Appl. Comput. Harmon. Anal. 62 (2023) 237–250

Contents lists available at ScienceDirect

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

On the frame set of the second-order B-spline

A. Ganiou D. Atindehou [a], Christina Frederick [b], Yébéni B. Kouagou [1], Kasso A. Okoudjou [c,*]

[a] *Département de Mathématiques, Université d'Abomey Calavi, 01 BP: 4521, Cotonou, Bénin*
[b] *Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 07102, United States of America*
[c] *Department of Mathematics, Tufts University, Medford, MA 02155, United States of America*

A R T I C L E I N F O

Article history:
Received 23 May 2022
Received in revised form 9 August 2022
Accepted 30 August 2022
Available online 6 September 2022
Communicated by Radu Balan

MSC:
primary 42C15
secondary 42C40

Keywords:
Gabor frames
Frame set
B-splines

A B S T R A C T

The frame set of a function $g \in L^2(\mathbb{R})$ is the set of all parameters $(a, b) \in \mathbb{R}^2_+$ for which the collection of time-frequency shifts of g along $a\mathbb{Z} \times b\mathbb{Z}$ form a Gabor frame for $L^2(\mathbb{R})$. Finding the frame set of a given function remains a challenging open problem in time-frequency analysis. In this paper, we establish new regions of the frame set of the second-order B-spline. Our method uses the compact support of this function to partition a subset of the putative frame set and finds an explicit dual window function in each subregion. Numerical evidence indicates the existence of further regions belonging to the frame set.

1. Introduction and main results

Given a window function $g \in L^2(\mathbb{R})$ and constants $a, b > 0$, the collection of time-frequency shifts

$$\mathcal{G}(g, a, b) = \left\{ M_{\ell b} T_{ka} g = e^{2\pi i \ell b \cdot} g(\cdot - ka) : (\ell, k) \in \mathbb{Z}^2 \right\}$$

is called a Gabor frame for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{\ell, k \in \mathbb{Z}} |\langle f, M_{\ell b} T_{ka} g \rangle|^2 \le B\|f\|_2^2,$$

* Corresponding author.
E-mail addresses: ganiou.atindehou@fast.uac.bj (A.G.D. Atindehou), christin@njit.edu (C. Frederick), kasso.okoudjou@tufts.edu (K.A. Okoudjou).

[1] Yébéni B. Kouagou suddenly passed away in 2018, a few weeks after the first version of this work was released. This version is dedicated to his memory.

https://doi.org/10.1016/j.acha.2022.08.007
1063-5203/

for all $f \in L^2(\mathbb{R})$. When $\mathcal{G}(g,a,b)$ is a Gabor frame, there exists a dual window $\gamma \in L^2(\mathbb{R})$ such that $\mathcal{G}(\gamma,a,b)$ is also a Gabor frame for $L^2(\mathbb{R})$ called the (canonical) dual to $\mathcal{G}(g,a,b)$. Consequently, for any $f \in L^2(\mathbb{R})$ we have the following reconstruction formulas:

$$f = \sum_{k\ell\in\mathbb{Z}} \langle f, M_{\ell b}T_{ka}\gamma\rangle M_{\ell b}T_{ka}g = \sum_{k\ell\in\mathbb{Z}} \langle f, M_{\ell b}T_{ka}g\rangle M_{\ell b}T_{ka}\gamma.$$

We refer to [3,10] for more on Gabor frame theory.

Despite the outstanding advances in the theory and applications of Gabor frames over the last three decades, the problem of characterizing the set of all points $(a,b) \in \mathbb{R}^2_+$ such that $\mathcal{G}(g,a,b)$ is a Gabor frame for a given window $g \in L^2(\mathbb{R})$ remains largely unresolved. This set is known as the *frame set* of g and will be denoted by $\mathcal{F}(g)$. The current state-of-the-art result in this direction states that if g is either:

(1) in $\{e^{-\pi x^2}, \frac{1}{\cosh x}, \chi_{[0,\infty)}e^{-x}, e^{-|x|}\}$, or
(2) a totally positive function of finite type, or
(3) a totally positive function of exponential type

then $\mathcal{F}(g) = \{(a,b) \in \mathbb{R}^2_+, ab < 1\}$ (see [11,13,18,15,17,16,21–23]). At the same time, the frame set of $g = \chi_{[0,c]}$, known as the Janssen's tie [17] has a more complex structure that was fully described by Dai and Sun [7,14].

In this paper we consider the frame set of the B-spline of order 2:

$$B_2(x) = \begin{cases} 1+x & x \in [-1,0] \\ 1-x & x \in [0,1]. \end{cases}$$

It is known that $\mathcal{F}(B_2)$ is an open set in $\mathbb{R}^2_+$ [9], but a full characterization of this set remains an open question. To date, it is known that

$$\left\{(a,b) \in \mathbb{R}^2_+ : ab < 1, 0 < a < 2, 0 < b \le \max_a \left(1, \frac{4}{2+3a}\right)\right\} \subset \mathcal{F}(B_2)$$

(see Fig. 1 for a sketch and [2,6,5,3,12,20,19] for details).

In an earlier work [2], we introduced a framework for determining the frame sets of compactly supported functions, including the B-splines of order $N \ge 2$. This framework unified many of the known results on frame sets of B-splines [6,5,3,12,20,19]. In this paper, we use a similar linear algebra based approach to shed new light on set $\mathcal{F}(B_2)$ and prove the following result

Theorem 1. *The frame set of the second-order B-spline contains the set $\Gamma_3 \cup \Lambda$, that is,*

$$\mathcal{F}(B_2) \supset \Gamma_3 \cup \Lambda, \qquad \Lambda := \bigcup_{m=3}^{\infty} \Lambda_m,$$

where

$$\Gamma_3 := \left\{(a,b) \in \mathbb{R}^2_+ : a \in \left(0, \frac{2}{9}\right] \cup \left(\frac{2}{7}, \frac{1}{2}\right), b \in \left(\frac{4}{2+3a}, \frac{2}{1+a}\right]\right\}$$

$$\Lambda_3 := \left\{(a,b) \in \mathbb{R}^2_+ : a \in \left[\frac{1}{2}, \frac{4}{5}\right], b \in \left(\frac{4}{2+3a}, \frac{6}{2+5a}\right], b > 1\right\},$$

and for $m \ge 4$

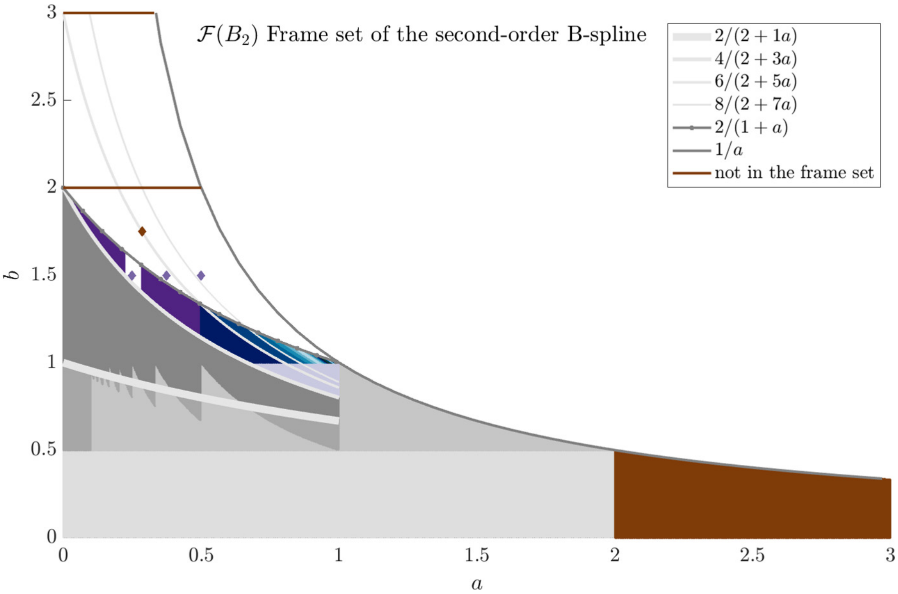

Fig. 1. Sketch depicting some of the known results on the frame set of B_2. The shaded regions in gray were proved to be in $\mathcal{F}(B_2)$ in [1,5,6,8,19,20] and the light purple region was established as a subset of $\mathcal{F}(B_2)$ in [2]. Points that are brown are known not to be in $\mathcal{F}(B_2)$ (the case $b = 2$ was shown in [12]). Theorem 1 asserts that the union of Γ_3 (dark purple) and $\cup_{m=3}^{\infty}\Lambda_m$ (shades of blue) belong to $\mathcal{F}(B_2)$. We show that the additional points marked with purple diamonds are in $\mathcal{F}(B_2)$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\Lambda_m = \Big\{(a,b) \in \mathbb{R}_+^2 : a \in \left[\frac{m-3}{m-2}, \frac{2(m-1)}{2m-1}\right] \\ b \in \left(\frac{2(m-1)}{2+(2m-3)a}, \min_a\left(\frac{2m}{2+(2m-1)a}, \frac{2}{1+a}\right)\right], b > 1\Big\}.$$

The dark purple regions in Fig. 1 illustrates Γ_3, and the sets Λ_m, $m \geq 3$, which form a partition of Λ, are shown in varying shades of blue. The sets Λ and Γ_3 appear naturally in the region $\{(a,b) \in \mathbb{R}_+^2 : ab < 1\}$. Any $(a,b) \in \Lambda$ is associated with a unique $m \geq 3$ so that $(a,b) \in \Lambda_m$, and as $ab \to 1^-$, with $(a,b) \in \Lambda$, this value of m increases. Furthermore, while the methods presented here do not fully characterize $\mathcal{F}(B_2)$, they allow us to analytically confirm putative points in the frame set and numerically suggest new regions to analyze. In addition, while we prove theory for second-order B-splines, the methods presented here also extend to higher-order B-splines.

The proof of Theorem 1 is divided in two parts: Theorem 2 (concerning the regions Λ_m) and Theorem 3 (concerning the region Γ_3), and is based on methods developed in [2, Proposition 2][2] which give a necessary and sufficient condition on two Bessel Gabor systems to be dual. We outline the method before giving its full details in the next two sections.

It is known that (see, [4, Theorem 2.10]) for $g, h \in L^2(\mathbb{R})$ and $a, b > 0$, the Bessel sequences $\mathcal{G}(g,a,b)$ and $\mathcal{G}(h,a,b)$ are dual frames for $L^2(\mathbb{R})$ if and only if

$$\sum_{k\in\mathbb{Z}} \overline{g(x-\ell/b+ka)}h(x+ka) = b\delta_{\ell,0}, \forall \ell \in \mathbb{Z} \text{ and a.e. } x \in \left[-\frac{a}{2}, \frac{a}{2}\right]. \tag{1}$$

In our case, we use $g = B_2$ and seek among the set of dual windows a bounded and compactly supported function h for which (1) holds. Subsequently, the heart of our proof uses the compact support of the window B_2 and this dual h to simplify both the infinite sum over $k \in \mathbb{Z}$ and also the condition $\forall \ell \in \mathbb{Z}$.

[2] The important assumption that $\frac{2(m-1)}{N+(2m-3)a} < b \leq \frac{2m}{N+(2m-1)a}$, while not explicitly stated, is automatically satisfied by the choice of $(a,b) \in T_m$ in [2, Proposition 2].

In particular, we assume that the bounded and real-valued function h is supported on $[-\frac{(2m-1)a}{2}, \frac{(2m-1)a}{2}]$, where $(a,b) \in \Lambda_m$ for $m \geq 3$. It follows that the Gabor systems $\mathcal{G}(B_2, a, b)$ and $\mathcal{G}(h, a, b)$ are dual frames for $L^2(\mathbb{R})$ if and only if

$$\sum_{k=1-m}^{m-1} B_2(x - \ell/b + ka)h(x + ka) = b\delta_{\ell,0}, \forall \ell \in \mathbb{Z} \text{ and a.e. } x \in \left[-\frac{a}{2}, \frac{a}{2}\right]. \tag{2}$$

Then, although in [4] the result is stated for all $\ell \in \mathbb{Z}$, in the present setting we can replace this with the condition $|\ell| \leq m-1$ using the following argument. Observe that $B_2(x) \neq 0$ if and only if $-1 < x < 1$, and therefore the left-hand side of (2) is non-zero if and only if $|\ell| < \frac{2+(2m-1)a}{2}b$. Since the condition $(a,b) \in \Lambda_m$ implies that $b \leq \frac{2m}{2+(2m-1)a}$, the left-hand side of (2) is non-zero if and only if $|\ell| < \frac{2+(2m-1)a}{2}b \leq m$. In summary, the Gabor systems $\mathcal{G}(B_2, a, b)$ and $\mathcal{G}(h, a, b)$ are dual frames for $L^2(\mathbb{R})$ if and only if

$$\sum_{k=1-m}^{m-1} B_2(x - \ell/b + ka)h(x + ka) = b\delta_{\ell,0}, |\ell| \leq m-1\,, \quad \text{a.e. } x \in [-\frac{a}{2}, \frac{a}{2}]. \tag{3}$$

The main goal of this paper is to show the existence of a bounded compactly supported function $h \in L^2(\mathbb{R})$ that solves (3) when $(a,b) \in \Lambda_m$, for $m \geq 3$. To do this, we rewrite (3) as a matrix-vector equation using the compact support of B_2,

$$G_m H_m = \begin{pmatrix} F_{m-1} & A_{m-1} \\ 0 & C_{m-1} \end{pmatrix} H_m = V \tag{4}$$

where G_m is a $(2m-1) \times (2m-1)$ matrix, and the column vectors H_m and V are given by

$$H_m(x) = [h(x + ka)]_{|k| \leq m-1}\,, \qquad V = [b\delta_{\ell,0}]_{|\ell| \leq m-1}.$$

Furthermore, C_{m-1} is a $(m-2) \times (m-2)$ upper triangular matrix with strictly positive diagonal entries, and F_{m-1} is the $(m+1) \times (m+1)$ tridiagonal matrix given by

$$F_{m-1}(x) = \left[B_2(x - \frac{\ell}{b} + ka)\right]_{1-m \leq \ell, k \leq 1}. \tag{5}$$

We observe that the diagonal entries of F_{m-1} are also strictly positive. Therefore, the existence of a dual is guaranteed when G_m has a strictly positive determinant, or equivalently (because of its structure), that the tridiagonal matrix F_{m-1} has a strictly positive determinant.

Although recursive formulas for computing the determinants of tridiagonal matrices exist, the present case requires more than showing these determinants are nonzero. As such, we exploit the properties of B_2 to prove recursively that the determinant of F_{m-1} is strictly positive. This allows us to solve explicitly for the bounded and compactly supported function h that generates a Bessel system and is dual to g. We carry out this analysis in Section 2, establishing the invertibility of the matrices when $(a,b) \in \Lambda$.

To deal with the region Γ_3, we effectively compute the determinant a 4×4 matrix that models (3). However, in this case the matrix is no longer tridiagonal. Nonetheless, we again exploit the structure of B_2 to show that the determinant of this matrix is also strictly positive which allows us to find a compactly supported and bounded function h which generates a dual Gabor frame to $\mathcal{G}(B_2, a, b)$. This is established in Section 3.

Notation. In the sequel, given a $p \times p$ matrix A and $E \subset \{1, 2, 3, \ldots, p\}$, we denote by A^E the $\#E \times \#E$ sub-matrix of A using rows and columns from E, and denote by $|A|$ the determinant of the matrix A. We also denote by $g_{l,k}$ the function

$$g_{l,k}(x) := B_2(x - \frac{l}{b} + ka).$$

2. $\mathcal{F}(B_2)$ contains Λ

In this section we prove the first part of Theorem 1 by establishing the following result.

Theorem 2. *For $m \geq 3$, let $(a, b) \in \Lambda_m$. Then, the Gabor system $\mathcal{G}(B_2, a, b)$ is a frame for $L^2(\mathbb{R})$, and there is a unique dual window $h \in L^2(\mathbb{R})$ such that $supp(h) \subseteq \left[-\frac{2m-1}{2}a, \frac{2m-1}{2}a\right]$. Furthermore, for each $(a, b) \in \Lambda$, the Gabor system $\mathcal{G}(B_2, a, b)$ is a frame for $L^2(\mathbb{R})$.*

To prove Theorem 2 we only need to show that (3) has a solution $h \in L^2(\mathbb{R})$ that is a bounded and compactly supported function. As mentioned earlier, the determinant of the (block) matrix G_m is

$$|G_m| = |F_{m-1}||C_{m-1}| = |F_{m-1}| \prod_{1-m}^{-2} g_{-k,-k}$$

where we used the fact C_{m-1} is an upper triangular matrix. Because for each $k = 1-m, \dots, -2$, $g_{-k,-k} > 0$ on $[-\frac{a}{2}, 0]$, we turn our attention to establishing that the determinant $|F_{m-1}|$ of this tridiagonal matrix is strictly positive when $(a, b) \in \Lambda_m$.

Proposition 1. *For $m \geq 3$, let $(a, b) \in \cup_{k=m}^{\infty} \Lambda_k$. Then $|F_{m-1}| > 0$ on $\left[-\frac{a}{2}; 0\right]$.*

We will prove the result by showing that $|F_{m-1}|$ never vanishes on $[-\frac{a}{2}, 0]$. Since this matrix is a tri-diagonal, we could rely on standard formulas to find its determinant. However, the challenging part is to establish that the determinant is never 0. We will do this by an induction argument on m, relying on the fact that B_2 is a compactly supported piecewise linear function. In Lemma 1 we prove the result for the base case $m = 3$, and complete the induction argument in Lemma 2.

Lemma 1. *Let $(a, b) \in \cup_{k=3}^{\infty} \Lambda_k$. Then, $|F_2| > 0$ on $[-\frac{a}{2}, 0]$.*

Proof. We note that for $(a, b) \in \cup_{k=3}^{\infty} \Lambda_k$

$$F_2 = \begin{pmatrix} g_{-2,-2} & g_{-2,-1} & 0 & 0 \\ g_{-1,-2} & g_{-1,-1} & g_{-1,0} & 0 \\ 0 & g_{0,-1} & g_{0,0} & g_{0,1} \\ 0 & 0 & g_{1,0} & g_{1,1} \end{pmatrix}. \tag{6}$$

In each case below, we write $[-\frac{a}{2}, 0]$ as a union of intervals and show that $|F_2| > 0$ on each of them. Fig. 2 shows plots of the four regions of $\cup_{k=3}^{\infty} \Lambda_k$ considered (left) as well as the partitions of the interval $[-\frac{a}{2}, 0]$ (right).

(I) For $a \in [\frac{1}{2}, \frac{2}{3})$ and $2 - \frac{3}{b} + a \leq 0$,

$$[-\frac{a}{2}, 0] = [-\frac{a}{2}, 1 - \frac{2}{b} + a) \cup [1 - \frac{2}{b} + a, -1 + \frac{1}{b}] \cup (-1 + \frac{1}{b}, 0].$$

On the first interval, $g_{1,0} = 0$, and since $g_{-1,-1} > g_{-1,-2}$, $g_{-1,-1} \geq g_{-1,0}$ and

$$\begin{vmatrix} g_{-2,-2} & 0 \\ 0 & g_{0,0} \end{vmatrix} > \begin{vmatrix} g_{-2,-2} & g_{-2,-1} \\ 0 & g_{0,-1} \end{vmatrix} + \begin{vmatrix} g_{-2,-1} & 0 \\ g_{0,-1} & g_{0,0} \end{vmatrix},$$

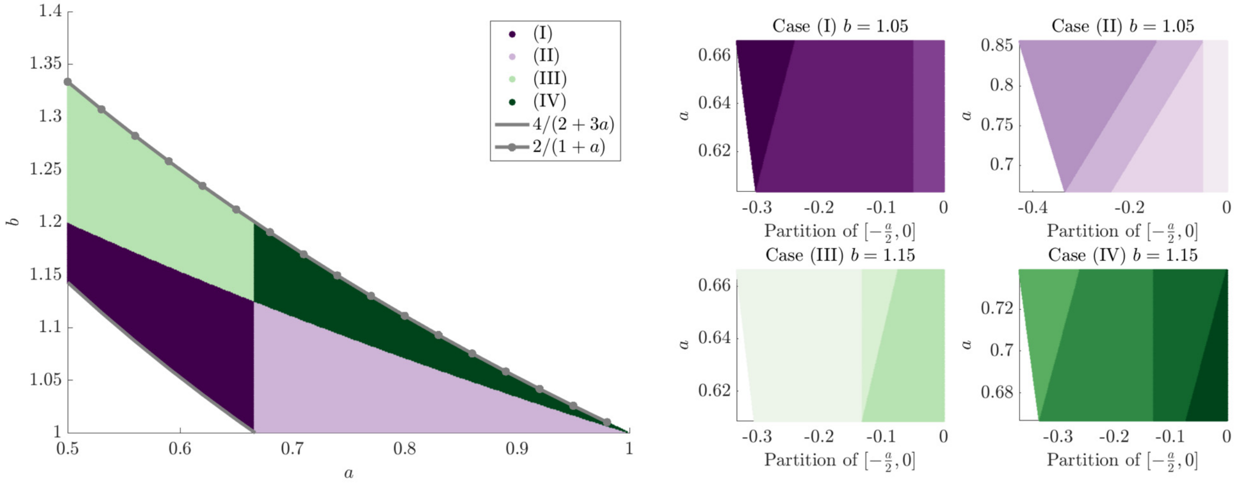

Fig. 2. The frame set of B_2 established in Lemma 1 by partitioning $\cup_{k=3}^{\infty}\Lambda_k$ into the four regions shown in the left plot. In each region, for each fixed b, the interval $[-\frac{a}{2},0]$ is subdivided into three intervals (Cases (I) and (III)) or four intervals (Cases (II) and (IV)) as depicted by the shading in the plots on the right.

it follows that $|F_2| = g_{1,1}|F_2^{\{1,2,3\}}| > 0$. On the second interval, $g_{2,1} = g_{1,0} = 0$, and since $g_{-1,-1} > g_{-1,0}$ and $g_{0,-1} < g_{0,0}$, it follows that $|F_2| = g_{-2,-2}g_{1,1}|F_2^{\{2,3\}}| > 0$. On the third interval, $g_{-2,-1} = 0$, and since $g_{0,0} > \max\{g_{0,-1}, g_{0,1}\}$, and

$$\begin{vmatrix} g_{-1,-1} & 0 \\ 0 & g_{1,1} \end{vmatrix} > \begin{vmatrix} g_{-1,-1} & g_{-1,0} \\ 0 & g_{1,0} \end{vmatrix} + \begin{vmatrix} g_{-1,0} & 0 \\ g_{1,0} & g_{1,1} \end{vmatrix},$$

it follows that $|F_2| = g_{-2,-2}|F_2^{\{2,3,4\}}| > 0$.

(II) For $a \in [\frac{2}{3}, 1]$ and $2 - \frac{3}{b} + a \le 0$,

$$[-\frac{a}{2}, 0] = [-\frac{a}{2}, a-1] \cup (a-1, 1-\frac{2}{b}+a) \cup [1-\frac{2}{b}+a, \frac{1}{b}-1] \cup (\frac{1}{b}-1, 0].$$

On the first interval, we have $g_{0,-1} = g_{1,0} = 0$, and since $g_{-2,-2} > g_{-2,-1}$, $g_{-1,-2} < g_{-1,-1}$, it follows that $|F_2| = \prod_{\ell=0}^{1} g_{\ell,\ell}|F_2^{\{1,2\}}| > 0$. The last three intervals are treated as in the previous case.

(III) For $[\frac{1}{2}, \frac{2}{3})$ and $2 - \frac{3}{b} + a > 0$,

$$[-\frac{a}{2}, 0] = [-\frac{a}{2}, -1+\frac{1}{b}] \cup (-1+\frac{1}{b}, 1-\frac{2}{b}+a) \cup [1-\frac{2}{b}+a, 0].$$

On the first interval, $g_{1,0} = 0$. As in (I), we conclude that $|F_2| = g_{1,1}|F_2^{\{1,2,3\}}| > 0$. On the second interval, we have $g_{0,0} > g_{0,-1}$, $g_{0,0} > g_{0,1}$, $|F_2^{\{1,2\}}| > 0$ and $|F_2^{\{1,2,4\}}| > g_{1,0}|F_2^{\{1,2\}}| + g_{-2,-2}g_{-1,0}g_{1,1}$, and therefore

$$|F_2| = \begin{vmatrix} g_{-2,-2} & g_{-2,-1} & 0 & 0 \\ g_{-1,-2} & g_{-1,-1} & g_{-1,0} & 0 \\ 0 & g_{0,-1} & g_{0,0} & g_{0,1} \\ 0 & 0 & g_{1,0} & g_{1,1} \end{vmatrix} > 0.$$

On the third interval, $g_{-2,-1} = 0$. As in part (I), we conclude $|F_2| = g_{-2,-2}|F_2^{\{2,3,4\}}| > 0$.

(IV) For $a \in [\frac{2}{3}, 1]$ and $2 - \frac{3}{b} + a > 0$,

$$[-\frac{a}{2}, 0] = [-\frac{a}{2}, -1+a] \cup (-1+a, -1+\frac{1}{b}] \cup (-1+\frac{1}{b}, 1-\frac{2}{b}+a) \cup [1-\frac{2}{b}+a, 0].$$

On the first interval, we have $g_{0,-1} = g_{1,0} = 0$, and therefore $|F_2| = g_{1,1}g_{0,0}|F_2^{\{1,2\}}| > 0$. The remaining three intervals can be treated as above. □

We present the induction step in the following lemma.

Lemma 2. *Suppose that for some $m \geq 3$, $|F_{m-1}| > 0$ on $[-\frac{a}{2}, 0]$ for each $(a, b) \in \Lambda_k$, for all $k \geq m$. Then $|F_m| > 0$ on $[-\frac{a}{2}, 0]$ for $(a, b) \in \Lambda_k$, and all $k \geq m + 1$.*

Proof. We first prove that $|F_m| > 0$ on $[-\frac{a}{2}, 0]$ for $(a, b) \in \Lambda_{m+1}$ in the following four cases:

I) For $a \in [\frac{m-2}{m-1}, \frac{4m-2}{4m+1}]$ and $2 - \frac{3}{b} + a \leq 0$, then $-1 - \frac{m-2}{b} + (m-1)a \leq -\frac{a}{2}$. In this case,

$$\left[-\frac{a}{2}, 0\right] = \left[-\frac{a}{2}, 1 - \frac{m}{b} + (m-1)a\right) \cup \left[1 - \frac{m}{b} + (m-1)a, 0\right].$$

On the first interval, $g_{-k,-(k+1)} = 0$ for all $k \in \{-1, ..., m-3\}$. Therefore, since $g_{-m+1,-m} < g_{-m+1,-m+1}, g_{-m+1,-m+1} > g_{-m+1,-m+2}$, and

$$\begin{vmatrix} g_{-m,-m} & 0 \\ 0 & g_{-m+2,-m+2} \end{vmatrix} > \begin{vmatrix} g_{-m,-m+1} & 0 \\ g_{-m+2,-m+1} & g_{-m+2,-m+2} \end{vmatrix} + \begin{vmatrix} g_{-m,-m} & g_{-m,-m+1} \\ 0 & g_{-m+2,-m+1} \end{vmatrix},$$

it follows that $|F_m| = \prod_{\ell=-1}^{m-3} g_{\ell,\ell}|F_m^{\{1,2,3\}}| > 0$. On the second interval, $g_{-m,-(m-1)} = 0$. The induction assumption implies that $|F_m| = g_{-m,-m}|F_{m-1}| > 0$.

II) For $a \in (\frac{2m-2}{2m-1}, \frac{2m}{2m+1}]$ and $2 - \frac{3}{b} + a \leq 0$, we have $-\frac{a}{2} \leq -1 - \frac{(m-2)}{b} + (m-1)a$, and

$$\begin{aligned}[-\frac{a}{2}, 0] = [&-\frac{a}{2}, -1 - \frac{m-2}{b} + (m-1)a] \\ &\cup (-1 - \frac{m-2}{b} + (m-1)a, 1 - \frac{m}{b} + (m-1)a) \\ &\cup [1 - \frac{m}{b} + (m-1)a, 0].\end{aligned}$$

On the first interval, $g_{-k,-(k+1)} = 0$ for all $k \in \{-1, ..., m-2\}$. Since $F_m^{\{1,2\}}$ is diagonally dominant, it follows that $|F_m| = \prod_{\ell=-1}^{m-2} g_{-\ell,-\ell}|F_m^{\{1,2\}}| > 0$. On the second interval, $g_{-k,-(k+1)} = 0$ for all $k \in \{-1, ..., m-3\}$. As in part (I), we conclude $|F_m| = \prod_{\ell=-1}^{m-3} g_{-\ell,-\ell}|F_m^{\{1,2,3\}}| > 0$. On $\left[1 - \frac{m}{b} + (m-1)a, 0\right]$, $g_{-m,-(m-1)} = 0$. Consequently $|F_m| = g_{-m,-m}|F_{m-1}| > 0$ by the induction assumption.

III) For $a \in (\frac{4m-2}{4m+1}, \frac{2m-2}{2m-1})$ and $2 - \frac{3}{b} + a \leq 0$, the quantity $-1 - (m-2)/b + (m-1)a + \frac{a}{2}$ can be either positive or negative, falling into the categories of (I) and (II).

IV) If $2 - \frac{3}{b} + a > 0$, then $a \in [\frac{m-2}{m-1}, \frac{4m-2}{4m+1}]$ and

$$\begin{aligned}\left[-\frac{a}{2}, 0\right] = &\left[-\frac{a}{2}, -1 - \frac{m-3}{b} + (m-2)a\right] \\ &\bigcup \left(-1 - \frac{m-3}{b} + (m-2)a, 1 - \frac{m}{b} + (m-1)a\right) \\ &\bigcup \left[1 - \frac{m}{b} + (m-1)a, 0\right].\end{aligned}$$

On the first interval, $g_{-k,-(k+1)} = 0$ for all $k \in \{-1, ..., m-3\}$. A in part (I), we have $|F_m| = \prod_{\ell=-1}^{m-3} g_{-\ell,-\ell}|F_m^{\{1,2,3\}}| > 0$. On the second interval, $g_{-k,-(k+1)} = 0$ for all $k \in \{-1, ..., m-4\}$. We then have $F_m^{\{1,2,3,4\}}(x) = F_2(x + \frac{m-2}{b} - (m-2)a)$, where F_2 is the 4×4 matrix (6). Hence,

$$|F_m| = \prod_{\substack{\ell=-1 \\ \ell-m\neq -3,-2,-1,0}}^{m} g_{-\ell,-\ell} |F_m^{\{1,2,3,4\}}| > 0.$$

On $\left[1 - \frac{m}{b} + (m-1)a, 0\right]$, $g_{-m,-(m-1)} = 0$. Hence, $|F_m| = g_{-m,-m}|F_{m-1}| > 0$ by the induction assumption.

To establish the result for $(a, b) \in \Lambda_k$, $k \geq m + 2$, we prove that $|F_m| > 0$ on each interval in a partition of $\left[-\frac{a}{2}, 0\right]$ and reduce the analysis to the case $(a, b) \in \Lambda_{m+1}$. We omit details of the proof, only indicating the relevant partitions of $[-\frac{a}{2}, 0]$.

For $2 - \frac{3}{b} + a \leq 0$ we have the following partition:

$$\left[\frac{k-3}{k-2}, \frac{2(k-1)}{2k-1}\right] = \left[\frac{k-3}{k-2}, \frac{4k-6}{4k-3}\right] \bigcup \left(\frac{4k-6}{4k-3}; \frac{2k-4}{2k-3}\right) \bigcup \left[\frac{2k-4}{2k-3}, \frac{2(k-1)}{2k-1}\right].$$

(I) If $a \in \left[\frac{k-3}{k-2}, \frac{4k-6}{4k-3}\right]$, then $-1 - \frac{k-3}{b} + (k-2)a \leq -\frac{a}{2}$, and we write

$$[-\frac{a}{2}, 0] = \bigcup_{\ell=1}^{k-2} (Q_\ell \cup T_\ell) \cup T_{k-1}$$

where $T_1 = (-1 + \frac{1}{b}, 0]$, $T_{k-1} = [-\frac{a}{2}, 1 - \frac{k-1}{b} + (k-2)a)$, and for $\ell = 2, \ldots, k-2$ we have $T_\ell = (-1 - \frac{\ell-2}{b} + (\ell-1)a, 1 - \frac{\ell}{b} + (\ell-1)a)$; while for $\ell = 1, \ldots, k-2$ we have $Q_\ell = [1 - \frac{\ell+1}{b} + \ell a, -1 - \frac{\ell-2}{b} + (\ell-1)a]$.

(II) If $a \in \left[\frac{2k-4}{2k-3}, \frac{2(k-1)}{2k-1}\right]$, then $-1 - \frac{k-3}{b} + (k-2)a \geq -\frac{a}{2}$, and we have the partition

$$[-\frac{a}{2}, 0] = \bigcup_{\ell=1}^{k-1} (Q_\ell \cup T_\ell)$$

where $T_1 = (-1 + \frac{1}{b}, 0]$, for $\ell = 2, \ldots, k-1$, $T_\ell = (-1 - \frac{\ell-2}{b} + (\ell-1)a, 1 - \frac{\ell}{b} + (\ell-1)a)$, while $\ell = 1, \ldots, k-2$, $Q_\ell = [1 - \frac{\ell+1}{b} + ka, -1 - \frac{\ell-2}{b} + (\ell-1)a]$, and $Q_{k-1} = [-\frac{a}{2}, -1 - \frac{k-3}{b} + (k-2)a]$.

(III) Finally, if $a \in \left(\frac{4k-6}{4k-3}; \frac{2k-4}{2k-3}\right)$, then the sign of $-1 - \frac{k-3}{b} + (k-2)a + \frac{a}{2}$ is not constant. However, by considering the case in which this expression is either positive or negative, we can reduce the analysis to one of the two previous cases.

For $2 - \frac{3}{b} + a > 0$, we have $a \in \left[\frac{k-3}{k-2}, \frac{4k-6}{4k-3}\right]$ and

$$\left[-\frac{a}{2}, 0\right] = \left(\bigcup_{\ell=1}^{k-1} \widetilde{Q}_\ell\right) \cup \left(\bigcup_{\ell=2}^{k-1} \widetilde{T}_\ell\right)$$

where

$$\widetilde{Q}_\ell = \left[1 - \frac{\ell+1}{b} + ka, -1 - \frac{\ell-3}{b} + (\ell-2)a\right] \text{ and } \widetilde{T}_\ell = (-1 - \frac{\ell-3}{b} + (\ell-2)a, 1 - \frac{\ell}{b} + (\ell-1)a),$$

with the convention that $\widetilde{Q}_{k-1} = [-\frac{a}{2}, -1 - \frac{k-4}{b} + (k-3)a]$ and $\widetilde{Q}_1 = \left[1 - \frac{2}{b} + a, 0\right]$. $\square$

We can now prove Proposition 1.

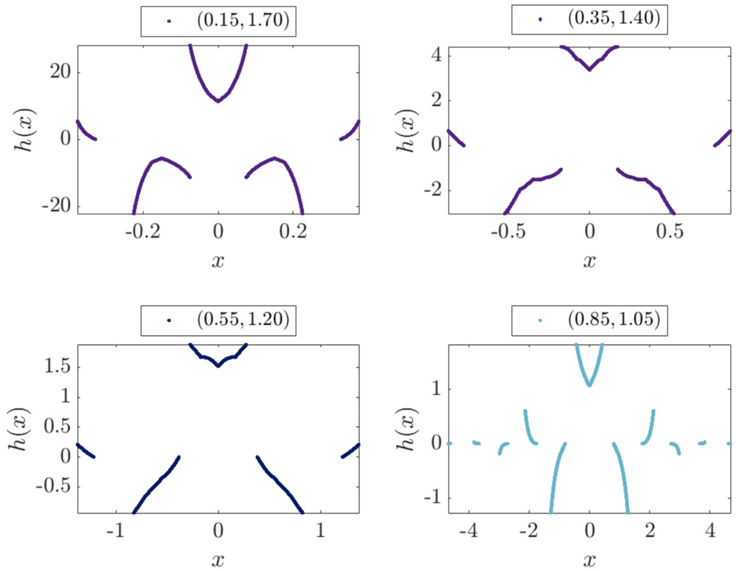

Fig. 3. Plots of nonzero values of the dual $h(x)$ of $\mathcal{G}(B_2, a, b)$ for the following (a, b): (0.15, 1.70) and (0.35, 1.40) in Γ_3; (0.55, 1.20) in Λ_3; and (0.85, 1.05) in Λ_6.

Proof of Proposition 1. The result is now proved by induction using Lemma 1 and Lemma 2. $\square$

We are now ready to prove Theorem 2.

Proof of Theorem 2. By Proposition 1 we know that G_m is invertible. Furthermore, $|G_m(-x)| = |G_m(x)|$ for all $x \in [-\frac{a}{2}, \frac{a}{2}]$, [2, Remark 2]. Let h be defined on $\mathbb{R}$ as follows: let $h(x) = 0$ if $x \in \mathbb{R} \setminus \left[-\frac{2m-1}{2}a, \frac{2m-1}{2}a\right]$ and for $x \in \left[-\frac{2m-1}{2}a, \frac{2m-1}{2}a\right]$ let $h(x)$ be the appropriate entry of $H_m(x) = b(G_m^{-1}(x))_m$, where

$$H_m(x) = [h(x + ka)]_{|k| \le m-1} \tag{7}$$

and $(G_m^{-1}(x))_m$ is the m^{th} column vector of the matrix $G_m^{-1}(x)$.

Then we can determine h on $(-\frac{a}{2} + ka, ka]$ where $k \in \{1 - m, ..., m - 1\}$. It can be shown that h is even and therefore can be defined on the interval $\left[-\frac{2m-1}{2}a, \frac{2m-1}{2}a\right]$ except at finitely many points (see Fig. 3). Since $|G_m| > 0$ on $\left[-\frac{a}{2}, \frac{a}{2}\right]$, we conclude that $\frac{1}{|G_m|}$ is a continuous, hence a bounded function on $\left[-\frac{a}{2}; \frac{a}{2}\right]$. Consequently, h is a compactly supported and bounded function for which $\mathcal{G}(h, a, b)$ is a Bessel sequence. By construction, it also follows that B_2 and h are dual windows. $\square$

3. $\mathcal{F}(B_2)$ contains Γ_3

We now turn to the second part of Theorem 1 by showing that the following result holds.

Theorem 3. *Let $(a, b) \in \Gamma_3$. Then, the Gabor system $\mathcal{G}(B_2, a, b)$ is a frame for $L^2(\mathbb{R})$, and there is a unique dual window $h \in L^2(\mathbb{R})$ such that supp$(h) \subseteq \left[-\frac{5a}{2}, \frac{5a}{2}\right]$.*

We observe that when $(a, b) \in \Gamma_3$, the matrix G_3 becomes

$$G_3 = \begin{pmatrix} D & v \\ 0 & g_{2,2} \end{pmatrix}, \tag{8}$$

where 0 is a 1×4 matrix of 0s, v is a column vector in $\mathbb{R}^4$ and D denotes the 4×4 matrix obtained by deleting the last row and the last column of G_3 and given by

$$D = \begin{pmatrix} g_{-2,-2} & g_{-2,-1} & 0 & 0 \\ g_{-1,-2} & g_{-1,-1} & g_{-1,0} & g_{-1,1} \\ g_{0,-2} & g_{0,-1} & g_{0,0} & g_{0,1} \\ g_{1,-2} & g_{1,-1} & g_{1,0} & g_{1,1} \end{pmatrix}.$$

Because $g_{2,2} > 0$ on $[-\frac{a}{2}, 0]$, G_3 is invertible on $[-\frac{a}{2}, 0]$ if and only if D is invertible. The following proposition shows that the matrix D is invertible for $(a,b) \in \Gamma_3 = \Gamma_3' \cup \Gamma_3''$, where

$$\begin{aligned} \Gamma_3' &:= \left\{(a,b) \in \mathbb{R}_+^2 : a \in \left(0, \frac{2}{9}\right], b \in \left(\frac{4}{2+3a}, \frac{2}{1+a}\right]\right\}, \text{ and} \\ \Gamma_3'' &:= \left\{(a,b) \in \mathbb{R}_+^2 : a \in \left(\frac{2}{7}, \frac{1}{2}\right), b \in \left(\frac{4}{2+3a}, \frac{2}{1+a}\right]\right\}. \end{aligned} \tag{9}$$

Proposition 2. *Let $(a,b) \in \Gamma_3 = \Gamma_3' \cup \Gamma_3''$. Then $|D| > 0$, and therefore $|G_3| > 0$, on $[-\frac{a}{2}, 0]$.*

Proof. We first consider the case $a \in (0, \frac{2}{13}]$. Then $g_{1,-2} > 0$ and $-\frac{a}{2} < 1 - \frac{2}{b} + a \leq 0$. A series of computations shows that $|D| > 0$, since

$$|D(x)| = \begin{cases} -\frac{4a}{b^2}(bx - b + 2)[(1-b)x + a(1-b)] & x \in [-\frac{a}{2}, 1 - \frac{2}{b} + a) \\ \frac{4a(b-1)(x+a)}{b} g_{-2,-2}(x) & x \in [1 - \frac{2}{b} + a, 0]. \end{cases} \tag{10}$$

Next, assume that $a \in (\frac{2}{13}, \frac{1}{5})$. Then $g_{1,-2}(x) > 0$ if and only if $x \in (-1 + \frac{1}{b} + 2a, 1 + \frac{1}{b} + 2a)$.

We first treat the subcase $-1 + \frac{1}{b} + 2a \geq 0$, then $g_{1,-2} = 0$. Therefore

$$|D(x)| = \begin{cases} I(x) & x \in [-\frac{a}{2}, 1 - \frac{2}{b} + a) \\ \frac{4a(b-1)(x+a)}{b} g_{-2,-2}(x) & x \in [1 - \frac{2}{b} + a, 0]. \end{cases} \tag{11}$$

We now prove that $I > 0$ on the interval $[-\frac{a}{2}, 1 - \frac{2}{b} + a)$ by showing that $I' > 0$ on $\left[-\frac{a}{2}, 0\right]$ and $I(-\frac{a}{2}) > 0$. It can be proved that $f(a,b) := I(-\frac{a}{2})$, as a function of (a,b) has no critical point in the interior of the domain $\left[\frac{2}{13}, \frac{1}{5}\right] \times \left[\frac{4}{2+3a}, \frac{2}{1+a}\right]$. Thus, the minimum value of $f(a,b)$ is achieved on the boundary of the domain. Furthermore, a series of calculations shows that f is positive on the boundary. Consequently $f(a,b) > 0$ for all $(a,b) \in \left[\frac{2}{13}, \frac{1}{5}\right] \times \left[\frac{4}{2+3a}, \frac{2}{1+a}\right]$. Similarly, we can show $L(x,a,b) := I'(x) > 0$ for all $(x,a,b) \in [-\frac{1}{10}, 0] \times [\frac{2}{13}, \frac{1}{5}] \times \left[\frac{20}{13}, \frac{26}{15}\right]$ which contains the compact set $\left[-\frac{a}{2}, 0\right] \times \left[\frac{2}{13}, \frac{1}{5}\right] \times \left[\frac{4}{2+3a}, \frac{2}{1+a}\right]$.

For the subcase $-1 + \frac{1}{b} + 2a < 0$ (with $a \in (\frac{2}{13}, \frac{1}{5})$) we note that $g_{1,-2} \geq 0$. In this case, a series of computations shows that $|D|$ is given either by (10) or (11). We proceed similarly to establish that the determinant is positive for $x \in [-\frac{a}{2}, 0]$.

Finally, assume $a \in [\frac{1}{5}, \frac{2}{9})$. It follows that $g_{1,-2} = 0$, and a series of computations shows that $|D|$ is given by (11), which can be shown to be positive. Consequently, for $(a,b) \in \Gamma_3'$, then $|D| > 0$ on $[-\frac{a}{2}, 0]$.

We now consider $(a,b) \in \Gamma_3''$. Assume that $a \in (\frac{2}{7}, \frac{1}{3}]$, $b \in (\frac{4}{2+3a}, \frac{3}{2}]$. If $-1 + \frac{1}{b} + a > 0$, then

$$|D(x)| = \begin{cases} J(x) & x \in [-\frac{a}{2}, 1 - \frac{1}{b} - a) \\ g_{-2,-2}(x)|D^{\{2,3,4\}}(x)| & x \in [1 - \frac{1}{b} - a, 0] \end{cases}. \tag{12}$$

As in the case of I, we prove that $J \neq 0$ and $|D^{\{2,3,4\}}| > 0$, since

$$\begin{vmatrix} g_{-1,0} & g_{-1,1} \\ g_{1,0} & g_{1,1} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{-1,-1} & g_{-1,0} \\ 0 & g_{1,0} \end{vmatrix} > 0, \text{and} \begin{vmatrix} g_{-1,-1} & g_{-1,1} \\ 0 & g_{1,1} \end{vmatrix} > \begin{vmatrix} g_{-1,0} & g_{-1,1} \\ g_{1,0} & g_{1,1} \end{vmatrix} + \begin{vmatrix} g_{-1,-1} & g_{-1,0} \\ 0 & g_{1,0} \end{vmatrix}.$$

The same decomposition is obtained for $b \in (\frac{4}{2+3a}, \frac{3}{2}]$, $-1 + \frac{1}{b} + a \geq 0$ and $a \in (\frac{2}{7}, \frac{1}{3}]$, $b \in (\frac{3}{2}; \frac{2}{1+a}]$.

Let $a \in (\frac{1}{3}, \frac{2}{5}]$ and $2a - \frac{1}{b} > 0$. Then,

$$|D(x)| = \begin{cases} J(x) & x \in [-\frac{a}{2}, 1 - \frac{1}{b} - a) \\ M(x) & x \in [1 - \frac{1}{b} - a, 1 - \frac{2}{b} + a) \\ g_{-2,-2}(x)|D^{\{2,3,4\}}(x)| & x \in [1 - \frac{2}{b} + a), 0]. \end{cases} \tag{13}$$

The previous determinants are obtained in the case $a \in (\frac{1}{3}, \frac{2}{5}]$ and $2a - \frac{1}{b} \leq 0$.

Let $a \in (\frac{2}{5}, \frac{1}{2})$ and $2 - \frac{2}{b} - a \leq 0$. Then

$$|D(x)| = \begin{cases} g_{1,1}(x)|D^{\{1,2,3\}}(x)| & x \in [-\frac{a}{2}, -1 + \frac{1}{b}] \\ N(x) & x \in (-1 + \frac{1}{b}, 1 - \frac{2}{b} + a) \\ g_{-2,-2}(x)|D^{\{2,3,4\}}(x)| & x \in [1 - \frac{2}{b} + a, 0] \end{cases}. \tag{14}$$

Let $a \in (\frac{2}{5}, \frac{1}{2})$ and $2 - \frac{2}{b} - a > 0$. Then

$$|D(x)| = \begin{cases} P(x) & x \in [-\frac{a}{2}, 2a - 1] \\ J(x) & x \in (2a - 1, 1 - \frac{1}{b} - a) \\ M(x) & x \in [1 - \frac{1}{b} - a, 1 - \frac{2}{b} + a) \\ g_{-2,-2}(x)|D^{\{2,3,4\}}(x)| & x \in [1 - \frac{2}{b} + a, 0] \end{cases}. \tag{15}$$

Similar techniques used for $|D^{\{2,3,4\}}|$ can be used to prove that $|D^{\{1,2,3\}}| > 0$, $N > 0$ and $P > 0$. We conclude that for $(a, b) \in \Gamma_3'''$, D is invertible on $[-\frac{a}{2}, 0]$. □

Remark 1. Using numerical simulations (Fig. 4) we observe that for $(a, b) \in \Gamma_3''' := \{(a, b) \in \mathbb{R}_+^2 : a \in (\frac{2}{9}, \frac{2}{7}], b \in (\frac{4}{2+3a})\}$, then $|D| > 0$. However, we have not yet been able to prove this.

Proof of Theorem 3. We recall from Proposition 2 that for $(a, b) \in \Gamma_3$, we have $|G_3| \neq 0$ on $[-\frac{a}{2}, 0]$. Since h is compactly supported, we know that $h(x) = 0$ for $|x| > \frac{5a}{2}$. In addition, on $\left[-\frac{5a}{2}, \frac{5a}{2}\right]$, h is determined by the entries of $H_3 = b(G_3^{-1})_3$, where H_3 is given by (7).

Then, we can determine h on $(-\frac{a}{2} + ka, ka]$ where $k = 0, \pm 1, \pm 2$. This, along with the symmetry of h, will define the function h everywhere except possibly at $\frac{ka}{2}$ for $k = \pm 1, \pm 2, \pm 3, \pm 5$. In particular, $h(x + 2a) = 0$ for $x \in (-\frac{a}{2}, 0)$. That is, $h = 0$ on $(-\frac{5a}{2}, -2a)$, and $(2a, \frac{5a}{2})$.

Consequently, h is a compactly supported and bounded function for which $\mathcal{G}(h, a, b)$ is a Bessel sequence. By construction, it also follows that g and h are dual windows. □

Remark 2.

(1) We observe that the dual window h constructed from Theorem 2 and Theorem 3 is discontinuous. This is proved in the same way as in [2, Remark 4].

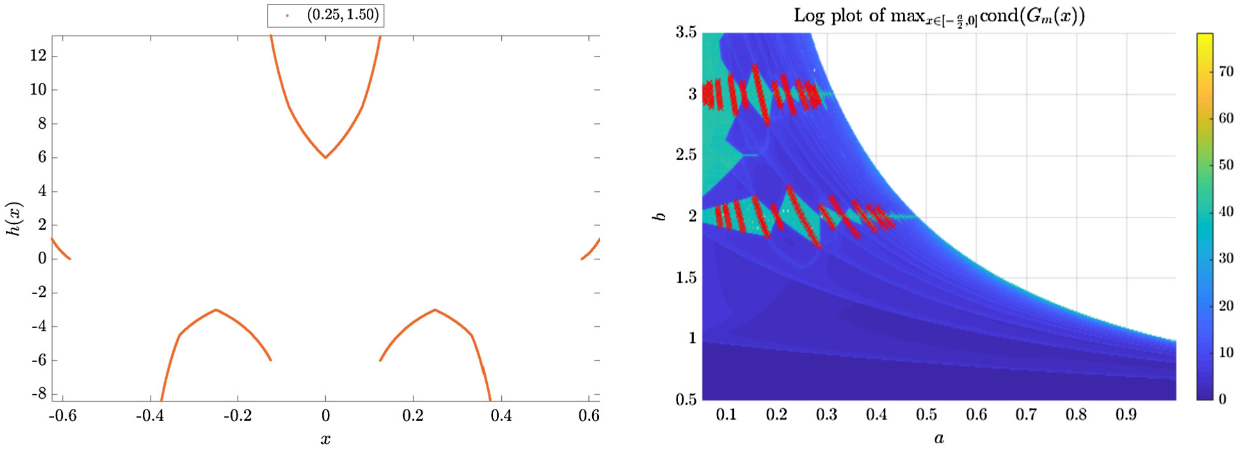

Fig. 4. Although the region $\{a \in [\frac{2}{9}, \frac{2}{7}], b \in [\frac{4}{2+3a}, \frac{2}{1+a}], b > 1\}$ is not covered by Theorem 1, numerical evidence suggests that this set is included in $\mathcal{F}(B_2)$. On the left is a plot of nonzero values of the dual h of $\mathcal{G}(B_2, a, b)$ corresponding to $a = 0.25, b = 1.50$, and on the right is a plot of the condition number of the matrices G_m, indicating the invertibility of the matrices in this region. The red marks indicate known points $(a, b) \notin \mathcal{F}(B_2)$ proved in [20].

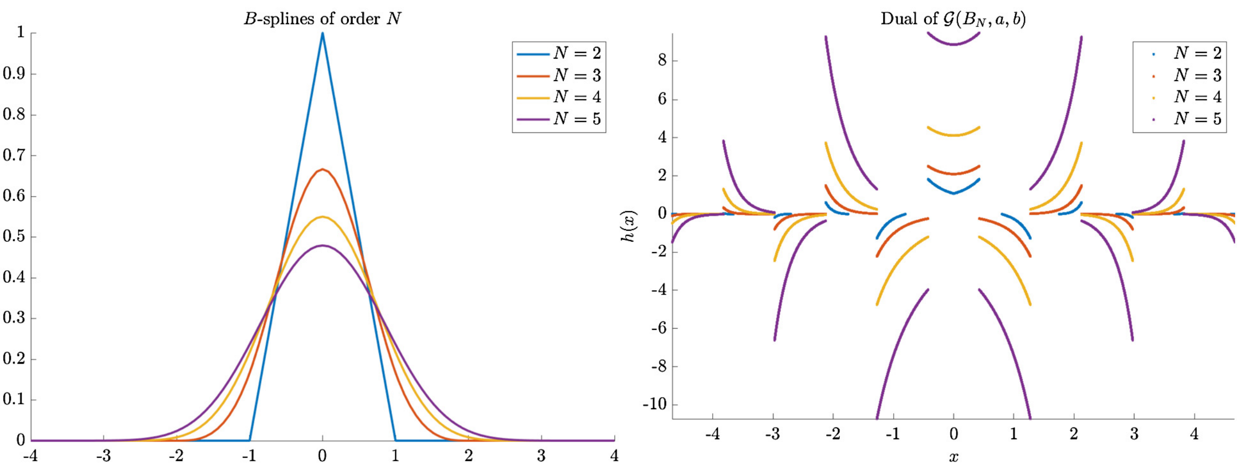

Fig. 5. The results presented here extend in a straightforward way to higher-order B-splines (left). On the right are plots of the nonzero values of the corresponding dual frames h of $\mathcal{G}(B_N, a, b)$ for $(a, b) = (0.85, 1.05)$ and $N = 2, 3, 4, 5$.

(2) On the line $b = \frac{3}{2}$, all of the dyadic points in the set $\{(\frac{1}{2^j}, \frac{3}{2})\}_{j=1}^{\infty} \cup \{(\frac{3}{2^j}, \frac{3}{2})\}_{j=3}^{\infty}$ belong to $\mathcal{F}(B_2)$, since most of these points belong Γ_3, except for $(\frac{1}{4}, \frac{3}{2}), (\frac{3}{8}, \frac{3}{2})$, and $(\frac{1}{2}, \frac{3}{2})$ for which we omit the proof.

(3) Our results extend to regions beyond the frame set established here and also to higher-order B-splines, as demonstrated in Figs. 4 and 5.

Acknowledgment

Part of this work was completed while the first-named author was a visiting graduate student in the Department of Mathematics at the University of Maryland during the Fall 2017 semester. He would like to thank the Department for its hospitality and the African Center of Excellence in Mathematics and Application (CEA-SMA) at the Institut de Mathématiques et de Sciences Physiques (IMSP) for funding his visit. K.A. Okoudjou was partially supported by a grant from the Simons Foundation # 319197, by ARO grant W911NF1610008, and the National Science Foundation grant DMS 1814253.

Appendix A

```matlab
function plotDual
a = .85;
b = 1.05;
N = 2;
g = @(x) fnval(spmak(-N:N,[0 1 0]),x);
x=-a/2:.001:a/2;

% Set up and solve the linear systems for h(x)
for j=1:length(x)
    [m, Gm]=G(x(j),g, a, b);
    if m==0
        H = NaN;
    else
        bv = zeros(2*m-1,1);
        bv(m) = b;
        H(:,j) = Gm\bv;
    end
end

% concatenate the components of H
X = []; h = [];
for k = -(m-1):(m-1)
    X = [X x+k*a];
    h = [h, H(k+m,:)];
end

figure;
plot(X(abs(h)>1e-10), h(abs(h)>1e-10), '.')
title(sprintf('Dual of $\\mathcal{G}(B_N)$ \n N=%d, a=%1.2f, b=%1.2f,',N, a, b))
end

function [m, Y] = G(x, g, a, b, m_max)
% Returns the (2m-1) x (2m-1) matrix Y = G_3(x) and the integer m between 1 and m_max such that (a,b) is in \Lambda_m.
% If (a,b) is not in \Lambda_m for any m\geq 1, Y=NaN.
% The window g is a function handle, m_max is a large integer and x, a, and b are real numbers.

if nargin<5
    m_max=50;
end

m = NaN;
for mm = 1:m_max
    if (b>2*(mm-1)/(2+(2*mm-3)*a)) && (b<2*mm/(2+(2*mm-1)*a)) && (b<2/(1+a)) && (b>1)
         m= mm;
    end
end
if ~isnan(m)
    l = meshgrid(-(m-1):(m-1));
    Y = g(x-l'/b+l*a);
else
    Y=NaN;
    fprintf('Error. Perhaps try again with larger m_max.')
end
end
```

Listing 1: MATLAB code for plotting the dual of $\mathcal{G}(B_N, a, b)$.

References

[1] Akram Aldroubi, Karlheinz Gröchenig, Beurling-Landau-type theorems for non-uniform sampling in shift invariant spline spaces, J. Fourier Anal. Appl. 6 (1) (2000) 93–103.

[2] A.G.D. Atindehou, Y.B. Kouagou, K.A. Okoudjou, Frame sets for a class of compactly supported continuous functions, Asian-Eur. J. Math. 5 (5) (2020) 2050093.

[3] O. Christensen, An Introduction to Frames and Riesz Bases, Applied and Numerical Harmonic Analysis, Birkhaüser, Boston, Inc., Boston, MA, 2003.

[4] O. Christensen, Six (seven) problems in frame theory, in: I. Zayed, G. Schmeisser (Eds.), New Perspectives on Approximation and Sampling Theory, in: Applied and Numerical Harmonic Analysis, Springer, 2014, pp. 337–358.

[5] O. Christensen, H.O. Kim, R.Y. Kim, On Gabor frames generated by sign-changing windows and B-splines, Appl. Comput. Harmon. Anal. 39 (3) (2015) 534–544.
[6] O. Christensen, H.O. Kim, R.Y. Kim, On Gabor frame set for compactly supported continuous functions, J. Inequal. Appl. (94) (2016) 94.
[7] X. Dai, Q. Sun, The abc-problem for Gabor systems, Mem. Am. Math. Soc. 244 (1152) (2016).
[8] I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27 (1986) 1271–1283.
[9] H. Feitchinger, N. Kaiblinger, Varying the time-frequency lattice of Gabor frames, Trans. Am. Math. Soc. 356 (5) (2004) 2001–2023.
[10] K. Gröchenig, Foundations of Time-Frequency Analysis, Applied and Numerical Harmonic Analysis, Springer-Birkhäuser, New York, 2001.
[11] K. Gröchenig, The mystery of Gabor frames, J. Fourier Anal. Appl. 20 (2014) 865–895.
[12] K. Gröchenig, A.J.E.M. Janssen, N. Kaiblinger, G. Pfander, Note on b-splines, wavelet scaling functions, and Gabor frames, IEEE Trans. Inf. Theory 49 (2003) 3318–3320.
[13] K. Gröchenig, J. Ströckler, Gabor frames and totally positive functions, Duke Math. J. 162 (2013) 1003–1031.
[14] Q. Gu, D. Han, When a characteristic function generates a Gabor frame?, Appl. Comput. Harmon. Anal. 24 (2008) 290–309.
[15] A.J.E.M. Janssen, Some Weyl-Heisenberg frame bound calculations, Indag. Math. 7 (1996) 165–183.
[16] A.J.E.M. Janssen, On generating tight Gabor frames at critical density, J. Fourier Anal. Appl. 9 (2) (2003) 175–214.
[17] A.J.E.M. Janssen, Zak transforms with few zeros and the tie, in: H.G. Feichtinger, T. Strohmer (Eds.), Advances in Gabor Analysis, Birkhaüser, Boston, Boston, MA, 2003, pp. 31–70.
[18] A.J.E.M. Janssen, T. Strohmer, Hyperbolic secants yield Gabor frames, Appl. Comput. Harmon. Anal. 12 (2) (2002) 259–267.
[19] T. Kloos, J. Stöckler, Zak transforms and Gabor frames of totally positive functions and exponential b-splines, J. Approx. Theory 184 (2014) 209–237.
[20] J. Lemvig, K. Haahr Nielsen, Counterexamples to the b-spline conjecture for Gabor frames, J. Fourier Anal. Appl. 22 (6) (2016) 1440–1451.
[21] Y. Lyubarskii, Frames in the Bargmann Space of Entire Functions, Entire and Subharmonic Functions, Adv. Sov. Math., vol. 11, Amer. Math. Soc., Providence, RI, 1992, pp. 167–180.
[22] K. Seip, Density theorems for sampling and interpolation in the Bargmann-Fock space, I, J. Reine Angew. Math. 429 (1992) 91–106.
[23] K. Seip, R. Wallstén, Density theorems for sampling and interpolation in the Bargmann-Fock space, II, J. Reine Angew. Math. 429 (1992) 107–113.