Far East Journal of Applied Mathematics
Volume 68, Number 1, 2012, Pages 21-28
Published Online: August 2012
Available online at http://pphmj.com/journals/fjam.htm
Published by Pushpa Publishing House, Allahabad, INDIA

KINEMATICAL BROWNIAN MOTION AND TIME DEPENDENT ENTROPY

M. Tchoffo[1], J. C. Ngana Kuetche[1], S. Massou[2], G. C. Fouokeng[1], L. C. Fai[1], A. A. Beilinson[3] and Jean-Pierre Kenne[4]

[1]Mesoscopic and Multilayer Structure Laboratory
Department of Physics
Faculty of Science
University of Dschang
Cameroon
e-mail: mtchoffo2000@yahoo.fr

[2]Département de Physique
Faculté des Sciences et Techniques
Université d'Abomey-Calavi
Benin

[3]Department of Theoretical Physics
Russian People's Friendship University
Moscow
Russian Federation

[4]Department of Mechanical Engineering
Laboratory of Integrated Production Technologies
University of Québec
Ecole de Technologie Supérieure
Canada

2010 Mathematics Subject Classification: 60J65, 60G15, 53D22, 94A17.

Keywords and phrases: Brownian motion, Wiener process, Jacobi's second equality, Fokker-Planck equation, joint entropy.

Received March 10, 2012

Abstract

We present the Jacobi's second equality in the form of stochastic equation. By the theorem of factorization of the solution of Fokker-Planck equation, the solution of Bloch equation is obtained and using the Matsubara formalism we determine the wave functions of a free particle and harmonic oscillator of the Euclidean quantum mechanic that permitted us to calculate the Leipnik entropy of ours particles. Showing that the joint entropy of the free particle increases with time and for the case of harmonic oscillator it fluctuates with time and frequency.

1. Introduction

We consider the Jacobi's second equality in the form of a stochastic equation as follows [1]:

$$\dot{x}_\tau + \frac{1}{m}\frac{\partial S_{cl}(x_s,\ \tau;\ x,\ t)}{\partial x_\tau} = \dot{\varphi}_\tau, \tag{1}$$

where $\varphi(\tau)$ is the Wiener process, $S_{cl}(x_\tau,\ \tau;\ x,\ t) = \int_\tau^t L(x_s,\ \dot{x}_s s)ds$ is the action functional along the classical trajectory of an arbitrary field with the Lagrangian $L(x_s,\ \dot{x}_s s)$.

The corresponding Fokker-Planck equation of the stochastic equation (1) has the form [2, 3]:

$$\hbar\frac{\partial W}{\partial \tau} - \frac{\hbar}{m}\frac{\partial}{\partial x_\tau}\left(\frac{\partial S_{cl}(x_\tau,\ \tau;\ x,\ t)}{\partial x_\tau}W\right) = \frac{\hbar^2}{2m}\frac{\partial^2 W}{\partial x_\tau^2}. \tag{2}$$

The fundamental solution of equation (2) can be obtained from (1) by the method of substitution of variables in Wiener integral in the following form:

$$W(x_0,\ 0;\ x_\tau,\ \tau;\ x,\ t)$$

$$= \int_C \exp\left[\int_0^\tau \left(\begin{array}{l} -\dfrac{m}{2\hbar}\left(\dot{x}_\tau + \dfrac{1}{m}\dfrac{\partial S_{cl}(x_s,\ s;\ x,\ t)}{\partial x_\tau}\right)^2 \\ +\dfrac{1}{2m}\dfrac{\partial S_{cl}(x_s,\ s;\ x,\ t)}{\partial x_\tau}\end{array}\right)ds\right]\prod_{s=0}^{\tau.}\frac{dx_s}{\sqrt{\dfrac{2\pi\hbar}{m}ds}}. \tag{3}$$

Using the factorization theorem this yield:

$$W(x_0,\ 0;\ x_\tau,\ \tau;\ x,\ t) = \frac{\exp\left[-\frac{1}{\hbar}S_{cl}(x_\tau,\ \tau;\ x,\ t)\right]}{\exp\left[-\frac{1}{\hbar}S_{cl}(x_0,\ 0;\ x,\ t)\right]} Z(x_0,\ 0;\ x_\tau,\ \tau), \quad (4)$$

where

$$Z(x_0,\ 0;\ x_\tau,\ \tau)$$
$$= \int_C \exp\left[-\int_0^\tau \left(V(x(s),\ s) - \frac{1}{2m}\frac{\partial^2 S_{cl}(x(s),\ s;\ x,\ t)}{\partial x_s^2}\right) ds\right] d_w x_s,$$
$$x(0) = x_0,\ x(t) = x,\ x(\tau) = x_\tau \quad (5)$$

is the Kăc's formula [4, 5].

From equation (5), we arrive at

$$\hbar \frac{\partial Z(x_0,\ 0;\ x,\ \tau)}{\partial \tau} = -\hat{H} Z(x_0,\ 0;\ x_\tau,\ \tau)$$

and

$$\frac{\hbar\partial \exp\left[-\frac{1}{\hbar}S_{cl}(x_\tau,\ \tau;\ x,\ t)\right]}{\partial \tau} = \hat{H}\exp\left[-\frac{1}{\hbar}S_{cl}(x_\tau,\ \tau;\ x,\ t)\right]. \quad (6)$$

We see from the evaluation that the Brownian motion is related to Euclidian quantum mechanics. This can be used to evaluate the wave function of any particle.

2. Wave Function of Free Particle and Harmonic Oscillator

Considering our theory, the probability density of a free particle is given by

$$W(x_0,\ 0;\ x_\tau,\ \tau;\ x,\ t)$$
$$= \frac{\exp\left[-\frac{m}{2\hbar}\frac{(x_\tau - x_0)^2}{\tau} - \frac{m}{2\hbar}\frac{(x - x_\tau)^2}{t-\tau} + \frac{m}{2\hbar}\frac{(x - x_0)^2}{t}\right]}{\sqrt{\frac{2\pi\hbar\tau(t-\tau)}{mt}}} \quad (7)$$

from where the solution of the Bloch equation is given by

$$Z(x_\tau,\ \tau;\ x,\ t) = \frac{\exp\left\{-\frac{m}{2\hbar}\frac{(x - x_\tau)^2}{t - \tau}\right\}}{\sqrt{\frac{m}{2\pi\hbar(t - \tau)}}}. \tag{8}$$

To obtain the solution of the Schrödinger equation $\psi(x_\tau,\ \tau;\ x,\ t)$, we use the Matsubara formalism, that is,

$$t \to it, \quad \tau \to i\tau.$$

Thus

$$\psi(x_\tau,\ \tau;\ x,\ t) = \frac{\exp\left\{-\frac{im}{2\hbar}\frac{(x - x_\tau)^2}{t - \tau}\right\}}{\sqrt{\frac{m}{2\pi i\hbar(t - \tau)}}}. \tag{9}$$

The dependent wave function at time τ in momentum space can be written easily as

$$\widetilde{\psi}(p_\tau,\ \tau,\ p,\ t) = -\sqrt{\frac{2\hbar}{\pi}}\frac{t - \tau}{m}\exp -i\left\{\left(\frac{t - \tau}{m}\right)^2 p_\tau^2 - \frac{1}{\hbar}p_\tau x\right\}. \tag{10}$$

In the same manner, the solution of the Bloch equation, the wave function in coordinate and momentum space of the harmonic oscillator are, respectively, given by

$$Z(x_\tau,\ \tau;\ x,\ t) = \frac{\exp\left\{\frac{m}{2\hbar}\frac{\omega}{\sin\omega(t - \tau)}[(x_\tau^2 - x^2)\cos\omega(t - \tau) - x_\tau x]\right\}}{\sqrt{\frac{m\pi\omega}{2\hbar\sin\omega(t - \tau)}}},$$

$$\psi(x_\tau,\ \tau;\ x,\ t) = \frac{\exp\left\{-i\frac{m\omega}{2\hbar sh\omega(t - \tau)}[(x_\tau^2 + x^2)ch\omega(t - \tau) - 2xx_\tau]\right\}}{\sqrt{\frac{\pi m\omega}{2i\hbar sh\omega(t - \tau)}}}, \tag{11}$$

$$\widetilde{\psi}(p_\tau,\ \tau;\ p,\ t) = \frac{2\hbar sh\omega(t-\tau)}{m\omega\sqrt{2\pi\hbar ch\omega(t-\tau)}}\exp - i$$

$$\cdot\left\{\frac{\left(p_\tau - \dfrac{m\omega}{sh\omega(t-\tau)}x\right)^2}{(m\omega cth\omega(t-\tau))^2} + \frac{m\omega x^2}{2\hbar}cth\omega(t-\tau)\right\}. \tag{12}$$

3. Calculation of the Joint Entropy of the Free Particle and the Harmonic Oscillator and their Numerical Result

The joint entropy is defined as follows [4, 6-8]:

$$S_j(t) = -\int dx|\,\psi(x,\ t)\,|^2 \ln|\,\psi(x,\ t)\,|^2$$

$$-\int dp|\,\widetilde{\psi}(p,\ t)\,|^2 \ln|\,\widetilde{\psi}(p,\ t)\,|^2 - \ln h^d. \tag{13}$$

The probabilities densities of the free particle in the coordinate and momentum space are given by the following equations:

$$|\,\psi(x_\tau,\ \tau,\ x,\ t)\,|^2 = \frac{2\hbar}{\pi m}\frac{x - x_\tau}{x_\tau - x_0}\tau$$

and

$$|\,\psi(p_\tau,\ \tau,\ p,\ t)\,|^2 = \frac{2}{\pi}\left(\frac{x - x_\tau}{x_\tau - x_0}\right)^2\tau^2. \tag{14}$$

Equations (14) into (13) give the joint entropy of a free particle as follows:

$$S_j^f(\tau) = \frac{\hbar}{\pi m}\frac{x - x_0}{t}(t-\tau)^2\left[\ln\left(\frac{2\hbar}{\pi m}\right)(t-\tau) - \frac{1}{2}\right] - \ln 2\pi\hbar. \tag{15}$$

For the case of harmonic oscillator the joint entropy is

$$S_j^{\omega}(\tau) = \frac{2\hbar}{m\pi\omega} x_0 sh\omega(t-\tau)$$

$$\cdot \left\{ cth\omega(t-\tau)\left(\ln b + \ln \frac{sh^2\omega(t-\omega)}{m\omega ch\omega(t-\tau)} - a(\ln b + \ln sh\omega(t-\tau)) \right) \right\},$$

$$a = \left(\cos\omega\tau - \frac{x_0\cos\omega t - x}{x_0\sin\omega t}\sin\omega\tau \right),$$

$$b = \frac{2\hbar}{\pi m\omega},$$

$$c = \left(\sin\omega\tau + \frac{x_0\cos\omega t - x_0}{x_0\sin\omega t}\cos\omega\tau \right) = \frac{da}{d\tau}. \qquad (16)$$

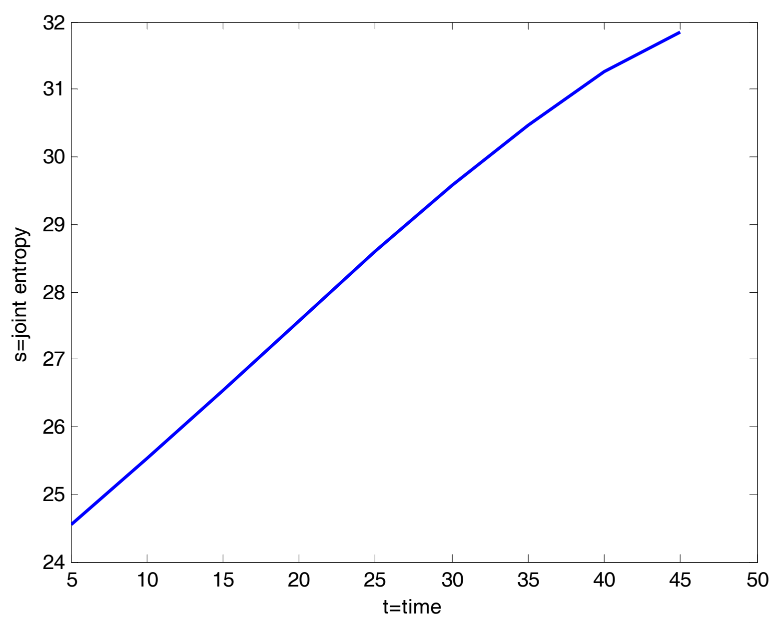

Figure 1. Joint entropy of a free particle versus time.

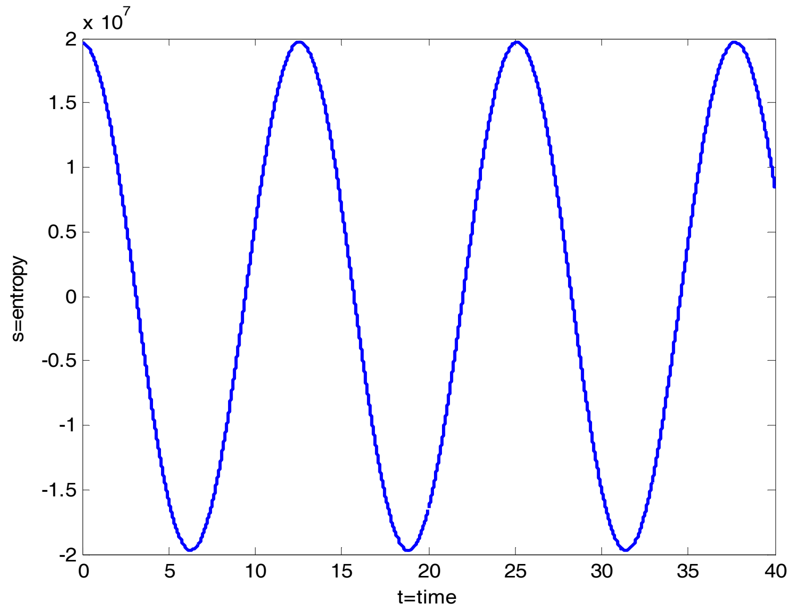

Figure 2. Joint entropy of harmonic oscillator versus time.

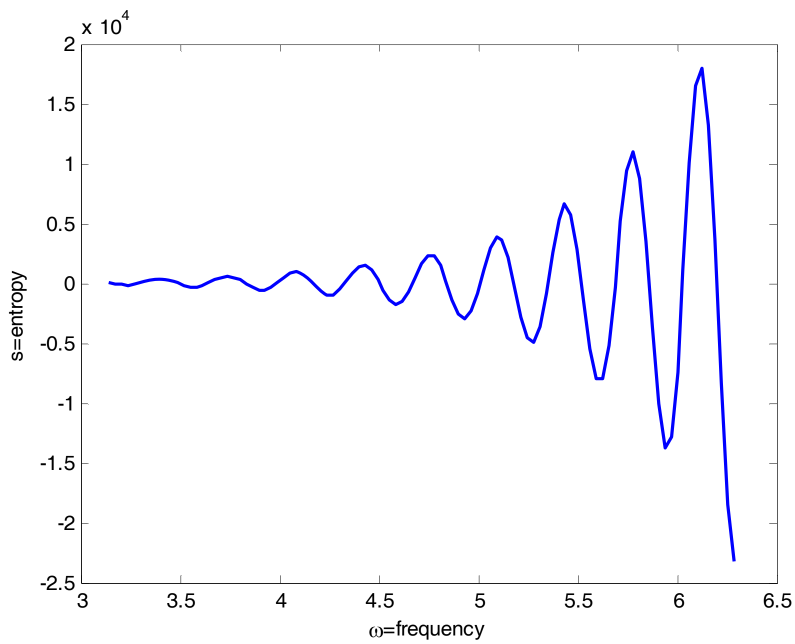

Figure 3. Joint entropy of harmonic oscillator versus frequency.

4. Conclusion

We obtained the wave functions of free particle and harmonic oscillator of Euclidean quantum mechanic using the Kinematics' Brownian motion due to the random canonical transformation. This helps us to determine the joint entropy of ours particles and to show that: the Leipnik entropy of a free particle increases with time and in the case of harmonic oscillator it fluctuates with time and frequency. This result indicates that the information entropy of free particle increases with time however it is periodically transferred between systems in the case of harmonic oscillator.

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