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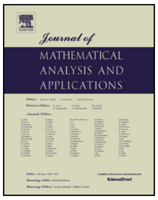

Asymmetric Steklov problems with sign-changing weights

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ABSTRACT

We study two asymmetric Steklov problems with indefinite weights involving the p-Laplacian operator. We prove the existence of a first nontrivial eigenvalue for the first problem and the second one serves as an application in the description of the beginning of the Fučik spectrum with weights. We thereby extend several known results related to these problems.

1. Introduction

Recently, progress was made in studying Steklov eigenvalue problems. The starting point was the modeling of an elastic membrane problem whose mass is concentrated on the boundary (see [14]) which has been known as the so-called problems of Steklov type. The present work is partly concerned with the following problem

$$(P_{V,m,n}) : \begin{cases} \Delta_p u = V(x)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain of class $C^{2,\alpha}$ $(0 < \alpha < 1)$ with outward unit normal ν on the boundary $\partial\Omega$, the p-Laplacian operator is defined as $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $p > 1$ and $\lambda \in \mathbb{R}$ is regarded as an eigenvalue. We assume that $m, n \in C^r(\partial\Omega)$ for some $0 < r < 1$. Finally, V is a given function in $L^\infty(\Omega)$ which may change sign and $u = u^+ - u^-$ where $u^\pm := \max\{\pm u, 0\}$. Problems like (1.1) are usually known in literature as asymmetric Steklov problems with weights, and the solutions of (1.1) are always

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understood in the weak sense, i.e., functions $u \in W^{1,p}(\Omega)$ with

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u.\nabla v dx + \int_{\Omega} V(x)|u|^{p-2}uv dx = \lambda \int_{\partial\Omega} \left[m(x)\left(u^{+}\right)^{p-1} - n(x)\left(u^{-}\right)^{p-1}\right] v d\sigma \tag{1.2}$$

for all $v \in W^{1,p}(\Omega)$ where $d\sigma$ is the $(N-1)$ dimensional Hausdorff measure and the Sobolev space $W^{1,p}(\Omega)$ is endowed with the norm

$$\|u\| := \left(\int_{\Omega} |\nabla u|^{p} dx + \int_{\partial\Omega} |u|^{p} d\sigma\right)^{1/p}. \tag{1.3}$$

It is well known that the quasilinear problem

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

has no principal eigenvalue (i.e. an eigenvalue whose associated eigenfunctions are sign-constant) when V has a negative part sufficiently large and m may change sign or vanishes in some part of Ω. Lately, an optimal condition was established in [8] in order to guarantee the existence of principal eigenvalues of (1.4). Recently, the ideas of [8] were extended in [10] for the following problem

$$(P_{V,m}) \begin{cases} \Delta_p u = V(x)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda m(x)|u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

In short, let

$$\beta(V,m) := \inf\left\{E_V(u);\ \|u\|^{p}_{L^{p}(\partial\Omega)} = 1 \text{ and } B(u) = 0\right\} \tag{1.6}$$

where

$$E_V(u) = \int_{\Omega} \left(|\nabla u|^{p} + V(x)|u|^{p}\right) dx, \qquad \|u\|^{p}_{L^{p}(\partial\Omega)} = \int_{\partial\Omega} |u|^{p} d\sigma \quad \text{and} \quad B(u) = \int_{\partial\Omega} m(x)|u|^{p} d\sigma$$

for $u \in W^{1,p}(\Omega)$. If m is sign-changing then

$$\lambda_{\pm 1}(V,m) := \inf_{B(u)=\pm 1} E_V(u)$$

are the principal eigenvalues of $(P_{V,m})$ if and only if $\beta(V,m) \geq 0$. Furthermore, if $\beta(V,m) < 0$ then $\lambda_{\pm 1}(V,m) = -\infty$. It is worth mentioning that the fact that V is sign-changing potential prevents the functional energy E_V associated to (1.5) from being coercive and then led the authors of [10] to embed the problem into a new eigenvalue problem for each fixed λ and to construct an eigenvalue curve as λ varies. Indeed,

$$\begin{cases} \Delta_p u = V(x)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \left(\lambda m(x) + \mu\right)|u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where λ is viewed as a fixed parameter and $\mu = \mu(\lambda)$ eigenvalue of (1.7) is considered and the following result holds:

Theorem 1.1. *(See [10].) Let Ω be a bounded smooth domain of class $C^{2,\alpha}$ $(0<\alpha<1)$ in $\mathbb{R}^N$ with boundary $\partial\Omega$. Assume that the weights $V \in L^\infty(\Omega)$ and $m \in C^r(\partial\Omega)$. Then for every $\lambda \in \mathbb{R}$, (1.7) admits a principal eigenvalue $\mu_1(\lambda)$, i.e. an eigenvalue associated to a one-signed eigenfunction if and only if*

$$\lambda_1^D(V) := \inf\left\{E_V(u);\ u \in W_0^{1,p}(\Omega) \text{ and } \int_\Omega |u|^p dx = 1\right\} > 0. \tag{1.8}$$

Moreover $\mu_1(\lambda)$ is simple (i.e. the associated eigenfunctions are a constant multiple of one another) and there is no other principal eigenvalue of (1.7).

As one can see, the zeros of μ_1 are the principal eigenvalues of (1.5). In this case, the associated eigenfunctions may be denoted by φ_λ with $\int_{\partial\Omega} \varphi_\lambda^p d\sigma = 1$ and $\varphi_\lambda > 0$ on $\overline{\Omega}$. Continuity, differentiability and asymptotic properties of the function $\lambda \in \mathbb{R} \longmapsto \mu_1(\lambda)$ are given in [10] and we recall these properties are useful in proving Theorem 1.3 below.

Proposition 1.2. *(See [10].)*

1. *The function $\lambda \in \mathbb{R} \mapsto \mu_1(\lambda)$ is concave and differentiable. Moreover*

$$\mu_1'(\lambda) = -\int_{\partial\Omega} m(x)\varphi_\lambda^p(x) d\sigma \quad \text{for all } \lambda \in \mathbb{R}. \tag{1.9}$$

2. *If $m^+ \not\equiv 0$ (resp. $m^- \not\equiv 0$) then $\lim_{\lambda\to+\infty} \mu_1(\lambda) = -\infty$ (resp. $\lim_{\lambda\to-\infty} \mu_1(\lambda) = -\infty$).*
3. *If $m^- \equiv 0$ then μ_1 is strictly decreasing.*
4. *$\sup_{\lambda\in\mathbb{R}} \mu_1(\lambda) = \beta(V,m)$ where*

$$\begin{aligned} &\beta(V,m) = \infty \quad \text{if } m(x) > 0 \text{ a.e. in } \partial\Omega \quad \text{or} \\ &\beta(V,m) \overset{\text{def}}{=} \inf_{u\in\mathcal{A}} E_V(u) \quad \text{if } 0 < \left|\partial\Omega_u^+\right| < |\partial\Omega| \end{aligned} \tag{1.10}$$

with

$$\mathcal{A} = \left\{u \in W^{1,p}(\Omega),\ \int_{\partial\Omega} |u|^p d\sigma = 1 \text{ and } \int_{\partial\Omega} m|u|^p d\sigma = 0\right\}. \tag{1.11}$$

Theorem 1.3. *(See [10].) Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain of class $C^{2,\alpha}$, $0<\alpha<1$, with boundary $\partial\Omega$. Assume that $V \in L^\infty(\Omega)$ and $m \in C^r(\partial\Omega)$ and that $\lambda_1^D(V) > 0$. One has:*

1. *If $m^- \equiv 0$ then there exists a principal eigenvalue of (1.5) if and only if $\beta(V,m) > 0$. In this case the principal eigenvalue is unique and it is defined by*

$$\lambda_1(V,m) = \inf_{u\in\mathcal{M}^+} E_V(u) \tag{1.12}$$

where

$$\mathcal{M}^+ \overset{\text{def}}{=} \left\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} m(x)|u|^p d\sigma = 1\right\} = \mathcal{M}_{m,m}.$$

2. *If* $m^- \not\equiv 0$ *then there exists a principal eigenvalue of* (1.5) *if and only if* $\beta(V,m) \geq 0$. *More precisely:*
   (a) *If* $\beta(V,m) > 0$ *then* (1.5) *admits exactly two principal eigenvalues* $\lambda_{-1}(V,m) < \lambda_1(V,m)$, *with* $\lambda_1(V,m)$ *defined above and*

$$\lambda_{-1}(V,m) = -\inf_{u\in\mathcal{M}^-} E_V(u) \tag{1.13}$$

   where

$$\mathcal{M}^- \overset{\text{def}}{=} \left\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} m|u|^p d\sigma = -1\right\}.$$

   (b) *If* $\beta(V,m) = 0$ *then* (1.5) *has a unique principal eigenvalue* $\lambda_1(V,m)$ *given by*

$$\lambda_1(V,m) = \inf_{u\in\mathcal{M}^+} E_V(u) = -\inf_{u\in\mathcal{M}^-} E_V(u).$$

   These infima are not achieved. Moreover any function $u \not\equiv 0$ *in* $W^{1,p}(\Omega)$ *satisfying*

$$E_V(u) = \int_{\partial\Omega} m(x)|u|^p d\sigma = 0 \tag{1.14}$$

   is an eigenfunction associated to $\lambda_1(V,m)$.

Turning back to our concern, the problem $(P_{V,m,n})$ is closely related to $(P_{V,m})$ and it can be regarded as an extension of this one. Indeed, $(P_{V,m,n})$ is reduced to $(P_{V,m})$ when $m \equiv n$. Note that every principal eigenvalue of $(P_{V,m})$ [and $(P_{V,n})$] is a principal eigenvalue of $(P_{V,m,n})$. The special cases $V \equiv 0$ and $V \equiv 1$ were considered in [1,3,2] respectively under different hypotheses on the space Ω and the weights. Some more general Steklov problems have also been analyzed, when different hypotheses are made on Ω, V, m and n (see [4,9,15]) and our purpose is to extend known results related.

We are also interested in the problem

$$(P_{V,A,B}) : \begin{cases} \Delta_p u = V(x)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = Am(x)(u^+)^{p-1} - Bn(x)(u^-)^{p-1} & \text{on } \partial\Omega \end{cases} \tag{1.15}$$

with A, B real constants, which in its turns extends $(P_{V,m,n})$. Recall that the Fučik spectrum with weights is defined as $\sum = \sum_{(m,n)} := \{(A,B) \in \mathbb{R}^2/(P_{V,m,n})$ has a nontrivial solution$\}$.

In [2], the authors studied $(P_{1,A,B})$ and they have shown that $\sum_{(m,n)}$ contains the lines $\{0\}\times\mathbb{R}$, $\mathbb{R}\times\{0\}$, $\{\lambda_1(1,m)\}\times\mathbb{R}$, $\mathbb{R}\times\{\lambda_1(1,n)\}$ and also possibly the lines $\{-\lambda_1(1,-m)\}\times\mathbb{R}$ and $\mathbb{R}\times\{-\lambda_1(1,-n)\}$. They have also found that $\mathcal{C} := \{(A(s),B(s)) = (c(m,sn), sc(m,sn)) \ : \ s > 0\}$ is the first curve in $\sum^*_{(m,n)} \cap (\mathbb{R}^+\times\mathbb{R}^+)$ where $\sum^*_{(m,n)}$ denotes the set $\sum_{(m,n)}$ without the lines mentioned above and

$$c(m,n) \overset{\text{def}}{=} \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \int_\Omega \left|\nabla\big(\gamma(t)\big)\right|^p dx. \tag{1.16}$$

Similar results have been obtained when considering (P_0, m, n) and so far, results about Fučik spectrum are in their early stage since the study of Fučik spectrum is still a challenge (see [2,5,11]). It is definitely an interesting fact to focus on (1.15) (as V is sign-changing potential) and the purpose here is therefore to find a first nontrivial curve in the Fučik spectrum of (1.15).

To conclude this introduction, let us give an overview of our work. In the preliminary Section 2, we gather some results relative to the well-known Steklov problem $(P_{V,m,m})$. Next, we devote Section 3 to not only the construction of the first nonprincipal eigenvalue $c(m,n,V)$ for the weighted asymmetric Steklov problem (1.1) but also its variational characterization. Useful properties of $c(m,n,V)$ with respect to its weights are investigated in Section 4 where we restrict ourselves to the case one can find principal eigenvalues belonging to some manifold. We then apply in Section 5 our results to the study of Fučik spectrum with weights through (1.15) and we end up with some properties of the first nontrivial curve of $\sum_{(m,n)}$.

2. Preliminaries and main assumptions

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain of class $C^{2,\alpha}$ $(0<\alpha<1)$ with outward unit normal ν on the boundary $\partial\Omega$. From now on, we make the assumptions:

$$(H_V): V \in L^\infty(\Omega);$$

$$(H_m): m \in C^r(\partial\Omega) \quad \text{for some } 0<r<1 \quad \text{and} \quad m^+ \not\equiv 0 \quad \text{a.e. in } \partial\Omega;$$

$$(H_n): n \in C^r(\partial\Omega) \quad \text{for some } 0<r<1 \quad \text{and} \quad n^+ \not\equiv 0 \quad \text{a.e. in } \partial\Omega.$$

To prove the existence of a first nonprincipal eigenvalue $c(m,n,V)$ for (1.1), we will review in a more general setting the approach used in [8,10,11] by introducing two values $\beta(V,m)$ and $\beta(V,n)$ which guarantee the boundedness of the energy E_V over the weighted manifold

$$\mathcal{M}_{m,n} \overset{\text{def}}{=} \left\{ u \in W^{1,p}(\Omega) : \int_{\partial\Omega} \left[m(x)\left(u^+\right)^p + n(x)\left(u^-\right)^p\right] d\sigma = 1 \right\}. \tag{2.1}$$

Remark 2.1. Using the fact that Ω is of class $C^{2,\alpha}$ for some $0<\alpha<1$ and $m,n \in C^r(\partial\Omega)$ with $0<r<1$, one derives from regularity results of [12], Theorem 6.1 in [10] (or Theorem A.1 in [7] for a more general result) that all weak solution of (1.1) belongs to $C^{1,\sigma}(\overline{\Omega})$ for some $0<\sigma<1$.

The construction of $c(m,n,V)$ relies on the application of two very different versions of the mountain pass theorem to the functional E_V restricted to the manifold $\mathcal{M}_{m,n}$. The Palais–Smale condition (PS) is needed in the classical case while the Palais–Smale–Cerami condition (PSC) will be applied to overcome difficulties that arise when (PS) breaks down. In order to state our main results, we are describing with more precision the general definitions of (PS) and (PSC). Let E be a real Banach space and let

$$M := \left\{ u \in E : g(u) = 1 \right\} \tag{2.2}$$

where $g \in C^1(E,\mathbb{R})$ and 1 is a regular value of g. Let $f \in C^1(E,\mathbb{R})$ and consider the restriction $\tilde{f}$ of f to M. The differential $\tilde{f}'$ at $u \in M$, has a norm which will be denoted by $\|\tilde{f}'(u)\|_\star$ and which is given by the norm of the restriction of $f'(u) \in E^\star$ to the tangent space of M at u

$$Tu(M) := \left\{ v \in E : \left\langle g'(u), v \right\rangle = 0 \right\},$$

where $\langle , \rangle$, denotes the pairing between $E^\star$ and E.

Definition 2.2. $\tilde{f}$ is said to satisfy the $(PS)_c$ condition (resp. $(PSC)_c$ condition) at level $c \in \mathbb{R}$ if for any sequence $u_k \in M$ such that $\tilde{f}(u_k) \to c$ and $\|\tilde{f}'(u_k)\|_\star \to 0$ (resp. $\tilde{f}(u_k) \to c$ and $(1+\|u_k\|_E)\|\tilde{f}'(u_k)\|_\star \to 0$), one has that u_k admits a convergent subsequence.

3. Existence of nonprincipal eigenvalues when principal eigenvalues exist

We present here the results concerning the construction of a non-trivial eigenvalue for the problem

$$(P_{V,m,n}) : \begin{cases} \Delta_p u = V(x)|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] & \text{on } \partial\Omega \end{cases}$$

with all of our basic hypotheses stated above but in addition, we assume throughout this paper that

$$\lambda_1^D(V) > 0, \qquad \beta(V,m) \geq 0 \quad \text{and} \quad \beta(V,n) \geq 0. \tag{3.1}$$

We recall that under these assumptions, it can be clearly seen that the main problem $(P_{V,m,n})$ has a non-trivial and one-signed solutions if and only if $\lambda = \lambda_1(V,m)$ with φ_m as one-signed associated eigenfunction or $\lambda = \lambda_1(V,n)$ with $-\varphi_n$ as one-signed associated eigenfunction. As we are interested in weak solutions of $(P_{V,m,n})$, it follows:

Definition 3.1. We say that $u \in W^{1,p}(\Omega)$ is a weak solution of $(P_{V,m,n})$ if for any $v \in W^{1,p}(\Omega)$ we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v dx + \int_\Omega V(x)|u|^{p-2}uv dx = \lambda \int_{\partial\Omega} \left[m(x)\left(u^+\right)^{p-1} - n(x)\left(u^-\right)^{p-1}\right] v d\sigma. \tag{3.2}$$

Let us formulate variationally $(P_{V,m,n})$ by considering the functionals

$$E_V(u) = \int_\Omega \left(|\nabla u|^p + V(x)|u|^p\right) dx$$

and

$$B_{m,n}(u) := \int_{\partial\Omega} \left[m(x)\left(u^+\right)^p + n(x)\left(u^-\right)^p\right] d\sigma \tag{3.3}$$

which are C^1 functionals on $W^{1,p}(\Omega)$. As a result, the relation (3.2) reads:

$$\left\langle E_V'(u), v\right\rangle = \left\langle B_{m,n}'(u), v\right\rangle, \quad \forall v \in W^{1,p}(\Omega). \tag{3.4}$$

By Lagrange's multiplier rule, our eigenvalue problem $(P_{V,m,n})$ can be transformed into the problem of looking for critical points and critical values of $\tilde{E}_V$ where $\tilde{E}_V$ is the restriction of E_V to the manifold

$$\mathcal{M}_{m,n} := \left\{u \in W^{1,p}(\Omega) : B_{m,n}(u) = 1\right\}. \tag{3.5}$$

Let us recall that for $u \in \mathcal{M}_{m,n}$, the tangent space is defined as

$$T_u\mathcal{M}_{m,n} := \left\{w \in W^{1,p}(\Omega) : \left\langle B_{m,n}'(u), w\right\rangle = 0\right\}. \tag{3.6}$$

Problem (1.1) will be approached by taking into account the two following cases:

$$\textbf{Case 1} \quad \left(\beta(V,m) > 0 \quad \text{and} \quad \beta(V,n) > 0\right)$$
$$\textbf{Case 2} \quad \left(\beta(V,m) = 0 \quad \text{or} \quad \beta(V,n) = 0\right).$$

Before we go further, we would like to give the following results as our guideline towards the construction of the first nonprincipal eigenvalue of $(P_{V,m,n})$.

Proposition 3.2. *(See [6].) Let $u, v \in M$ with $u \neq v$ and suppose that*

$$H := \{h \in C([0,1], M) : h(0) = u \text{ and } h(1) = v\} \tag{3.7}$$

is nonempty. Assume also that

$$c := \inf_{h\in H} \max_{w\in h([0,1])} f(w) > \max\{f(u), f(v)\} \tag{3.8}$$

and that $\tilde{f}$ satisfies (PS) condition on M. Then c is a critical value of $\tilde{f}$.

Theorem 3.3. *(See [5].) Let K be a compact metric space, $K_0 \subset K$ and $h_0 \in C(K_0, M)$. Consider the family of extensions of h_0:*

$$H := \{h \in C(K, M) : h \,|\, K_0 = h_0\}. \tag{3.9}$$

Assume that H is nonempty as well as the following condition

$$\max_{t\in K_0} f(h_0(t)) < \max_{t\in K} f(h(t)), \quad \forall h \in H.$$

Define

$$c := \inf_{h\in H} \max_{t\in K} f(h(t)). \tag{3.10}$$

Assume that $\tilde{f}$ satisfies $(PSC)_c$ for c given in (3.10). Then c is a critical value of $\tilde{f}$.

Remark 3.4. The two previous results are known as versions of mountain pass theorem. The first one is the so-called classical mountain pass theorem which shall be applied when considering the principal eigenvalues $\lambda_1(V, m)$ and $\lambda_1(V, n)$ with their one-signed associated eigenfunctions φ_m and φ_n respectively in case $\beta(V, m) > 0$ and $\beta(V, n) > 0$. It is worth mentioning explicitly that a lot of well-known facts break down when we pass from the previous case to the one in which either $\beta(V, m) = 0$ or $\beta(V, n) = 0$. For instance, the mountain pass procedure of minimizing E_V over paths going from φ_0 to $-\varphi_0$ is no longer suitable since $\varphi_0 \notin \mathcal{M}^+$. Therefore we shall adopt the approach given by [5] which is depicted in Theorem 3.3.

Lemma 3.5.

1. *Let $w \in L^{p'}(\partial\Omega)$ such that $w > 0$. Then there exist two positive constants C_1 and C_2 such that for all $u \in W^{1,p}(\Omega)$,*

$$\int_\Omega |\nabla u|^p dx \le C_1 E_V(u) + C_2 \int_{\partial\Omega} w|u|^p d\sigma \tag{3.11}$$

2. *Suppose $\lambda_1^D(V) > 0$ and $\beta(V, m) > 0$. If $V_k \to V$ in $L^\infty(\Omega)$, $m_k \to m$ in $L^q(\partial\Omega)$ and u_k is a sequence such that $E_{V_k}(u_k)$ and $\int_{\partial\Omega} m_k|u_k|^p d\sigma$ are bounded then u_k is bounded.*
3. *Assume $m^+ \not\equiv 0$, $m^- \not\equiv 0$ and $\beta(V, m) = 0$. If $u_k \ge 0$ is a sequence in $\mathcal{M}_{m,n}$ such that $|x \in \Omega : u_k(x) > 0| \longrightarrow_{k\to+\infty} 0$ and $E_V(u_k)$ is bounded then u_k is bounded.*

Proof. We adapt the proof made in [8].

1. To prove the first point, we check the following claim:

$$\forall \varepsilon > 0, \exists M_\varepsilon > 0 \Big/ \|u\|^p_{L^p(\Omega)} \le \varepsilon \int_\Omega |\nabla u|^p dx + M_\varepsilon \int_{\partial\Omega} w|u|^p d\sigma. \tag{3.12}$$

Let assume by contradiction that $\exists\varepsilon_0 > 0$ and a sequence $u_n \in W^{1,p}(\Omega)$ such that

$$\|u_n\|^p_{L^p(\Omega)} > \varepsilon_0 \int_\Omega |\nabla u_n|^p dx + n \int_{\partial\Omega} w|u_n|^p d\sigma.$$

Then, setting $v_n := \frac{u_n}{\|u_n\|_{L^p(\Omega)}}$, one writes

$$\|v_n\|_{L^p(\Omega)} = 1 \quad \text{and} \quad \varepsilon_0 \int_\Omega |\nabla v_n|^p dx + n \int_{\partial\Omega} w|v_n|^p d\sigma < 1.$$

As v_n is bounded in $W^{1,p}(\Omega)$, up to a subsequence, v_n converges weakly to some v_0 in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Passing to the limit, $\|v_0\|^p = 1$ and $\int_{\partial\Omega} w|v_n|^p d\sigma < 0$ which yields to a contradiction with the assumption $w > 0$. Our claim (3.12) is thus proved. Using now Hölder inequality we get

$$\left|\int_\Omega V|u|^p dx\right| \le \|V\|_{L^\infty(\Omega)}\|u\|^p_{L^p(\Omega)}. \tag{3.13}$$

Combining (3.12) and (3.13), we have for all $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\left|\int_\Omega V|u|^p dx\right| \le \varepsilon\|V\|_{L^\infty(\Omega)} \int_\Omega |\nabla u|^p dx + M_\varepsilon\|V\|_{L^\infty(\Omega)} \int_{\partial\Omega} w|u|^p d\sigma \tag{3.14}$$

and then

$$\int_\Omega |\nabla u|^p dx - E_V(u) \le \varepsilon\|V\|_{L^\infty(\Omega)} \int_\Omega |\nabla u|^p dx + M_\varepsilon\|V\|_{L^\infty(\Omega)} \int_{\partial\Omega} w|u|^p d\sigma \tag{3.15}$$

that is

$$\left(1 - \varepsilon\|V\|_{L^\infty(\Omega)}\right) \int_\Omega |\nabla u|^p dx \le E_V(u) + M_\varepsilon\|V\|_{L^\infty(\Omega)} \int_{\partial\Omega} w|u|^p d\sigma. \tag{3.16}$$

Letting $\varepsilon > 0$ be such that $(1 - \varepsilon\|V\|_{L^\infty(\Omega)}) > 0$, the result follows.

2. Let assume by contradiction that u_k is unbounded that is $\|u_k\| \to \infty$ and set $v_k = \frac{u_k}{\|u_k\|}$. Then there exists $v_0 \in W^{1,p}(\Omega)$ such that v_k converges weakly to v_0 and strongly in $L^p(\Omega)$. In addition, a simple computation shows that $\int_{\partial\Omega} m|v_0|^p d\sigma = 0$ and $E_V(v_0) = 0$. Furthermore

$$\begin{aligned}
1 + \int_\Omega V(x)|v_0|^p dx &= \lim_{k\to+\infty} 1 + \int_\Omega V(x)|v_k|^p dx \\
&= \lim_{k\to+\infty} \left(\int_\Omega |\nabla v_k|^p dx + \int_{\partial\Omega} |v_k|^p d\sigma + \int_\Omega V(x)|v_k|^p dx\right) \\
&= \lim_{k\to+\infty} \left(E_V(v_k) + \int_{\partial\Omega} |v_k|^p d\sigma\right)
\end{aligned}$$

$$1 + \int_\Omega V(x)|v_0|^p dx = \int_{\partial\Omega} |v_0|^p d\sigma.$$

Consequently $v_0 \not\equiv 0$ in Ω. If $v_0 \equiv 0$ in $\partial\Omega$ then $v_0 \in W_0^{1,p}(\Omega)$ and we deduce that

$$\lambda_1^D(V) \int\limits_{\Omega} |v_0|^p dx \leq E_V(v_0) = 0,$$

a contradiction since $\lambda_1^D(V) > 0$. Hence $v_0 \not\equiv 0$ in $\partial\Omega$ and $\frac{v_0}{\|v_0\|_{L^p(\partial\Omega)}}$ is an admissible function in the definition of $\beta(V,m)$ and consequently

$$\beta(V,m) \leq E_V\left(\frac{v_0}{\|v_0\|_{L^p(\partial\Omega)}}\right) = 0.$$

This leads to a contradiction with the assumption $\beta(V,m) > 0$ and we get expected result.

3. We suppose by contradiction that u_k is unbounded. Proceeding as in the previous case, we reach $0 = \beta(V,m) \leq E_V(\frac{v_0}{\|v_0\|_{L^p(\partial\Omega)}}) \leq 0$ and by Theorem 1.3, $v_0 > 0$ in Ω which contradicts $|x \in \Omega : u_k(x) > 0| \longrightarrow_{k\to+\infty} 0$. □

Proposition 3.6. *If $\lambda_1^D(V) > 0$, $\beta(V,m) > 0$ and $\beta(V,n) > 0$ then φ_m and $-\varphi_n$ are strict local minima of $\tilde{E}_V$ with corresponding critical values $\lambda_1(V,m)$ and $\lambda_1(V,n)$, respectively.*

Proof. The proof is partly adapted from an analogous result in [6] and [11]. Let us show that φ_m is a strict minimum for $\tilde{E}_V$ (similar argument for $-\varphi_n$). Assume by contradiction the existence of a sequence $u_k \in \mathcal{M}_{m,n}$ with $u_k \neq \varphi_m$, $u_k \to \varphi_m$ in $W^{1,p}(\Omega)$ and $\tilde{E}_V(u_k) \leq \lambda_1(V,m)$. We first observe that u_k changes sign for k sufficiently large. Indeed, since $u_k \to \varphi_m$ with $\varphi_m > 0$ then $u_k > 0$ somewhere in Ω. If we assume that $u_k \geq 0$ in Ω, then

$$\tilde{E_V}(u_k) = \int\limits_{\Omega} \left(|\nabla u_k|^p + V(x)|u_k|^p\right) dx > \lambda_1(V,m) \int\limits_{\partial\Omega} m(x)|u_k|^p d\sigma = \lambda_1(V,m)$$

since $u_k \neq \pm\varphi_m$, but this contradicts $\tilde{E}_V(u_k) \leq \lambda_1(V,m)$. Hence u_k changes sign for k sufficiently large. On the other hand, the hypotheses $\tilde{E}_V(u_k) \leq \lambda_1(V,m)$ and $B_{m,n}(u_k) = 1$ imply

$$\begin{aligned} \lambda_1(V,m) &\geq \int\limits_{\Omega} \left(|\nabla u_k|^p + V(x)|u_k|^p\right) dx \\ &= \int\limits_{\Omega} \left(\left|\nabla u_k^+\right|^p + V(x)\left|u_k^+\right|^p\right) dx + \int\limits_{\Omega} \left(\left|\nabla u_k^-\right|^p + V(x)\left|u_k^-\right|^p\right) dx \\ \lambda_1(V,m) &\geq E_V\left(u_k^+\right) + E_V\left(u_k^-\right). \end{aligned} \tag{3.17}$$

One may think of taking into account separately the cases $B_{m,n}(u_k^+) = 0$, $B_{m,n}(u_k^+) < 0$ and $B_{m,n}(u_k^+) > 0$.

- If $B_{m,n}(u_k^+) = 0$ then one can renormalize u_k^+ to show that $\beta(V,m) \leq 0$ and get a contradiction.
- If $B_{m,n}(u_k^+) > 0$ then $\frac{u_k^+}{(B_{m,n}(u_k^+))^{1/p}}$ can be seen as an admissible function for the definition of $\lambda_1(V,m)$ and therefore $E_V(u_k^+) \geq \lambda_1(V,m)B_{m,n}(u_k^+)$.
- If $B_{m,n}(u_k^+) < 0$, it is rather $\frac{u_k^+}{(-B_{m,n}(u_k^+))^{1/p}}$ that is admissible for $\lambda_{-1}(V,m)$ and we get

$$\lambda_1(V,m)B_{m,n}\left(u_k^+\right) \leq \lambda_{-1}(V,m)B_{m,n}\left(u_k^+\right) \leq E_V\left(u_k^+\right)$$

Either $B_{m,n}(u_k^+) < 0$ or $B_{m,n}(u_k^+) > 0$, we conclude that

$$E_V\left(u_k^+\right) \geq \lambda_1(V,m) B_{m,n}\left(u_k^+\right) \tag{3.18}$$

Merging (3.17) and (3.18), one writes

$$E_V\left(u_k^-\right) \leq \lambda_1(V,m) B_{m,n}\left(u_k^-\right). \tag{3.19}$$

We now consider the three cases that arise on $B_{m,n}(u_k^-)$:

- **Case** $B_{m,n}(u_k^-) = 0$
  In this case, $v_k := \frac{u_k^-}{(\int_{\partial\Omega} |u_k^-|^p d\sigma)^{1/p}}$ is admissible for $\beta(V,n)$ and then

$$\beta(V,n) \leq E_V(v_k) = \frac{E_V(u_k^-)}{\int_{\partial\Omega} |u_k^-|^p d\sigma} \leq 0 \tag{3.20}$$

  which leads to a contradiction with $\beta(V,n) > 0$.
- **Case** $B_{m,n}(u_k^-) > 0$
  Setting $v_k := \frac{u_k^-}{(B_{m,n}(u_k^-))^{1/p}}$, it reads:

$$B_{m,n}(v_k) = 1 \quad \text{and} \quad \lambda_1(V,n) \leq E_V(v_k) \leq \lambda_1(V,m)$$

  by taking into account the definition of $\lambda_1(V,n)$ and (3.19). Then, $E_V(v_k)$ is bounded and from Lemma 3.5, we conclude that v_k as a sequence in $W^{1,p}(\Omega)$ is bounded. Since

$$\frac{1}{\|v_k\|} = \frac{B_{m,n}(u_k^-)}{\|u_k^-\|^p} \leq \frac{B_{m,n}(u_k^-)}{\int_\Omega |\nabla u_k^-|^p dx}, \tag{3.21}$$

  we then reach a contradiction as the right-hand side of (3.21) → 0 according to Lemma 3.7 below.
- **Case** $B_{m,n}(u_k^-) < 0$
  Taking $v_k := \frac{u_k^-}{(-B_{m,n}(u_k^-))^{1/p}}$, one has $B_{m,n}(v_k) = -1$ and as n is sign-changing, it follows

$$-\lambda_{-1}(V,n) \leq E_V(v_k) \leq -\lambda_{-1}(V,m).$$

  As in the previous case, we reach a contradiction.

All in all we have proved that φ_m is a strict minimum for $\tilde{E}_V$ (similar argument for $-\varphi_n$). □

Lemma 3.7. *(See [3].) Let $(v_k)_k$ be a sequence in $W^{1,p}(\Omega)$ with $v_k \geq_{\neq 0}$ and mes$\{x \in \Omega\ v_k(x) > 0\} \to 0$. Let $(n_k)_k$ be bounded $\in L^q(\partial\Omega)$. Then*

$$\frac{\int_{\partial\Omega} n_k(x)(v_k)^p d\sigma}{\int_\Omega |\nabla v_k|^p dx} \to 0 \quad \textit{when } k \to +\infty.$$

Lemma 3.8. *Consider the family paths in $\mathcal{M}_{m,n}$*

$$\Gamma := \left\{h \in C\left([0,1], \mathcal{M}_{m,n}\right) : h(0) = \varphi_m \textit{ and } h(1) = -\varphi_n\right\}.$$

Then Γ is nonempty.

Proof. We borrow construction ideas from [10] by recalling the open sets

$$\partial\Omega_m^{\pm} = \{x \in \partial\Omega, m(x) \gtrless 0\} \quad \text{and} \quad \partial\Omega_n^{\pm} = \{x \in \partial\Omega, n(x) \gtrless 0\}.$$

It is worth mentioning the fact that we deal with two sign-changing weights on $\partial\Omega$. This makes construction ideas quite different from the one we are adapting and requires some discussions as one can think of the different case of intersection of open sets defined above. For simplicity, we shall only deal with the case where $\partial\Omega^+ = \partial\Omega_m^+ \cap \partial\Omega_n^+$ is nonempty (similar procedure is applied when considering other nonempty intersection) in which one can find two disjoint compact sets K_1 and K_2 and construct two functions u_1 and u_2 in $\mathcal{D}(\partial\Omega^+)$ such that

$$\begin{aligned} &u_1 \equiv 1 \quad \text{on } K_1; \quad 0 \le u_1 \le 1 \quad \text{on } \partial\Omega^+ \\ &u_2 \equiv 1 \quad \text{on } K_2; \quad 0 \le u_2 \le 1 \quad \text{on } \partial\Omega^+. \end{aligned}$$

Extending u_1 and u_2 by zero on $\partial\Omega\backslash\partial\Omega^+$, we get $u_1, u_2 \in \mathcal{D}(\partial\Omega)$ and thereby a function in $\mathcal{D}(\partial\Omega)$ defined as $u_0 = u_1 - u_2$ satisfying

$$\int_{\partial\Omega} m(x)\big(u_0^+\big)^p d\sigma > 0 \quad \text{and} \quad \int_{\partial\Omega} n(x)\big(u_0^-\big)^p d\sigma > 0.$$

Let

$$\gamma_1(t) = u_0^+ - tu_0^-, \quad t \in [0,1]$$

be a path which joining u_0 to u_0^+ and

$$\gamma_2(t) = \big[t\big(u_0^+\big)^p + (1-t)(\varphi_m)^p\big]^{1/p}, \quad t \in [0,1]$$

be the one joining u_0^+ to φ_m. Using the fact that u_0^+ and u_0^- have disjoint supports, one obtains

$$B_{m,n}\big(\gamma_1(t)\big) = \int_{\partial\Omega} m\big(u_0^+\big)^p d\sigma + t^p \int_{\partial\Omega} n\big(u_0^-\big)^p d\sigma > 0, \quad \forall t \in [0,1]$$

and

$$\begin{aligned} B_{m,n}\big(\gamma_2(t)\big) &= t\int_{\partial\Omega} m\big(u_0^+\big)^p d\sigma + (1-t)\int_{\partial\Omega} m(\varphi_m)^p d\sigma \\ &\ge \min\left\{\int_{\partial\Omega} m(x)\big(u_0^+\big)^p d\sigma, \int_{\partial\Omega} n(x)\big(u_0^-\big)^p d\sigma\right\} \\ &> 0, \quad \forall t \in [0,1]. \end{aligned}$$

Consequently, one has a path γ joining u_0 and φ_m by considering γ_1 and γ_2 which satisfies $B_{m,n}(\gamma(t)) > 0$ for all $t \in [0,1]$. In a very similar way, we get a path γ' from u_0 to $-\varphi_n$ satisfying $B_{m,n}(\gamma'(t)) > 0$ for all $t \in [0,1]$ which allows us by putting it together with γ to get a well-defined path after renormalization that belongs to Γ. $\square$

One of our main results is as follows

Theorem 3.9. *Assume* $\lambda_1^D(V) > 0$, $\beta(V,m) > 0$ *and* $\beta(V,n) > 0$ *and let*

$$\Gamma := \big\{\gamma \in C\big([0,1], \mathcal{M}_{m,n}\big) : \gamma(0) = \varphi_m \text{ and } \gamma(1) = -\varphi_n\big\}.$$

Then

$$c(m,n,V) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} \tilde{E_V}(u) \tag{3.22}$$

is a nonprincipal eigenvalue for $(P_{V,m,n})$ *which satisfies*

$$c(m,n,V) > \max\{\lambda_1(V,m), \lambda_1(V,n)\}.$$

Moreover $c(m,n,V)$ *is the first nonprincipal eigenvalue of* $(P_{V,m,n})$ *in sense that there is no other eigenvalue of* $(P_{V,m,n})$ *between* $\max\{\lambda_1(V,m), \lambda_1(V,n)\}$ *and* $c(m,n,V)$.

As stated above, we are applying Proposition 3.2 and to demonstrate Theorem 3.9 we need two preliminary results which concern the (PS) condition and the geometry of E_V near the strict local minima φ_m and $-\varphi_n$. We formulate these required conditions as lemmas and propositions.

Proposition 3.10. *We assume that* $\beta(V,m) > 0$, $\beta(V,n) > 0$ *and* $\lambda_1^D(V) > 0$. *Then* $\tilde{E}_V$ *satisfies Palais–Smale* (PS) *condition on* $\mathcal{M}_{m,n}$.

Proof. Let $(u_k)_k \in \mathcal{M}_{m,n}$ be a Palais–Smale sequence, i.e.

$$E_V(u_k) \longrightarrow c \quad \text{as } k\to+\infty$$

for some $c\in\mathbb{R}$, and there is $(\varepsilon_k)_k$ with $\varepsilon_k\to 0$ such that:

$$\left|\left\langle E_V'(u_k), \xi\right\rangle\right| \le \varepsilon_k\|\xi\| \tag{3.23}$$

for all $\xi\in T_{u_k}\mathcal{M}_{m,n}$. We show that for all $v\in W^{1,p}(\Omega)$,

$$a_k(v) := v - \left\langle B_{m,n}'(u_k), v\right\rangle u_k \in T_{u_k}\mathcal{M}_{m,n}.$$

Thus for $\xi = a_k(v)$ in (3.23) we obtain

$$\left|\left\langle E_V'(u_k), v\right\rangle - \left\langle B_{m,n}'(u_k), v\right\rangle E_V(u_k)\right| \le \varepsilon_k\left|v - \left\langle B_{m,n}'(u_k), v\right\rangle u_k\right|. \tag{3.24}$$

Let us say that u_k remains bounded in $W^{1,p}(\Omega)$. Indeed assume by contradiction that u_k is unbounded. Then, up to a subsequence, $v_k = \frac{u_k}{\|u_k\|}$ converges to some v_0 weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. Hence, passing to the limit, we deduce that

$$\left(\int_\Omega |\nabla v_0|^p dx + \int_\Omega V(x)|v_0|^p dx\right) \le \liminf E_V(v_k) = \liminf \frac{E_V(u_k)}{\|u_k\|^p} = 0. \tag{3.25}$$

On the other hand, $B_{m,n}(u_k) = 1$ thus $B_{m,n}(v_k) \longrightarrow 0$ as $k\to+\infty$ that is

$$B_{m,n}(v_0) = \int_{\partial\Omega} m\left(v_0^+\right)^p d\sigma + \int_{\partial\Omega} n\left(v_0^-\right)^p d\sigma = 0.$$

This gives $B_{m,n}(v_0^+) \ne 0$ and $B_{m,n}(v_0^-) \ne 0$. Indeed, assume by contradiction that $B_{m,n}(v_0^+) = 0$ or $B_{m,n}(v_0^-) = 0$. For simplicity, we shall only deal with $B_{m,n}(v_0^+) = 0$ (the same argument holds when considering $B_{m,n}(v_0^-) = 0$). We have

$$\begin{aligned}1+\int_{\Omega} V(x)\left|v_0^+\right|^p dx &= \lim_{k\to+\infty} 1+\int_{\Omega} V(x)\left|v_k^+\right|^p dx\\ &= \lim_{k\to+\infty}\left(\int_{\Omega}|\nabla v_k|^p dx+\int_{\partial\Omega}\left|v_k^+\right|^p d\sigma+\int_{\Omega} V(x)\left|v_k^+\right|^p dx\right)\\ &= \lim_{k\to+\infty}\left(E_V\left(v_k^+\right)+\int_{\partial\Omega}\left|v_k^+\right|^p d\sigma\right)\\ 1+\int_{\Omega} V(x)\left|v_0^+\right|^p dx &= \int_{\partial\Omega}\left|v_0^+\right|^p d\sigma\end{aligned}$$

and consequently $v_0^+ \not\equiv 0$ in Ω. If $v_0^+ \equiv 0$ in $\partial\Omega$ then $v_0^+ \in W_0^{1,p}(\Omega)$ and we deduce from (3.25) that

$$\lambda_1^D(V)\int_{\Omega}\left|v_0^+\right|^p dx \le \left(\int_{\Omega}\left|\nabla v_0^+\right|^p dx+\int_{\Omega} V\left|v_0^+\right|^p dx\right)\le 0,$$

a contradiction since $\lambda_1^D(V) > 0$. Hence $v_0^+ \not\equiv 0$ in $\partial\Omega$ and $\frac{v_0^+}{\|v_0^+\|_{L^p(\partial\Omega)}}$ is an admissible function in the definition of $\beta(V,m)$ and consequently

$$\beta(V,m)\le E_V\left(\frac{v_0^+}{\|v_0^+\|_{L^p(\partial\Omega)}}\right)\le 0.$$

This leads to a contradiction with the assumption $\beta(V,m) > 0$ (a similar argument for $B_{m,n}(v_0^-) = 0$ would lead to a contradiction with the assumption $\beta(V,n) > 0$) and we conclude that $B_{m,n}(v_0^+) \ne 0$ and $B_{m,n}(v_0^-) \ne 0$. As $B_{m,n}(v_0) = B_{m,n}(v_0^+)+B_{m,n}(v_0^-) = 0$, one has either $B_{m,n}(v_0^+) < 0$ or $B_{m,n}(v_0^-) < 0$. Assuming $B_{m,n}(v_0^+) < 0$, the functions $\frac{v_0^+}{[-B_{m,n}(v_0^+)]^{\frac{1}{p}}}$ and $\frac{v_0^-}{[B_{m,n}(v_0^-)]^{\frac{1}{p}}}$ are admissible functions in the definitions of $\lambda_{-1}(V,m)$ and $\lambda_1(V,n)$ respectively. As a result

$$-\lambda_{-1}(V,m)\le\frac{E_V(v_0^+)}{-B_{m,n}(v_0^+)}$$

and

$$\lambda_1(V,n)\le\frac{E_V(v_0^-)}{B_{m,n}(v_0^-)}.$$

We then obtain $\lambda_1(V,n)-\lambda_{-1}(V,m)\le\frac{E_V(v_0)}{B_{m,n}(v_0^-)} = 0$ that is $\lambda_1(V,n)\le\lambda_{-1}(V,m) < \lambda_1(V,m)$ for any n and m. This fails to hold in the particular case $m \equiv n$ and we then reach a contradiction. From all above we reach the conclusion that u_k is a bounded sequence in $W^{1,p}(\Omega)$. Up to a subsequence $u_k \to u_0$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$. Choosing $v = u_k - u_0$ in (3.24) and passing to the limit we obtain

$$\left\langle B'_{m,n}(u_k), u_k-u_0\right\rangle \longrightarrow 0$$

and then

$$\lim_{k\to+\infty}\left(\int_{\Omega}|\nabla u_k|^{p-2}\nabla u_k\nabla(u_k-u_0)dx+\int_{\Omega} V(x)|u_k|^{p-2}u_k(u_k-u_0)dx\right)=0.$$

Using Hölder inequality, we get

$$\begin{aligned}\int\limits_{\Omega} V(x)|u_k|^{p-2}u_k(u_k-u_0)dx &\leq \|V\|_{L^\infty}\int\limits_{\Omega}|u_k|^{p-1}|u_k-u_0|dx\\ &\leq \|V\|_{L^\infty}\Bigg(\int\limits_{\Omega}|u_k|^p dx\Bigg)^{\frac{p-1}{p}}\Bigg(\int\limits_{\Omega}|u_k-u_0|^p dx\Bigg)^{\frac{1}{P}}\\ &= \|V\|_{L^\infty}\|u_k\|^{p-1}_{L^p(\Omega)}\|u_k-u_0\|_{L^p(\Omega)}\end{aligned}$$

and as u_k converges to u_0 in $L^p(\Omega)$, it follows

$$\int\limits_{\Omega} V(x)|u_k|^{p-2}u_k(u_k-u_0)dx \longrightarrow 0$$

and consequently,

$$\lim_{k\to+\infty}\int\limits_{\Omega}|\nabla u_k|^{p-2}\nabla u_k\nabla(u_k-u_0)dx = 0.$$

Moreover,

$$\int\limits_{\Omega}\big(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u_0|^{p-2}\nabla u_0\big)(\nabla u_k-\nabla u_0)dx \longrightarrow 0.$$

Applying the (S^+) property stated in Lemma 3.11 and Hölder inequality, one easily derives that $\nabla u_k \to \nabla u_0$ in $L^p(\Omega)$ and consequently $u_k \to u_0$ in $W^{1,p}(\Omega)$. □

Lemma 3.11 *((S^+) property). (See [13].) For all $x, y \in \mathbb{R}^N$, we have*

$$|x-y|^p \leq c\big[\big(|x|^{p-2}x-|y|^{p-2}y\big)(x-y)\big]^{s/2}\big[|x|^p+|y|^p\big]^{1-s/2}$$

with $c = c(p)$, $s = p$ if $1 < p < 2$ and $s = 2$ if $p \geq 2$.

Lemma 3.12. *Let $m_k \in L^q(\partial\Omega)$ be a sequence with $m_k^+ \not\equiv 0$. If $m_k^+ \to 0$ in $L^q(\partial\Omega)$, then $\lambda_1(V, m_k) \to \infty$.*

Proof. Let assume by contradiction that $\lambda_1(V, m_k)$ is achieved at φ_k with $\varphi_k > 0$. We then have

$$1 = \int\limits_{\partial\Omega} m_k\varphi_k^p d\sigma \leq \int\limits_{\partial\Omega} m_k^+\varphi_k^p d\sigma \leq \big\|m_k^+\big\|_q \|\varphi_k\|^p_{pq'},$$

so that $\|\varphi_k\|_{pq'} \longrightarrow \infty$. As $\lambda_1(V, m_k) = E_V(\varphi_k)$ is bounded then $(\varphi_k)_k$ by Lemma 3.5 is bounded which leads to a contradiction and we conclude that $\lambda_1(V, m_k) \to \infty$. □

Proposition 3.13. *Let E, g, M, f and $\tilde{f}$ be as considered previously in (2.2). Let u_0 be a strict local minima of $\tilde{f}$, i.e. for some $\varepsilon_0 > 0$,*

$$\tilde{f}(u_0) < \tilde{f}(u) \tag{3.26}$$

for all $u \in M$ *with* $u \neq u_0$ *and* $\|u - u_0\|_E < \varepsilon_0$. *Assume that* $\tilde{f}$ *satisfies the* (PS) *condition on* M. *Then, for any* $0 < \varepsilon < \varepsilon_0$,

$$\tilde{f}(u_0) < \inf\{\tilde{f}(u) : u \in M \text{ and } \|u - u_0\|_E = \varepsilon\}. \tag{3.27}$$

Proof. A version of the proof that uses Ekeland's variational principle can be found in [6]. □

Lemma 3.14. *(See [6].) Let* E, g, M, f *and* $\tilde{f}$ *be as considered previously in (2.2). Assume that* $\tilde{f}$ *is bounded from below on* M *and satisfies the* (PS) *condition on* M. *Let* $r \in \mathbb{R}$ *and consider*

$$\mathcal{O} := \{u \in M : \tilde{f}(u) < r\}.$$

Then any (nonempty) component $\mathcal{O}_1$ *of* $\mathcal{O}$ *contains a critical point of* $\tilde{f}$.

We are now ready to prove Theorem 3.9.

Proof of Theorem 3.9. First of all, Γ is nonempty from Lemma 3.8. Furthermore the (PS) condition and the geometric assumption (3.22) are satisfied by Proposition 3.10 and Proposition 3.13. In a nutshell, it remains to show that $c(m,n,V)$ is the first nonprincipal eigenvalue of $(P_{V,m,n})$ in sense that there is no other eigenvalue of $(P_{V,m,n})$ between $\max\{\lambda_1(V,m), \lambda_1(V,n)\}$ and $c(m,n,V)$. We are using the same approach as in the proof of Theorem 3.1 (see [6]). We then assume by contradiction that there exists an eigenvalue λ that verifies

$$\max\{\lambda_1(V,m), \lambda_1(V,n)\} < \lambda < c(m,n,V)$$

and let $u \in \mathcal{M}_{m,n}$ be the corresponding eigenfunction as a solution of $(P_{V,m,n})$. Hence u changes sign (since $\lambda > \max\{\lambda_1(V,m), \lambda_1(V,n)\}$). In addition

$$\int_{\partial\Omega} m(x)|u^+|^p d\sigma > 0 \quad \text{and} \quad \int_{\partial\Omega} n(x)|u^-|^p d\sigma > 0.$$

Indeed if for instance

$$\int_{\partial\Omega} m(x)|u^+|^p d\sigma = 0 \quad \left(\text{a similar proof leads to } \int_{\partial\Omega} n(x)|u^-|^p d\sigma > 0\right)$$

then $\frac{u^+}{\|u^+\|_p}$ is an admissible function in the definition of $\beta(V,m)$. So

$$\beta(V,m) \leq E_V\left(\frac{u^+}{\|u^+\|_p}\right) = \lambda \int_{\partial\Omega} m(x)\left(\frac{u^+}{\|u^+\|_p}\right)^p d\sigma = 0,$$

a contradiction. Now if

$$\int_{\partial\Omega} m(x)|u^+|^p d\sigma < 0$$

then

$$\frac{u^+}{(-\int_{\partial\Omega} m(x)|u^+|^p d\sigma)^{1/p}}$$

is an admissible function for $\lambda_{-1}(V,m)$ and consequently

$$-\lambda_{-1}(V,m) \le E_V\left(\frac{u^+}{(-\int_{\partial\Omega} m(x)|u^+|^p d\sigma)^{1/p}}\right).$$

But

$$E_V\left(u^+\right) = -\lambda\left(-\int\limits_{\partial\Omega} m(x)\left|u^+\right|^p d\sigma\right)$$

then

$$-\lambda = E_V\left(\frac{u^+}{(-\int_{\partial\Omega} m(x)|u^+|^p d\sigma)^{1/p}}\right) \ge -\lambda_{-1}(V,m) > -\lambda_1(V,m),$$

a contradiction. Therefore

$$B_{m,n}\left(u^+ - tu^-\right) = \int\limits_{\partial\Omega} m(x)\left|u^+\right|^p d\sigma + t^p \int\limits_{\partial\Omega} n(x)\left|u^-\right|^p d\sigma > 0$$

and one can define

$$\gamma_1(t) := \frac{u^+ - tu^-}{(B_{m,n}(u^+ - tu^-))^{1/p}}, \quad t \in [0,1]$$

which is a path in $\mathcal{M}_{m,n}$ joining

$$v := \frac{u^+}{(B_{m,n}(u^+))^{1/p}} \quad \text{to } u.$$

We then note that

$$E_V\left(\gamma_1(t)\right) = \lambda \quad \text{for all } t \in [0,1].$$

Indeed, firstly

$$E_V\left(\gamma_1(t)\right) = \frac{E_V(u^+) + t^p E_V(u^-)}{B_{m,n}(u^+ - tu^-)}$$

and secondly

$$\begin{cases} E_V(u^+) = \lambda \int\limits_{\partial\Omega} m(x)\left|u^+\right|^p d\sigma \\ t^p E_V(u^-) = \lambda t^p \int\limits_{\partial\Omega} n(x)\left|u^-\right|^p d\sigma. \end{cases}$$

By combining both equations in the previous set, we get:

$$E_V\left(u^+\right) + t^p E_V\left(u^-\right) = \lambda\left(\int\limits_{\partial\Omega} m(x)\left|u^+\right|^p d\sigma + t^p \int\limits_{\partial\Omega} n(x)\left|u^-\right|^p d\sigma\right)$$

i.e.

$$E_V\big(u^+\big) + t^p E_V\big(u^-\big) = \lambda B_{m,n}\big(u^+ - tu^-\big).$$

Lastly

$$E_V\big(\gamma_1(t)\big) = \lambda, \quad \forall t \in [0,1].$$

By reasoning completely analogous to the previous,

$$\gamma_2(t) := \frac{(1-t)u^+ - u^-}{(B_{m,n}((1-t)u^+ - u^-))^{1/p}}, \quad t \in [0,1]$$

defines a path in $\mathcal{M}_{m,n}$ joining

$$u \quad \text{to} \; \frac{-u^-}{(B_{m,n}(-u^-))^{1/p}}$$

such that

$$E_V\big(\gamma_2(t)\big) = \lambda \quad \text{for all } t \in [0,1].$$

We shall now describe the construction of two paths in $\mathcal{M}_{m,n}$ which stay at levels $\leq \lambda$, one goes from φ_m to v and the other from $\frac{-u^-}{(B_{m,n}(-u^-))^{1/p}}$ to $-\varphi_n$. Combining everything together, we get the desired path from φ_m to $-\varphi_n$.

We first construct the path from φ_m to v by considering the manifold $\mathcal{M}_{m,m}$. Clearly $v \in \mathcal{M}_{m,m}$ and the critical points of the restriction $\tilde{E}_V$ of E_V to $\mathcal{M}_{m,m}$ are the normalized eigenfunctions of Δ_p for the weight m. Since v does not change sign and vanishes on a set of positive measure, v is not a critical point of this restriction of E_V to $\mathcal{M}_{m,m}$. Consequently there exists a C^1 path $\nu :]-\varepsilon, +\varepsilon[\longrightarrow \mathcal{M}_{m,m}$ with $\nu(0) = v$ and $\frac{d}{dt}E_V(\nu(t))|_{t=0}$. Following a little portion of this path ν in the negative or positive direction (this portion is called ν_1), we move from v to w by a path in $\mathcal{M}_{m,m}$ which, with the exception of its starting point v where $E_V(v) = \lambda$, lies at levels $< \lambda$. The path $\gamma_3(t) := |\nu_1(t)|$ then lies in $\mathcal{M}_{m,m}$, goes from v to $v_1 := |w|$ and remains, with the exception of its starting point v where $E_V(v) = \lambda$, at levels $< \lambda$ (since $E_V(\nu_1(t)) = E_V(|\nu_1(t)|)$). We now construct a path from v_1 to φ_m and for that purpose, we are constructing by using Lemma 3.12, a weight $\hat{n} \in L^q(\partial\Omega)$ such that $(\hat{n})^+ \not\equiv 0$, $\lambda_1(V,\hat{n}) > \lambda$ and $\hat{n} \leq m$ on $\partial\Omega$. When $m^- \not\equiv 0$, it suffices to take $\hat{n} = \varepsilon m^+ - m^-$ in this construction with $\varepsilon > 0$ sufficiently small and $\hat{n} = \varepsilon m - k\chi_B$ if $m^- \equiv 0$ i.e. $m \geq 0$ with $k > 0$ sufficiently high and χ_B denotes the characteristic function of the ball $B \subset \partial\Omega$ such that $m^+ \not\equiv 0$ on $\partial\Omega \backslash B$. We now consider the manifold $\mathcal{M}_{m,\hat{n}}$ and the subset

$$\mathcal{O} := \big\{u \in \mathcal{M}_{m,\hat{n}} : E_V(u) < \lambda\big\}.$$

We have v_1 and φ_m which belong to $\mathcal{O}$. Moreover the only critical point in $\mathcal{O}$ of the restriction $\tilde{E}_V$ of E_V to $\mathcal{M}_{m,\hat{n}}$ is φ_m (because the first two critical levels $\lambda_1(V,m)$ and $\lambda_1(\hat{n},V)$ of $\tilde{E}_V$ verify $\lambda_1(V,m) < \lambda < \lambda_1(\hat{n},V)$ by the choice of $\hat{n}$). Applying Lemma 3.14 to the component of $\mathcal{O}$ which contains v_1 and using the fact that any open connected subset of a manifold is arcwise connected, we get a path γ_4 in $\mathcal{O}$ from v_1 to φ_m. By the choice of $\hat{n}$, we write

$$1 = \int_{\partial\Omega} \big(m(x)\big(\gamma_4(t)^+\big)^p + \hat{n}\big(\gamma_4(t)^-\big)^p\big)d\sigma$$

$$\begin{aligned}&\le \int\limits_{\partial\Omega} \left(m(x)\left(\gamma_4(t)^+\right)^p + m(x)\left(\gamma_4(t)^-\right)^p\right) d\sigma \\ &= \int\limits_{\partial\Omega} m(x)\left|\gamma_4(t)\right|^p d\sigma, \quad \forall t \in [0,1] \end{aligned} \tag{3.28}$$

which guarantees the well-defined of

$$\gamma_5(t) := \frac{|\gamma_4(t)|^p}{(\int_{\partial\Omega} m(x)|\gamma_4(t)|^p d\sigma)^{1/p}}$$

as a path in the original manifold $\mathcal{M}_{m,n}$ going from v_1 to φ_m. Finally

$$E_V\left(\gamma_5(t)\right) = \frac{E_V(\gamma_4(t))}{\int_{\partial\Omega} m(x)|\gamma_4(t)|^p d\sigma} < \lambda$$

since $\gamma_4(t) \in \mathcal{O}$ and by (3.28),

$$\int\limits_{\partial\Omega} m(x)\left|\gamma_4(t)\right|^p d\sigma \ge 1.$$

The path γ_5 allows us to join v_1 to φ_m, by staying at levels $< \lambda$. By a similar argument, we get to a path which goes from $\frac{-u^-}{(B_{m,n}(-u^-))^{1/p}}$ to $-\varphi_n$ and by putting these paths together, we ultimately define a path γ from φ_m to $-\varphi_n$ which verifies $E_V(\gamma(t)) < c(m,n,V)$. This contradicts the definition of $c(m,n,V)$ in (3.22). □

This result extends obtained results in [10, Proposition 4.2]. Indeed

Corollary 3.15. *In particular case $m \equiv n$, one obtains a variational characterization of the second eigenvalue $\lambda_2(V,m)$ of problem $(P_{V,m,m})$ with*

$$\lambda_2(V,m) = c(m,m,V) = \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} \tilde{E_V}(u)$$

where

$$\Gamma := \left\{\gamma \in C\left([0,1], \mathcal{M}_{m,m}\right) : \gamma(0) = \varphi_m \text{ and } \gamma(1) = -\varphi_m\right\}.$$

Remark 3.16. In the last step of the proof of Theorem 3.9, we conclude that for $u \in \mathcal{M}_{m,n}$ with $u \ge 0$ and $E_V < \mu$ for some μ, there exists a path in $\mathcal{M}_{m,n}$ which joins u to φ_m and made up of nonnegative functions but remains at levels $< \mu$ as well. A similar remark can be found in [6] and [10].

As an application of Theorem 3.9 one can enlarge the family of paths in (3.22) and keep the same minimax level. This is expressed in the proposition below.

Lemma 3.17.

$$\Gamma_0 := \left\{\gamma \in C\left([0,1], \mathcal{M}_{m,n}\right) : \gamma(0) \ge 0 \text{ and } \gamma(1) \le 0\right\}$$

is nonempty.

Proof. From the proof of Lemma 3.8, one can find $u \in W^{1,p}(\Omega)$ such that

$$\int_{\partial\Omega} m(x)\big(u^+\big)^p d\sigma > 0 \quad \text{and} \quad \int_{\partial\Omega} n(x)\big(u^-\big)^p d\sigma > 0.$$

Defining $\gamma_1(t) := (1-t)^{1/p}u^+ - t^{1/p}u^-$, $t \in [0,1]$ and recalling the fact that u^+ and u^- have disjoint supports, we get

$$\begin{aligned} B_{m,n}\big(\gamma_1(t)\big) &= (1-t)\int_{\partial\Omega} m(x)\big(u^+\big)^p d\sigma + t\int_{\partial\Omega} n(x)\big(u^-\big)^p d\sigma \\ &\geq \min\Big\{\int_{\partial\Omega} m(x)\big(u^+\big)^p d\sigma, \int_{\partial\Omega} n(x)\big(u^-\big)^p d\sigma\Big\} > 0, \quad \forall t \in [0,1]. \end{aligned}$$

The path

$$\gamma_2(t) := \frac{\gamma_1(t)}{(B_{m,n}(\gamma_1(t)))^{1/p}}, \quad t \in [0,1]$$

is then well-defined in Γ_0. □

Proposition 3.18.

$$c(m,n,V) = \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} E_V\big(\gamma(t)\big) \tag{3.29}$$

where

$$\Gamma_0 = \big\{\gamma \in C\big([0,1], \mathcal{M}_{m,n}\big) : \gamma(0) \geq 0 \textit{ and } \gamma(1) \leq 0\big\}.$$

Proof. The right-hand side of (3.29) that shall be called d is finite due to Lemma 3.17 and $d \leq c(m,n,V)$ since $\Gamma \subset \Gamma_0$. Let us assume by contradiction that $d < c(m,n,V)$ and choose μ with $d < \mu < c(m,n,V)$. Let γ be a path in Γ_0 which remains at levels $< \mu$ and proceed as in [6] by considering a path in Γ that will remain at levels $< \mu$. This shall contradict the definition of $c(m,n,V)$ in (3.22). To construct such path we first go from φ_m to $\gamma(0)$ by using Remark 3.16, then we follow γ from $\gamma(0)$ to $\gamma(1)$ and finally we join $\gamma(1)$ to $-\varphi_n$ by still using Remark 3.16. □

Turning to the case $\beta(V,m) = 0$ or $\beta(V,n) = 0$ where lots of well-known facts break down, we shall in practice apply Theorem 3.3 with $K = [0,1]$, $K_0 = \{0,1\}$, $f = E_V$, $E = W^{1,p}(\Omega)$, $g = B_{m,n}$ and $H = \Gamma_0$ as defined below. We recall that when $\beta(V,m) = 0$ or $\beta(V,n) = 0$, one can deduce from Theorem 1.3 that the infima in (1.12) and (1.13) are not achieved. Moreover eigenfunction associated to $\lambda_1(V,m)$ belongs to $\mathcal{A}$ defined in (1.11). This fact is a handicap to the use of classical mountain pass theorem.

Proposition 3.19. *Assume that* $m^- \not\equiv 0$, $n^- \not\equiv 0$, $\lambda_1^D(V) > 0$ *and* $\beta(V,m) = 0$ *or* $\beta(V,n) = 0$. *Then* $c(m,n,V) > \max\{\lambda_1(V,m), \lambda_1(V,n)\}$.

Proof. The proof is similar to the proof of Proposition 18 in [8]. One observes first that $c(m,n,V) \geq \max\{\lambda_1(V,m), \lambda_1(V,n)\}$ and we will only prove that $c(m,n,V) > \lambda_1(V,m)$ since the proof for $c(m,n,V) > \lambda_1(V,n)$ is similar. Thus assume by contradiction that $c(m,n,V) = \lambda_1(V,m)$. Then there exists γ_k in Γ_0 such that

$$\max_{t\in[0,1]} E_V\big(\gamma_k(t)\big) \longrightarrow \lambda_1(V,m) \quad \text{when } k \to \infty.$$

Let $\gamma_k \in \varGamma_0$ and define

$$f(t) := B_{m,n}\big(\gamma(t)^+\big) - B_{m,n}\big(\gamma(t)^-\big) \quad \text{for } t \in [0,1]. \tag{3.30}$$

As f is a continuous map with $f(0) = B_{m,n}(\gamma(0)) = 1$ and $f(1) = -B_{m,n}(\gamma(1)) = -1$, one can find some $s \in (0,1)$ satisfying $f(s) = 0$. Hence

$$B_{m,n}\big(\gamma(s)^+\big) = B_{m,n}\big(\gamma(s)^-\big) \quad \text{and} \quad 1 = B_{m,n}\big(\gamma(s)\big) = B_{m,n}\big(\gamma(s)^+\big) + B_{m,n}\big(\gamma(s)^-\big).$$

Therefore

$$B_{m,n}\big(\gamma(s)^+\big) = B_{m,n}\big(\gamma(s)^-\big) = \frac{1}{2}.$$

As a result, for every k, there exists $t_k \in [0,1]$ such that

$$B_{m,n}\big(\gamma(t_k)^+\big) = B_{m,n}\big(\gamma(t_k)^-\big) = \frac{1}{2}. \tag{3.31}$$

Setting $u_k := \gamma_k(t_k)$, we deduce from (3.31) that $2^{\frac{1}{p}} u_k^+ \in \mathcal{M}_{m,n}$ and $2^{\frac{1}{p}} u_k^- \in \mathcal{M}_{m,n}$ so that

$$E_V\big(u_k^\pm\big) \geq \frac{1}{2}\lambda_1(V,m).$$

We now state that

$$\lim_{k\to+\infty} E_V\big(u_k^\pm\big) = \frac{1}{2}\lambda_1(V,m). \tag{3.32}$$

Indeed,

$$\begin{aligned}
\frac{1}{2}\lambda_1(V,m) &\leq E_V\big(u_k^\pm\big) = E_V(u_k) - E_V\big(u_k^\mp\big) \\
&\leq \max_{t\in[0,1]} E_V\big(\gamma_k(t)\big) - \frac{1}{2}\lambda_1(V,m) \longrightarrow \frac{1}{2}\lambda_1(V,m).
\end{aligned}$$

Assume by contradiction that u_k as a sequence is unbounded in $W^{1,p}(\varOmega)$ then we set $v_k = \frac{u_k}{\|u_k\|}$ and up to a subsequence, v_k converges weakly to some v_0 in $W^{1,p}(\varOmega)$, strongly in $L^p(\varOmega)$ and in $L^p(\partial\varOmega)$. Clearly, $v_0 \not\equiv 0$ and we have

$$\begin{aligned}
\int_{\partial\varOmega} m(x)|u_k|^p d\sigma &= \int_{\partial\varOmega} m(x)\big(\big|u_k^+\big| + \big|u_k^-\big|\big)^p d\sigma \\
&= \int_{\partial\varOmega} m(x)\big|u_k^+\big|^p d\sigma + \int_{\partial\varOmega} m(x)\big|u_k^-\big|^p d\sigma \\
&= \int_{\partial\varOmega} m(x)\big|\gamma_k(t_k)^+\big|^p d\sigma + \int_{\partial\varOmega} m(x)\big|\gamma_k(t_k)^-\big|^p d\sigma \\
&= \frac{1}{2} + \frac{1}{2}
\end{aligned}$$

i.e.

$$\int_{\partial\Omega} m(x)|u_k|^p d\sigma = 1,$$

it follows that

$$\int_{\partial\Omega} m(x)|v_k|^p d\sigma = \frac{\int_{\partial\Omega} m(x)|u_k|^p d\sigma}{\|u_k\|^p} = \frac{1}{\|u_k\|^p}$$

and going to the limit, one obtains

$$\int_{\partial\Omega} m(x)|v_0|^p d\sigma = 0.$$

Then

$$0 = \beta(V,m) \le E_V\left(\frac{v_0}{\|v_0\|_p}\right) \le \frac{1}{\|v_0\|_p^p} \lim_{k\to+\infty} E_V(v_k) \le \lim_{k\to+\infty} \frac{\lambda_1(V,m)}{\|u_k\|^p \|v_0\|_p^p} = 0$$

which means $E_V(v_0) = 0$. Recalling Theorem 1.3, we say that v_0 is an eigenfunction associated to $\lambda_1(V,m)$ and as a result, either $|x \in \Omega : u_k^+(x) > 0| \longrightarrow_{k\to+\infty} 0$ or $|x \in \Omega : u_k^-(x) > 0| \longrightarrow_{k\to+\infty} 0$. Without loss of generality, let assume that $|x \in \Omega : u_k^-(x) > 0| \longrightarrow_{k\to+\infty} 0$. Combining (3.31), (3.32) and Lemma 3.5 we conclude that u_k^- is bounded in $W^{1,p}(\Omega)$ and up to a subsequence, we may assume that u_k^- converges weakly to some u_0^- in $W^{1,p}(\Omega)$ satisfying

$$E_V\left(u_0^-\right) = \frac{1}{2}\lambda_1(V,m) \quad \text{and} \quad \int_{\partial\Omega} m(x)\left|u_0^-\right|^p d\sigma = \frac{1}{2}.$$

Thus $2^{\frac{1}{p}} u_k^- \in \mathcal{M}_{m,n}$ realizes $\lambda_1(V,m)$ and therefore $u_0^- > 0$ which contradicts the fact that $|x \in \Omega : u_k^-(x) > 0| \longrightarrow_{k\to+\infty} 0$. To sum up, the sequence u_k is bounded in $W^{1,p}(\Omega)$ and arguing as above for u_k we reach a contradiction. □

Lemma 3.20. *There exist $u_1 \ge 0$ and $u_2 \le 0$ in $\mathcal{M}_{m,n}$ such that*

$$E_V(u_1) < c(m,n,V) \quad \textit{and} \quad E_V(u_2) < c(m,n,V).$$

Proof. An easy adaptation of Corollary 19 in [8] based upon Proposition 3.19. □

The main result about the existence of a first nonprincipal eigenvalue of $(P_{V,m,n})$ in case where we need (PSC) condition to be applied is as follows.

Theorem 3.21. *Consider $u_1 \ge 0$ and $u_2 \le 0$ in $\mathcal{M}_{m,n}$ such that*

$$E_V(u_1) < c(m,n,V) \quad \textit{and} \quad E_V(u_2) < c(m,n,V).$$

Define

$$\overline{\Gamma}_0 := \left\{\gamma \in C\left([0,1], \mathcal{M}_{m,n}\right) : \gamma(0) = u_1 \textit{ and } \gamma(1) = u_2\right\}.$$

Then

$$\bar{c}(m,n,V) := \inf_{\gamma\in\bar{\Gamma}_0} \max_{t\in[0,1]} E_V\big(\gamma(t)\big) \tag{3.33}$$

is a nonprincipal eigenvalue for $(P_{V,m,n})$*. Moreover*

$$\bar{c}(m,n,V) = c(m,n,V).$$

Corollary 3.22. $c(m,n,V)$ *is the first nonprincipal eigenvalue of* $(P_{V,m,n})$ *in sense that there is no other eigenvalue of* $(P_{V,m,n})$ *between* $\max\{\lambda_1(V,m),\lambda_1(V,n)\}$ *and* $c(m,n,V)$*.*

The rest of this section is devoted to the proof of Theorem 3.21.

Proposition 3.23. *Assume that* $\lambda_1^D(V) > 0$ *and* $\beta(V,m) = 0$ *or* $\beta(V,n) = 0$*. The functional* E_V *satisfies the* $(PSC)_c$ *condition on* $\mathcal{M}_{m,n}$ *for every* $c > \max\{\lambda_1(V,m),\lambda_2(V,n)\}$*.*

Proof. Let $c > \max\{\lambda_1(V,m),\lambda_1(V,n)\}$ and $u_k \in \mathcal{M}_{m,n}$ be a (PSC) for $\tilde{E}_V$. Then there exists a sequence ε_k with $\varepsilon_k \to 0$ such that

$$E_V(u_k) \longrightarrow c \quad \text{as } k \to +\infty$$

and

$$\big|\big\langle E_V'(u_k),\xi\big\rangle\big| \le \frac{\varepsilon_k}{1+\|u_k\|}\|\xi\|, \quad \forall\xi \in T_{u_k}\mathcal{M}_{m,n}, \tag{3.34}$$

where

$$\|\xi\| = \int_{\Omega} |\nabla\xi|^p dx + \int_{\partial\Omega} |\xi|^p dx$$

is an equivalent usual norm of $W^{1,p}(\Omega)$. By arguing as in the proof of Proposition 3.10, we assume by contradiction that u_k is unbounded and setting $v_k = \frac{u_k}{\|u_k\|}$. Up to a subsequence, there is some v_0 such that $v_k \rightharpoonup v_0$ in $W^{1,p}(\Omega)$ and v_k converges strongly towards v_0 in $L^{\frac{pq}{q-1}}(\Omega)$ and in $L^{\frac{pq}{q-1}}(\partial\Omega)$. One defines

$$w := (v_k - v_0) - \Bigg(\int_{\partial\Omega} \big(m(x)\big(u_k^+\big)^{p-1} - n(x)\big(u_k^-\big)^{p-1}\big)(v_k - v_0)d\sigma\Bigg)u_k \in T_{u_k}\mathcal{M}_{m,n}.$$

We choose $\xi = w$ in (3.34) and divide it by $\|u_k\|^{p-1}$ to obtain

$$\begin{aligned}
&\Bigg|\big\langle E_V'(v_k), v_k - v_0\big\rangle - \Bigg(\int_{\partial\Omega} \big(m(x)\big(v_k^+\big)^{p-1} - n(x)\big(v_k^-\big)^{p-1}\big)(v_k - v_0)d\sigma\Bigg)E_V(u_k)\Bigg| \\
&\quad\le \frac{\varepsilon_k\|u_k\|}{1+\|u_k\|}\Bigg\|\frac{(v_k - v_0)}{\|u_k\|^p} - \Bigg(\int_{\partial\Omega} \big(m(x)\big(v_k^+\big)^{p-1} - n(x)\big(v_k^-\big)^{p-1}\big)(v_k - v_0)d\sigma\Bigg)v_k\Bigg\|
\end{aligned} \tag{3.35}$$

The right-hand side of (3.35) $\to 0$ as $k \to +\infty$ and as $\int_{\partial\Omega}(m(x)(v_k^+)^{p-1} - n(x)(v_k^-)^{p-1})(v_k - v_0)d\sigma \longrightarrow 0$ while $k \to +\infty$, it follows that $\langle E_V'(v_k), v_k - v_0\rangle \longrightarrow 0$ when $k \to +\infty$. By using the S^+ property of $-\Delta_p$ depicted in Lemma 3.11, Hölder inequality and the same technic as in the proof of Proposition 3.10, we conclude that $v_k \to v_0$ in $W^{1,p}(\Omega)$. Moreover

$$\|v_k\| = 1, \qquad E_V(v_k) = \frac{E_V(u_k)}{\|u_k\|^p} \quad \text{and} \quad B_{m,n}(v_k) = \frac{B_{m,n}(u_k)}{\|u_k\|^p}.$$

Thus

$$v_0 \not\equiv 0, \qquad E_V(v_0) = 0 \quad \text{and} \quad B_{m,n}(v_0) = 0. \tag{3.36}$$

On the other hand, for every $w \in W^{1,p}(\Omega)$,

$$a_k(w) := w - \Big(\int_{\partial\Omega} \big(m(x)\big(u_k^+\big)^{p-1} - n(x)\big(u_k^-\big)^{p-1}\big) w \, d\sigma\Big) u_k \in T_{u_k}\mathcal{M}_{m,n}$$

and taking $\xi = a_k(w)$ in (3.34), it follows

$$\begin{aligned} &\Big| \big\langle E'_V(u_k), w \big\rangle - \Big(\int_{\partial\Omega} \big(m(x)\big(u_k^+\big)^{p-1} - n(x)\big(u_k^-\big)^{p-1}\big) w \, d\sigma\Big) E_V(u_k) \Big| \\ &\quad \le \frac{\varepsilon_k}{1 + \|u_k\|} \Big\| w - \Big(\int_{\partial\Omega} \big(m(x)\big(u_k^+\big)^{p-1} - n(x)\big(u_k^-\big)^{p-1}\big) w \, d\sigma\Big) u_k \Big\|. \end{aligned} \tag{3.37}$$

We then divide (3.37) by $\|u_k\|^{p-1}$ and we get

$$\begin{aligned} &\Big| \big\langle E'_V(v_k), w \big\rangle - \Big(\int_{\partial\Omega} \big(m(x)\big(v_k^+\big)^{p-1} - n(x)\big(v_k^-\big)^{p-1}\big) w \, d\sigma\Big) E_V(u_k) \Big| \\ &\quad \le \frac{\varepsilon_k \|u_k\|}{1 + \|u_k\|} \Big\| \frac{w}{\|u_k\|^p} - \Big(\int_{\partial\Omega} \big(m(x)\big(v_k^+\big)^{p-1} - n(x)\big(v_k^-\big)^{p-1}\big) w \, d\sigma\Big) v_k \Big\| \end{aligned} \tag{3.38}$$

and passing to limit, relation (3.38) becomes

$$\int_{\Omega} \big(|\nabla v_0|^{p-2} \nabla v_0 \nabla w + V(x)|v_0|^{p-2} v_0 w\big)\, dx = c \int_{\partial\Omega} \big(m(x)\big(v_0^+\big)^{p-1} - n(x)\big(v_0^-\big)^{p-1}\big) w \, d\sigma,$$

which means v_0 is a weak solution of problem

$$\begin{cases} \Delta_p v_0 = V|v_0|^{p-2} v_0 & \text{in } \Omega \\ |\nabla v_0|^{p-2} \dfrac{\partial v_0}{\partial \nu} = \lambda\big[m\big(v_0^+\big)^{p-1} - n\big(v_0^-\big)^{p-1}\big] & \text{on } \partial\Omega \end{cases}$$

We then distinguish two cases:

1. **Case v_0 is sign-constant**
   In this case, $\int_{\partial\Omega} m(x)(v_0^+)^p d\sigma = 0$ or $\int_{\partial\Omega} n(x)(v_0^-)^p d\sigma = 0$, and v_0 is therefore an eigenfunction associated to c, which contradicts [Theorem 1.3, 2.(b)].
2. **Case v_0 changes sign**
   Here, as $B_{m,n}(v_0) = \int_{\partial\Omega} m(x)(v_0^+)^p d\sigma + \int_{\partial\Omega} n(x)(v_0^-)^p d\sigma = 0$ then $\int_{\partial\Omega} m(x)(v_0^+)^p d\sigma \le 0$ or $\int_{\partial\Omega} n(x)(v_0^-)^p d\sigma \le 0$. Assume for instance

$$\int_{\partial\Omega} m(x)\big(v_0^+\big)^p d\sigma \le 0$$

and consider the two following subcases:

(a) If $\int_{\partial\Omega} m(x)(v_0^+)^p d\sigma < 0$ then $-\lambda_{-1}(V,m) \le \frac{E_V(v_0^+)}{-\int_{\partial\Omega} m(x)(v_0^+)^p d\sigma} = -c$, which is impossible since $\lambda_{-1}(V,m) < \lambda_1(V,m) < c$.

(b) If $\int_{\partial\Omega} m(x)(v_0^+)^p d\sigma = 0$ then $\int_{\partial\Omega} n(x)(v_0^-)^p d\sigma = 0$. Thus, $v_0^+ = d\varphi_0$ and $v_0^- = d'\varphi_0$ for $(d,d') \neq (0,0)$. As a result, v_0 is sign-constant and according to the first case, we reach a contradiction.

Thanks to the two previous subcases, we conclude that $\int_{\partial\Omega} m(x)|v_0^+|^p d\sigma > 0$ and in a similar way, we also prove $\int_{\partial\Omega} n(x)|v_0^-|^p d\sigma > 0$, which goes against the fact that $B_{m,n}(v_0) = 0$ in (3.36).

All in all, we claim that u_k is bounded and by repeating the same argument used for v_k, we reach the conclusion that, up to a subsequence, u_k converges. □

With these data in hand, we are ready now to prove Theorem 3.21.

Proof of Theorem 3.21. After applying the results above to Theorem 3.3, there is no doubt anymore that $\bar{c}(m,n,V)$ is a nonprincipal eigenvalue for $(P_{V,m,n})$. It is now left with the fact that $\bar{c}(m,n,V) = c(m,n,V)$. Summarizing one has $c(m,n,V) \le \bar{c}(m,n,V)$ since $\bar{\Gamma}_0 \subset \Gamma_0$. To show the reverse inequality $\bar{c}(m,n,V) \le c(m,n,V)$, let $\varepsilon > 0$ be sufficiently small and $\gamma \in \bar{\Gamma}_0$ such that $\max_{u\in\gamma([0,1])} E_V(u) \le c(m,n,V) + \varepsilon$. We set $u_0 = \gamma(0)$ and claim that either there exists a path $\tilde{\gamma}$ in $\mathcal{M}_{m,n}$ from u_0 to u_1 such that E_V remains at levels $< c(m,n,V) + \varepsilon$ or u_1 and u_2 can be connected by a path which remains at levels $< c(m,n,V) + \varepsilon$ as well. In both cases, the conclusion would be $\bar{c}(m,n,V) \le c(m,n,V) + \varepsilon$ and the proof will be ended.

From [8, Lemma 23, $\tilde{V} \equiv 1$], the functions $\beta_m(t) := \beta(V+t,m)$ and $\beta_n(t) := \beta(V+t,n)$ are differentiable with $\beta_m'(0) = \int_{\partial\Omega} |\varphi_{0,m}|^p d\sigma > 0$ and $\beta_n'(0) = \int_{\partial\Omega} |\varphi_{0,n}|^p d\sigma > 0$ where $\varphi_{0,m}$ and $-\varphi_{0,n}$ are normalized eigenfunctions associated to $\lambda_1(V,m)$ and $\lambda_1(V,n)$ respectively which satisfy (1.14). As a result, $\beta(V+t,m) > 0$ and $\beta(V+t,n) > 0$ for every $t > 0$ small enough so that $\max\{E_{V+t}(u_0), E_{V+t}(u_2)\} < c(m,n,V) + \varepsilon$. We now consider the open set $\mathcal{O}' := \{u \in \mathcal{M}_{m,n} : E_{V+t}(u) < c(m,n,V) + \varepsilon\}$ and observe that if $t > 0$ is small enough, then $c(m,n,V) < c(m,n,V+t)$. This comes from the fact that if γ realizes $c(m,n,V+t)$ then $c(m,n,V) \le \max_{u\in\gamma([0,1])} E_V(u) < \max_{u\in\gamma([0,1])} E_{V+t}(u) = c(m,n,V+t)$ (taking a look at the last part of the proof of Theorem 3.9). Hence $\mathcal{O}'$ has at most two arcwise connected components (see [6, Lemma 14]) because $\varphi_m(V+t,m)$ and $-\varphi_n(V+t,n)$ are the only critical points of $\tilde{E}_{V+t}$ in $\mathcal{O}'$. If u_1 and u_2 belong to the same arcwise connected component then $\bar{c}(m,n,V) \le c(m,n,V) + \varepsilon$. On the contrary either u_0 and u_1 or u_0 and u_2 can be connected. □

4. Asymptotic behaviors and monotonicity properties

Here, we present some useful properties which will be used in the next section to construct the first curve of Fučik spectrum with weights. These properties are well-known in the literature and can be easily adapted in the current case of interest. As one can note, $\beta(V,.)$, $\lambda_1(V,.)$ and $c(m,n,V)$ are strongly depending on one another and we shall therefore show how each value varies with respect to its weights and its link with others.

Proposition 4.1. *Let $m,n \in L^q(\partial\Omega)$ and V be in $L^\infty(\Omega)$ such that $\frac{N-1}{p-1} < q < \infty$ if $p < N$ and $q \ge 1$ if $p \ge N$. Assume that $m_k \rightharpoonup m$, $n_k \rightharpoonup n$ and $V_k \rightharpoonup V$ as $k \to +\infty$. Then the following assertions hold:*

1. *$m^- \not\equiv 0$ (resp. $n^- \not\equiv 0$) implies that $\beta(V_k,m_k) \to \beta(V,m)$ as $k \to +\infty$ (resp. $\beta(V_k,n_k) \to \beta(V,n)$ as $k \to +\infty$).*
2. *$m^- \equiv 0$ (resp. $n^- \equiv 0$) implies that $\beta(V_k,m_k) \to \beta(V,m)$ and $\liminf \beta(V_k,m_k) \ge \beta(V,m)$ (resp. $\beta(V_k,n_k) \to \beta(V,n)$ as $k \to +\infty$ and $\liminf \beta(V_k,n_k) \ge \beta(V,n)$).*

Proof. Since the proofs for the boundary weights are similar, we are only proving the assertions with the weight m and the same arguments can be written to deduce the results when n is considered. Let us take m and V as in the hypotheses of Proposition 4.1 and assume that $m_k \rightharpoonup m$ and $V_k \rightharpoonup V$ when $k \to +\infty$.

1. Let us suppose $m^- \not\equiv 0$ and show that $\beta(V_k, m_k) \to \beta(V, m)$.
   We get
   $$\int\limits_{\partial\Omega_+} m_k(x)d\sigma \longrightarrow \int\limits_{\partial\Omega_+} m(x)d\sigma > 0 \quad \text{and} \quad \int\limits_{\partial\Omega_-} m_k(x)d\sigma \longrightarrow \int\limits_{\partial\Omega_-} m(x)d\sigma < 0 \quad \text{as } k \to +\infty.$$
   Thus for k sufficiently large
   $$\partial\Omega_{k,+} := \big\{x \in \partial\Omega \;:\; m_k(x) > 0\big\} \neq \phi \quad \text{and} \quad \partial\Omega_{k,-} := \big\{x \in \partial\Omega \;:\; m_k(x) < 0\big\} \neq \phi$$
   and using Theorem 1.3, there exists on the one hand, $\lambda \in \mathbb{R}$ and $u \in W^{1,p}(\Omega)$ such that:
   $$\int\limits_{\partial\Omega} |u|^p d\sigma = 1, \qquad \int\limits_{\partial\Omega} m(x)|u|^p d\sigma = 0 \quad \text{and} \quad \beta(V, m) = \mu_1(\lambda) = E_V(u)$$
   and on the other hand, for each k, there exists $\lambda_k \in \mathbb{R}$ and $u_k \in W^{1,p}(\Omega)$ such that:
   $$\int\limits_{\partial\Omega} |u_k|^p d\sigma = 1, \qquad \int\limits_{\partial\Omega} m(x)|u_k|^p d\sigma = 0 \quad \text{and} \quad \beta(V_k, m_k) = \mu_1(\lambda_k) = E_{V_k}(u_k).$$
   From Lemma 3.5 one gets that $\beta(V_k, m_k) = \mu_1(\lambda_k)$ is bounded from below. It is left showing that $(\lambda_k)_k$ is a bounded sequence. Let then assume by contradiction that up to a subsequence, $\lambda_k \longrightarrow +\infty$ when $k \longrightarrow +\infty$. Let $w \in W^{1,p}(\Omega)$ such that:
   $$\int\limits_{\partial\Omega} |w|^p d\sigma = 1 \quad \text{and} \quad \int\limits_{\partial\Omega} m(x)|w|^p d\sigma > 0,$$
   then
   $$\beta(V_k, m_k) \leq E_V(w) - \lambda_k \int\limits_{\partial\Omega} m_k(x)|w|^p d\sigma \longrightarrow -\infty, \tag{4.1}$$
   which is impossible since $\beta(V_k, m_k)$ is bounded from below. A similar argument holds when considering $\lambda_k \longrightarrow -\infty$ and one can conclude that the sequence $(\lambda_k)_k$ is bounded. From (4.1) we deduce that $\beta(V_k, m_k)$ is bounded from above. As a result, $(u_k)_k$ is bounded in $W^{1,p}(\Omega)$ and converges not only to $u \in W^{1,p}(\Omega)$, up to a subsequence but also in $L^{pq'}(\Omega)$. This results in,
   $$E_V(u) \leq \liminf E_{V_k}(u_k)$$
   and then,
   $$\beta(V_k, m_k) = \mu_1(\lambda_k) \leq \beta(V, m) + \int\limits_{\Omega} (V_k - V)(x)|u|^p dx - \lambda_k \int\limits_{\partial\Omega} m_k(x)|u|^p d\sigma$$

which leads to:

$$\limsup \beta(V_k, m_k) \leq \beta(V, m).$$

In conclusion, $\beta(V_k, m_k) \longrightarrow \beta(V, m)$ as $k \to +\infty$.

2. Let us assume that $m^- \equiv 0$ and prove that $\beta(V_k, m) \to \beta(V, m)$ and $\liminf \beta(V_k, m_k) \geq \beta(V, m)$. Let $w \in W^{1,p}(\Omega)$ such that:

$$\int\limits_{\partial\Omega} |w|^p d\sigma = 1 \quad \text{and} \quad \int\limits_{\partial\Omega} m(x)|w|^p d\sigma = 0.$$

Then

$$\begin{aligned} \beta(V_k, m) \leq E_{V_k}(w) &= \int\limits_{\Omega} \big(|\nabla w|^p + V_k(x)|w|^p\big) dx \\ &\leq \big(1 + \|V_k\|_\infty\big)\|w\|^p \leq (1 + M)\|w\|^p \end{aligned} \tag{4.2}$$

where $\|V_k\|_\infty \leq M$ for all k. Let $\xi_k, \xi \in W^{1,p}(\Omega)$ realize $\beta(V_k, m)$ and $\beta(V, m)$ respectively. From Lemma 3.5,

$$\|\xi_k\|^p \leq C_1 E_V(\xi_k) + C_2 = C_1\beta(V_k, m) + C_2, \tag{4.3}$$

and then, the sequence $(\xi_k)_k$ is bounded. Using (1.10), one gets:

$$\beta(V_k, m) \leq \int\limits_{\Omega} |\nabla \xi|^p dx + \int\limits_{\Omega} V_k(x)|\xi|^p dx$$

and

$$\beta(V, m) \leq \int\limits_{\Omega} |\nabla \xi_k|^p dx + \int\limits_{\Omega} V(x)|\xi_k|^p dx$$

i.e.

$$\beta(V_k, m) - \int\limits_{\Omega} V_k(x)|\xi|^p dx \leq \beta(V, m) - \int\limits_{\Omega} V(x)|\xi|^p dx \tag{4.4}$$

and

$$\beta(V, m) - \int\limits_{\Omega} V(x)|\xi_k|^p dx \leq \beta(V_k, m) - \int\limits_{\Omega} V(x)|\xi_k|^p dx. \tag{4.5}$$

From both (4.4) and (4.5), we deduce that:

$$\int\limits_{\Omega} (V - V_k)(x)|\xi|^p dx \leq \beta(V, m) - \beta(V_k, m) \leq \int\limits_{\Omega} (V - V_k)(x)|\xi_k|^p dx$$

and since ξ_k is bounded in $L^{pq'}(\Omega)$, we conclude $\beta(V_k, m) \longrightarrow \beta(V, m)$.

We are now going to show that

$$\liminf \beta(V_k, m_k) \geq \beta(V, m).$$

As we know from Lemma 3.5 that $\beta(V_k, m_k)$ is bounded from below, up to a subsequence, one can find $u \in W^{1,p}(\Omega)$ such that $u_k \rightharpoonup u$ and $u_k \longrightarrow u$ in $L^{pq'}(\Omega)$, where $\beta(V_k, m_k)$ is reached at u_k. We also know that:

$$\int\limits_{\partial\Omega} m(x)|u|^p d\sigma = \lim_{k\to+\infty} \int\limits_{\partial\Omega} m_k(x)|u_k|^p = 0$$

and

$$\int\limits_{\partial\Omega} |u|^p d\sigma = 1,$$

which imply $\beta(V, m) \leq \liminf \beta(V_k, m_k)$ and the conclusion follows. □

Proposition 4.2. *Let $m, n \in L^q(\partial\Omega)$ and $V \in L^\infty(\Omega)$. Assume that (3.1) is verified with $m_k \rightharpoonup m$, $n_k \rightharpoonup n$ and $V_k \rightharpoonup V$ as $k \to +\infty$. If $\beta(V, m) = 0$ (resp. $\beta(V, n) = 0$), we then assume in addition that $\beta(V_k, m_k) \geq 0$ for all $k \in \mathbb{N}$ and $m^- \not\equiv 0$ (resp. $\beta(V_k, n_k) \geq 0$ for all $k \in \mathbb{N}$ and $n^- \not\equiv 0$). Hence, the following relations hold:*

1. *$\lambda_1(V_k, m_k) \longrightarrow \lambda_1(V, m)$ as $k \to +\infty$ (resp. $\lambda_1(V_k, n_k) \longrightarrow \lambda_1(V, n)$ as $k \to +\infty$).*
2. *$c(m_k, n_k, V_k) \longrightarrow c(m, n, V)$ as $k \to +\infty$.*

Proof. The ideals follow those in [8, Proposition 26] but we present them here because of the slight difference that occurs.

1. We will show that $\lambda_1(V_k, m_k) \longrightarrow \lambda_1(V, m)$ as $k \to +\infty$ and by arguing in the same way, one can also conclude that $\lambda_1(V_k, n_k) \longrightarrow \lambda_1(V, n)$ as $k \to +\infty$. We then distinguish two cases:
   (a) Case $\beta(V, m) > 0$.
   In this case, from Proposition 4.1, one writes:

$$\beta(V_k, m_k) > 0 \quad \text{for } k \text{ sufficiently large.}$$

Let φ_k and φ_m be the eigenfunctions associated to $\lambda_1(V_k, m_k)$ and $\lambda_1(V, m)$ respectively and let $w \in \mathcal{M}_{m,m}$. Since

$$\int\limits_{\partial\Omega} m_k(x)|w|^p d\sigma \longrightarrow 1 \quad \text{as } k \to +\infty,$$

one gets:

$$\left|\lambda_1(V_k, m_k)\right| \leq \frac{|E_{V_k}(w)|}{\int_{\partial\Omega} m_k(x)|w|^p d\sigma} \leq \frac{\|w\|^p + \|V_k\|_\infty \|w\|_p^p}{\int_{\partial\Omega} m_k(x)|w|^p d\sigma}.$$

Thus, $\lambda_1(V_k, m_k)$ is bounded and from Lemma 3.5, we conclude φ_k is bounded. In addition,

$$\lambda_1(V_k, m_k) \leq E_{V_k}\left(\frac{\varphi_m}{(\int_{\partial\Omega} m_k(x)|\varphi_m|^p d\sigma)^{1/p}}\right)$$

which implies:

$$\lambda_1(V_k,m_k)\int\limits_{\partial\Omega} m_k(x)|\varphi_m|^p d\sigma \leq \int\limits_{\Omega} |\nabla\varphi_m|^p dx + \int\limits_{\Omega} V_k(x)|\varphi_m|^p dx$$

i.e.

$$\lambda_1(V_k,m_k)\int\limits_{\partial\Omega} m_k(x)|\varphi_m|^p d\sigma \leq \lambda_1(V,m) - \int\limits_{\Omega} V(x)|\varphi_m|^p dx + \int\limits_{\Omega} V_k(x)|\varphi_m|^p dx$$

or

$$\lambda_1(V,m) - \lambda_1(V_k,m_k)\int\limits_{\partial\Omega} m_k(x)|\varphi_m|^p d\sigma \geq \int\limits_{\Omega} (V-V_k)(x)|\varphi_m|^p dx. \tag{4.6}$$

Moreover

$$\lambda_1(V,m) \leq E_V\left(\frac{\varphi_k}{(\int_{\partial\Omega} m(x)|\varphi_k|^p)^{1/p}}\right)$$

leads to

$$\lambda_1(V,m)\int\limits_{\partial\Omega} m(x)|\varphi_k|^p d\sigma \leq \int\limits_{\Omega} |\nabla\varphi_k|^p dx + \int\limits_{\Omega} V(x)|\varphi_k|^p dx$$

i.e.

$$\lambda_1(V,m)\int\limits_{\partial\Omega} m(x)|\varphi_k|^p d\sigma \leq \lambda_1(V_k,m_k) - \int\limits_{\Omega} V_k(x)|\varphi_k|^p dx + \int\limits_{\Omega} V(x)|\varphi_k|^p dx$$

or

$$\int\limits_{\Omega} (V-V_k)(x)|\varphi_k|^p dx \geq \lambda_1(V,m)\int\limits_{\partial\Omega} m(x)|\varphi_k|^p d\sigma - \lambda_1(V_k,m_k). \tag{4.7}$$

Both (4.6) and (4.7) imply:

$$\begin{aligned}\int\limits_{\Omega} (V-V_k)(x)|\varphi_m|^p dx &\leq \lambda_1(V,m) - \lambda_1(V_k,m_k)\int\limits_{\partial\Omega} m_k(x)|\varphi_m|^p d\sigma \\ &\leq \frac{\int_{\Omega}(V-V_k)(x)|\varphi_k|^p dx + \lambda_1(V_k,m_k)}{\int_{\partial\Omega} m(x)|\varphi_k|^p d\sigma} \\ &\quad - \lambda_1(V_k,m_k)\int\limits_{\partial\Omega} m_k(x)|\varphi_m|^p d\sigma. \end{aligned}\tag{4.8}$$

To sum up,

$$\lambda_1(V_k,m_k) - \lambda_1(V,m) \longrightarrow 0 \quad \text{as } k \to +\infty$$

i.e.

$$\lambda_1(V_k,m_k) \longrightarrow \lambda_1(V,m) \quad \text{as } k \to +\infty.$$

(b) Case $\beta(V, m) = 0$.
Here,

$$\beta(V_k, m_k) \geq 0 \quad \text{for all } k \in \mathbb{N}.$$

Then,

$$\beta(V_k, m_k) > 0 \quad \text{for all } k \in \mathbb{N} \quad \text{(up to a subsequence)}$$

or

$$\beta(V_k, m_k) = 0 \quad \text{for all } k \in \mathbb{N}$$

In the former case $\beta(V_k, m_k) > 0$ for all $k \in \mathbb{N}$, we consider, for all $\varepsilon > 0$, a function

$$u_\varepsilon := \frac{\varphi_{\lambda_0} + \frac{\psi}{\varepsilon}}{(\int_{\partial\Omega} m(x)|\varphi_{\lambda_0} + \frac{\psi}{\varepsilon}|^p d\sigma)^{\frac{1}{p}}} \tag{4.9}$$

(taking from the proof of Theorem 1.3 in [10]) for some $0 < \psi \in C^\infty(\Omega)$ and such that:

$$E_V(u_\varepsilon) \leq \lambda_1(V, m) + \varepsilon.$$

One has:

$$\begin{aligned} \int\limits_{\Omega} (V - V_k)(x)|u_\varepsilon|^p dx - \varepsilon \leq \lambda_1(V, m) - \lambda_1(V_k, m_k) \int\limits_{\partial\Omega} m_k(x)|u_\varepsilon|^p d\sigma \\ \leq \frac{\int_\Omega (V - V_k)(x)|\varphi_k|^p dx + \lambda_1(V_k, m_k)}{\int_{\partial\Omega} m(x)|\varphi_k|^p d\sigma} \\ - \lambda_1(V_k, m_k) \int\limits_{\partial\Omega} m_k(x)|u_\varepsilon|^p d\sigma. \end{aligned} \tag{4.10}$$

The result follows as in (4.8) when $k \to +\infty$ ($\varepsilon > 0$ is arbitrary).
Let us now consider the case $\beta(V_k, m_k) = 0$ for all $k \in \mathbb{N}$ and assume that $\beta(V_k, m_k)$ is reached at ξ_k and $\beta(V, m)$ at ξ. As $m^- \not\equiv 0$, one gets:

$$\beta(V_k, m_k) \longrightarrow \beta(V, m)$$

and it reads:

$$\xi_k \longrightarrow \xi \quad \text{when } k \to +\infty.$$

Let $v \in W^{1,p}(\Omega)$ such that:

$$\int\limits_{\partial\Omega} m(x)\xi^{p-1} v d\sigma > 0 \quad \text{(which is impossible)}.$$

Consequently, for k sufficiently large,

$$\int\limits_{\partial\Omega} m_k(x)\xi^{p-1} v d\sigma > 0.$$

Thus,

$$\lambda_1(V_k, m_k) = \frac{\langle E'_{V_k}(\xi_k), v\rangle}{\int_{\partial\Omega} m_k(x)\xi^{p-1} v d\sigma} \longrightarrow \frac{\langle E'_V(\xi), v\rangle}{\int_{\partial\Omega} m(x)\xi^{p-1} v d\sigma} = \lambda_1(V, m).$$

2. Let us now show that:

$$c(m_k, n_k, V_k) \longrightarrow c(m, n, V) \quad \text{when } k \to +\infty.$$

For this purpose, we will need first to prove the upper semicontinuity and at last the lower semicontinuity. Let $\varepsilon > 0$ and $\gamma \in \Gamma$ such that:

$$\max_{t\in[0,1]} E_V\big(\gamma(t)\big) < c(m, n, V) + \varepsilon.$$

Using continuity, we reach for k sufficiently large,

$$\max_{t\in[0,1]} \frac{E_{V_k}(\gamma(t))}{\int_{\partial\Omega}(m_k(x)(\gamma(t)^+)^p + n_k(x)(\gamma(t)^-)^p)d\sigma} < c(m, n, V) + \varepsilon. \tag{4.11}$$

From Proposition 3.29, it comes

$$c(m_k, n_k, V_k) \le \max_{t\in[0,1]} \frac{E_{V_k}(\gamma(t))}{\int_{\partial\Omega}(m_k(x)(\gamma(t)^+)^p + n_k(x)(\gamma(t)^-)^p)d\sigma} \quad \forall k \in \mathbb{N},$$

and consequently,

$$\limsup_{k\to+\infty} c(m_k, n_k, V_k) \le c(m, n, V) + \varepsilon.$$

We conclude the upper semicontinuity since ε is arbitrary. Returning to the lower semicontinuity, we assume by contradiction that

$$c(m_k, n_k, V_k) \longrightarrow c_0 \quad \text{when } k \to +\infty \text{ with } c_0 < c(m, n, V).$$

We take into account the following two cases:

(a) Case $\beta(V, m) > 0$ and $\beta(V, n) > 0$.

Let $\phi_k \in \mathcal{M}_{m,n}$ be an eigenfunction associated to $c(m_k, n_k, V_k)$. From Lemma 3.5, we then deduce that the sequence $(\phi_k)_k$ is bounded and up to a subsequence, $\phi_k \rightharpoonup \phi_0$ in $W^{1,p}(\Omega)$ for some $\phi_0 \in W^{1,p}(\Omega)$ and $\phi_k \to \phi_0$ in $L^{pq'}(\Omega)$. Setting $v = \phi_k - \phi_0$ in (3.2) and passing to the limit, one gets:

$$\lim_{k\to+\infty} \int_\Omega |\nabla\phi_k|^{p-2}\nabla\phi_k\nabla(\phi_k - \phi_0)dx = 0$$

and using both (S^+) property Lemma 3.11 and Hölder inequality, one easily follows ideas of proof of Proposition 3.10 and shows that $\phi_k \longrightarrow \phi_0$ in $W^{1,p}(\Omega)$. Finally, ϕ_0 solves $(P_{V,m,n})$ for $\lambda = c_0 < c(m, n, V)$ and we get,

$$c_0 = \lambda_1(V, m) \quad \text{and} \quad \phi_0 = \varphi_m$$

or

$$c_0 = \lambda_1(V, n) \quad \text{and} \quad \phi_0 = -\varphi_n.$$

Let suppose without loss of generality that $c_0 = \lambda_1(V, m)$ and $\phi_0 = \varphi_m$. Since $\phi_k \longrightarrow \phi_0$ in $L^{pq'}(\Omega)$, then

$$mes\{x \in \Omega \ : \ \phi_k^-(x) > 0\} \longrightarrow 0 \quad \text{as } k \to +\infty.$$

This implies through Lemma 3.7 that

$$\frac{\int_{\partial\Omega} n_k(x)(\phi_k^-)^p d\sigma}{E_{V_k}(\phi_k^-)} \longrightarrow 0.$$

Besides,

$$\begin{aligned}\int\limits_{\Omega} |\nabla\phi_k|^{p-2}\nabla\phi_k\nabla v\,dx + \int\limits_{\Omega} V_k(x)|\phi_k|^{p-2}\phi_k v\,dx \\ = c(m_k, n_k, V_k)\int\limits_{\partial\Omega}\left[m_k(x)\left(\phi_k^+\right)^{p-1} - n_k(x)\left(\phi_k^-\right)^{p-1}\right]v\,d\sigma\end{aligned} \tag{4.12}$$

for all $v \in W^{1,p}(\Omega)$. Taking $v = \phi_k^-$ in (4.12) leads to:

$$\int\limits_{\Omega}\left|\nabla\phi_k^-\right|^p dx + \int\limits_{\Omega} V_k(x)\left|\phi_k^-\right|^p dx = c(m_k, n_k, V_k)\int\limits_{\partial\Omega} n_k(x)\left(\phi_k^-\right)^p d\sigma$$

i.e.

$$\frac{1}{c(m_k, n_k, V_k)} = \frac{\int_{\partial\Omega} n_k(x)(\phi_k^-)^p d\sigma}{E_{V_k}(\phi_k^-)} \longrightarrow 0 \quad \text{as } k \to +\infty,$$

a contradiction since $1/c_0 \neq 0$.

(b) Case $\beta(V, m) = 0$ or $\beta(V, n) = 0$.
Let us assume for instance that $\beta(V, m) = 0$. Then $(\psi_k)_k \in W^{1,p}(\Omega)$ defined as

$$\psi_k := \frac{\phi_k}{\|\phi_k\|}$$

is bounded in $W^{1,p}(\Omega)$ and up to a subsequence, $(\psi_k)_k$ converges to $\psi_0 \in W^{1,p}(\Omega)$. In addition,

$$E_V(\psi_k) = E_{V_k}(\psi_k) + \int\limits_{\Omega}(V - V_k)(x)|\psi_k|^p dx$$

hence

$$E_V(\psi_k) \longrightarrow 0 \quad \text{as } k \to +\infty.$$

Up to a constant, $\beta(V, m)$ is reached at ψ_0, i.e. ψ_0 solves $(P_{V,m,n})$. Consequently,

$$mes\{x \in \Omega \ : \ \phi_k(x) < 0\} \longrightarrow 0 \quad \text{as } k \to +\infty$$

and we then reach a contradiction as in the previous case and the right conclusion follows.

The two cases above guarantee the lower semicontinuity and as a final conclusion,

$$c(m_k, n_k, V_k) \longrightarrow c(m, n, V) \quad \text{as } k \to +\infty. \qquad \square$$

Proposition 4.3. *If $m \leq \hat{m}$ and $n \leq \hat{n}$ then $c(m,n,V) \geq c(\hat{m},\hat{n},V)$.*

Proof. Let γ be an admissible path for the definition of $c(m,n,V)$ given by (3.22). Then

$$1 = \int_{\partial\Omega} \big(m(x)\big(\gamma(t)^+\big)^p + \hat{n}(x)\big(\gamma(t)^-\big)^p\big)d\sigma \leq \int_{\partial\Omega} \big(\hat{m}(x)\big(\gamma(t)^+\big)^p + \hat{n}(x)\big(\gamma(t)^-\big)^p\big)d\sigma$$

i.e.

$$B_{\hat{m},\hat{n}}\big(\gamma(t)\big) = \int_{\partial\Omega} \big(\hat{m}(x)\big(\gamma(t)^+\big)^p + \hat{n}(x)\big(\gamma(t)^-\big)^p\big)d\sigma \geq 1$$

and consequently,

$$\hat{\gamma}(t) := \frac{\gamma(t)}{(B_{\hat{m},\hat{n}}(\gamma(t)))^{1/p}}, \quad t \in [0,1]$$

is well-defined and admissible for $c(\hat{m},\hat{n},V)$ given by (3.29). In addition,

$$E_V\big(\hat{\gamma}(t)\big) = \frac{E_V(\gamma(t))}{B_{\hat{m},\hat{n}}(\gamma(t))} \leq E_V\big(\gamma(t)\big) \quad \forall t \in [0,1]$$

i.e. $c(\hat{m},\hat{n},V) \leq c(m,n,V)$. □

Proposition 4.4. *If $m \leq \hat{m}$, $n \leq \hat{n}$ in $\partial\Omega$ and*

$$\int_{\partial\Omega} (\hat{m}-m)\big(u^+\big)^p d\sigma + \int_{\partial\Omega} (\hat{n}-n)\big(u^-\big)^p d\sigma > 0 \tag{4.13}$$

for at least one eigenfunction u associated to $c(m,n,V)$, then $c(m,n,V) > c(\hat{m},\hat{n},V)$.

Proof. The proof can be easily adapted from the one given in [6, Proposition 25]. □

Some sort of separate sub-homogeneity that holds in [2,5,6] also holds with our issue $(P_{V,m,n})$ and it reads:

Proposition 4.5. *If $0 < s < \hat{s}$ then*

$$c(sm,n,V) > c(\hat{s}m,n,V) \quad \text{and} \quad c(m,sn,V) > c(m,\hat{s}n,V). \tag{4.14}$$

Proof. We will prove the first inequality of (4.14) and by similar arguments one can also prove the second inequality of (4.14). Let u be an eigenfunction in $\mathcal{M}_{sm,n}$ associated to $c(sm,n,V)$ and γ the path joining φ_{sm} and $-\varphi_n$ in $\mathcal{M}_{sm,n}$ constructed from u as in the last part of the proof of Theorem 3.9. The path

$$\hat{\gamma}(t) := \left(\frac{s}{\hat{s}}\right)^{1/p} \gamma(t)^+ - \gamma(t)^-, \quad t \in [0,1]$$

is then admissible for $c(\hat{s}m,n,V)$ given by (3.22) and we have:

$$\begin{aligned} E_V\big(\hat{\gamma}(t)\big) &= \frac{s}{\hat{s}} \int_{\Omega} \big(\big|\nabla\gamma(t)^+\big|^p + V(x)\big|\gamma(t)^+\big|^p\big)dx + \int_{\Omega} \big(\big|\nabla\gamma(t)^-\big|^p + V(x)\big|\gamma(t)^-\big|^p\big)dx \\ &\leq E_V\big(\gamma(t)\big) \end{aligned}$$

with strict inequality if $\gamma(t)^+ \not\equiv 0$. Hence the path $\hat{\gamma}$ goes from $\varphi_{\hat{s}m}$ to $-\varphi_n$ in $\mathcal{M}_{\hat{s}m,n}$ and remains at levels $< c(sm,n,V)$ except $v := -u/(B_{sm,n}(-u))^{1/p}$ where E_V is at the same level as $c(sm,n,V)$. Consequently, $c(\hat{s}m,n,V) \le c(sm,n,V)$ and let us assume by contradiction that $c(\hat{s}m,n,V) = c(sm,n,V)$. Applying [6, Lemma 26] to the path $\hat{\gamma}$ in $\mathcal{M}_{\hat{s}m,n}$, one can conclude that v must be a critical point of $\tilde{E}_V$ on $\mathcal{M}_{\hat{s}m,n}$ with $c(\hat{s}m,n,V)$ as an associated eigenvalue which cannot be held since v is one-signed function. This puts an end to the proof. □

Remark 4.6. From (3.22), one easily checks that $c(m,n,V)$ is homogeneous of degree -1 with respect to the weights m and n i.e.

$$c(sm,sn,V) = s^{-1}c(m,n,V), \quad \forall s > 0. \tag{4.15}$$

5. Fučik spectrum with weights

In this section, we carry on the following problem

$$(P_{V,A,B}) : \begin{cases} \Delta_p u = V|u|^{p-2}u & \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = Am(u^+)^{p-1} - Bn(u^-)^{p-1} & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

and our purpose is to look for $(A,B) \in \mathbb{R}^2$ such that $(P_{V,A,B})$ has a nontrivial solution. This set which is denoted by $\sum_{(m,n)}$ is the so-called Fučik spectrum associated to $(P_{V,A,B})$. This problem is related to $(P_{V,m,n})$ since it reduces to it when $A = B$.

One easily checks that $\sum_{(m,n)}$ contains the lines $\{\lambda_1(V,m)\} \times \mathbb{R}$, $\mathbb{R} \times \{\lambda_1(V,n)\}$ and also possibly the lines $\{-\lambda_1(V,-m)\} \times \mathbb{R}$ and $\mathbb{R} \times \{-\lambda_1(V,-n)\}$. Let us start looking for the part of $\sum_{(m,n)}$ that lies in $]\lambda_1(V,m),+\infty[\times]\lambda_1(V,n),+\infty[$. This leads us to denote by $\sum^*_{(m,n)}$ the set $\sum_{(m,n)}$ without the lines enumerated above and $(A,B) \in \sum_{(m,n)} \cap (]\lambda_1(V,m),+\infty[\times]\lambda_1(V,n),+\infty[)$ implies that $A > \lambda_1(V,m)$ and $B > \lambda_1(V,n)$. One might notice $\sum^*_{(m,n)}$ does not necessary lie in $\mathbb{R}^+ \times \mathbb{R}^+$ due to the fact that $\lambda_1(V,m)$ and $\lambda_1(V,n)$ are not necessary positive. We thus have the following results.

Theorem 5.1. *Assume that $m^+ \not\equiv 0$, $n^+ \not\equiv 0$, $\lambda_1^D(V) > 0$, $\beta(V,m) > 0$ and $\beta(V,n) > 0$. For $s > 0$, the line $B = sA$ intersects $\sum_{(m,n)} \cap (]\lambda_1(V,m),+\infty[\times]\lambda_1(V,n),+\infty[)$. Moreover the first point in this intersection is given by $A(s) = c(m,sn,V)$, $B(s) = sA(s)$ where $c(m,n,V)$ is defined as in (3.22).*

Proof. From Theorem 3.9, $(A,B) \in \sum_{(m,n)} \cap (]\lambda_1(V,m),+\infty[\times]\lambda_1(V,n),+\infty[)$ is such that no element of $\sum^*_{(m,n)}$ belongs to the segment $[(\lambda_1(V,m),\lambda_1(V,n)),(A,B)[$ if and only if $A = A(s) = c(m,sn,V)$. □

We get a first curve $\mathcal{C} := \{(\alpha(s),\beta(s)) = (c(m,sn,V), sc(m,sn,V)) \ : \ s > 0\}$ when we let s vary in $]0,+\infty[$. Some properties of this curve are as follows:

Proposition 5.2.

1. *The functions $A(s)$ and $B(s)$ in Theorem 5.1 are continuous,*
2. *$\alpha(s)$ is strictly decreasing,*
3. *$\beta(s)$ is strictly increasing.*

Proof.

1. The continuity of $A(s)$ and $B(s)$ is a consequence of the continuity of $c(.,.,V)$ (see Proposition 4.2).
2. Let $0 < s < s'$ then from Proposition 4.5, one writes

$$c(m,sn,V) > c(m,s'n,V)$$

that is

$$A(s) > A(s').$$

3. Recall that $B(s) = sc(m, sn, V)$ and from (4.15), $B(s) = c(\frac{m}{s}, n, V)$. Let $0 < s < s'$ then $\frac{1}{s'} < \frac{1}{s}$ and applying again Proposition 4.5 we get

$$c\left(\frac{m}{s'}, n, V\right) > c\left(\frac{m}{s}, n, V\right)$$

that is

$$B(s') > B(s). \qquad \square$$

Following the notations of [11], we set

$$A_0 = \lim_{s\to 0} A(s), \qquad B_0 = \lim_{s\to\infty} B(s), \qquad A_\infty = \lim_{s\to\infty} A(s), \qquad B_\infty = \lim_{s\to 0} B(s),$$
$$\bar{A} = \inf\{E_V(u^+) : B_{m,n}(u^+) = 1, B_{m,n}(u^-) > 0\},$$

and

$$\bar{B} = \inf\{E_V(u^-) : B_{m,n}(u^-) = 1, B_{m,n}(u^+) > 0\}.$$

The following result depicts the asymptotic behavior of $\mathcal{C}$.

Proposition 5.3. *Assume that* $m^+ \not\equiv 0$, $n^+ \not\equiv 0$, $\lambda_1^D(V) > 0$, $\beta(V, m) > 0$ *and* $\beta(V, n) > 0$.

1. $A_0 = B_0 = \infty$,
2. $A_\infty = \bar{A}$ *and* $B_\infty = \bar{B}$.

Proof. Confer [11]. $\square$

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