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Exact Solutions For Axial And Transverse Boundary Layers In The Case of Steady Flow Past A Horizontal Plate Embedded In A Saturated Porous Medium

[1]Villevo Adanhounme, [2]Alain Adomou and [3]Francois de Paule Codo

[1]International Chair of Mathematical Physics and Applications, (ICMPA-UNESCO Chair), Université d'Abomey-Calavi, 072 B.P. 50 Cotonou, Republic of Benin.
[2]International Chair of Mathematical Physics and Applications, (ICMPA-UNESCO Chair), Université d'Abomey-Calavi, 072 B.P. 50 Cotonou, Republic of Benin.
[3]International Chair of Mathematical Physics and Applications, (ICMPA-UNESCO Chair), Université d'Abomey-Calavi, 072 B.P. 50 Cotonou, Republic of Benin.

Abstract: The present paper is concerned with the interesting situation of the existence of transverse velocity and thermal boundary layers in the case of flow past a horizontal plate embedded in a saturated porous medium.The analysis shows that the transverse and thermal boundary layers are thicker than the axial boundary layers. The obtained expressions for both velocity and temperature associated with axial and transverse boundary layers are exact solutions and the expressions for the shear-stress at the wall and the specific heat flux are obtained. The study allows one to impose the constraint on the velocity of the free stream .

Key words: Approximation error, Temperature, Thickness of boundary layer.

Nomenclature:

A quantity which involves boundary layer thickness is defined as

$$A = \frac{\nu}{u_0 \delta \frac{d\delta}{dx}}$$

1. B the porous parameter defined by $B = \frac{\delta^2}{K}$
2. C_p specific heat of the convective fluid
3. K permeability of the porous medium
4. P_r Prandtl number $P_r = \frac{\nu}{\alpha}$
5. $q(x)$ specific heat flux: $q(x) = \lambda \frac{dT}{dy}\Big|_{y=0}$
6. Re local Reynolds number $\mathrm{Re} = \frac{xu_0}{\nu}$
7. T temperature
8. T_0 temperature of the free stream
9. T_w temperature of the plate
10. u velocity in the x direction
11. u_0 velocity of the free stream
12. v - velocity in the y-direction
13. x horizontal co-ordinate
14. y vertical co-ordinate
15. α thermal diffusivity
16. δ thickness of the velocity boundary layer in the x direction
17. δ_T thickness of the temperature boundary layer in the x direction
18. Δ thickness of the velocity boundary layer in the y direction

Corresponding Author: Villevo Adanhounme, International Chair of Mathematical Physics and Applications, (ICMPA-UNESCO Chair), Université d'Abomey-Calavi, 072 B.P. 50 Cotonou, Republic of Benin.
E-mail: adanhounm@yahoo.fr

19. ξ the ratio of boundary layer thickness, $\xi = \frac{\Delta}{\delta}$

20. ξ_T dimensionless variable $\xi_T = \frac{\delta_T}{\delta}$

21. ν kinematic viscosity of the fluid
22. ρ density of the fluid
23. Γ shear stress

INTRODUCTION

The problem of flow past a flat plate is one of the interesting problems in fluid mechanics, which was first solved by Blasius (1908) by assuming a series solutions. Later, numerical methods were used (Howarth, L., 1938) to obtain the solution of the boundary layer equation. In all the above analysis, it is presumed that a boundary layer of variable thickness δ is formed about the plate and in this layer the stream-wise component of the flow velocity u varies from zero on the plate to the free stream value u_0. However, no mention is being made of the boundary layer associated with the transverse component v which turns out to have considerable influence on the motion of a viscous fluid past a plate. In this direction Kanakov (1960) investigated the formation of two boundary layers associated with two components of velocity u and v ; in the first, of thickness δ , the component u varies from 0 to u_0 ; in the second, of thickness Δ , the component v varies from 0 at the plate to 0 in the free stream.

The present problem is addressed to the question of whether such a situation arises in a steady flow past a flat plate embedded in a saturated porous medium and other related aspects. The purpose of the present study is to determine the exact solutions for the governing equations of motion for two dimensional boundary layers and for the energy equation. In section 2 we formulate the problem; in section 3 we obtain exact solutions for governing equations and we show that the transverse boundary layer is thicker than the axial boundary layer. Furthermore we have computed the shear-tress at the wall and the specific heat flux. In section 4 we obtain the temperature distribution and in section 5 we have estimated the approximation error when we replace the exact solution for the equation of motion by the approximate solution in (Chandrasekhara, B.C., 1986) which allows to impose the constraint on the velocity of the free stream.

Mathematical Formulation:

The physical model consists of a flat plate parallel to the x axis with its leading edge at $x = 0$ and infinitely long down stream. For mathematical analysis we assume the properties of the fluid and the porous medium such as viscosity, permeability and conductivity, to a first approximation, are constant. Under these assumptions the governing equations of motion for two dimensional boundary layers, following (Chandrasekhara, B.C., 1986) and (Choudhary, M., M. Propster and J. Szekely, 1976) are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\nu}{K}u \tag{2}$$

where K the coefficient of permeability of the porous medium.

The following analysis is based on the notion of a second laminar boundary layer of thickness Δ being formed on the plate (Lu, L., C.R. Doering and F.H. Busse, 2004).

The boundary conditions of the problem expressed in terms of Δ have the form

$$u = v = 0 \ , \ y = 0 \tag{3}$$

$$u = u_0 \ , \ v = 0 \ , \ y = \Delta \tag{4}$$

$$\frac{\partial u}{\partial y} = 0 \ , \ y = \Delta \tag{5}$$

Analytical solution:

Taking into account the stream function ψ the components u and v of the velocity can be written as follows

$$u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x} \tag{6}$$

Putting $\psi(x, y) = f(\eta)$, where $\eta = \frac{y}{\delta}$ and δ a thickness depending on x the equation (2) is transformed to the form

$$f''' - \frac{\delta^2}{K} f' + \frac{1}{\nu}\frac{d\delta}{dx} f'^2 = 0 \tag{7}$$

which produces the solution f in the form

$$f(\eta) = \frac{3\nu\delta}{\frac{d\delta}{dx}\sqrt{K}} \frac{1 - e^{\frac{\delta}{\sqrt{K}}(\xi-\eta)}}{1 + e^{\frac{\delta}{\sqrt{K}}(\xi-\eta)}} \tag{8}$$

where $\xi = \frac{\Delta}{\delta}$ is the ratio of the boundary layer thickness and the stream function ψ has the form

$$\psi(x, y) = f(\eta) \tag{9}$$

Taking into account the relation (8) and the boundary conditions (3-5) we obtain the solutions u, v in the following form

$$u(x, y) = \begin{cases} \frac{1}{\delta} f'(\eta) & 0 < y < \Delta \\ 0 & y = 0 \\ u_0 & y = \Delta \end{cases} \tag{10}$$

$$v(x, y) = \begin{cases} \frac{\eta}{\delta}\frac{d\delta}{dx} f'(\eta) & 0 \le y < \Delta \\ 0 & y = \Delta \end{cases} \tag{11}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\delta^2} f''(\eta) \tag{12}$$

Where

$$f'(\eta) = \frac{6\nu\delta^2}{\frac{d\delta}{dx}K} \frac{e^{\frac{\delta}{\sqrt{K}}(\xi-\eta)}}{\left(1 + e^{\frac{\delta}{\sqrt{K}}(\xi-\eta)}\right)^2} \tag{13}$$

$$f''(\eta) = \frac{6\nu\delta^3}{K^{\frac{3}{2}}\frac{d\delta}{dx}} \frac{e^{\frac{2\delta}{\sqrt{K}}(\xi-\eta)} - e^{\frac{\delta}{\sqrt{K}}(\xi-\eta)}}{\left(1 + e^{\frac{\delta}{\sqrt{K}}(\xi-\eta)}\right)^3} \tag{14}$$

Following (Chandrasekhara, B.C., 1986) we consider that

$$\delta = \sqrt{\frac{x\nu}{u_0}\frac{1}{4}(18+B)(2+B)} \quad ; \quad \frac{d\delta}{dx} = \sqrt{(18+B)(2+B)}\frac{1}{4\sqrt{\mathrm{Re}}} \tag{15}$$

where $B = \frac{\delta^2}{K}$ is the porous parameter.

We assume that the velocity profiles of u with the co-ordinate y_Δ in the second boundary layer are similar to the analogous profiles in the first boundary layer. In this case the variable η of u is defined as

$$\eta = \frac{y_\Delta}{\Delta} = \frac{y_\Delta}{\xi\delta} \tag{16}$$

Substituting (16) into (10) we obtain the expression for u in terms of the original variable $\eta = \frac{y}{\delta}$ and ξ as

$$u(x,y) = \frac{1}{\delta} f'\left(\frac{\eta}{\xi}\right) \tag{17}$$

In the above equation, the variable η varies from 0 to ξ as in (11) ; the expression for the transverse velocity v, which satisfies the boundary conditions at $\eta = 0$ and $\eta = \xi$, is obtained , using (11) in the form

$$v(x,y) = \begin{cases} \frac{\eta}{2x} f'(\eta) & 0 < \eta < \xi \\ 0 & \eta = \xi \end{cases} \tag{18}$$

Where δ is defined (15).

The dimensionless shear-stress at the wall is defined by

$$\Gamma = \frac{\nu}{u_O^2}\left(\frac{\partial u}{\partial y}\right)_\omega \tag{19}$$

$$= \frac{\nu}{u_0^2}\frac{6\nu\delta}{\xi K^{\frac{3}{2}}\frac{d\delta}{dx}}\frac{e^{\frac{2\delta}{\sqrt{K}}\xi} - e^{\frac{\delta}{\sqrt{K}}\xi}}{\left(1+e^{\frac{\delta}{\sqrt{K}}(\xi-\eta)}\right)^3} \tag{20}$$

$$= \frac{3}{2}\frac{\nu^2\delta}{\xi u_0^2 K^{\frac{3}{2}}\frac{d\delta}{dx}}\frac{\sin\left(\frac{\sqrt{B}}{2}\xi\right)}{\cosh^3\left(\frac{\sqrt{B}}{2}\xi\right)} \tag{21}$$

Which produces using (15)

$$\Gamma = \frac{3}{\xi}\left(\frac{4B}{(18+B)(2+B)}\right)^{\frac{3}{2}}\frac{1}{\sqrt{\mathrm{Re}}}\tanh\left(\frac{\sqrt{B}}{2}\xi\right)\left(1-\tanh^2\left(\frac{\sqrt{B}}{2}\xi\right)\right) \tag{22}$$

where Re is the Reynolds number based on the free stream velocity u_0. The values of the dimensionless shear stress Γ ,the thickness δ, B and ξ are presented in the table 1

	B	ξ	δ	Γ	
	0.1	$0.3244\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}$	$3.0826\sqrt{\frac{x\nu}{u_0}}$	$\frac{0.0099}{\Delta}\sqrt{\frac{x\nu}{u_0}}\frac{1}{\sqrt{\mathrm{Re}}}\tanh\left(0.051\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\left(1-\tanh^2\left(0.051\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\right)$	
	0.3	$0.3082\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}$	$3.2438\sqrt{\frac{x\nu}{u_0}}$	$\frac{0.0468}{\Delta}\sqrt{\frac{x\nu}{u_0}}\frac{1}{\sqrt{\mathrm{Re}}}\tanh\left(0.0844\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\left(1-\tanh^2\left(0.0844\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\right)$	
	0.5	$0.2940\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}$	$3.4003\sqrt{\frac{x\nu}{u_0}}$	$\frac{0.0917}{\Delta}\sqrt{\frac{x\nu}{u_0}}\frac{1}{\sqrt{\mathrm{Re}}}\tanh\left(0.1039\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\left(1-\tanh^2\left(0.1039\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\right)$	
	0.7	$0.2815\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}$	$3.5528\sqrt{\frac{x\nu}{u_0}}$	$\frac{0.1391}{\Delta}\sqrt{\frac{x\nu}{u_0}}\frac{1}{\sqrt{\mathrm{Re}}}\tanh\left(0.1177\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\left(1-\tanh^2\left(0.1177\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\right)$	
	0.9	$0.2701\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}$	$3.7016\sqrt{\frac{x\nu}{u_0}}$	$\frac{0.1869}{\Delta}\sqrt{\frac{x\nu}{u_0}}\frac{1}{\sqrt{\mathrm{Re}}}\tanh\left(0.1281\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\left(1-\tanh^2\left(0.1281\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\right)$	
	1.0	$0.2649\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}$	$3.7749\sqrt{\frac{x\nu}{u_0}}$	$\frac{0.2105}{\Delta}\sqrt{\frac{x\nu}{u_0}}\frac{1}{\sqrt{\mathrm{Re}}}\tanh\left(0.1324\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\left(1-\tanh^2\left(0.1324\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}}\right)\right)$	

It is observed from the table 1 that the ratio ξ decreases with the increase in the values of B and if $\frac{\Delta}{\sqrt{\frac{x\nu}{u_0}}} > 3.7749$ then the thickness Δ of the second boundary layer is greater than the thickness of the first boundary layer δ for all values of B

Temperature Distribution and Heat Transfer:

In what follows, we investigate the temperature distribution about a flat plate embedded in a saturated porous medium when a steady laminar viscous fluid flows past the plate maintained at a constant temperature T_w. For mathematical analysis we introduce the notion of two temperature boundary layers being formed about the plate. In the first layer, of thickness δ_T, the component ρCTu varies from zero at the plate to a value of $\rho CT_0 u_0$ at the outer edge of the layer. In the second layer of thickness Δ_T, the component ρCTv varies from zero at the plate to a value $\rho CT_0 u_0$ at the outer edge of the layer. We assume that the temperature boundary layer of thickness Δ_T is formed within the velocity boundary layer of thickness Δ and the convective fluid and the porous medium are everywhere in local thermodynamic equilibrium. Under these assumptions, the energy equation has the form

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{23}$$

where $\alpha = k_m / \rho C_p$ is the thermal diffusivity: $k_m = \varepsilon k_f + (1-\varepsilon)k_s$ is the coefficient of thermal conductivity of the porous medium. k_f is the thermal conductivity of the fluid, ε is the porosity of the medium, k_s is the conductivity of the solid matrix.

The boundary conditions on temperature are :

$$T = T_w, \qquad y = 0 \tag{24}$$

$$T = T_0, \qquad y = \Delta \tag{25}$$

$$\frac{\partial T}{\partial y} = 0 \quad , \quad y = \Delta \tag{26}$$

Consider

$$T = T\left(\eta_T\right) \tag{27}$$

where $\eta_T = \dfrac{y}{\delta_T}$ and δ_T is the thickness of the first thermal boundary layer. Denoting the ratio $\dfrac{\delta_T}{\delta}$ by ξ_T we can write:

$$\delta_T = \delta \xi_T \tag{28}$$

We assume that $\dfrac{1}{\delta_T}\Delta_T \approx \dfrac{\Delta}{\delta}$; then the above assumption leads to the relation $\eta_T = \eta \xi_T$. Taking into account the quantities

$$\frac{\partial T}{\partial x} = -\frac{dT}{d\eta_T}\frac{\eta_T}{\delta_T}\frac{d\delta_T}{dx}, \qquad \frac{\partial T}{\partial y} = \frac{1}{\delta_T}\frac{dT}{d\eta_T} \qquad \frac{\partial^2 T}{\partial y^2} = \frac{1}{\delta_T{}^2}\frac{d^2 T}{d\eta_T{}^2} \tag{29}$$

we transform the equation (23) to the form

$$\left[-\frac{\eta_T}{u_0}u + \frac{v}{u_0}\frac{1}{\frac{d\delta_T}{dx}} \right]\frac{dT}{d\eta_T} = \frac{A}{P_r}\frac{d^2T\left(\eta_T\right)}{d\eta_T{}^2} \tag{30}$$

which is equivalent to the following equation

$$\left[-\frac{\eta_T}{u_0}u + \frac{v}{u_0}\frac{1}{\frac{d\delta_T}{dx}} \right]\frac{dT}{d\eta_T}\left[\frac{T - T_w}{T_0 - T_w}\right] = \frac{A}{P_r}\frac{d^2T\left(\eta_T\right)}{d\eta_{T^2}}\left[\frac{T - T_w}{T_0 - T_w}\right] \tag{31}$$

that yields the solution satisfying the boundary conditions (24)-(26)

$$\frac{T - T_w}{T_0 - T_w} = \begin{cases} \dfrac{\eta_T}{\overline{\eta}_T} , & 0 \le \eta_T \le \sigma \\ \dfrac{\eta_T}{\overline{\eta}_T}\varepsilon\left(\overline{\eta}_T - \eta_T\right) + 1 , & \sigma < \eta_T \le \overline{\eta}_T \end{cases} \tag{32}$$

Where

$$\frac{A}{P_r} = \frac{\alpha}{u_0 \delta_T \frac{d\delta_T}{dx}}; \quad \sigma \in \left]0; \overline{\eta}_T\right[; \quad \overline{\eta}_T = \frac{\Delta}{\delta_T} \tag{33}$$

$$\varepsilon\left(t\right) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases} \tag{34}$$

u, v are defined in the formulas (10-11).

Following (Chandrasekhara, B.C., 1986) we consider that

$$\xi_T = \left(\frac{2}{(2+B)P_r} \right)^{\frac{1}{3}} \tag{35}$$

To obtain an expression for the temperature distribution in the second temperature boundary layer we assume that the variation of the temperature difference $T - T_w$ is similar as in the first boundary layer, which means that (32) is also valid for the boundary layer thickness Δ_T. For this case

$$\eta = \frac{y}{\Delta_T} = \frac{y}{\xi \delta_T}$$

and the expression for the variation of temperature difference $T - T_w$ becomes

$$\frac{T - T_w}{T_0 - T_w} = \begin{cases} \frac{\eta}{\xi \xi_T \bar{\eta}_T} \\ \frac{\eta}{\xi \xi_T \bar{\eta}_T} \varepsilon \left(\bar{\eta}_T - \frac{\eta}{\xi \xi} \right) + 1 \end{cases} \tag{36}$$

where $\eta = \frac{y}{\delta}$.The above expression is valid for the temperature distribution in the temperature boundary layer of thickness Δ_T in which η varies from 0 to $\xi \xi_T$.

The important quantity of interest in engineering application is the specific heat flux $q(x)$ which is defined by:

$$q(x) = \lambda \left(\frac{\partial T}{\partial y} \right) \bigg|_{y=0} = \frac{\lambda}{\delta \xi \xi_T \bar{\eta}_T} (T_0 - T_w) = \frac{1}{\bar{\eta}_T} \left(\frac{\Pr(2+B)}{2} \right)^{\frac{1}{3}} \left(\frac{1}{\xi} \right) \left(\frac{\lambda}{x} \right) \frac{2\sqrt{\mathrm{Re}}}{\sqrt{(18+B)(2+B)}} (T_0 - T_w) \tag{37}$$

negative sign has been omitted from the right hand side of (37) because the direction of $q(x)$ is opposite to that of η.

In this section we arrive at the following result: we obtain exact solution for the equations (1,2), the temperature distribution (36) of the equation (23) in the temperature boundary layer of thickness Δ_T in which η varies from 0 to $\xi \xi_T$ and the specific heat flux $q(x)$ (37).

Estimation Error:

In this section we estimate the approximation error replacing the exact solution u (10) for the equations (1), (2) by the approximate solution $\tilde{u}$ in (Chandrasekhara, B.C., 1986)

$$\tilde{u}(x, y) = u_0 \left(\frac{a_1}{\xi} \eta + \frac{B a_1}{6 \xi^3} \eta^3 - \frac{a_1^2}{24 A \xi^4} \eta^4 \right) \tag{38}$$

$$u(x, y) = \frac{6 \nu \delta}{K \frac{d\delta}{dx}} \frac{e^{\frac{\delta}{\sqrt{K}} \left(\xi - \frac{\eta}{\xi} \right)}}{\left(1 + e^{\frac{\delta}{\sqrt{K}} \left(\xi - \frac{\eta}{\xi} \right)} \right)^2} = \frac{48 u_0 B}{(18+B)(2+B)} \frac{e^{\sqrt{B} (\xi - \frac{\eta}{\xi})}}{\left(1 + e^{\sqrt{B} (\xi - \frac{\eta}{\xi})} \right)^2} \tag{39}$$

Where $B = \delta^2 / K : a_1 = \frac{24}{18+B}; A = \frac{8}{(18+B)(2+B)}$

	η	$\left|\tilde{u} - u\right|$
	0.2	$0.094u_0$
	0.4	$0.2577u_0$
	0.6	$0.488u_0$
	0.8	$0.6617u_0$
	1.0	$0.7968u_0$
	1.2	$0.8681u_0$
	1.4	$0.8447u_0$

Table 2 for $B = 0,5; A = 0.1730;\ a_1 = 1.297 : \xi = 1.2599$

It is observed from the table 2 that the approximation is efficient if the error is less than $10^{-n}, n \in \mathbb{N}^*$, which allows to write $u_0 < \frac{1}{q}10^{-n}$ where q is the coefficient of u_0 in the table 2.

We arrive at the following result: the condition

$$u_0 < \frac{1}{q}10^{-n} \qquad n \in \mathbb{N}^* \tag{40}$$

is sufficient to approximate the exact solution u by the approximate solution $\tilde{u}$ in (Chandrasekhara, B.C., 1986).

Conclusion:

In this paper,we have investigated the exact solutions for the equations (1),(2) , (23) and we have obtained that the transverse boundary layer is thicker than the axial boundary layer under suitable assumption. Furthermore we have estimated the error of approximation of the exact solution (10) by the approximate solution in (Chandrasekhara, B.C., 1986) which allows to impose the constraint on the velocity of the free stream.

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