Pedagogical comments about nonperturbative Ward-constrained melonic renormalization group flow

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This paper, in addition to our recent works, intends to improve and give in detail the behavior of the Wetterich flow equations in the physical theory space. We focus on the local potential approximation and present a new framework of investigation, namely the effective vertex expansion coupled with Ward's identities for quartic melonic interactions, allowing us to consider infinite sectors rather than finite dimensional subspaces of the full theory space. The flow behavior in the vicinity of the Gaussian fixed point and the correction given by the effective vertex expansion is also analyzed. The dynamical constrained flow allows us to identify the influence of the number of melonic interactions related to the existence of divergence at which the possible first order phase transition may be identified is studied. Comparing our results using the formalism of Callan Symanzik equation is also scrutinized.

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I. INTRODUCTION

Recently, several investigations have been done concerning the study of the functional renormalization group (FRG) applied to tensor models (TM) and group field theory (GFT). The study of the FRG applied to tensorial group field theory (TGFT) is motivated by its close relations with the fluctuation problem of quantum gravity phenomenon [1–17]. TGFTs are celebrated for their nonlocal behavior in the interactions and the difficulties related to combinatorics. Thus new computation tools are needed to address the question of the renormalization group for TGFT [1–12]. First insights have been gained by nonperturbative Wetterich equation and, in particular by an investigation of the leading order melonic interactions, with a new method called effective vertex expansion (EVE) [1–5]. EVE described the FRG without truncation as approximation and will certainly become a promising way in investigating nonperturbative field theory. A lot of possible phase transitions which are identified near the fixed point are shown to be nonphysical due to the violation of the Ward identities (WI) [3]. The WI is an additional constraint on the flow and therefore should not be overlooked in the study of renormalization group. Note that the study of phase transitions is deeply related to the classification of all possible universality classes of the exact coincidence of the critical exponents. This universality is broken by the Ward constraint driving by the EVE like by the truncation method. The nontrivial form of the Ward identity for the TGFT with nontrivial propagator in the functional actions is not a consequence of the regulator r_k but rather is due to the violation of the kinetic term under $\mathrm{U}(N)$ symmetry. Let us remark that for standard gauge invariant theories like QED see [18] the regulator breaks generally the explicit invariance of the kinetic term and leads to a new non trivial Ward identity that depends on the regulator r_k. This is not the case for TGFT models for which the kinetic term intrinsically violates the $\mathrm{U}(N)$ symmetry. Therefore the appearance of the regulator generalizes the definition of the theory but does not add any new information concerning the Ward identity. In the symmetric phase we showed that, apart from the fact that no physical fixed point may be observed, the possible existence of first order phase transition in the reduced subspace of theory space can be given (see [2]).

In the present paper, as a complement to our recent works concerning the FRG applied to TGFT, and by taking into account the combinatorial factors allows to count all the corresponding Feynman graph, allows to give in depth explanation the behavior of the Wetterich flow equation by using a new alternative way, considering together the two dynamics of the average effective action. The first dynamic is given by the Wetterich flow equation [19,20] and the second by the Ward identity [21,22]. The description of the flow over the constrained theory space denoted by $\mathcal{E}_C$ i.e., the reduced theory space controlled by the constraint is given. Taking into account the influence of the number of melonic interactions in the functional action and the local potential approximation, we provide the relation between this number and the existence of divergence that occurs in

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the flow and which is related to the starting point of first-order phase transitions [23]. The behavior of the flow around the Gaussian fixed point is also studied. Finally we compared our result by using other formalism of the FRG called the Callan Symanzik equation.

The paper is organized as follows. In Sec. II we provide in details useful ingredients for the description of the FRG to TGFT. In the Sec. III the EVE is derived to thank about the FRG with a new alternative way without truncation. The corresponding flow equations which improves the truncation method are given. Section IV describes our new proposal to merge the Wetterich equation and the Ward identity in the constrained melonic phase space $\mathcal{E}_{\mathcal{C}}$ of theory space. In Sec. V we investigate the local potential approximation. Section VI is devoted to other point of views in the computation of the scale dynamics called Callan Symanzik equation. In the last section, Sec. VII, we give our conclusion.

II. PRELIMINARIES

A *group field* φ is a field, complex or real, defined over d-copies of a group manifold G rather than on space time:

$$\varphi\colon\ \mathrm{G}^d \to \mathbb{R},\mathbb{C}. \tag{1}$$

Standard choices to make contact with physics are SU(2) and SO(4) [13,14]. In this paper, we focus only on the nonlocal aspects of the interactions, and consider the Abelian version of the theory, setting $\mathrm{G}=\mathrm{U}(1)$. For this choice, the field may be equivalently described on the Fourier dual group $\mathbb{Z}^d$ by a *tensor field* $T\colon\ \mathbb{Z}^d \to \mathbb{C}$. We consider a theory for two complexes fields φ and $\bar{\varphi}$, requiring two complex tensors fields T and $\bar{T}$. The allowed configurations are then constrained by the choice of a specific action, completing the definition of the GFT. At the classical level, for free fields we choose the familiar form:

$$S_{\mathrm{kin}}[T,\bar{T}] \coloneqq \sum_{\vec{p}\in\mathbb{Z}^d} \bar{T}_{p_1\cdots p_d}(\vec{p}^{\,2}+m^2)T_{p_1\cdots p_d}, \tag{2}$$

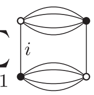
with the standard notation $\vec{p}^{\,2} \coloneqq \sum_i p_i^2$, $\vec{p} \coloneqq (p_1,\ldots,p_d)$. For the rest of this paper we use the short notation $T_{\vec{p}} \equiv T_{p_1\cdots p_d}$. The equation (2) defines the bare propagator $C^{-1}(\vec{p}) \coloneqq \vec{p}^{\,2}+m^2$. Among the natural transformations that we can consider for a pair of complex tensor fields, the unitary transformations play an important role. They provide the principle that allows to build the interactions, which are chosen to be invariant under such a transformation. Denoting by N the *size* of the tensor field, restricting the domain of the indices p_i into the window $[\![-N,N]\!]$, we require invariance with respect to independent transformations along each of the d indices of the tensors:

$$T'_{p_1\cdots p_d} = \sum_{\vec{q}\in\mathbb{Z}^d}\left[\prod_{i=1}^d U^{(i)}_{p_iq_i}\right] T_{q_1\cdots q_d}, \tag{3}$$

with $U^{(i)}(U^{(i)})^\dagger = \mathrm{id}$. Define $\mathbb{U}(N)$ as the set of unitary symmetries of size N, a transformation for tensors is then a set of d independent elements of $\mathbb{U}(N)$, $\mathcal{U} \coloneqq (U_1,\ldots,U_d) \in \mathbb{U}(N)^d$, one per index of the tensor fields. The unitary symmetries admitting an inductive limit for arbitrary large N, we will implicitly consider the limit $N\to\infty$ in the rest of this paper [24]. We call *bubble* all the invariant interactions which cannot be factorized into two or more smaller bubbles. Observe that because the transformations are independent, the bubbles are not local in the usual sense over the group manifold G^d. However, locality does not make sense without physical content. In standard field theory for instance, or in physics in general, the locality is defined by the way following which the fields or particles interact together, and as for tensors, this choice reflects invariance with respect to some transformations like translations and rotations. With this respect, the transformation rule (3) define both the nature of the field (a tensor) and the corresponding locality principle. To summarize:

Definition 1. Any interaction bubble is said to be local. By extension, any functions expanding as a sum of bubble will be said local.

This locality principle called *traciality* in the literature has some good properties of the usual ones. In particular it allows to define local counterterms and to follow the standard renormalization procedure for interacting quantum fields with UV divergences. In this paper, we focus on the quartic melonic model in rank $d=5$, describing by the classical interaction:

$$S_{\mathrm{int}}[T,\bar{T}] = g\sum_{i=1}^d$$  $, \tag{4}$

g denoting the coupling constant and where we adopted the standard graphical convention [24] to picture the interaction bubble as d-colored bipartite regular connected graphs. The black (resp. white) nodes corresponding to T (resp. $\bar{T}$) fields, and the colored edges fixing the contractions of their indices. Note that, because we contract indices of the same color between T and $\bar{T}$ fields, the unitary symmetry is ensured by construction. The model that we consider has been showed to be *just renormalizable* is the usual sense, that is to say, all the UV divergences can be subtracted with a finite set of counterterms, for mass, coupling and field strength. From now on, we will consider m^2 and g as the bare couplings, sharing their counterterms, and we introduce explicitly the wave function renormalization Z replacing the propagator C^{-1} by

$$C^{-1}(\vec{p}) = Z\vec{p}^2 + m^2. \tag{5}$$

The Eqs. (2) and (4) define the classical model, without fluctuations. We quantize using path integral formulation, and define the partition function integrating over all configurations, weighted by e^{-S}:

$$\mathcal{Z}(J,\bar{J}) \coloneqq \int dTd\bar{T}e^{-S[T,\bar{T}]+\langle \bar{J},T\rangle+\langle \bar{T},J\rangle}, \tag{6}$$

the sources being tensor fields themselves $J,\bar{J}\colon \mathbb{Z}^d \to \mathbb{C}$ and $\langle \bar{J},T\rangle \coloneqq \sum_{\vec{p}} \bar{J}_{\vec{p}} T_{\vec{p}}$. Note that the quantization procedure provide a canonical definition of what is UV and what is IR. The UV theory corresponding to the classical action $S = S_{\text{kin}} + S_{\text{int}}$ whereas the IR theory corresponds to the standard effective action defined as the Legendre transform of the free energy $\mathcal{W} \coloneqq \ln(\mathcal{Z}(J,\bar{J}))$.

Renormalization in standard field theory allows to subtract divergences, and it has been showed that quantum GFT can be renormalized in the usual sense. With respect to the quantization procedure moreover, the renormalization group allows to describe quantum effects *scale by scale*, through more and more effective models, defining a path from UV to IR by integrating out fluctuation of increasing size.

Recognizing this path from UV to IR as an element of the quantization procedure itself, we substitute to the global quantum description (6) a set of models $\{\mathcal{Z}_k\}$ indexed by a referent scale k. This scale define what is UV, and integrated out and what is IR, and frozen out from the long distance physics. The set of scales may be discrete or continuous, and in this paper we choose a continuous description $k \in [0,\Lambda]$ for some fundamental UV cutoff Λ. There are several ways to build what we call functional renormalization group. We focus on the Wetterich-Morris approach [19,20], $\mathcal{Z}_k(J,\bar{J})$ being defined as:

$$\mathcal{Z}_k(J,\bar{J}) \coloneqq \int dTd\bar{T}e^{-S_k[T,\bar{T}]+\langle \bar{J},T\rangle+\langle \bar{T},J\rangle}, \tag{7}$$

with: $S_k[T,\bar{T}] \coloneqq S[T,\bar{T}] + \sum_{\vec{p}} \bar{T}_{\vec{p}} r_k(\vec{p}^2) T_{\vec{p}}$. The momentum dependent mass term $r_k(\vec{p}^2)$ called *regulator* vanish for UV fluctuations $\vec{p}^2 \gg k^2$ and becomes very large for the IR ones $\vec{p}^2 \ll k^2$. Some additional properties for $r_k(\vec{p}^2)$ may be found in standard references [25,26]. Without explicit mentions, we focus on the Litim's modified regulator:

$$r_k(\vec{p}^2) \coloneqq Z(k)(k^2 - \vec{p}^2)\theta(k^2 - \vec{p}^2), \tag{8}$$

where θ designates the Heaviside step function and $Z(k)$ is the running wave function strength. The renormalization group flow equation, describing the trajectory of the RG flow into the full theory space is the so called Wetterich equation [19,20], which for our model writes as:

$$\frac{\partial}{\partial k}\Gamma_k = \sum_{\vec{p}} \frac{\partial r_k}{\partial k}(\vec{p})(\Gamma_k^{(2)} + r_k)^{-1}_{\vec{p}\,\vec{p}}, \tag{9}$$

where $(\Gamma_k^{(2)})_{\vec{p}\vec{p}'}$ is the second derivative of the *average effective action* Γ_k with respect to the classical fields M and $\bar{M}$:

$$(\Gamma_k^{(2)})_{\vec{p}\vec{p}'} = \frac{\partial^2 \Gamma_k}{\partial M_{\vec{p}} \partial \bar{M}_{\vec{p}'}}, \tag{10}$$

where $M_{\vec{p}} = \partial \mathcal{W}_k / \partial \bar{J}_{\vec{p}}$, $\bar{M}_{\vec{p}} = \partial \mathcal{W}_k / \partial J_{\vec{p}}$ and:

$$\Gamma_k[M,\bar{M}] + \sum_{\vec{p}} \bar{M}_{\vec{p}} r_k(\vec{p}^2) M_{\vec{p}} \coloneqq \langle \bar{M},J\rangle + \langle \bar{J},M\rangle - \mathcal{W}_k(M,\bar{M}), \tag{11}$$

with $\mathcal{W}_k = \ln(\mathcal{Z}_k)$.

The flow equation (9) is a consequence of the variation of the propagator, indeed

$$\frac{\partial r_k}{\partial k} = \frac{\partial C_k^{-1}}{\partial k}, \tag{12}$$

for the *effective covariance* $C_k^{-1} \coloneqq C^{-1} + r_k$. But the propagator has other source of variability. In particular, it is not invariant with respect to the unitary symmetry of the classical interactions (4). Focusing on an infinitesimal transformation: $\delta_1 \coloneqq (\text{id} + \epsilon, \text{id}, \ldots, \text{id})$ acting nontrivially only on the color 1 for some infinitesimal anti-Hermitian transformations ϵ, the transformation rule for the propagator follows the Lie bracket:

$$\mathcal{L}_{\delta_1} C_k^{-1} = [C_k^{-1}, \epsilon]. \tag{13}$$

The source terms are non invariant as well. However, due to the translation invariance of the Lebesgue measure $dTd\bar{T}$ involved in the path integral (7), we must have $\mathcal{L}_{\delta_1}\mathcal{Z}_k = 0$. Translating this invariance at the first order in ϵ provide a nontrivial *Ward-Takahashi identity* for the quantum model:

Theorem 1. ***(Ward identity*.)** The noninvariance of the kinetic action with respect to unitary symmetry induce nontrivial relations between $\Gamma^{(n)}$ and $\Gamma^{(n+2)}$ for all n, summarized as:

$${\sum_{\vec{p}_\perp,\vec{p}'_\perp}}' \left\{ [C_k^{-1}(\vec{p}) - C_k^{-1}(\vec{p}')] \left[\frac{\partial^2 \mathcal{W}_k}{\partial \bar{J}_{\vec{p}'} \partial J_{\vec{p}}} + \bar{M}_{\vec{p}} M_{\vec{p}'} \right] - \bar{J}_{\vec{p}} M_{\vec{p}'} + J_{\vec{p}'} \bar{M}_{\vec{p}} \right\} = 0. \tag{14}$$

where $\sum'_{\vec{p}_\perp,\vec{p}'_\perp} \coloneqq \sum_{\vec{p}_\perp,\vec{p}'_\perp} \delta_{\vec{p}\vec{p}'_\perp}$.

In this statement we introduced the notations $\vec{p}_\perp \coloneqq (p_2,\ldots,p_d) \in \mathbb{Z}^{d-1}$ and $\delta_{\vec{p}\vec{p}'_\perp} = \prod_{j\neq 1} \delta_{p_j p'_j}$. Equations (9) and (14) are two formal consequences of the path integral (7), coming both from the nontrivial variations of the propagator. Therefore, there are no reason to treat these two equations separately. This formal proximity is highlighted in their expanded forms, comparing Eqs. (23)–(24) and (34)–(35). Instead of a set of partition function, the quantum model may be alternatively defined as an (infinite) set of effective vertices $\mathcal{Z}_k \sim \{\Gamma_k^{(n)}\} =: \mathfrak{h}_k$. RG equations dictate how to move from $\mathfrak{h}_k \to_{\mathrm{RG}} \mathfrak{h}_{k+\delta k}$ whereas Ward identities dictate how to move in the momentum space, along $\mathfrak{h}_k$.

III. EFFECTIVE VERTEX EXPANSION

This section essentially summarize the state of the art in [2–5]. The exact RG equation cannot be solved except for very special cases. The main difficulty is that the Wetterich equation (9) split as a infinite hierarchical system, the derivative of $\Gamma^{(n)}$ involving $\Gamma^{(n+2)}$, and so one. Appropriate approximation schemes are then required to extract an information on the exact solutions. The effective vertex expansion (EVE) is a recent technique allowing to build an approximation considering infinite sectors rather than crude truncations on the full theory space. We focus on the *melonic sector*, sharing all the divergences of the model and then dominating the flow in the UV. One recall that melonic diagrams are defined as the diagram with optimal degree of divergence. Fixing some fundamental cutoff Λ, we consider the domain $\Lambda \ll k \ll 1$, so far from the deep UV and the deep IR regime. At this time, the flow is dominated by the renormalized couplings, have positive or zero *flow dimension* (see [4]). We recall that the flow dimension reflect the behavior of the RG flow of the corresponding quantity, and discriminate between essential, marginal and inessential couplings just like standard dimension in quantum field theory.[1] Because our theory is just-renormalizable, one has necessarily $[m^2] = 2$ and $[g] = 0$, denoting as $[X]$ the flow dimension of X.

Note that we focus on the strictly local potential approximation, in which the EVE method work well. As recalled in the previous section, locality for TGFT means that we can be expanded as a sum (eventually infinite) of interaction bubbles. In order to make contact the effective vertex formalism of this section, we have to recall the notion of boundary graphs. Indeed, effective vertex expand generally as a sum of connected diagrams, but what is relevant for locality is boundary locality, and the boundary map $\mathcal{B}$ is defined as follow:

Definition 2. Let $\mathfrak{G}_p$ the set of bubbles with at most p black nodes. The boundary map $\mathcal{B}\colon \mathfrak{G}_p \to (\mathfrak{G}_p)^{\times p}$ is defined as follows. For any connected Feynman graph $\mathcal{G}$, $\mathcal{B}(\mathcal{G})$ is the subset of external nodes, linking together with colored edges, according to their connectivity path in the graph.

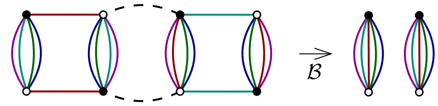


FIG. 1. Illustration of the manner that the boundary map work on a connected Feynman graph. The dotted edges correspond to Wick contractions.

We recall that external nodes are nodes hooked to external edges. Figure 1 provide an illustration of the definition. Any Feynman graph $\mathcal{G}$ contributing to a local effective vertex are then such that $\mathcal{B}(\mathcal{G})$ is a interaction bubble. The basic strategy of the EVE is to close the hierarchical system coming from (9) using the analytic properties of the effective vertex functions[2] and the rigid structure of the melonic diagrams. More precisely, the EVE express all the melonic effective vertices $\Gamma^{(n)}$ having negative flow dimension (that is for $n > 4$) in terms of effective vertices with positive or null flow dimension, that is $\Gamma^{(2)}$ and $\Gamma^{(4)}$, and their flow is entirely driven by just-renormalizable couplings. As recalled, in this way we keep the entirety of the melonic sector and the full momentum dependence of the effective vertices.

We work into the *symmetric phase*, i.e., in the interior of the domain where the vacuum $M = \bar{M} = 0$ make sense. This condition ensure that effective vertices with an odd number of external points, or not the same number of black and white external nodes have to be discarded from the analysis. These ones being called *assorted functions*. Moreover, due to the momentum conservation along the boundaries of faces, $\Gamma_k^{(2)}$ must be diagonal:

$$\Gamma^{(2)}_{k,\vec{p}\,\vec{q}} = \Gamma_k^{(2)}(\vec{p})\delta_{\vec{p}\,\vec{q}}. \tag{15}$$

We denote as G_k the effective 2-point function $G_k^{-1} := \Gamma_k^{(2)} + r_k$.

The main assumption of the EVE approach is the existence of a finite analyticity domain for the leading order effective vertex functions, in which they may be identified with the resumed perturbative series. For the melonic vertex function, the existence of a such analytic domain is ensured, melons can be mapped as trees and easily summed. Moreover, these resumed functions satisfy the Ward-Takahashi identities, written without additional assumption than cancellation of odd and assorted effective vertices. One then expect that the symmetric phase entirely cover the perturbative domain. Among the properties of the melonic diagrams, we recall the following statement:

[1]For ordinary quantum field theory, the dimension is fixed by the background itself. Without background, this is the behavior of the RG flow which fix the canonical dimension.

[2]Melonic diagrams may be easily counted as "trees," and the (renormalized) melonic perturbation series is easy to sum.

Proposition 1. Let $\mathcal{G}_N$ be a $2N$-point 1PI melonic diagrams build with more than one vertices for a purely quartic melonic model. We call external vertices the vertices hooked to at least one external edge of $\mathcal{G}_N$ has:

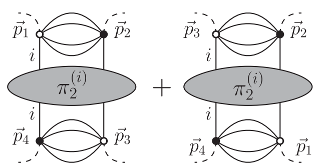
(i) Two external edges per external vertices, sharing $d-1$ external faces of length one.
(ii) N external faces of the same color running through the interior of the diagram.

Due to this proposition, the melonic effective vertex $\Gamma_k^{(n)}$ decompose as d functions $\Gamma_k^{(n),i}$, labeled with a color index i:

$$\Gamma^{(n)}_{k,\vec{p}_1,\ldots,\vec{p}_n} = \sum_{i=1}^{d} \Gamma^{(n),i}_{k,\vec{p}_1,\ldots,\vec{p}_n}. \tag{16}$$

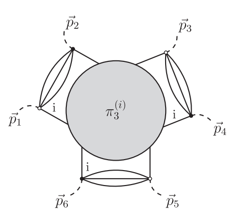
The Feynman diagrams contributing to the perturbative expansion of $\Gamma^{(n,i)}_{k,\vec{p}_1,\ldots,\vec{p}_n}$ fix the relations between the different indices. For $n=4$ for instance, we get, from proposition 1:

$$\Gamma^{(4),i}_{\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4} = \quad + \quad , \tag{17}$$



Where the half dotted edges correspond to the amputated external propagators, and the reduced vertex functions $\pi_2^{(i)}:\ \mathbb{Z}^2 \to \mathbb{R}$ denotes the sum of the interiors of the graphs, excluding the external nodes and the colored edges hooked to them. In the same way, one expect that the melonic effective vertex $\Gamma^{(6),i}_{\text{melo}\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4,\vec{p}_5,\vec{p}_6}$ is completely determined by a reduced effective vertex $\pi_3^{(i)}:\ \mathbb{Z}^3 \to \mathbb{R}$ hooked to a boundary configuration such as:

$$\Gamma^{(6),i}_{\vec{p}_1,\vec{p}_2,\vec{p}_3,\vec{p}_4,\vec{p}_5,\vec{p}_6} = \quad + \text{perm}, \tag{18}$$



and so one for $\Gamma^{(n),i}_{k,\vec{p}_1,\ldots,\vec{p}_n}$, involving the reduced vertex $\pi_n^{(i)}: \mathbb{Z}^n \to \mathbb{R}$. In the last expression, perm denote the permutation of the external edges like in (17). The reduced vertices $\pi_2^{(i)}$ can be formally resumed as a geometric series [2–4]:

$$\begin{aligned}\pi^{(1)}_{2,pp} &= 2(g - 2g^2\mathcal{A}_{2,p} + 4g^3(\mathcal{A}_{2,p})^2 - \cdots) \\ &= \frac{2g}{1+2g\mathcal{A}_{2,p}},\end{aligned} \tag{19}$$

where $\pi^{(1)}_{2,pp}$ is the diagonal element of the matrix $\pi_2^{(1)}$ and:

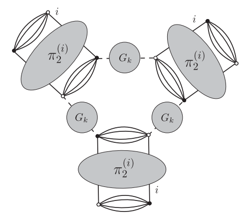


FIG. 2. Interne structure of the 1PI 6-points graphs.

$$\mathcal{A}_{n,p} \coloneqq \sum_{\vec{p}} G_k^n(\vec{p})\delta_{pp_1}. \tag{20}$$

The reduced vertex $\pi^{(1)}_{2,pp}$ depend implicitly on k, and the renormalization conditions defining the *renormalized coupling* g_r are such that:

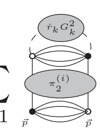
$$\pi^{(i)}_{2,00}|_{k=0} \eqqcolon 2g_r. \tag{21}$$

For arbitrary k, the zero momentum value of the reduced vertex define the effective coupling for the local quartic melonic interaction: $\pi^{(i)}_{2,00} \eqqcolon 2g(k)$. The explicit expression for $\pi_3^{(1)}$ may be investigated from the proposition 1. The constraint over the boundaries and the recursive definition of melonic diagram enforce the internal structure pictured on Fig. 2 below [1–5].[3] Explicitly:

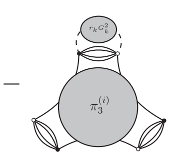
$$\pi^{(i)}_{3,ppp} = 2(\pi^{(i)}_{2,pp})^3\mathcal{A}_{3,p}, \tag{22}$$

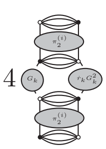
The two orientations of the external effective vertices being took into account in the definition of $\pi^{(i)}_{2,pp}$. Expanding the exact flow equation (9), and keeping only the relevant contraction for large k, one get the following relevant contributions for $\dot{\Gamma}_k^{(2)}$ and $\dot{\Gamma}_k^{(4)}$:

$$\dot{\Gamma}_k^{(2)}(\vec{p}) = -\sum_{i=1}^{d} \tag{23}$$



$$\dot{\Gamma}_k^{(4),i} = - \quad + 4 \tag{24}$$





where $\dot{X} \coloneqq k\partial X/\partial k$. The computation require the explicit expression of $\Gamma_k^{(2)}$. In the melonic sector, the self-energy obey to a closed equation, reputed difficult to solve. We approximate the exact solution by considering only the first term in the derivative expansion in the interior of the windows of momenta allowed by $\dot{r}_k$:

[3]Note that the result is differs of a factor two with respect to the results given in the Ref. [5].

$$\Gamma_k^{(2)}(\vec{p}) \coloneqq Z(k)\vec{p}^2 + m^2(k), \tag{25}$$

where $Z(k) \coloneqq \partial\Gamma_k^{(2)}/\partial p_1^2(\vec{0})$ and $m^2(k) \coloneqq \Gamma_k^{(2)}(\vec{0})$ are both renormalized and effective field strength and mass. From the definition (8), and with some calculation (see [4]), we obtain the following statement:

Proposition 2. In the UV domain $\Lambda \ll k \ll 1$ and in the symmetric phase, the leading order flow equations for essential and marginal local couplings are given by:

$$\begin{cases} \beta_m = -(2+\eta)\bar{m}^2 - 10\bar{g}\frac{\pi^2}{(1+\bar{m}^2)^2}(1+\frac{\eta}{6}), \\ \beta_g = -2\eta\bar{g} + 4\bar{g}^2\frac{\pi^2}{(1+\bar{m}^2)^3}\left(1+\frac{\eta}{6}\right)\left[1-\pi^2\bar{g}\left(\frac{1}{(1+\bar{m}^2)^2}+\left(1+\frac{1}{1+\bar{m}^2}\right)\right)\right]. \end{cases} \tag{26}$$

With:

$$\eta = 4\bar{g}\pi^2\frac{(1+\bar{m}^2)^2 - \frac{1}{2}\bar{g}\pi^2(2+\bar{m}^2)}{(1+\bar{m}^2)^2\Omega(\bar{g},\bar{m}^2) + \frac{(2+\bar{m}^2)}{3}\bar{g}^2\pi^4}, \tag{27}$$

and

$$\Omega(\bar{m}^2,\bar{g}) \coloneqq (\bar{m}^2+1)^2 - \pi^2\bar{g}. \tag{28}$$

Where in this proposition $\beta_g \coloneqq \dot{\bar{g}}$, $\beta_m \coloneqq \dot{\bar{m}}^2$ and the effective-renormalized mass and couplings are defined as: $\bar{g} \coloneqq Z^{-2}(k)g(k)$ and $\bar{m}^2 \coloneqq Z^{-1}(k)k^{-2}m^2(k)$. For the computation, note that we made use of the approximation (25) only for absolutely convergent quantities, and into the windows of momenta allowed by $\dot{r}_k$. As pointed out in [2–4], taking into account the full momentum dependence of the effective vertex $\pi_2^{(i)}$ in (19) drastically modify the expression of the anomalous dimension η with respect to crude truncation. In particular, the singularity line discussed in [3] disappears below the singularity $\bar{m}^2 = -1$. Moreover, because all the effective melonic vertices only depend on $\bar{m}^2$ and $\bar{g}$, any fixed point for the system (26) is a global fixed point for the melonic sector. Note that to compute η, we required the knowledge of the derivative of the effective vertex with respect to the external momenta:

$$\frac{d}{dp^2}\pi_{2,pp}^{(i)}|_{p=0}, \tag{29}$$

which can be computed from (22) and Ward identity (35), or directly from (19). The system (26) admits two fixed points for $p_1 \coloneqq (\bar{g}_*;\bar{m}_*^2) \approx (0.002;-0.56)$ and $p_2 \approx (1.11, 1.97)$. The first one has been investigated in [5].[4] It has one relevant and one irrelevant direction, with critical exponents

$$\theta_1^{(1)} \approx -4.23, \qquad \theta_2^{(1)} \approx 0.78, \tag{30}$$

and is interpreted as an IR fixed point of Wilson-Fisher type, reminiscent to a second order phase transition. The second fixed point has critical exponents:

$$\theta_1^{(2)} \approx -2.8 - 4.2i, \qquad \theta_2^{(2)} \approx -2.8 + 4.2i, \tag{31}$$

and corresponds to an infrared attractor, pictured on Fig. 3. Note that the arrows on this diagrams are oriented toward UV scales. The anomalous dimensions for the two fixed points are respectively $\eta_1 \approx 0.57$ and $\eta_2 \approx -4.05$, and it is easy to show that these values are insufficient to promote irrelevant interactions at the rank of marginal or relevant ones. Indeed, in full generality, the canonical dimension d_b (i.e., the flow dimension in the vicinity of the Gaussian fixed point) of any interaction bubble b in the quartic melonic just renormalizable sector is given by (see [4] for an extended discussion):

$$d_b \coloneqq 2 - \max_b \omega(b), \tag{32}$$

where b denote any two point function obtained from the connected interaction b, and $\omega(b)$ is the corresponding divergence degree. Explicitly, we get

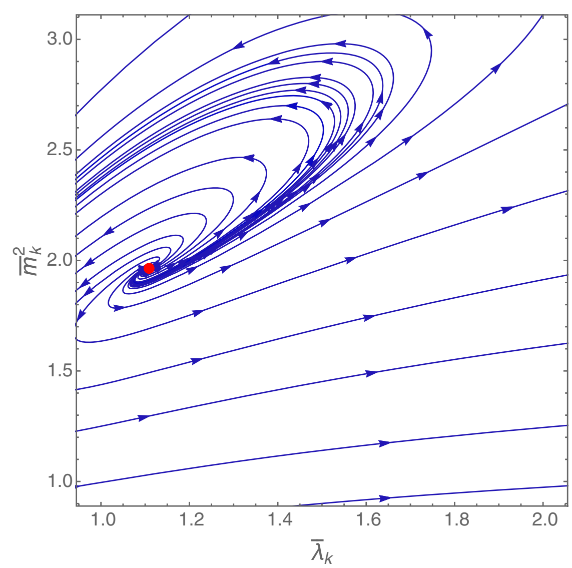


FIG. 3. The numerical RG flow in the vicinity of the non-Gaussian infrared attractor p_2 (in red).

[4]Note that the minor correction to the 6-point function pointed out in the footnote 3 does not modify the result significantly.

$$d_b = 4 - 2p, \tag{33}$$

where p denote the number of black nodes of the bubble b. Denoting as g_b the coupling corresponding to b, the dimensionless version is $\bar{g}_b \coloneqq k^{-d_b} Z^{-p} g_b$. At the vicinity of any non-Gaussian fixed point with anomalous dimension η^*, the power counting is then improved from the quantity $p\eta^*$, and becomes $d_b^* = 4 - (2-\eta^*)p$. For $\eta^* < 0$, as for the second fixed point, couplings with negative dimensions remains inessential. For $\eta^* = 0.57$, it is easy to check that $4 - (2-\eta^*)p$ becomes negative for $p > 2$. These conclusions are obviously in accordance with the values of the critical exponents.

IV. THE (LOCAL) MELONIC CONSTRAINED FLOW

To close the hierarchical system derived from (9) and obtain the autonomous set (26), we made use of the explicit expressions (19) and (22). In this derivation we mentioned the Ward identity but they do not contribute explicitly. In this section we take into account their contribution, and show that they introduce a strong constraint over the RG trajectories.

Deriving successively the Ward identity (14) with respect to external sources, and setting $J = \bar{J} = 0$ at the end of the computation, we get the two following relations involving $\Gamma_k^{(4)}$ and $\Gamma_k^{(6)}$ see [3][5]:

$$\pi_{2,00}^{(1)} \mathcal{L}_{2,k} = -\frac{\partial}{\partial p_1^2} (\Gamma_k^{(2)}(\vec{p}) - Z\vec{p}^2)|_{\vec{p}=\vec{0}}, \tag{34}$$

$$(\pi_{3,00}^{(1)} \mathcal{L}_{2,k} - 2(\pi_{2,00}^{(1)})^2 \mathcal{L}_{3,k}) = -\frac{d}{dp_1^2} \pi_{2,p_1p_1}^{(1)}|_{p_1=0}, \tag{35}$$

where:

$$\mathcal{L}_{n,k} \coloneqq \sum_{\vec{p}_\perp} \left(Z + \frac{\partial r_k}{\partial p_1^2}(\vec{p}_\perp) \right) G_k^n(\vec{p}_\perp). \tag{36}$$

Note that, in order to investigate the local potential approximation, we kept only the contributions with connected boundaries. We will return to this point at the end of Sec. V.

It can be easily checked that the structure equations (19) and (22) satisfy the second Ward identity (35) see [2–4] and also [27–30]. In the same way the first Ward identity (34) has been checked to be compatible with the equation (19) and the melonic closed equation for the 2-point function. However, the last condition does not exhaust the information contained in (19). Indeed, with the same level of approximation as for the computation of the flow equations (26), the first Ward identity can be translated locally as a constraint between beta functions see [2–4]:

$$\mathcal{C} \coloneqq \beta_g + \eta \bar{g} \frac{\Omega(\bar{g}, \bar{m}^2)}{(1+\bar{m}^2)^2} - \frac{2\pi^2 \bar{g}^2}{(1+\bar{m}^2)^3} \beta_m = 0. \tag{37}$$

This relation hold in the deep UV limit only, that, for large k. Inessential contributions have been discarded, which play an important role in the IR sector $k \approx 0$, where one expect that Ward identity reduces to its unregularized form, i.e., for $r_k = 0$. Moreover, note that in this limit, and depending on the choice of the regulator, it may happen that additional inessential operators have to be added to the original action to recover the true Ward identities, see Sec. VI.

Generally, the solutions of the system (26) do not satisfy the constraint $\mathcal{C} = 0$. We call *constrained melonic phase space* and denote as $\mathcal{E}_\mathcal{C}$ the subspace of the melonic theory space satisfying $\mathcal{C} = 0$. A attempt to describe this space has been expressed in [2]. In particular, we showed that there are no global fixed point of (26) which satisfy the constraint $\mathcal{C} = 0$.

In the description of the physical flow over $\mathcal{E}_\mathcal{C}$, we substituted the explicit expressions of β_g, β_m and η, translating the relations between velocities as a complicated constraint on the couplings $\bar{g}$ and $\bar{m}^2$ providing a systematic projection of the RG trajectories. Explicitly, solving this constraint, we get two equations for this constrained subspace $\mathcal{E}_\mathcal{C}$, defining respectively $\mathcal{E}_{\mathcal{C}0}$ and $\mathcal{E}_{\mathcal{C}1}$:

$$\bar{g} = 0, \quad \text{and} \quad \bar{g} = f(\bar{m}^2), \tag{38}$$

where:

$$f(\bar{m}^2) \coloneqq \frac{(1+\bar{m}^2)^2 (3\bar{m}^2(\bar{m}^2+3) - 10)}{\pi^2(\bar{m}^2(\bar{m}^2+7)+2)}. \tag{39}$$

Equation (40) is pictured on Fig. 4(a) below. The solution $\bar{g} = 0$ leads to the beta function $\beta_m = -2\bar{m}^2$, and the flow reach the purely Gaussian region $\bar{g} = \bar{m} = 0$. The second solution $\bar{g} = f(\bar{m}^2)$ leads to the beta function:

$$\beta_m = \frac{4\bar{m}^2(\bar{m}^2(3\bar{m}^2(\bar{m}^2(\bar{m}^2+6)-1)-128)-34)+400}{\bar{m}^2(3\bar{m}^2(\bar{m}^2(\bar{m}^2+6)+1)-56)+68}. \tag{40}$$

This function and the corresponding anomalous dimension have been pictured on Fig. 4(b) below. Interestingly, they exhibit the same singularities, for $\bar{m}_\infty^2 = -3.23$ and $\bar{m}_\infty^2 = -4.75$, which, like in [2], we interpret as a pathology of the expansion around vanishing vacuum. Moreover, the singularity occurring at $\bar{m}^2 = -1$ in the unconstrained flow (23) disappears, due to the factor $(1+\bar{m}^2)^2$ in the numerator of f. This fact has been pointed out in [2], and may be viewed as a indication that the physical RG flow may be continued for negative values smaller that -1, above the singular point

[5] Note that a factor 2 was omitted in the first versions of the reference papers, which compensated the missed factor 2 in front of the Eq. (22) see arXiv:1809.00247, arXiv:1809.06081, arXiv:1812.00905.

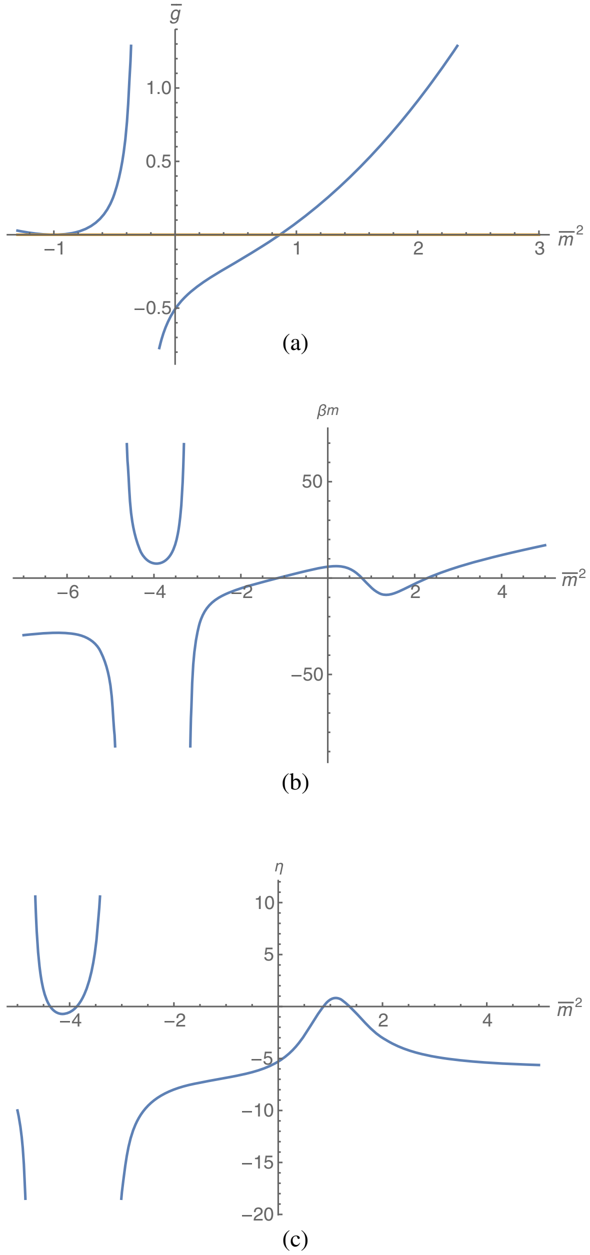


FIG. 4. The constrained melonic flow $\bar{g} = f(\bar{m}^2)$ ($\mathcal{E}_{\mathcal{C}1}$) in blue and $\bar{g} = 0$ ($\mathcal{E}_{\mathcal{C}0}$) in brown (a); the corresponding beta function $\beta_m(\bar{m}^2, \bar{g} = f(\bar{m}^2))$ (b) and the anomalous dimension $\eta(\bar{m}^2, \bar{g} = f(\bar{m}^2))$ (c).

$\bar{m}_\infty^2 = -3.23$.[6] Note that the beta function for the coupling $\bar{g}$ has an additional singularity for $\bar{m}_*^2 \approx -0.29$, which in fact correspond to the singular point of the function f, as pictured on Fig. 4(a). We expect that this singularity is not a singularity of the RG flow itself (driving by β_m, which is no singular at this point), but a discontinuity of the classical action Γ_k, which is undefined at this point. On left,

[6]This argument, obviously is subordinated to our approximations.

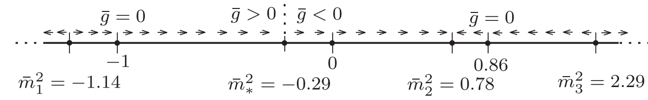


FIG. 5. Qualitative behavior of the RG flow over the constrained subspace $\bar{g} = f(\bar{m}^2)$ and the relevant points.

$f(\bar{m}_*^2 - \epsilon) > 0$, while on right $f(\bar{m}_*^2 + \epsilon) < 0$. Such a pathology is generically reminiscent of a first order phase transition, an interpretation supported by the existence of a purely repulsive fixed point for $\bar{m}_1^2 \approx -1.14$.

We may investigate more precisely the phase space structure of the RG flow. On Fig. 5 below, we pictured all the significant points, and the direction of the RG flow, toward UV scales.

From left to right (and toward deep UV scales), we find a repulsive fixed point for $\bar{m}_1^2 \approx -1.14$. The left part of the flow goes inexorably toward the singularity at $\bar{m}_\infty^2 = -3.23$. The right part however progress toward positive values of $\bar{m}^2$, until to reach the point $\bar{m}^2 = -1$, at which $\bar{g}$ vanish. At this point $\mathcal{E}_{\mathcal{C}0}$ meet $\mathcal{E}_{\mathcal{C}1}$, and the RG map could be discontinuous, because on $\mathcal{E}_{\mathcal{C}0}$, $\beta_m = 2$ and on $\mathcal{E}_{\mathcal{C}1}$, $\beta_m = 6/7$. We assume that the RG map cannot be discontinuous, and discard in our analysis the possibility to reach the region $\mathcal{E}_{\mathcal{C}0}$ from the region $\mathcal{E}_{\mathcal{C}1}$ at this point. The same phenomenon occurs at the point $\bar{m}^2 = 0.86$, where once again two values for β_m are allowed. With this *jumping censure*, the two subspaces $\mathcal{E}_{\mathcal{C}0}$ and $\mathcal{E}_{\mathcal{C}1}$ are dynamically disconnected. The RG flow then progress toward positive mass, and cross the discontinuity of Γ_k at $\bar{m}_*^2 = 0.29$ without disturbing, and reach the positive region. Nothing is happening at $\bar{m}^2 = 0$, but the flow continues on this way to reach the second purely attractive fixed point, at $\bar{m}_2^2 = 0.78$. At this point, the right part of the flow comes from a third purely repulsive fixed point at $\bar{m}_3^2 = 2.29$. This description play in favor of an asymptotic safety scenario, the flow being strongly attracted in the deep UV toward the fixed point $\bar{m}_2^2$; completed with a first order phase transition toward IR scales, in the way to reach the IR attractive fixed point $\bar{m}_1^2$.

V. MELONIC STRICTLY LOCAL POTENTIAL APPROXIMATION

Despite the fact that this strategy is difficult to extend for models involving higher order interactions, even for the quartic melonic model some difficulty appear, the multi-branch phenomenon discussed previously being one example. Moreover, some objections can be addressed to this method. First of all, our computation concern only the melonic sector, and a more complete analysis should be done, using the method explained in [4,5]. Second, the computation is based on the approximation (25), keeping only the relevant terms in the derivative expansion. This approximation is used to compute the sums involved in the flow equations, into the windows of momenta allowed by $\dot{r}_k$. Note that, because this windows is the same as $\partial r_k/\partial p_1^2$, no additional approximation is used to compute the Ward constraint (37). Indeed, the undefined term:

$$2g\dot{\mathcal{A}}_{2,0}, \tag{41}$$

is fixed in term of the rate $\dot{\lambda}$, from Eq. (19):

$$\dot{g} = -2g^2\dot{\mathcal{A}}_{2,0}. \tag{42}$$

However, the computation of the flow equations has required to compute superficially convergent sums using the approximation (25), out of the windows of momenta allowed by $\dot{r}_k$. This was concerned π_3 defined by (22), and the derivative of the 4-point vertex (29). We checked in [4,5] the compatibility of this approximation with the second Ward identity (35). This is a strong constraint, in favor of the reliability of our approximation. But not a well justification, without exact computation. Such an exact computation is expected to be very hard. However, we may use of the constraint, not to find where our approximation make sense in the investigated region of the full phase space, but to fix these undefined sums. This is the strategy that we will describe now, keeping the approximation (25) only for the sums in the domain $\vec{p}^2 < k^2$. As we will see, this alternative description of the interacting sector of the theory strongly simplify the description of the constrained space, that we call $\mathcal{E}_C$ as well, and can be easily extended for models with higher order interactions.

Another objection could concern the Ward identities themselves, or more precisely the form of Ward identities that we used to compute the constraint (37). As discussed in the previous paragraph, we discarded the effective vertices with disconnected boundaries from our analysis. However, disconnected interactions does not modify the Ward identities for connected interactions, Eqs. (34) and (35). They have their own Ward identities, but what we called Ward constraint, Eq. (37) holds even if we consider disconnected pieces. A simple way to check this point is to remark that, as pointed out in [4,5], the relation (34) may be deduced from the structure of the melonic diagrams rather than a consequence of the symmetry breaking of the kinetic action. Indeed, in the melonic sector, the self-energy satisfies a closed equation [see [27] and Eq. (59)]; which combined with the melonic expansion (19) provides exactly the same relation than (34) (lemma 1 and 2 are specific realization of this general feature). Moreover, as explained in the previous paragraph, the second relation (35) work as well, at least in the first order in the derivative expansion to compute the derivative of the effective vertex (29). Indeed, as checked in [4,5], Appendix B[7]:

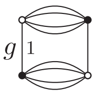
$$\frac{d}{dp_1^2}\pi^{(1)}_{2,p_1p_1}|_{p_1=0} = -4g^2(k)\frac{d}{dp_1^2}\mathcal{A}_{2,p_1}|_{p_1=0}$$
$$= 8g^2(k)\sum_{\vec{p}_\perp}\left(Z(k)+\frac{\partial r_k}{\partial p_1^2}(\vec{p}_\perp)\right)G_k^3(\vec{p}_\perp)$$
$$= 2(\pi_{00}^{(1)})^2\left[\sum_{\vec{p}_\perp}(Z(k)-Z)G_k^3(\vec{p}_\perp)+\mathcal{L}_{3,k}\right]$$

where we used (25) for the second line and (36) for the last line. Finally, from the first Ward identity (34), or mixing (19) and the closed equation (59), $(Z(k)-Z) = -Z\pi_{00}^{(1)}\mathcal{L}_{2,k}$; such that because (22) we recover exactly the Ward identity (35). Alternatively, we may view this accordance as an indication that the symmetric phase coincide with the region where the perturbative expansion converge.

The reliability of our approximations for the Ward constraint however cannot be extended for the RG flow equations themselves, and in the next two subsection, we investigate two heuristic ways to discuss the robustness of our conclusions.

A. Influence of the number of interactions

A simple way of investigation for the robustness of our conclusions about the constrained phase space is to reduce the number of interactions. Indeed, configurations as pictured on Fig. 1 disappears if we restrict the number of interactions to one, replacing the original model (4) by:

$$S_{\text{int}}[T,\bar{T}] = g$$  . (43)

In the point of view of the EVE method, Eqs. (26), the only change with respect to the fully interacting model concern the beta function β_m, the factor 10 being in fact $2\times d$, for $d=5$. For a single color, we get a single fixed point, for the values:

$$p_2^{(1)} = (0.34, 0.42), \tag{44}$$

with strongly negative anomalous dimension $\eta_2^{(1)} \approx -4.3$ and critical exponents:

$$(\theta_1^{(1)}, \theta_2^{(1)}) \approx (-11.8, -3.3), \tag{45}$$

the higher index 1 referring to the number of interactions. This purely attractive fixed point is reminiscent of the fixed point that we called p_2 in Sec. III. The Ward constraint (37) however remains unchanged, and inserting the flow equations following the strategy described in Sec. IV, we get:

[7]Due to the missed factors 2 recalled in footnote 3 and 5, we reproduce the proof.

$$\bar{g} = 0, \;\text{ or }\; \bar{g}(\bar{m}^2) = \frac{(\bar{m}^2+1)^2(3\bar{m}^2(\bar{m}^2+3)+14)}{\pi^2(\bar{m}^2+2)(\bar{m}^2+5)}. \tag{46}$$

The new function f, as the corresponding beta function and the anomalous dimension are pictured on Fig. 6. This time, the function $f(\bar{m}^2) \coloneqq g$ has singularities for $\bar{m}^2 = -2$ and $\bar{m}^2 = -5$. The first singularity is in fact common to β_m, η and β_λ, and may be viewed as a pathology of the approach. Among the fixed point, only the repulsive fixed point for positive mass survives, this time for the value $\bar{m}^2 \approx 0.66$.

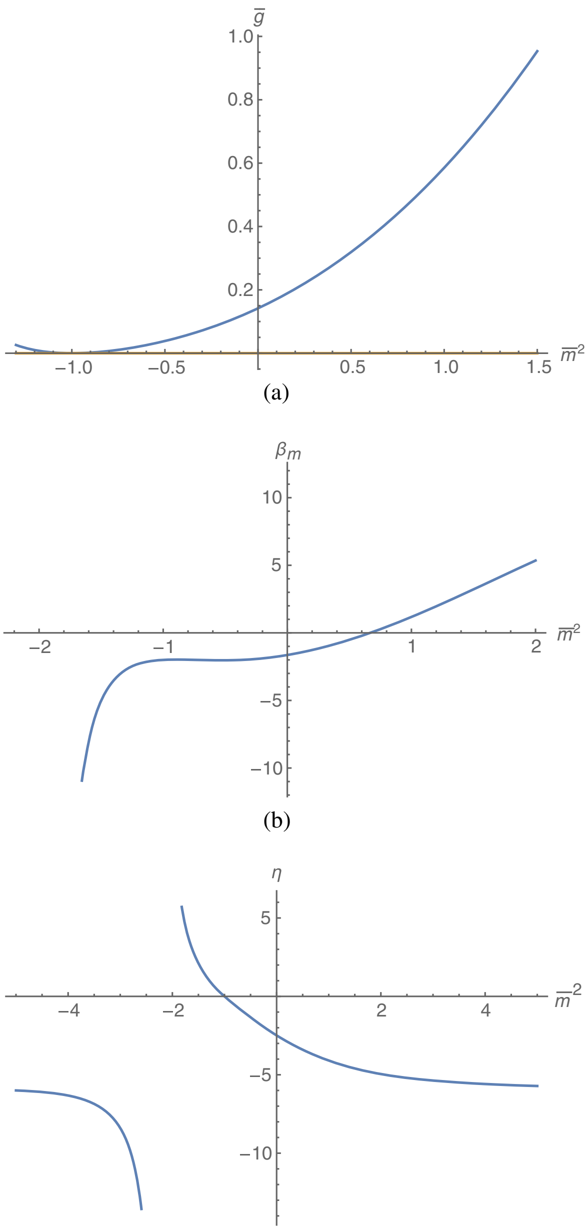


FIG. 6. The constrained melonic flow $\bar{g} = f(\bar{m}^2)$ ($\mathcal{E}_{\mathcal{C}1}$) in blue and $\bar{g} = 0$ ($\mathcal{E}_{\mathcal{C}0}$) in brown for a single interaction (a); the corresponding beta function $\beta_m(\bar{m}^2, \bar{g} = f(\bar{m}^2))$ (b) and the anomalous dimension $\eta(\bar{m}^2, \bar{g} = f(\bar{m}^2))$ (c).

Below this fixed point, the RG flow run slowly through $\bar{m}^2 = 0$ to reach negative values, toward the singular region, without crossing any singularity for the effective action. These conclusions seems to indicate that the Wilson-Fisher fixed point that we called p_1 and the first order phase transition over the constrained phase space depend strongly on the number of interactions. This can be viewed as the sign that our local potential approximation break down. Indeed, in the singular region for f, the magnitude of the coupling becomes very large, and even if they do not modifies the Ward identities, as discussed previously, disconnected contributions may play a role in this strongly nonperturbative regime. Note that this argument is in competition with an other effect, which can explained the disagreement with the fully interacting theory. The $2d$ factor that comes in the interaction strongly enhance the weight of the beta function β_m, especially in the Ward constraint (37); and the occurring of the singularity as well as of the Wilson-Fisher fixed point could be related to this weight, in the LPA. For 2 interactions, the results remains qualitatively the same, and the corresponding beta function β_m is pictured on Fig. 7 No singularity for f and no Wilson-Fisher fixed point occur in the physical domain, bounded by a common singularity for the beta functions, at the value $\bar{m}^2 \approx -1.26$. This divergence is in fact protected by the existence of an UV attractive fixed point, for $\bar{m}^2 = -1$ (once again, in the constrained phase space, the singularity $\bar{m}^2 = -1$ disappears). Over the unconstrained phase space, only the purely positive fixed point occurs, $p_2^{(2)} \approx (0.5, 0.8)$, becoming an IR attractor, with complex critical exponents:

$$(\theta_1^{(2)}, \theta_2^{(2)}) \approx (-5.2 - 2.2i, -5.2 + 2.2i). \tag{47}$$

The convergence toward the values corresponding to p_2 for five interaction continues for three, and four interactions, and we close here our comments for this fixed point. For three interactions, the Wilson-Fisher fixed point appears, for $p_1^{(3)} \approx (0.003, -0.7)$, with essentially the same

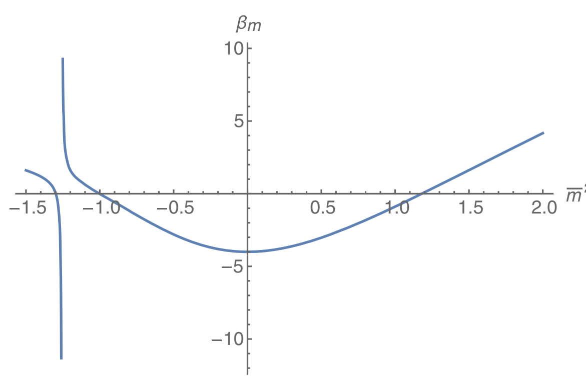


FIG. 7. The beta function β_m over the constrained phase space for two interactions.

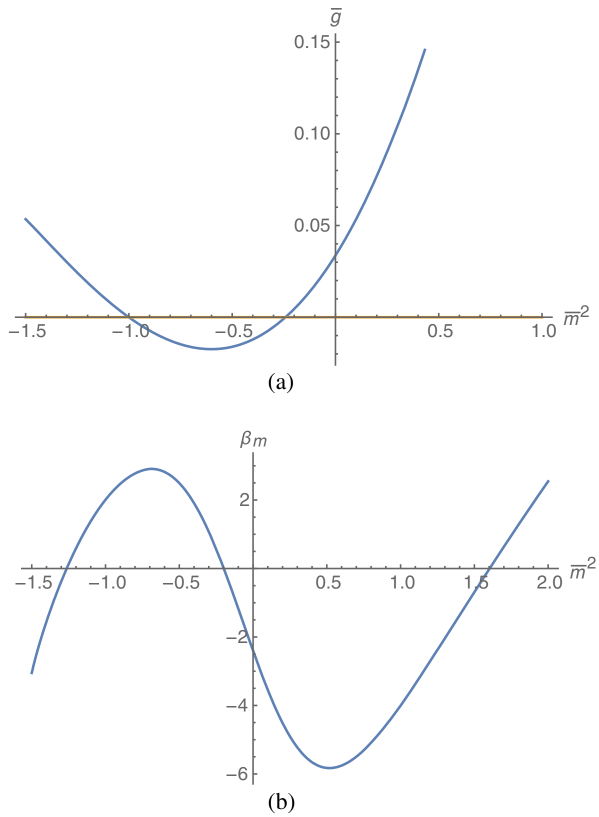


FIG. 8. The constrained melonic flow $\bar{g} = f(\bar{m}^2)$ ($\mathcal{E}_{C1}$) in blue and $\bar{g} = 0$ ($\mathcal{E}_{C0}$) in brown for three interactions (a); the corresponding beta function $\beta_m(\bar{m}^2, \bar{g} = f(\bar{m}^2))$ (b).

characteristics than p_1, anomalous dimension $\eta_1^{(3)} \approx 1.36$ and critical exponents $(\theta_1^{(3)}, \theta_2^{(3)}) \approx (-7.6, 1.6)$. Once again, the convergence continues with four interaction. Function f and beta functions are given on Fig. 8. A singularity occurs for β_m, β_g and η for $\bar{m}^2 = -2$, bounded the physical domain. However, no singularity occurs for f. We only observe that f change of sign, and become negative into the interval $[-1, -0.24]$; just before the attractive IR fixed point at $\bar{m}^2 \approx -0.2$. Another attractive UV fixed point occurs for $\bar{m}^2 \approx -1.26$, protected the RG flow in the deep UV from the singularity.

With four interactions, the picture becomes close to the fully interacting model. The singularity of f appears, for $\bar{m}^2 \approx -0.63$, and the behavior of β_m is qualitatively the same, except for the positions of its zeros. This analysis show two important things. Firstly, the singularity occurs from four interactions. If the singularity was the sign of a strong breaking of the strictly local potential approximation, we would like expected that this divergence occurs from two interactions, where disconnected diagrams appears perturbatively. This is not what we observe. This does not mean that disconnected diagrams do not play a role, but simply that their contributions is not really responsible of the singularity, even if their contribution does not appears in competition with the enhancement of β_m increasing the number of interactions. Our position on this point is that investigations beyond strictly LPA have to be conducted; requiring more sophistical methods, but no very strong disagreement with the picture provided by strictly LPA is expected, especially in regard to the existence of the singularity of the effective action.

B. Dynamical constrained flow

As an alternative way of investigation, we propose to fix $\pi_3^{(i)}$, and then the sum $\mathcal{A}_{3,0}$ by the flow itself, rather through a specific approximation for $\Gamma_k^{(2)}$, out of the windows of momenta allowed by $\dot{r}_k$. Our procedure is schematically the following:

(1) We keep β_m and fix β_g from the Eq. (37):

$$\begin{cases} \beta_m = -(2+\eta)\bar{m}^2 - \frac{10\pi^2\bar{g}}{(1+\bar{m}^2)^2}\left(1+\frac{\eta}{6}\right), \\ \beta_g = -\eta\bar{g}\frac{\Omega(\bar{g},\bar{m}^2)}{(1+\bar{m}^2)^2} + \frac{2\pi^2\bar{g}^2}{(1+\bar{m}^2)^3}\beta_m. \end{cases} \tag{48}$$

(2) We fix $\pi_{3,00}^{(i)}$ dynamically from the flow equation (24):

$$\beta_g = -2\eta\bar{g} - \frac{1}{2}\bar{\pi}_3^{(1)}\frac{\pi^2}{(1+\bar{m}^2)^2}\left(1+\frac{\eta}{6}\right) + 4\bar{g}^2\frac{\pi^2}{(1+\bar{m}^2)^3}\left(1+\frac{\eta}{6}\right) \tag{49}$$

(3) We compute $\frac{d}{dp_1^2}\pi_{2,00}^{(i)}$ from Eq. (35), and finally deduce an equation for the anomalous dimension η. The computation require the sums $\mathcal{L}_{2,k}$ and $\mathcal{L}_{3,k}$. $\mathcal{L}_{2,k}$ has a vanishing power counting, and contain the undefined sum $\mathcal{A}_{2,0}$. However, $\mathcal{L}_{2,k}$ may be expressed in term of $Z(k)$ and $g(k)$ from Eq. (34). Indeed, setting $k = 0$ and fixing the renormalization condition such that $Z(k=0) = 1$,[8] we get that, in the continuum limit $\Lambda \to \infty$, $Z \to 0$. To summarize, in the same limit, (34) reduces to $-2g(k)\mathcal{L}_{2,k} = Z(k)$, and from (35):

$$\frac{d}{dp_1^2}\pi_{2,00}^{(1)} = \left(Z(k)\frac{\pi_{3,00}^{(1)}}{2g(k)} + 2(\pi_{2,00}^{(1)})^2\mathcal{L}_{3,k}\right). \tag{50}$$

To compute $\mathcal{L}_{3,k}$, we have to note that, because all the quantities are renormalized, only the superficial divergences survives. As a result, $\mathcal{A}_{3,p}$ does not diverge, such that in the continuum limit, $Z_{-\infty}\mathcal{A}_{3,p}$ must vanish, and we get straightforwardly:

[8]This condition may be refined, see [4], but this point has no consequence on our discussion.

$$\mathcal{L}_{3,k} = -\frac{1}{2Z^2(k)k^2}\frac{\pi^2}{(1+\bar{m}^2)^3}. \tag{51}$$

Note that this term was computed using (25) in the interior of the domain $\vec{p}^2 < k^2$.

From Eq. (48) and (49), we get (we omit the indices 0 to simplify the notations):

$$-\frac{1}{2\bar{g}}\pi_3^{(1)}\frac{\pi^2}{(1+\bar{m}^2)^2}\left(1+\frac{\eta}{6}\right) = \eta + \eta\frac{\pi^2\bar{g}}{(1+\bar{m}^2)^2} - \frac{2\pi^2\bar{g}\bar{m}^2}{(1+\bar{m}^2)^3}(2+\eta) - \frac{20\pi^4\bar{g}^2}{(1+\bar{m}^2)^5}\left(1+\frac{\eta}{6}\right) - 4\bar{g}\frac{\pi^2}{(1+\bar{m}^2)^3}\left(1+\frac{\eta}{6}\right).$$

Interestingly, it is easy to see, using the perturbative expansion that this formula is in accordance with the formula (22). Indeed, from (27),

$$\eta \approx 4\pi^2\bar{g}(1-2\bar{m}^2) + \cdots, \tag{52}$$

canceling all the $\bar{g}\bar{m}^2$ terms at the leading order. Then, from Eqs. (50), (49) and from the flow equation (23), it is easy to get an explicit relation, fixing η for a given value of the pair $(\bar{m}^2, \bar{g})$ along the flow. This relation takes the form:

$$\eta[\Omega_1(\bar{m}^2, \bar{g})\bar{g}] = 12\pi^2\bar{g}^2, \tag{53}$$

where:

$$\Omega_1(\bar{m}^2, \bar{g}) \coloneqq (1+\bar{m}^2) - 2\pi^2\bar{g}. \tag{54}$$

Equation (53) determine η in the constrained space $\mathcal{E}_{\mathcal{C}}$ excepts for $\bar{g} = 0$ and $\Omega_1 = 0$, where it is undefined. In the vicinity of the Gaussian fixed point, the behavior of the flow is far as from the perturbative one. In fact, we get:

$$\eta \approx 12\pi^2\bar{g}, \qquad \beta_g = -12\pi^2\bar{g}^2. \tag{55}$$

in strong disagreement with the perturbative solution:

$$\eta \approx 4\pi^2\bar{g}, \qquad \beta_g \approx -4\pi^2\bar{g}^2. \tag{56}$$

At this order of approximation, the accordance match only for $\bar{g} = 0$; but a question may be addressed. What is the portion of the phase space where the perturbative expansion match with the constrained η? A simple way of investigation is to equalize (53) with the anomalous dimension provided by EVE. Figure 9(a) show the solution for $\bar{g}$ as a function of the renormalized mass $\bar{m}^2$. Explicitly:

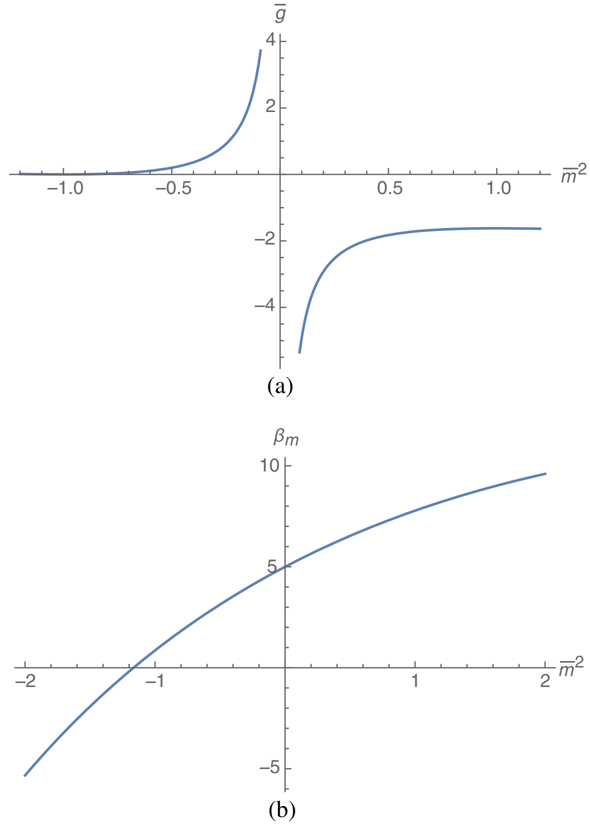


FIG. 9. The constrained melonic flow $\bar{g} = f(\bar{m}^2)$ in the dynamical point of view (a); the corresponding beta function $\beta_m(\bar{m}^2, \bar{g} = f(\bar{m}^2))$ (b).

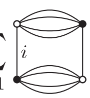
$$\bar{g}(\bar{m}^2) = -\frac{4(\bar{m}^2+1)^2}{\pi^2\bar{m}^2}, \tag{57}$$

which is not so far from the expression of f obtained from our previous heuristic method. A singularity occurs for $\bar{m}^2 = 0$, without disturbing the smoothness of β_m, showed Figure 9(b). Once again, we recover qualitatively the behavior of fully constrained flow. From UV to IR, the flow progress from positive to negative mass without discontinuity, the discontinuity only concern f and therefore the effective action.

Interestingly, only the divergence, and the IR attractive fixed point for negative mass survive. This can be interpreted as well as an argument in favor of our first order scenario than as the signal that our approximations break down very early in the history of the RG flow. In regard with our discussion about Ward constraint, we may view our result as a strong indication that unconnected interactions do not play a significance role in the disappearance of the singularity. Indeed, fixing π_3 in this way, we may expect to keep more than strictly local contributions; which have to be treated on the same footing. In this regard, the improvement coming from this method, fixing the undefined integral from the Ward identities along the flow should rather be seen as a recipe: How to build a local flow, which is compatible with Ward's identities by making as few assumptions as possible about the behavior of integrals outside the domain defined by $\dot{r}_k$.

VI. ANOTHER POINT OF VIEW ON THE CONSTRAINT

The existence of such strong constraints in the theory space can seems to be very surprising. Indeed, it is natural to expect that initial conditions can be chosen freely in the theory space, but our result seems to indicate that is wrong. Why ? We have to keep in mind that $\bar{g}(k)$ and $\bar{m}^2(k)$ are the renormalized couplings at scale k, as it is clear, for instance, from the renormalization condition (21) in the deep infrared limit. In the deep UV limit, that is to say, for $k \to \Lambda$, where the initial conditions are chosen; $\bar{g}(\Lambda)$ and $\bar{m}^2(\Lambda)$ reduce to the bare couplings g_b and m_b^2, including nontrivial counterterms to ensures UV finiteness of the theory. Indeed, the theory being just renormalizable, and their exist a set of three counterterms, Z_m, Z_0, and Z_g such that for the quantum model defined with the classical action[9]:

$$S_{\rm r} = \sum_{\vec{p}\in\mathbb{Z}^d} \bar{T}_{\vec{p}}\left(Z_0\vec{p}^{\,2} + Z_m m_r^2\right) T_{\vec{p}} + Z_g g_r \sum_{i=1}^{d}$$  ,

all the UV divergences, in the $\Lambda \to \infty$ limit have to be removed; the subscript r being for "renormalized." The initial conditions are therefore $\bar{m}^2(\Lambda) = Z_m m_r^2$ and $\bar{g}(\Lambda) = Z_g g_b$, such that the constraint on the initial condition simply reveal the existence of nontrivial relations between counterterms, holding to all order in the perturbative expansion. Such a relation has been investigated in [1,3,4], and to make this section clear, we begin to repeat shortly the arguments of the reference.

In the melonic sector, the 2-point self energy $\Sigma(\vec{p})$ decomposes as a sum over colors:

$$\Sigma(\vec{p}) =: \sum_{i=1}^{d} \tau(p_i), \tag{58}$$

where $\tau(p)$ satisfy a complicated closed equation:

$$\tau(p) = -2Z_g g_r \sum_{\vec{q}} \delta_{q_1 p} \frac{1}{Z_0\vec{q}^{\,2} + Z_m m_b^2 - \sum_i \tau(q_i)}. \tag{59}$$

Even if since this point we explicitly mention the existence of an UV cutoff Λ, referring to the deep UV sector, we follows [5], and regularize our sums using dimensional regularization rather than a crude cutoff in momentum space, which introduce nontrivial boundary contributions. Basically, the strategy is to change $U(1)$ for $[U(1)]^D$, and to use the analyticity of the melonic amplitude with respect to the group-dimension D.

Deriving the closed equation with respect to p_1^2, and setting $p_1 = 0$, we get:

$$\tau' = 2Z_g g_r \mathcal{A}_{2,0}(Z_0 - \tau'), \tag{60}$$

where $\tau' \equiv \tau'(0)$, and where we used of the definition (20) for $\mathcal{A}_{2,0}$. Explicitly, the 2 point function for small momenta writes as:

$$\Gamma_k^{(2)}(\vec{p}) = (Z_0 - \tau')\vec{p}^{\,2} + (m_b^2 - d\tau) + \mathcal{O}(\vec{p}^{\,2}), \tag{61}$$

where $\tau \equiv \tau(0)$. Then, if we fix the renormalization conditions such that, for small $\vec{p}^{\,2}$, $\Gamma_k^{(2)}(\vec{p}) \sim \vec{p}^{\,2} + m_r^2$, we get the relations between bare and renormalized quantities:

$$m_b^2 \coloneqq m_r^2 + d\tau, \qquad Z_0 = 1 + \tau', \tag{62}$$

such that, from (60)

$$Z_0 = 1 + 2Z_g g_r \mathcal{A}_{2,0}. \tag{63}$$

The 4-point melonic graphs may be expressed in term of the 2-point function in the same way, using the recursive definition of melonic diagrams. Using the notation of the Sec. III, like for Eq. (19), we get:

$$\pi_{00}^{(1)} = \frac{2Z_g g_r}{1 + 2Z_g g_r \mathcal{A}_{2,0}} \equiv 2g_r, \tag{64}$$

the last equality being a renormalization condition, fixing Z_g:

$$Z_g = \frac{1}{1 - 2g_r \mathcal{A}_{2,0}}, \tag{65}$$

and from (63), it is easy to check that:

$$Z_g = Z_0. \tag{66}$$

As a result, the wave function and the vertex counterterm has to be equals, to all orders of the perturbative expansion. This relation can be turned in a renormalization point of view, considering the way to which the finite parts of the counterterms move, in order to keep the relation (66) rigid, and compare them to the Ward constraint (37).

A. Minimal subtraction

In a first time, and as an illustration, let us consider the minimal subtraction scheme, where only the divergent parts of the divergent functions are subtracted. For 2 point functions, we denote them respectively as τ_∞ and τ'_∞. In

[9]In this section we left the regulator r_k.

that RG scheme, mass and wave function counterterms are then defined as:

$$Z_0 = 1 + \tau'_\infty, \qquad Z_m = 1 + d\tau_\infty/m_r^2. \tag{67}$$

Let us denote as $G_r(\vec{p})$ the renormalized 2-point function, related to the bare 2-point function as,

$$G_r(\vec{p}) = \frac{1}{Z_0\vec{p}^2 + Z_m m_r^2 - \Sigma(\vec{p})} = \frac{1}{\vec{p}^2 + m_r^2 - \sum_i \tau_r(p_i)}, \tag{68}$$

where the function $\tau_r(\vec{p})$ has no divergences in the limit $D \to 1$, $\tau_r(\vec{p}) - d\tau_\infty - \tau'_\infty$, and for small $\vec{p}$, the effective mass and field strength renormalization, z_m and z_0 are such that:

$$G_r(\vec{p}) \sim \frac{1}{z_0\vec{p}^2 + z_m m_r^2}, \tag{69}$$

explicitly: $z_0 ≔ 1 - \tau'_r$ and $z_m m_r^2 ≔ m_r^2 - d\tau_r$. In the same way, one get for Z_g, $Z_g^{-1} = 1–2g_r\mathcal{A}_{2,\infty}$, where once again the subscript ∞ means that we keep only the divergent terms in the limit $D \to 1$. From that relation, and using the definitions (67) and (60), we see that even in this regularization the relation $Z_g = Z_0$ hold. Moreover:

$$\pi_{00}^{(1)} = \frac{2g_r}{1 + 2g_r\mathcal{A}_{2,r}}, \tag{70}$$

with $\mathcal{A}_{2,r} ≔ \mathcal{A}_{2,0} - \mathcal{A}_{2,\infty}$. The effective coupling g_{eff} is then defined as:

$$g_{\text{eff}} ≔ z_0^{-2}\frac{1}{2}\pi_{00}^{(1)}. \tag{71}$$

Finally, we have the following statement:

Lemma 1. The finite field strength z_0 is equal to the finite vertex renormalization:

$$\pi_{00}^{(2)} = 2g_r z_0. \tag{72}$$

Proof.—Let us start from the equation for τ' (60). Splitting into infinite and renormalized parts:

$$\tau'_\infty + \tau'_r = 2Z_g g_r(\mathcal{A}_{2,\infty} + \mathcal{A}_{2,r})(1 - \tau'_r), \tag{73}$$

where we replaced $Z_0 - \tau' = 1 - \tau'_r$. From the definition (67) of the counterterms, we can rewrite the equation as:

$$Z_0 - z_0 = 2Z_g g_r(\mathcal{A}_{2,\infty} + \mathcal{A}_{2,r})z_0. \tag{74}$$

Using the equality $Z_0 = Z_g$, and dividing the previous equation with respect to Z_g, we get:

$$1 = (Z_g^{-1} + 2g_r(\mathcal{A}_{2,\infty} + \mathcal{A}_{2,r}))z_0. \tag{75}$$

From the explicit expression of Z_g^{-1}, we cancel the divergent term, to keep:

$$1 = (1 + 2g_r\mathcal{A}_{2,r})z_0. \tag{76}$$

Finally, from (70), we proved the lemma. □

B. Renormalization group equations

In this section we investigate the renormalization group properties of the melonic sector through Callan Symanzik (CS) formalism [31–34].

In the regularization procedure, the coupling take a nonzero dimension, and scales like $\mu^{2(1-D)}$ for some referent scale μ used to fix the renormalization conditions; and the dependence of the effective coupling on the finite part of the counter-terms imply the existence of a global parametrization invariance:

$$\Gamma_\mu^{(N)}(m^2(\mu), g(\mu)) \approx \frac{Z^{N/2}(\mu)}{Z^{N/2}(\mu')}\Gamma_{\mu'}^{(N)}(m^2(\mu'), g(\mu')). \tag{77}$$

This global parametrization invariance may be translated as a partial differential equation, which writes as:

$$\left[\frac{\partial}{\partial t} + \beta(\mu)\frac{\partial}{\partial g} + \gamma_m(\mu)m^2\frac{\partial}{\partial m^2} + \frac{N}{2}\gamma(\mu)\right]\Gamma_\mu^{(N)} = 0. \tag{78}$$

To a given subtraction scale μ, the renormalization conditions are fixed such that:

$$\tau'(p = \mu) = 0, \qquad \pi_{\mu\mu}^{(2)} = 2g(\mu). \tag{79}$$

As a consequence, from the melonic structure equations, the following statement holds:

Proposition 3. To all order of the perturbative expansion, and in the deep UV sector, the anomalous dimension $\gamma(\mu)$ and the beta function $\beta(\mu)$

$$\dot{g}(\mu) \equiv \beta(g(\mu)) = -\gamma(\mu)g(\mu). \tag{80}$$

Proof.–To prove this proposition, let us define $\chi(\mu) ≔ \ln(z(\mu))$. From the renormalization conditions:

$$\pi_{pp}^{(2)} = \frac{2g(\mu)}{1 + 2g(\mu)(\mathcal{A}_{2,p} - \mathcal{A}_{2,\mu})}, \tag{81}$$

fixing the counterterm Z_g as:

$$Z_g^{-1} = (g_r/g(\mu))(1 - 2g(\mu)\mathcal{A}_{2,\mu}). \tag{82}$$

To fix the mass and wave function counterterms, let us expand the unrenormalized 2-point function in the vicinity of $p_i = \mu$:

$$G_\mu^{-1}(\vec{p}) = (Z_0 - \tau'(\mu))\vec{p}^2 + (Z_m m_r^2 - d\tau(\mu) + d\mu^2\tau'(\mu)) - \sum_i \tau_\mu(p_i), \tag{83}$$

where we defined the subtracted self-energy $\tau_\mu(p)$ as

$$\tau_\mu(p) = \tau(p) - \tau(\mu) - (p^2 - \mu^2)\tau'(\mu). \tag{84}$$

From the renormalization conditions (79), it follows that $Z_0 - \tau'(\mu)$ have to be equal to 1, and we define the effective mass at scale μ as:

$$m_\mu^2 \coloneqq Z_m m_r^2 - d\tau(\mu) + d\mu^2\tau'(\mu). \tag{85}$$

It is however more convenient to fix the renormalization condition for vanishing momentum, that is, at scale $\mu = 0$:

$$G_\mu^{-1}(\vec{p} = \vec{0}) \eqqcolon m^2(\mu) = m_\mu^2 - d\tau_\mu(0), \tag{86}$$

depending on μ through the other renormalized parameters, and where we used the definition of $\tau_\mu(p)$ for the last equality. Expanding $G_\mu^{-1}(\vec{p})$ in the vicinity of $\vec{p} = 0$, we then get:

$$G_\mu^{-1}(\vec{p}) \approx z(\mu)\vec{p}^2 + m^2(\mu) + \mathcal{O}(\vec{p}^2), \tag{87}$$

with:

$$z(\mu) = 1 - \tau'_\mu(0), \tag{88}$$

and we have the equivalent-lemma of the lemma 1:

Lemma 2. For arbitrary renormalization conditions at scale μ; the effective wave function $z(\mu)$ and the effective melonic vertex $\pi_{00}^{(0)}$ are related as:

$$\pi_{00}^{(0)} = 2g(\mu)z(\mu). \tag{89}$$

Proof.–The proof follows the one of the lemma 1. From the Eq. (60), setting $p = \mu$, we get:

$$\tau'(\mu) = 2Z_g g_r \mathcal{A}_{2,\mu}(Z_0 - \tau'(\mu)). \tag{90}$$

The term in brackets on the right-hand side is equal to 1 from the renormalization condition. Therefore:

$$Z_0 = 1 + 2Z_g g_r \mathcal{A}_{2,\mu} \Rightarrow Z_g = \frac{g(\mu)}{g_r} Z_0. \tag{91}$$

Then, for $p = 0$,

$$\tau'_\mu(0) = \tau'(0) - \tau'(\mu) = 1 + \tau' - Z_0, \tag{92}$$

leading to:

$$\tau'(0) = Z_0 - z(\mu) = 2Z_g g_r \mathcal{A}_{2,0}(Z_0 - \tau'(0)). \tag{93}$$

From definition of $z(\mu)$, and because of the relation between Z_g and Z_0 given by (91), we get:

$$1 = z(\mu)[1/Z_0 + 2g(\mu)\mathcal{A}_{2,0}]. \tag{94}$$

Multiplying the two hands by $2g(\mu)$, and using Eqs. (82) and (91) to compute $1/Z_0$, we find:

$$2g(\mu)z(\mu) = \frac{2g(\mu)}{1 + 2g(\mu)(\mathcal{A}_{2,0} - \mathcal{A}_{2,\mu})}. \tag{95}$$

□

The basis ingredients of the proposition (3) are follows from this key lemma. Indeed, the CS equation (78) may be concisely written as $[D + n\gamma/2]\Gamma_\mu^{(N)} = 0$, D denoting collectively all the linear derivatives with respect to t, g and m. One one hand, for the 2 point function, keeping only the linear terms in $\vec{p}^2$, we get $Dz(\mu) + \gamma z(\mu) = 0$. One the second hand, from lemma 2, we get for $\Gamma_\mu^{(4)}$: $g(\mu)Dz(\mu) + z(\mu)Dg(\mu) + 2\gamma z(\mu)g(\mu) = 0$. As a result, we must has necessarily, for nonvanishing $z(\mu)$:

$$Dg(\mu) + \gamma g(\mu) = 0, \tag{96}$$

and $Dg(\mu) \equiv \beta(g(\mu))$. □

The RG coefficients β, γ_m and γ can be computed in terms of melonic equations. Indeed, from:

$$\Gamma^{(2)}(\vec{p}) = z_0\vec{p}^2 + z_m m_r^2 + \mathcal{O}(\vec{p}^2), \tag{97}$$

inserted into the CS equation, we get two independent equations:

$$\chi_{m,t} + \beta\chi_{m,g} + \gamma_m(1 + \chi_{m,m^2}) + \gamma = 0, \tag{98}$$

$$\chi_{,t} + \beta\chi_{,g} + \gamma_m\chi_{,m^2} + \gamma = 0, \tag{99}$$

where $\chi \coloneqq \ln(z_0)$, $\chi_m = \ln(z_m)$ and $t = \ln(\mu)$; and where we used of the notation $\partial f/\partial x \eqqcolon f_{,x}$. Substituting $\beta = -\gamma g$, we can solve these equations for γ and γ_m. What is important for our purpose is that proposition 3 corresponds to what we called Ward constraint in the previous sections, Eq. (37). The two relations are the same at one and two loops, neglecting contributions arising from mass couplings. These contributions, involving β_m arise from the regulation procedure, as a consequence of the relevant nature of the mass coupling. Once again, in this derivation, we do not use of the Ward identity, but only of the melonic structure of the leading order graphs, highlighting the complementarity of the two approach, especially in regard to disconnected contributions, discussed at the beginning of this section.

VII. CONCLUSION AND DISCUSSION

In this paper we have addressed a short presentation and an extended discussion about an approximate solution of the renormalization group flow in the melonic local potential approximation. In contrast with other methods like truncation for the most popular; effective vertex expansion allows to keep into account sector of infinite size, exploiting a parametrization of all irrelevant interactions in terms of the relevant ones. Translating the Ward identity on the melonic sector in a local form along the flow, we showed the existence of a strong relation between beta functions. Projecting our flow equations along this constraint, the provide a complete description of the constraint phase space, disconnected from the Gaussian region, and exhibiting a singularity reminiscent of a first order phase transition, as well as one purely UV attractive nontrivial fixed point. In complement of this analysis, we stressed the limitations of our approach, and the different way of investigation for future works. In particular, we show that some result crudely depends on the number of interactions involved in the original action. This in particular the case for the strong divergence of the effective action; a possible indication that strictly local potential approximation fail to describe this region. Going beyond this approximation remain a difficult challenge for effective vertex expansion, requiring more sophistical methods remaining in progress.

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