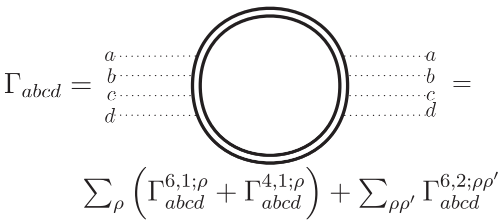

Figure 9. Schwinger–Dyson equation of a rank 4 tensor model.

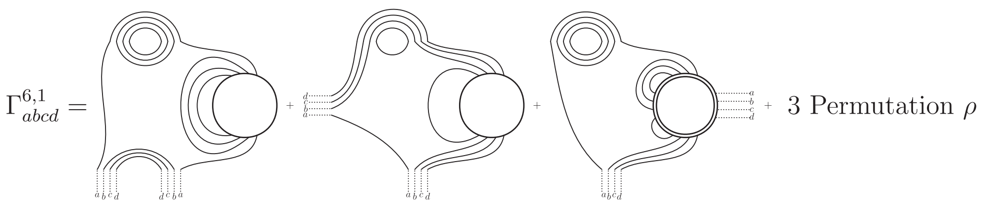

Figure 10. Definition of $\Gamma^{6,1}_{abcd}$.

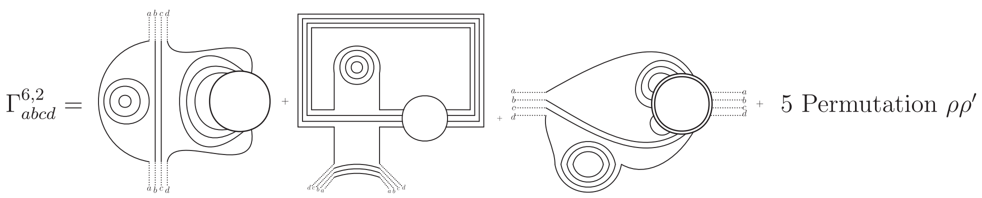

Figure 11. Definition of $\Gamma^{6,2}_{abcd}$.

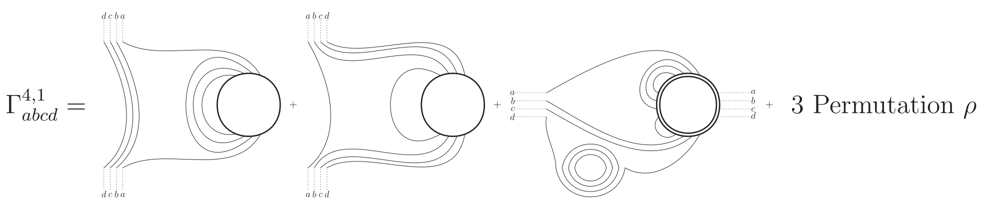

Figure 12. Definition of $\Gamma^{4,1}_{abcd}$.

$$\Sigma^{4,\,1;\,1}_{abcd} = Z^2\lambda_{4,1;1}\sum_p G^{-1}_{abcd}G^{\rm ins}_{[ap]bcd}, \qquad T^{4,\,1;\,1}_{abcd} = Z^2\lambda_{6,1;1}\sum_{p,q,r} G_{pqra}, \tag{96}$$

and then

$$\begin{aligned}\Gamma^1_{abcd} = Z^2C_{abcd}\lambda_{6,1;1}\left[\sum_p G^{-1}_{abcd}G^{\rm ins}_{[ap]bcd} + \sum_{p,q,r} G_{pqra}\right] + Z^2C_{abcd}\lambda_{6,2;14}\left[\sum_p G^{-1}_{abcd}G^{\rm ins}_{[ap]bcd}\right.\\ \left. + \sum_{p,q,r} G_{pqra}\right] + Z^2\lambda_{4,1;1}\left[\sum_p G^{-1}_{abcd}G^{\rm ins}_{[ap]bcd} + \sum_{p,q,r} G_{pqra}\right].\end{aligned} \tag{97}$$

We set $\lambda_{6,1;\rho} = \lambda_{6,1}$, $\lambda_{6,2;\rho\rho'} = \lambda_{6,2}$, and $\lambda_{4,1;\rho} = \lambda_{4,1}$. Note that the connected two-point functions can be expressed as $G^{-1}_{abcd} = M_{abcd} - \Gamma_{abcd}$. Then we get

$$\begin{aligned}\Gamma^1_{abcd} &= Z^2M^{-1}_{abcd}\lambda_{6,1}\left[\sum_p G^{-1}_{abcd}\frac{G_{pbcd} - G_{abcd}}{Z\left(a^2 - p^2\right)} + \sum_{p,q,r} G_{pqra}\right]\\ &\quad +Z^2M^{-1}_{abcd}\lambda_{6,2}\left[\sum_p G^{-1}_{abcd}\frac{G_{pbcd} - G_{abcd}}{Z\left(a^2 - p^2\right)} + \sum_{p,q,r} G_{pqra}\right]\\ &\quad +Z^2\lambda_{4,1}\left[\sum_p G^{-1}_{abcd}\frac{G_{pbcd} - G_{abcd}}{Z\left(a^2 - p^2\right)} + \sum_{p,q,r} G_{pqra}\right]\\ &= Z^2M^{-1}_{abcd}\lambda_{6,1}\left[\sum_p\left(\frac{1}{M_{pbcd} - \Gamma_{pbcd}} - \frac{1}{M_{pbcd} - \Gamma_{pbcd}}\frac{\Gamma_{abcd} - \Gamma_{pbcd}}{Z\left(a^2 - p^2\right)}\right)\right.\\ &\quad \left. + \sum_{p,q,r}\frac{1}{M_{pqra} - \Gamma_{pqra}}\right]\\ &\quad +Z^2M^{-1}_{abcd}\lambda_{6,2}\left[\sum_p\left(\frac{1}{M_{pbcd} - \Gamma_{pbcd}} - \frac{1}{M_{pbcd} - \Gamma_{pbcd}}\frac{\Gamma_{abcd} - \Gamma_{pbcd}}{Z\left(a^2 - p^2\right)}\right)\right.\\ &\quad \left. + \sum_{p,q,r}\frac{1}{M_{pqra} - \Gamma_{pqra}}\right]\\ &\quad +Z^2\lambda_{4,1}\left[\sum_p\left(\frac{1}{M_{pbcd} - \Gamma_{pbcd}} - \frac{1}{M_{pbcd} - \Gamma_{pbcd}}\frac{\Gamma_{abcd} - \Gamma_{pbcd}}{Z\left(a^2 - p^2\right)}\right)\right.\\ &\quad \left. + \sum_{p,q,r}\frac{1}{M_{pqra} - \Gamma_{pqra}}\right].\end{aligned} \tag{98}$$

Now we use the Taylor expansion that allows us to pass to the renormalized quantity as

$$\Gamma^1_{abcd} = Zm^2 - m_0^2 + (Z-1)\left(a^2 + b^2 + c^2 + d^2\right) + \Gamma^{\rm phys}_{abcd}, \tag{99}$$

with conditions $\Gamma_{0000} = 0$ and $\partial\Gamma_{0000} = 0$. This implies that

$$G^{-1}_{abcd} = a^2 + b^2 + c^2 + d^2 + m^2 - \Gamma^{\text{ren}}_{abcd}. \tag{100}$$

Then we get the following proposition.

Proposition 7. *The closed equation of the two-point functions of a four-dimensional tensor model is given by*

$$\begin{aligned}
&(Z-1)\Big(a^2+b^2+c^2+d^2\Big) + \Gamma^{\text{phys}}_{abcd}\\
&= M^{-1}_{abcd}\lambda_{6,1}\Bigg\{\sum_p\Bigg[\frac{Z}{p^2+b^2+c^2+d^2+m_0^2-\Gamma^{\text{phys}}_{pbcd}} - \frac{1}{m_0^2}\frac{M_{abcd}}{\Big(p^2+m_0^2-\Gamma^{\text{phys}}_{p000}\Big)}\\
&-\frac{Z}{p^2+b^2+c^2+d^2+m_0^2-\Gamma^{\text{phys}}_{pbcd}}\frac{\Gamma^{\text{phys}}_{abcd}-\Gamma^{\text{phys}}_{pbcd}}{\Big(a^2-p^2\Big)} + \frac{1}{m_0^2}\frac{M_{abcd}}{\Big(p^2+m_0^2-\Gamma^{\text{phys}}_{p000}\Big)}\frac{\Gamma^{\text{phys}}_{p000}}{p^2}\Bigg]\\
&+\sum_{p,q,r}\Bigg[\frac{Z^2}{p^2+q^2+r^2+a^2+m_0^2-\Gamma^{\text{phys}}_{pqra}} - \frac{1}{m_0^2}\frac{ZM_{abcd}}{\Big(p^2+q^2+r^2+m_0^2-\Gamma^{\text{phys}}_{pqr0}\Big)}\Bigg]\Bigg\}\\
&+M^{-1}_{abcd}\lambda_{6,2}\Bigg\{\sum_p\Bigg[\frac{Z}{p^2+b^2+c^2+d^2+m_0^2-\Gamma^{\text{phys}}_{pbcd}} - \frac{1}{m_0^2}\frac{M_{abcd}}{\Big(p^2+m_0^2-\Gamma^{\text{phys}}_{p000}\Big)}\\
&-\frac{Z}{p^2+b^2+c^2+d^2+m_0^2-\Gamma^{\text{phys}}_{pbcd}}\frac{\Gamma^{\text{phys}}_{abcd}-\Gamma^{\text{phys}}_{pbcd}}{\Big(a^2-p^2\Big)} + \frac{1}{m_0^2}\frac{M_{abcd}}{\Big(p^2+m_0^2-\Gamma^{\text{phys}}_{p000}\Big)}\frac{\Gamma^{\text{phys}}_{p000}}{p^2}\Bigg]\\
&+\sum_{p,q,r}\Bigg[\frac{Z^2}{p^2+q^2+r^2+a^2+m_0^2-\Gamma^{\text{phys}}_{pqra}} - \frac{1}{m_0^2}\frac{ZM_{abcd}}{\Big(p^2+q^2+r^2+m_0^2-\Gamma^{\text{phys}}_{pqr0}\Big)}\Bigg]\Bigg\}\\
&+\lambda_{4,1}\Bigg\{\sum_p\Bigg[\frac{Z}{p^2+b^2+c^2+d^2+m_0^2-\Gamma^{\text{phys}}_{pbcd}} - \frac{Z}{p^2+m_0^2-\Gamma^{\text{phys}}_{p000}}\\
&-\frac{Z}{p^2+b^2+c^2+d^2+m_0^2-\Gamma^{\text{phys}}_{pbcd}}\frac{\Gamma^{\text{phys}}_{abcd}-\Gamma^{\text{phys}}_{pbcd}}{\Big(a^2-p^2\Big)} + \frac{Z}{p^2+m_0^2-\Gamma^{\text{phys}}_{p000}}\frac{\Gamma^{\text{phys}}_{p000}}{p^2}\Bigg]\\
&+\sum_{p,q,r}\Bigg[\frac{Z^2}{p^2+q^2+r^2+a^2+m_0^2-\Gamma^{\text{phys}}_{pqra}} - \frac{Z^2}{p^2+q^2+r^2+m_0^2-\Gamma^{\text{phys}}_{pqr0}}\Bigg]\Bigg\}.
\end{aligned} \tag{101}$$

Proof. Equation (101) can be simply obtained using the relation $Zm_0^2 - m^2_{\text{phys}}$ in the same way that we saw in the last section:

$$Zm^2 - m_0^2 = ZM_{0000}^{-1}\lambda_{6,1}\Bigg[\sum_p\left(\frac{1}{p^2+m_0^2-\Gamma_{p000}^{\rm phys}} - \frac{1}{p^2+m_0^2-\Gamma_{p000}^{\rm phys}}\frac{\Gamma_{p000}^{\rm phys}}{p^2}\right)$$
$$+\sum_{p,q,r}\frac{Z}{p^2+q^2+r^2+m_0^2-\Gamma_{pqr0}^{\rm phys}}\Bigg] + ZM_{0000}^{-1}\lambda_{6,2}\Bigg[\sum_p\Bigg(\frac{1}{p^2+m_0^2-\Gamma_{p000}^{\rm phys}}$$
$$-\frac{1}{p^2+m_0^2-\Gamma_{p000}^{\rm phys}}\frac{\Gamma_{p000}^{\rm phys}}{p^2}\Bigg)+\sum_{p,q,r}\frac{Z}{p^2+q^2+r^2+m_0^2-\Gamma_{pqr0}^{\rm phys}}\Bigg]$$
$$+Z\lambda_{4,1}\Bigg[\sum_p\left(\frac{1}{p^2+m_0^2-\Gamma_{p000}^{\rm phys}} - \frac{1}{p^2+m_0^2-\Gamma_{p000}^{\rm phys}}\frac{\Gamma_{p000}^{\rm phys}}{p^2}\right)$$
$$+\sum_{p,q,r}\frac{Z}{p^2+q^2+r^2+m_0^2-\Gamma_{pqr0}^{\rm phys}}\Bigg],\quad M_{0000}^{-1}=\frac{1}{Zm_0^2}. \tag{102}$$

Then (101) takes the form by replacing relation (102) into the rhs of equation (98). □

Let us note that the continuous limit of equation (101) can be built. We identify the sum as $\sum_p = 2\int_0^\infty \mathrm{d}p$ and $\sum_{p,q,r} = 2\int_0^\infty p^2\mathrm{d}p$. We also impose the cutoff, p_Λ, in the UV and change the variables as

$$a^2 = m_0^2\frac{\alpha}{1-\alpha},\quad b^2 = m_0^2\frac{\beta}{1-\beta},\quad c^2 = m_0^2\frac{\gamma}{1-\gamma},$$
$$d^2 = m_0^2\frac{\epsilon}{1-\epsilon},\quad p^2 = m_0^2\frac{\rho}{1-\rho},\quad p_\Lambda^2 = m_0^2\frac{\Lambda}{1-\Lambda}. \tag{103}$$

Now let us define the two quantities, $s(\alpha,\beta,\gamma,\epsilon)$ and $p(\alpha,\beta,\gamma,\epsilon)$, as

$$s(\alpha,\beta,\gamma,\epsilon) = 1-\alpha\beta-\alpha\gamma-\alpha\epsilon-\beta\gamma-\beta\epsilon-\gamma\epsilon$$
$$+2\alpha\beta\gamma+2\alpha\beta\epsilon+2\alpha\gamma\epsilon+2\beta\gamma\epsilon-3\alpha\beta\gamma\epsilon \tag{104}$$

and

$$p(\alpha,\beta,\gamma,\epsilon) = (1-\alpha)(1-\beta)(1-\gamma)(1-\epsilon). \tag{105}$$

Equation (101) is re-expressed as

$$m_0^2(Z-1)\frac{p(\alpha,\beta,\gamma,\epsilon)}{(1-\alpha)(1-\beta)(1-\gamma)(1-\epsilon)} + m_0^2\frac{\Gamma_{\alpha\beta\gamma\epsilon}}{(1-\alpha)(1-\beta)(1-\gamma)(1-\epsilon)}$$
$$=2M_{\alpha\beta\gamma\epsilon}^{-1}\left(\lambda_{6,1}+\lambda_{6,2}\right)\Bigg\{\int_0^\Lambda \frac{1}{2m_0}\sqrt{\frac{1-\rho}{m_0\rho}}\,\frac{\mathrm{d}\rho}{(1-\rho)^2}\Bigg[\frac{Z(1-\rho)(1-\beta)(1-\gamma)(1-\epsilon)}{s(\rho,\beta,\gamma,\epsilon)-\Gamma_{\rho\beta\gamma\epsilon}}$$
$$-\frac{1}{m_0^2}\frac{M_{\alpha\beta\gamma\epsilon}(1-\rho)}{1-\Gamma_{\rho000}} - \frac{Z(1-\rho)}{\left(s(\rho,\beta,\gamma,\epsilon)-\Gamma_{\rho\beta\gamma\epsilon}\right)}\frac{(1-\rho)\Gamma_{\alpha\beta\gamma\epsilon}-(1-\alpha)\Gamma_{\rho\beta\gamma\epsilon}}{(\alpha-\rho)}$$

$$+\frac{1}{m_0^2}\frac{M_{\alpha\beta\gamma\epsilon}(1-\rho)}{\left(1-\Gamma_{\rho000}\right)}\frac{\Gamma_{\rho000}}{\rho}\Bigg]+\int_0^\Lambda\frac{1}{2m_0}\sqrt{\frac{m_0\rho}{1-\rho}}\frac{\mathrm{d}\rho}{(1-\rho)^2}\Bigg[\frac{Z^2(1-\rho)^3(1-\alpha)}{s(\rho,\rho,\rho,\alpha)-\Gamma_{\rho\rho\rho\alpha}}$$

$$-\frac{1}{m_0^2}\frac{ZM_{\alpha\beta\gamma\epsilon}(1-\rho)^3}{\left(2\rho^3-3\rho^2+1-\Gamma_{\rho\rho\rho0}\right)}\Bigg]\Bigg\}$$

$$+\lambda_{4,1}\Bigg\{\int_0^\Lambda\frac{1}{2m_0}\sqrt{\frac{1-\rho}{m_0\rho}}\frac{\mathrm{d}\rho}{(1-\rho)^2}\Bigg[\frac{Z(1-\rho)(1-\beta)(1-\gamma)(1-\epsilon)}{s(\rho,\beta,\gamma,\epsilon)-\Gamma_{\rho\beta\gamma\epsilon}}$$

$$-\frac{Z(1-\rho)}{1-\Gamma_{\rho000}}-\frac{Z(1-\rho)}{\left(s(\rho,\beta,\gamma,\epsilon)-\Gamma_{\rho\beta\gamma\epsilon}\right)}\frac{(1-\rho)\Gamma_{\alpha\beta\gamma\epsilon}-(1-\alpha)\Gamma_{\rho\beta\gamma\epsilon}}{(\alpha-\rho)}$$

$$+\frac{Z(1-\rho)}{\left(1-\Gamma_{\rho000}\right)}\frac{\Gamma_{\rho000}}{\rho}\Bigg]+\int_0^\Lambda\frac{1}{2m_0}\sqrt{\frac{m_0\rho}{1-\rho}}\frac{\mathrm{d}\rho}{(1-\rho)^2}\Bigg[\frac{Z^2(1-\rho)^3(1-\alpha)}{s(\rho,\rho,\rho,\alpha)-\Gamma_{\rho\rho\rho\alpha}}$$

$$-\frac{Z^2(1-\rho)^3}{\left(2\rho^3-3\rho^2+1-\Gamma_{\rho\rho\rho0}\right)}\Bigg]\Bigg\}. \tag{106}$$

The wave function, Z, can be also deduced as

$$Z=\frac{1-\frac{2}{m_0^2}\left(\lambda_{6,1}+\lambda_{6,2}\right)\int_0^\Lambda\frac{\mathrm{d}\rho}{2m_0^3}\sqrt{\frac{1-\rho}{m_0\rho}}\left(G_{\rho000}+\frac{G'_{\rho000}}{\rho}\right)}{1+\lambda_{4,1}\int_0^\Lambda\frac{\mathrm{d}\rho}{2m_0^3}\sqrt{\frac{1-\rho}{m_0\rho}}\left(G_{\rho000}+\frac{G'_{\rho000}}{\rho}\right)}, \tag{107}$$

where

$$s(\alpha,\beta,\gamma,\epsilon)-\Gamma_{\alpha\beta\gamma\epsilon}=\frac{s(\alpha,\beta,\gamma,\epsilon)}{G_{\alpha\beta\gamma\epsilon}}, \tag{108}$$

and

$$M_{\alpha\beta\gamma\epsilon}=Zm_0^2\frac{s(\alpha,\beta,\gamma,\epsilon)}{p(\alpha,\beta,\gamma,\epsilon)}. \tag{109}$$

Finally, by replacing the expressions (107) and (108) in equation (106), we obtain the closed equation in the continuous limit, which will also be fully addressed in forthcoming work.

5. Conclusions and remarks

In this paper, we have presented a perturbative calculation of two-point correlation functions of rank 3 TGFT. As discussed, the correlation functions are given by combining Ward–Takahashi identities and Schwinger–Dyson equations, which allow us to establish the appropriate closed equation. The closed equation in the 4D case is also given.

We proved that the nonperturbative techniques, as developed in [27–29] can be reported to the tensor situation. Indeed, although we only solve our closed-form equations for the two-point functions at initial orders, it is very promising to see that we can obtain even solutions in this highly combinatoric case. In future investigations, we can now undertake a calculation of the general solution at all orders of the coupling constants for both rank 3 and 4 models.

Let us note that we take the coupling constants, $\lambda =: \lambda_\rho, \quad \rho = 1, 2, 3$, and we pass to the renormalized quantities as ($\lambda_\rho = \lambda_\rho^{\text{phys}} + \mathcal{O}(\lambda_\rho^{\text{phys}})$). For further explanation, see [15], in which the one-loop computation of the β-function is given. This approximation leads to the finite coupling constant and therefore the solution of equation (78) is well given. The more complicated cases will take into account the bare coupling as the effective series of λ^{phys}; that is,

$$\lambda^{\text{phys}} = \sum_{n\geqslant 0} a_n \lambda^n, \quad \text{or} \quad \lambda = \sum_{n\geqslant 0} b_n \left(\lambda^{\text{phys}}\right)^n. \tag{110}$$

This will take form after writing the closed equation of the four-point function, and by using the relation

$$\Gamma^4_{000000} = \lambda^{\text{phys}}, \tag{111}$$

where Γ^4_{abcdef} represents the 1PI four-point functions.

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