

Harmonic oscillator in twisted Moyal plane: Eigenvalue problem and relevant properties

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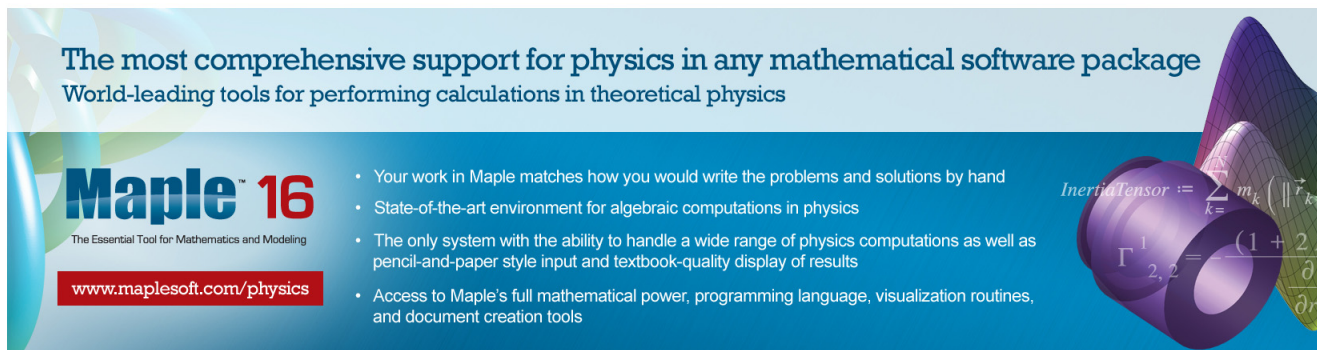
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$$\text{InertiaTensor} := \sum_{k=1}^n m_k \left(\|\vec{r}_k\|^2 \mathbf{1} - \vec{r}_k \vec{r}_k^T \right)$$
$$\Gamma_{2,2}^1 = \frac{(1+2)}{\partial r}$$

$$(f \star g)(x) = m\{e^{i\Theta^{\rho\sigma}/2\partial_\rho\partial_\sigma}f(x) \otimes g(x)\}, \quad x \in \mathbf{R}_\Theta^D \quad \forall f, g \in C^\infty(\mathbf{R}_\Theta^D), \quad (2)$$

m is the ordinary multiplication of functions, $C^\infty(\mathbf{R}_\Theta^D)$ is the space of suitable smooth functions on \mathbf{R}_Θ^D , and \mathbf{R}_Θ^D is the D -dimensional Moyal space. This product can be generalized under the form

$$(f \star g)(x) = m\{e^{i\Theta^{ab}/2X_a \otimes X_b}f(x) \otimes g(x)\}, \quad (3)$$

where $X_a = e_a^\mu(x)\partial_\mu$ are vector fields. The commutation relation of coordinates then becomes $[x^\mu, x^\nu]_\star = i\Theta^{ab}e_a^\mu(x)e_b^\nu(x) := i\tilde{\Theta}^{\mu\nu}(x)$ engendering a twisted scalar field theory where e_a^μ and hence the \star -product itself appear dynamical.

In general, the noncoordinate base has a nonvanishing Lie bracket and satisfies

$$[X_a, X_b] = e_c^\nu[e_a^\mu\partial_\mu e_b^\nu - e_b^\mu\partial_\mu e_a^\nu]X_c = C_{ab}^c X_c. \quad (4)$$

The particular condition $[X_a, X_b]=0$ (i.e., the vector fields are commuting) implies constraints on e_a^μ , namely, $e_{[a}^\mu\partial_\mu e_{b]}^\nu=0$, that can be solved off-shell in terms of D scalar fields ϕ^a . Supposing that the square matrix e_a^μ has an inverse e^μ_a everywhere, so that the X_a are linearly independent, then the above condition becomes $\partial_{[\mu} e^\alpha_{\nu]}=0$ which is satisfied by $e^\alpha_\nu = \partial_\nu \phi^\alpha$. Since $X_a \phi^b = \delta_a^b$, the field ϕ^b can be seen as new coordinates along the X_a directions. See Refs. 1–12 for more details. Besides, the Leibniz rule extends to the commuting fields X_a as follows:

$$X_a(f \star g) = (X_a f) \star g + f \star (X_a g). \quad (5)$$

Recently,¹ a formulation of dynamical noncommutativity, which allows for a consistent interpretation of position measurement and the solution of the problem of a noncommutative well, has been put forward. In their approach, the authors required that the vector fields X_a commute, ensuring the associativity of star product (3). This work addresses a study of a harmonic oscillator properties in the twisted Moyal plane. The associativity property does not play any specific role in the formulation of a noncoordinate base¹⁴ as used here and therefore is not required. Besides, it is worth noticing that such hypothesis made in Ref. 1 (consisting to assume the associativity of the star product or, equivalently, the commutativity of the fields X_a) does not change the results of our investigations as shown in Appendix A.

Furthermore, the use of noncoordinate base leads to consider a curve geometry, more general and richer than a flat one. This could be also of some importance in the study of quantum gravity.

The paper is organized as follows. In Sec. II, using appropriate matrix basis and deforming the issue of a twisted product, we solve the resulting eigenvalue problem to find the states and the energy spectrum of the harmonic oscillator Hamiltonian. These states are infinitely degenerate. Some concluding remarks are pointed out in Sec. III.

II. HARMONIC OSCILLATOR IN TWISTED MOYAL SPACE

As a prelude to the construction of a matrix basis appropriate for this study, let us set up main algebraic relations pertaining to twisted noncommutative coordinate transformations.

A. Useful relations

We consider the following infinitesimal affine transformation:

$$e_a^\mu(x) = \delta_a^\mu + \omega_{ab}^\mu x^b, \quad \omega_{ab}^\mu =: -\omega_{ba}^\mu, \quad \text{and} \quad |\omega^\mu| \ll 1. \quad (6)$$

In the sequel, we restrict the discussion to $D=2$, where e_a^μ and Θ^{ab} can be expressed as follows:

$$(e)_a^\mu = \begin{pmatrix} 1 + \omega_{12}^1 x^2 & \omega_{12}^2 x^2 \\ -\omega_{12}^1 x^1 & 1 - \omega_{12}^2 x^1 \end{pmatrix} \quad \text{and} \quad (\Theta)^{ab} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} = \theta(J)^{ab}, \quad (7)$$

where $J^{12} = -J^{21} = 1$, $J^{11} = J^{22} = 0$. There follow the relations

$$e^{-1} =: \det(e_a^\mu) = 1 + \omega_{12}^1 x^2 - \omega_{12}^2 x^1, \tag{8}$$

$$e =: \det(e_\mu^a) = 1 - \omega_{12}^1 x^2 + \omega_{12}^2 x^1. \tag{9}$$

Before going further in the development, let us immediately recall that all results of our investigation remain also valid even when the considered vector fields commute. See Appendix A for details.

The \star -product of two Schwartz functions on \mathbb{R}_Θ^2 can be written under the form

$$(f \star g)(x) = m \left[\exp \left(\frac{i}{2} \theta e^{-1} J^{\mu\nu} \partial_\mu \otimes \partial_\nu \right) (f \otimes g)(x) \right], \tag{10}$$

where $\mu, \nu = 1, 2$ and $\partial_\mu =: \frac{\partial}{\partial x^\mu}$. Using twisted star product (10), one can see that

$$e^{ikx} \star e^{iqx} = e^{i(k+q)x} e^{-i/2 \theta e^{-1} kJq}. \tag{11}$$

The Fourier transform of $f, g \in \mathcal{S}(\mathbb{R}_\Theta^2)$ can be written as

$$\tilde{f}(k) = \int d^2x e^{-ikx} f(x), \quad \tilde{g}(q) = \int d^2x e^{-iqx} g(x), \tag{12}$$

with the function inverse transform given by

$$f(x) = \frac{1}{(2\pi)^2} \int d^2k e^{ikx} \tilde{f}(k), \quad g(x) = \frac{1}{(2\pi)^2} \int d^2q e^{iqx} \tilde{g}(q). \tag{13}$$

We can then redefine the twisted star product of two Schwartz functions f, g as

$$\begin{aligned} (f \star g)(x) &= \frac{1}{(2\pi)^4} \int d^2k d^2q \tilde{f}(k) \tilde{g}(q) e^{ikx} \star e^{iqx} = \frac{1}{(2\pi)^4} \int d^2k d^2q \tilde{f}(k) \tilde{g}(q) e^{i(k+q)x} e^{-i/2 \theta e^{-1} kJq} \\ &= \frac{1}{(2\pi)^4} \int d^2k d^2q \int d^2y d^2z f(y) g(z) \times e^{ik(x-y-1/2 \theta e^{-1} Jq)} e^{iq(x-z)}. \end{aligned} \tag{14}$$

Using the identity

$$\int d^2k e^{ik(x-y-1/2 \theta e^{-1} Jq)} = (2\pi)^2 \delta^{(2)} \left(x - y - \frac{1}{2} \theta e^{-1} Jq \right) \tag{15}$$

and the variable change q to $q' = \frac{1}{2} \theta e^{-1} Jq$, we arrive at the adapted form for the proof of the next Proposition 2.1,

$$\begin{aligned} (f \star g)(x) &= \left(\frac{e}{\pi\theta} \right)^2 \int d^2y d^2z f(y) g(z) e^{-2ei\theta(x-y)J(x-z)} = \left(\frac{e}{\pi\theta} \right)^2 \int d^2y d^2z f(x-y) g(x-z) e^{-2ei\theta yJz} \\ &= \int d^2z \frac{d^2t}{(2\pi)^2} f \left(x - \frac{1}{2} \theta e^{-1} t \right) g(x-z) e^{-itJz}. \end{aligned} \tag{16}$$

Proposition 2.1: If f and g are two Schwartz functions on \mathbb{R}_Θ^2 , then $f \star g$ is also a Schwartz function on \mathbb{R}_Θ^2 .

Proof: It is immediate by induction on formulas (5) and (16) using integration by parts. ■

The tensor $\tilde{\Theta}^{\mu\nu}$ can be explicit as

$$(\tilde{\Theta})^{\mu\nu} = (\Theta)^{\mu\nu} - (\Theta)^{a[\mu} \omega_{ab}^{v]} x^b = \begin{pmatrix} 0 & \theta e^{-1} \\ -\theta e^{-1} & 0 \end{pmatrix}. \tag{17}$$

The twisted Moyal product of fields generates some basic properties such as the Jacobi identity,

$$[x^\mu, [x^\nu, x^\rho]_\star]_\star + [x^\rho, [x^\mu, x^\nu]_\star]_\star + [x^\nu, [x^\rho, x^\mu]_\star]_\star = \Theta^{b\mu} \Theta^{d[\nu} \omega_{bd}^{\rho]} = 0, \quad (18)$$

conferring a Lie algebra structure to the defined twisted Moyal space, and

$$x^\mu \star f = x^\mu f + \frac{i}{2} \Theta^{ab} e_a^\mu e_b^\rho \partial_\rho f \quad \text{and} \quad f \star x^\mu = x^\mu f - \frac{i}{2} \Theta^{ab} e_a^\mu e_b^\rho \partial_\rho f. \quad (19)$$

The star brackets (anticommutator and commutator) of x^μ and f can be immediately deduced as follows: $\{x^\mu, f\}_\star = 2x^\mu f$, $[x^\mu, f]_\star = i\Theta^{ab} e_a^\mu e_b^\rho \partial_\rho f$. Relations (19) can be detailed for x^μ , $\mu=1, 2$ as

$$x^1 \star f = x^1 f + \frac{i}{2} \theta e^{-1} \partial_2 f, \quad f \star x^1 = x^1 f - \frac{i}{2} \theta e^{-1} \partial_2 f, \quad (20)$$

$$x^2 \star f = x^2 f - \frac{i}{2} \theta e^{-1} \partial_1 f, \quad f \star x^2 = x^2 f + \frac{i}{2} \theta e^{-1} \partial_1 f, \quad (21)$$

giving rise to the creation and annihilation functions,

$$a = \frac{x^1 + ix^2}{\sqrt{2}}, \quad \bar{a} = \frac{x^1 - ix^2}{\sqrt{2}}, \quad (22)$$

with the commutation relation $[a, \bar{a}]_\star = \theta e^{-1}$. It then becomes a matter of algebra to use the transformations of the vector fields ∂_1 and ∂_2 into $\partial_a =: \frac{\partial}{\partial a}$ and $\partial_{\bar{a}} =: \frac{\partial}{\partial \bar{a}}$ and vice versa to infer $e^{-1} = 1 - a\omega - \bar{a}\bar{\omega}$ and $e = 1 + a\omega + \bar{a}\bar{\omega}$, where

$$\omega =: \frac{\omega_{12}^2 + i\omega_{12}^1}{\sqrt{2}} \quad \text{and} \quad \bar{\omega} =: \frac{\omega_{12}^2 - i\omega_{12}^1}{\sqrt{2}}, \quad (23)$$

leading to useful relations

$$\frac{\partial e^{-1}}{\partial a} = -\omega, \quad \frac{\partial e^{-1}}{\partial \bar{a}} = -\bar{\omega} \quad \text{and for } k \in \mathbb{Z}, \quad \omega e^k = \omega, \quad \bar{\omega} e^k = \bar{\omega}. \quad (24)$$

Expressing twisted \star -product (10) in terms of vectors fields ∂_a and $\partial_{\bar{a}}$ as

$$(f \star g)(a, \bar{a}) = m \left[\sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^{n-k}}{k!(n-k)!} \left(\frac{1}{2} \theta e^{-1} \right)^n \times (\partial_a \otimes \partial_{\bar{a}})^k (\partial_{\bar{a}} \otimes \partial_a)^{n-k} (f \otimes g)(a, \bar{a}) \right] \quad (25)$$

and using Eqs. (20) and (21) [or independently (25)] yield

$$a \star f = \left(a + \frac{\theta e^{-1}}{2} \frac{\partial}{\partial \bar{a}} \right) f, \quad \bar{a} \star f = \left(\bar{a} - \frac{\theta e^{-1}}{2} \frac{\partial}{\partial a} \right) f, \quad (26)$$

$$f \star a = \left(a - \frac{\theta e^{-1}}{2} \frac{\partial}{\partial \bar{a}} \right) f, \quad f \star \bar{a} = \left(\bar{a} + \frac{\theta e^{-1}}{2} \frac{\partial}{\partial a} \right) f. \quad (27)$$

Provided the above definitions, we can now introduce the notions of right and left harmonic oscillator states denoted by f_{m0}^R and f_{0n}^L , respectively.

B. The right and left states

Let $f_{00}^R \in L^2(\mathbb{R}_\Theta^2)$ be the ho “right” fundamental state, such that

$$a \star f_{00}^R =: 0 \quad \text{with} \quad f_{00}^R = 2e^{-2a\bar{a}/\theta e^{-1}} \left(1 - \frac{1}{2} \bar{a}\bar{\omega}\right). \quad (28)$$

Then, f_{00}^R solves the eigenvalue problem $H \star f_{00}^R = \mathcal{E}_{00}^R f_{00}^R$ with the corresponding right fundamental eigenvalue $\mathcal{E}_{00}^R = \frac{\theta}{2}(1 - 2\bar{a}\bar{\omega})$ of the self-adjoint unbounded twisted ho Hamiltonian operator,

$$\begin{aligned} H \star (\cdot) =: \bar{a}a \star (\cdot) &= \left[\bar{a} \star a + \frac{\theta e^{-1}}{2} \right] \star (\cdot) = \left[a \star \bar{a} - \frac{\theta e^{-1}}{2} \right] \star (\cdot) = \frac{1}{2} \left[(x^1)^2 + (x^2)^2 \right. \\ &\quad \left. + \left(i\theta e^{-1} x^1 - \frac{\theta^2}{4} \omega_{12}^1 \right) \partial_2 - \left(i\theta e^{-1} x^2 - \frac{\theta^2}{4} \omega_{12}^2 \right) \partial_1 - \frac{\theta^2}{4} e^{-2} (\partial_1^2 + \partial_2^2) \right] \equiv \frac{1}{2} \mu_1, \end{aligned} \quad (29)$$

defined in the domain

$$\mathcal{D}(H \star) = \left\{ f \in L^2(\mathbb{R}_\theta^2) \mid f, f_{x^1}, f_{x^2} \in \mathcal{AC}_{loc}(\mathbb{R}_\theta^2); \frac{\mu_1}{2} f \in L^2(\mathbb{R}_\theta^2) \right\}. \quad (30)$$

$\mathcal{AC}_{loc}(\mathbb{R}_\theta^2)$ denotes the set of the locally absolutely continuous functions on \mathbb{R}_θ^2 . Similarly, the fundamental left state $f_{00}^L \in L^2(\mathbb{R}_\theta^2)$ defined such that

$$f_{00}^L \star \bar{a} =: 0 \quad \text{with} \quad f_{00}^L = 2e^{-\frac{2a\bar{a}}{\theta e^{-1}}(1-1/2a\omega)} \quad (31)$$

solves the eigenvalue problem $f_{00}^L \star H = \mathcal{E}_{00}^L f_{00}^L$ with the left fundamental eigenvalue $\mathcal{E}_{00}^L = \frac{\theta}{2}(1 - 2a\omega)$ of the self-adjoint unbounded twisted ho Hamiltonian operator,

$$\begin{aligned} (\cdot) \star H =: (\cdot) \star \bar{a}a &= (\cdot) \star \left[\bar{a} \star a + \frac{\theta e^{-1}}{2} \right] = (\cdot) \star \left[a \star \bar{a} - \frac{\theta e^{-1}}{2} \right] = \frac{1}{2} \left[(x^1)^2 + (x^2)^2 \right. \\ &\quad \left. - \left(i\theta e^{-1} x^1 - \frac{\theta^2}{4} \omega_{12}^1 \right) \partial_2 + \left(i\theta e^{-1} x^2 - \frac{\theta^2}{4} \omega_{12}^2 \right) \partial_1 - \frac{\theta^2}{4} e^{-2} (\partial_1^2 + \partial_2^2) \right] \equiv \frac{1}{2} \mu_2, \end{aligned} \quad (32)$$

defined in the domain

$$\mathcal{D}(\star H) = \left\{ f \in L^2(\mathbb{R}_\theta^2) \mid f, f_{x^1}, f_{x^2} \in \mathcal{AC}_{loc}(\mathbb{R}_\theta^2); \frac{\mu_2}{2} f \in L^2(\mathbb{R}_\theta^2) \right\}. \quad (33)$$

Then, the other states follow from the next statement.

Proposition 2.2: The vectors $f_{m0}^R \in L^2(\mathbb{R}_\theta^2)$ given for any $m \in \mathbb{N}$ by

$$f_{m0}^R = \frac{1}{\sqrt{m! \theta^m}} \left[2^m \bar{a}^m \left(1 + \frac{m a \omega}{2} - \frac{m \bar{a} \bar{\omega}}{4} \right) - \frac{U_m \theta \omega \bar{a}^{m-1}}{2} \right] f_{00}^R \quad (34)$$

solve the eigenvalue problem $H \star f_{m0}^R = \mathcal{E}_{m0}^R f_{m0}^R$ with

$$\mathcal{E}_{m0}^R = \frac{\theta}{2} \left[2m + 1 - m a \omega - (3m + 2) \bar{a} \bar{\omega} - \frac{m^2 \theta \omega}{4 \bar{a}} + \frac{\theta \omega U_m}{2^m \bar{a}} \right], \quad m \in \mathbb{N}, \quad (35)$$

where

$$U_m = (m - 1) 2^{m-2} + \sum_{k=0}^{m-3} (k + 1) 2^{k+1}, \quad m \geq 3, \quad U_{i \leq 1} = 0, \quad U_2 = 1. \quad (36)$$

Proof: The results are immediate by induction, performing similar analysis as in Ref. 8 to construct the right states f_{m0}^R , such that $\bar{a} \star f_{m0}^R = \sqrt{\theta(m+1)} f_{m+1,0}^R$. ■

Similarly, the study of the ho left states provides the following result.

Proposition 2.3: The vectors $f_{0n}^L \in L^2(\mathbb{R}_\theta^2)$ given for any $n \in \mathbb{N}$ by

$$f_{0n}^L = \frac{1}{\sqrt{n! \theta^n}} \left[2^n a^n \left(1 + \frac{n \bar{a} \bar{\omega}}{2} - \frac{n a \omega}{4} \right) - \frac{U_n \theta \bar{\omega} a^{n-1}}{2} \right] f_{00}^L \tag{37}$$

solve the eigenvalue problem $f_{0n}^L \star H = \mathcal{E}_{0n}^L f_{0n}^L$ with

$$\mathcal{E}_{0n}^L = \frac{\theta}{2} \left[2n + 1 - n \bar{a} \bar{\omega} - (3n + 2) a \omega - \frac{n^2 \theta \bar{\omega}}{4a} + \frac{\theta \bar{\omega} U_n}{2^n a} \right], \quad n \in \mathbb{N}. \tag{38}$$

Proof: It uses the same procedure as previously, but with the construction of the left states f_{0n}^L , such that $f_{0n}^L \star a = \sqrt{\theta(n+1)} f_{0,n+1}^L$. ■

Besides $\lim_{\omega, \bar{\omega} \rightarrow 0} \mathcal{E}_{m0}^R = \theta(m + \frac{1}{2})$ and $\lim_{\omega, \bar{\omega} \rightarrow 0} \mathcal{E}_{0n}^L = \theta(n + \frac{1}{2})$ corresponding to the usual Moyal \star -product spectrum of the harmonic oscillator Hamiltonian H .

All these results show that the ho right and left states as well as their respective energy spectrum are expressible in terms of the space deformation constant θ and of an additional piece inherent to the nature of the induced infinitesimal transformation through the parameter ω and its conjugate.¹⁵⁻¹⁷ Besides, a noteworthy feature of these states is the following.

Proposition 2.4: The right and left fundamental states defined by $f_{00}^{(m)R} =: a^{m+1} \star f_{m0}^R$ and $f_{00}^{(n)L} =: f_{0n}^L \star \bar{a}^{n+1}$ are given by the following expressions:

$$f_{00}^{(m)R} = - \frac{\sqrt{m! \theta^{m+2}}}{8} \sum_{j=1}^m \frac{(m + 4j + 1)}{(m - j)!} \bar{\omega} f_{00}^R \tag{39}$$

and

$$f_{00}^{(n)L} = f_{00}^{(n)R}(\bar{a} \leftrightarrow a, \bar{\omega} \leftrightarrow \omega) = - \frac{\sqrt{n! \theta^{n+2}}}{8} \sum_{j=1}^n \frac{(n + 4j + 1)}{(n - j)!} \omega f_{00}^L, \tag{40}$$

which, in the usually Moyal product case, are reduced to 0. Besides, the twisted harmonic oscillator states f_{m0}^R and f_{0n}^L are degenerate with respect to the rules,

$$a^{m+2} \star f_{m0}^R = 0 \quad \text{and} \quad f_{0m}^L \star \bar{a}^{m+2} = 0 \quad \forall \quad m \geq 1. \tag{41}$$

The proof is straightforward. See Appendix B.

Remark 2.1:

- (1) f_{00}^R and f_{00}^L are the twisted fundamental states restoring, in the limit of ordinary Moyal space, the fundamental state given by $2e^{-2a\bar{a}/\theta}$. For the analysis purpose, we call f_{00}^R and f_{00}^L the normal twisted fundamental states.
- (2) The states f_{m0}^R correspond to twisted right $m+1$ particles states, reducing, in the usual case, to right m particle states, while the states f_{0n}^L represent the twisted left $n+1$ particles states.
- (3) There are an infinite number of twisted right $m-k$ particle states and an infinite number of twisted left $n-k$ particle states given by $a^{k+1} \star f_{m0}^R$ and $f_{0n}^L \star \bar{a}^{k+1}$, respectively.

C. Matrix basis of the theory

The usual construction of a matrix basis⁸ exploits the \star -multiplication of f_{m0}^R with f_{0n}^L , i.e.,

$$L^2(\mathbb{R}_\theta^2) \ni b_{mn}^{(2)} =: \chi(\omega, \bar{\omega}, \theta, m, n) f_{m0}^R \star f_{0n}^L = \chi(\omega, \bar{\omega}, \theta, m, n) \frac{(\bar{a}^n \star f_{00}^R) \star (f_{00}^L \star a^n)}{\sqrt{m! n! \theta^{m+n}}}. \tag{42}$$

Without loss of generality, we set the normalization constant $\chi(\omega, \bar{\omega}, \theta, m, n) =: 1$ by convention. The corresponding eigenvalue problems are given by

$$H \star b_{mn}^{(2)} = \mathcal{E}_{m0}^R b_{mn}^{(2)} \quad \text{and} \quad b_{mn}^{(2)} \star H = \mathcal{E}_{0n}^L b_{mn}^{(2)}, \tag{43}$$

while the \star -actions of the annihilation and creation functions a and \bar{a} are reproduced as follows: $\bar{a} \star b_{mn}^{(2)} = \sqrt{\theta(m+1)} b_{m+1,n}^{(2)}$ and $b_{mn}^{(2)} \star a = \sqrt{\theta(n+1)} b_{m,n+1}^{(2)}$; with the basis fundamental state $b_{00}^{(2)} = f_{00}^R \star f_{00}^L$ satisfying the expected requirements $a \star b_{00}^{(2)} = 0$, $b_{00}^{(2)} \star \bar{a} = 0$. Given the (1,1)-particle states defined by $L^2(\mathbb{R}_\theta^2) \ni \Lambda_{mn}^{1,1} =: (a^m \star f_{m0}^R) \star (f_{0n}^L \star \bar{a}^n)$, their twisted spectra can be computed from the eigenvalue problems $H \star \Lambda_{mn}^{1,1} = \mathcal{E}_{\Lambda_{m0}^R} \Lambda_{mn}^{1,1}$ and $\Lambda_{mn}^{1,1} \star H = \mathcal{E}_{\Lambda_{0n}^L} \Lambda_{mn}^{1,1}$ to get, depending on the right and left Hamiltonian \star -actions,

$$\mathcal{E}_{\Lambda_{m0}^R} = \frac{\theta}{2} \left[1 - \frac{\bar{a}\bar{\omega}}{2} \left(\sum_{j=1}^m \frac{m+4j+1}{(m-j)!} + 4 \right) \right], \quad m > 0, \tag{44}$$

$$\mathcal{E}_{\Lambda_{0n}^L} = \mathcal{E}_{\Lambda_{n0}^R}(\bar{a} \leftrightarrow a, \bar{\omega} \leftrightarrow \omega) = \frac{\theta}{2} \left[1 - \frac{a\omega}{2} \left(\sum_{j=1}^n \frac{n+4j+1}{(n-j)!} + 4 \right) \right], \quad n > 0. \tag{45}$$

For $\omega=0$ and $\bar{\omega}=0$, these energies are reduced to the usual Moyal space matrix basis right and left fundamental energies. As needed, the Wick rotation can be used to ensure the real value of the energy. In the same vein, one can define the single twisted $(m-k+1)$ right particle states by $a^k \star f_{m0}^R =: \Lambda_{m0}^{m-k+1} \in L^2(\mathbb{R}_\theta^2)$ corresponding to the energy values obtained from the right Hamiltonian \star -action by

$$\mathcal{E}_{\Lambda_{m0}^{m-k+1}}^R = \frac{\theta}{2} \left\{ 2m - 2k + 1 - (m-k)a\omega + \frac{\bar{a}\bar{\omega}}{2} \left[(m-k-1)(m-k)! \sum_{j=1}^k \frac{m+4j+1}{(m-j)!} - (m-k)(m+4k+6) - 4 \right] - \frac{(m-k)(m-2k)\theta\omega}{4\bar{a}} + \frac{(m-k+1)(m-k)\theta\omega U_m}{m2^{m+1}\bar{a}} \right\}, \quad m \geq k > 0. \tag{46}$$

By analogy, the single twisted $(n-l+1)$ left particle states $f_{0n}^L \star \bar{a}^l =: \Lambda_{0n}^{n-l+1} \in L^2(\mathbb{R}_\theta^2)$ are associated with the left action energy values,

$$\mathcal{E}_{\Lambda_{0n}^{n-l+1}}^L = \mathcal{E}_{\Lambda_{n0}^R}(\bar{\omega} \leftrightarrow \omega, \bar{a} \leftrightarrow a) = \frac{\theta}{2} \left\{ 2n - 2l + 1 - (n-l)\bar{a}\bar{\omega} + \frac{a\omega}{2} \left[(n-l-1)(n-l)! \sum_{j=1}^l \frac{n+4j+1}{(n-j)!} - (n-l)(n+4l+6) - 4 \right] - \frac{(n-l)(n-2l)\theta\bar{\omega}}{4a} + \frac{(n-l+1)(n-l)\theta\bar{\omega}U_n}{n2^{n+1}a} \right\}, \quad n \geq l > 0. \tag{47}$$

Proposition 2.5: Energy spectra (46) and (47) of the mixed twisted $(m-k+1)$ right and $(n-l+1)$ left particle states $L^2(\mathbb{R}_\theta^2) \ni \Lambda_{mn}^{m-k+1,n-l+1} =: (a^k \star f_{m0}^R) \star (f_{0n}^L \star \bar{a}^l)$ solve the following respective eigenvalue problems:

$$H \star \Lambda_{mn}^{m-k+1,n-l+1} = \mathcal{E}_{\Lambda_{m0}^R} \Lambda_{mn}^{m-k+1,n-l+1}, \tag{48}$$

$$\Lambda_{mn}^{m-k+1,n-l+1} \star H = \mathcal{E}_{\Lambda_{0n}^L} \Lambda_{mn}^{m-k+1,n-l+1}. \tag{49}$$

We readily recover spectra (44) and (45) by replacing $m=k$ and $n=l$ in relations (46) and (47). Of course, in the limit regime, these energies also well reproduce the ordinary Moyal plane $(m-k)$ right and $(n-l)$ left particles energies,

$$\lim_{\omega, \bar{\omega} \rightarrow 0} \mathcal{E}_{\Lambda_{m0}^R} = \frac{\theta}{2} [2(m-k) + 1], \tag{50}$$

$$\lim_{\omega, \bar{\omega} \rightarrow 0} \mathcal{E}_{\Lambda_{0n}^{n-l+1}}^L = \frac{\theta}{2} [2(n-l) + 1], \quad (51)$$

respectively.

III. CONCLUDING REMARKS

We have investigated the main properties of harmonic oscillator in the framework of a dynamical noncommutativity realized through a twisted Moyal product. The construction of the appropriate matrix basis has introduced an x -dependence in the definition of the star product. We have computed the states and energies of the twisted harmonic oscillator. The degeneracy states and their energy have been explicitly derived. All examined quantities easily acquire good physical properties when ω_{12}^2 and x^1 are transformed into $i\omega_{12}^2$ and ix^1 , respectively. Furthermore, the ordinary Moyal space tools are well recovered as expected by setting $\omega=0$ and $\bar{\omega}=0$.

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APPENDIX A: CASE OF COMMUTING VECTOR FIELDS

Consider the coordinates base $e_a^\mu = \delta_a^\mu + \omega_{ab}^\mu x^b$ and the symmetric tensor (between the index a and b) ω_{ab}^μ . Then, the twisted star product is associative and

$$[X_a, X_b] = \omega_{ba}^\mu \partial_\mu - \omega_{ab}^\mu \partial_\mu = 0. \quad (A1)$$

The matrix representation of e_a^μ is given by

$$(e)_a^\mu = \begin{pmatrix} 1 + \omega_{11}^1 x^1 + \omega_{12}^1 x^2 & \omega_{11}^2 x^1 + \omega_{12}^2 x^2 \\ \omega_{12}^1 x^1 + \omega_{22}^1 x^2 & 1 + \omega_{12}^2 x^1 + \omega_{22}^2 x^2 \end{pmatrix} \quad (A2)$$

and

$$(e)_\mu^a = \begin{pmatrix} 1 - \omega_{11}^1 x^1 - \omega_{12}^1 x^2 & -\omega_{11}^2 x^1 - \omega_{12}^2 x^2 \\ -\omega_{12}^1 x^1 - \omega_{22}^1 x^2 & 1 - \omega_{12}^2 x^1 - \omega_{22}^2 x^2 \end{pmatrix}. \quad (A3)$$

Then,

$$e^{-1} = \det(e_a^\mu) = 1 + (\omega_{11}^1 + \omega_{12}^2)x^1 + (\omega_{22}^2 + \omega_{12}^1)x^2 = 1 + a\omega + \bar{a}\bar{\omega}$$

$$e = \det(e_\mu^a) = 1 - (\omega_{11}^1 + \omega_{12}^2)x^1 - (\omega_{22}^2 + \omega_{12}^1)x^2 = 1 - a\omega - \bar{a}\bar{\omega}, \quad (A4)$$

where $\omega = \frac{1}{\sqrt{2}}((\omega_{11}^1 + \omega_{12}^2) - i(\omega_{22}^2 + \omega_{12}^1))$ and $a = \frac{1}{\sqrt{2}}(x^1 + ix^2)$. The noncommutative tensor is given by $(\bar{\Theta})^{\mu\nu} = \theta e^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then, we can immediately perform the computation and check that the eigenvalue problem of the harmonic oscillator yields the same solution as for the case of the skew symmetric tensor ω_{ab}^μ , but now defining ω and $\bar{\omega}$ as

$$\omega = \frac{1}{\sqrt{2}}((\omega_{11}^1 + \omega_{12}^2) - i(\omega_{22}^2 + \omega_{12}^1)), \quad \bar{\omega} = \frac{1}{\sqrt{2}}((\omega_{11}^1 + \omega_{12}^2) + i(\omega_{22}^2 + \omega_{12}^1)). \quad (A5)$$

Besides, the matrix e_μ^a can be written as $e_\mu^a = \delta_\mu^a + \omega_{\mu b}^{ab} x_b$, where $\omega_{\mu}^{ab} = -\omega_{ab}^\mu$. Finally, the solution of the field equation $e_\mu^a = \partial_\mu \phi^a$ is well given by $\phi^a = x^a + \frac{1}{2} \omega_{\mu}^{ab} x_b x^\nu \partial_\nu^\mu$. From there follow all results obtained in the core of this work.

APPENDIX B: RIGHT AND LEFT \star -ACTIONS OF THE CREATION AND ANNIHILATION FUNCTIONS ONTO THE HO STATES

In this part, we derive the right and left \star -actions of the creation and annihilation functions onto the ho states.

$$\begin{aligned}
 a \star f_{m0}^R &= \frac{m2^{m-1}\bar{a}^{m-1}}{\sqrt{m!\theta^{m-2}}} \left(1 + \frac{m-2}{2}a\omega - \frac{m+5}{4}\bar{a}\bar{\omega} \right) f_{00}^R - \frac{(m-1)\theta\omega U_m \bar{a}^{m-2}}{4\sqrt{m!\theta^{m-2}}} f_{00}^R, \\
 a^2 \star f_{m0}^R &= \frac{m(m-1)2^{m-2}\bar{a}^{m-2}}{\sqrt{m!\theta^{m-4}}} \left[1 + \frac{m-4}{2}a\omega - \frac{(m+9)(m-1)+m+5}{4(m-1)}\bar{a}\bar{\omega} \right] f_{00}^R \\
 &\quad - \frac{(m-1)(m-2)\theta\omega U_m \bar{a}^{m-3}}{8\sqrt{m!\theta^{m-4}}} f_{00}^R, \vdots \\
 a^k \star f_{m0}^R &= \frac{m(m-1)\cdots(m-k+1)2^{m-k}\bar{a}^{m-k}}{\sqrt{m!\theta^{m-2k}}} \left[1 + \frac{m-2k}{2}a\omega - \frac{\bar{a}\bar{\omega}}{4}(m-k)! \sum_{j=1}^k \frac{(m+4j+1)}{(m-j)!} \right] f_{00}^R \\
 &\quad - \frac{(m-1)(m-2)\cdots(m-k)\theta\omega U_m \bar{a}^{m-k-1}}{2^{k+1}\sqrt{m!\theta^{m-2k}}} f_{00}^R, \text{ where } k \leq m, \\
 &\quad \vdots \\
 a^m \star f_{m0}^R &= \frac{m!}{\sqrt{m!\theta^m}} \left[1 - \frac{ma\omega}{2} - \frac{\bar{a}\bar{\omega}}{4} \sum_{j=1}^m \frac{(m+4j+1)}{(m-j)!} \right] f_{00}^R, \\
 a^{m+1} \star f_{m0}^R &= -\frac{\sqrt{m!\theta^{m+2}}\bar{\omega}}{8} \sum_{j=1}^m \frac{(m+4j+1)}{(m-j)!} f_{00}^R \propto f_{00}^R, \tag{B1} \\
 a^{m+2} \star f_{m0}^R &= 0. \tag{B2}
 \end{aligned}$$

Similarly, $f_{0n}^L \star \bar{a}, f_{0n}^L \star \bar{a}^2, \dots, f_{0n}^L \star \bar{a}^n, f_{0n}^L \star \bar{a}^{n+1}$ can be computed as

$$\begin{aligned}
 f_{0n}^L \star \bar{a} &= \frac{n2^{n-1}a^{n-1}}{\sqrt{n!\theta^{n-2}}} \left(1 + \frac{n-2}{2}\bar{a}\bar{\omega} - \frac{n+5}{4}a\omega \right) f_{00}^L - \frac{(n-1)\theta\bar{\omega} U_n a^{n-2}}{4\sqrt{n!\theta^{n-2}}} f_{00}^L, \\
 f_{0n}^L \star \bar{a}^2 &= \frac{n(n-1)2^{n-2}a^{n-2}}{\sqrt{n!\theta^{n-4}}} \left[1 + \frac{n-4}{2}\bar{a}\bar{\omega} - \frac{(n+9)(n-1)+n+5}{4(n-1)}a\omega \right] f_{00}^L \\
 &\quad - \frac{(n-1)(n-2)\theta\bar{\omega} U_n a^{n-3}}{8\sqrt{n!\theta^{n-4}}} f_{00}^L, \\
 &\quad \vdots \\
 f_{0n}^L \star \bar{a}^k &= \frac{n(n-1)\cdots(n-k+1)2^{n-k}a^{n-k}}{\sqrt{n!\theta^{n-2k}}} \left[1 + \frac{n-2k}{2}\bar{a}\bar{\omega} - \frac{a\omega}{4}(n-k)! \sum_{j=1}^k \frac{(n+4j+1)}{(n-j)!} \right] f_{00}^L \\
 &\quad - \frac{(n-1)(n-2)\cdots(n-k)\theta\bar{\omega} U_n a^{n-k-1}}{2^{k+1}\sqrt{n!\theta^{n-2k}}} f_{00}^L,
 \end{aligned}$$

where $k \leq n$,

⋮

$$f_{0n}^L \star \bar{a}^n = \frac{n!}{\sqrt{n!} \theta^{-n}} \left[1 - \frac{n\bar{a}\bar{\omega}}{2} - \frac{a\omega}{4} \sum_{j=1}^n \frac{(n+4j+1)}{(n-j)!} \right] f_{00}^L,$$

$$f_{0n}^L \star \bar{a}^{n+1} = -\frac{\sqrt{n!} \theta^{n+2} \omega}{8} \sum_{j=1}^n \frac{(n+4j+1)}{(n-j)!} f_{00}^L \propto f_{00}^L, \quad (\text{B3})$$

$$f_{0n}^L \star \bar{a}^{n+2} = 0. \quad (\text{B4})$$

APPENDIX C: USEFUL IDENTITIES

$$\partial_a^k f_{00}^R = \left[k(k-1)\omega \left(-\frac{2\bar{a}}{\theta}\right)^{k-1} + \left(-\frac{2\bar{a}}{\theta e^{-1}}\right)^k \left(1 + ka\omega - \frac{k\bar{a}\bar{\omega}}{2}\right) \right] f_{00}^R, \quad (\text{C1})$$

$$\partial_{\bar{a}}^k f_{00}^R = \left[\frac{k(k-1)}{2} \bar{\omega} \left(-\frac{2a}{\theta}\right)^{k-1} + \left(-\frac{2a}{\theta e^{-1}}\right)^k \right] f_{00}^R, \quad (\text{C2})$$

$$\partial_a^k f_{00}^L = \left[k(k-1)\bar{\omega} \left(-\frac{2a}{\theta}\right)^{k-1} + \left(-\frac{2a}{\theta e^{-1}}\right)^k \left(1 + k\bar{a}\bar{\omega} - \frac{ka\omega}{2}\right) \right] f_{00}^L, \quad (\text{C3})$$

$$\partial_{\bar{a}}^k f_{00}^L = \left[\frac{k(k-1)}{2} \omega \left(-\frac{2\bar{a}}{\theta}\right)^{k-1} + \left(-\frac{2\bar{a}}{\theta e^{-1}}\right)^k \right] f_{00}^L, \quad (\text{C4})$$

$$\partial_a^k \left(-\frac{2a}{\theta e^{-1}}\right)^l = kl\omega \frac{l!}{(l-k+1)!} \left(-\frac{2}{\theta}\right)^{k-1} \left(-\frac{2a}{\theta}\right)^{l-k+1} + \frac{l!}{(l-k)!} \left(-\frac{2}{\theta}\right)^k \left(-\frac{2a}{\theta}\right)^{l-k} e^l, \quad (\text{C5})$$

$$\partial_{\bar{a}}^k \left(-\frac{2\bar{a}}{\theta e^{-1}}\right)^l = kl\bar{\omega} \frac{l!}{(l-k+1)!} \left(-\frac{2}{\theta}\right)^{k-1} \left(-\frac{2\bar{a}}{\theta}\right)^{l-k+1} + \frac{l!}{(l-k)!} \left(-\frac{2}{\theta}\right)^k \left(-\frac{2\bar{a}}{\theta}\right)^{l-k} e^l. \quad (\text{C6})$$

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