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Einstein–Weyl structures on lightlike hypersurfaces

Research Article

Cyriaque Atindogbe[1*], Lionel Bérard-Bergery[2†], Carlos Ogouyandjou[1‡]

1 *Institut de Mathematiques et de Sciences Physiques, Université d'Abomey-Calavi, Benin, 01 BP 613 Porto-Novo, Benin*

2 *Institut Elie Cartan, Université Henri Poincaré, Nancy I, B.P. 239 54506 Vandœuvre-lès Nancy Cedex, France*

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Abstract: We study Weyl structures on lightlike hypersurfaces endowed with a conformal structure of certain type and specific screen distribution: the Weyl screen structures. We investigate various differential geometric properties of Einstein–Weyl screen structures on lightlike hypersurfaces and show that, for ambient Lorentzian space $\mathbb{R}_1^{n+2}$ and a totally umbilical screen foliation, there is a strong interplay with the induced (Riemannian) Weyl-structure on the leaves.

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1. Introduction

Pseudo-Riemannian manifolds (M, g) with $\dim M = n > 4$ and $\operatorname{sgn} g = (n-1, 1)$ are natural generalizations of (4-dimensional model) spacetime of general relativity. Lightlike hypersurfaces in (M, g) are models of different types of horizons separating domains of (M, g) with different physical properties. As it is well known, contrary to timelike and spacelike hypersurfaces, the geometry of lightlike hypersurfaces is different and rather difficult since the normal bundle and the tangent bundle have nonzero intersection.

Being lightlike manifold is invariant under conformal change of the metric, along with many geometric objects. Thus, it is reasonable to believe that it would be more relevant to study the geometry of lightlike (sub-)manifolds in a conformal class of degenerate metrics. In this context, taking into account the Riemannian case, one of natural structures of which one can think of is that of Einstein–Weyl.

* *E-mail: atincyr@imsp-uac.org*
† *E-mail: berard@iecn.u-nancy.fr*
‡ *E-mail: ogouyandjou@imsp-uac.org*

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In a Riemannian setting, manifolds M^n with conformal structure $[g]$ and torsion-free connection D, such that parallel translation induces conformal transformations, are called Weyl manifolds. They are said to be Einstein–Weyl if the symmetric trace-free part of the Ricci tensor of the (Weyl) connection D vanishes. If D is locally the Levi-Civita connection of a compatible metric in $[g]$, the structure is said to be closed, and the (D-compatible) metric is locally Einstein [5, 6, 9].

In [4], Duggal and Bejancu introduced a main tool in studying the geometry of a lightlike hypersurface: the screen distributions. The latter is used to construct a lightlike transversal vector bundle which is nonintersecting to the lightlike tangent bundle. A suitable choice of screen distribution has produced an important result in lightlike geometry [1, 4]. In Section 2 we provide basic information on normalizations, induced geometric objects [4] and pseudo-inversion of degenerate metrics [2]. In Section 3, we define *Weyl screen structure*, Definition 3.5, and prove a result on model space of Weyl screen structures on the (conformal) lightlike hypersurface. Thereafter, we study and relate curvature and Ricci tensors of the Weyl connection, along with its scalar curvature to their respective analoges for a given representative element in the conformal class. In Section 4, we consider Einstein–Weyl screen structures and establish a necessary and sufficient condition for a Weyl screen structure to be Einstein–Weyl. Section 6 is devoted to a special case of total umbilicity of the screen foliation involved in Definition 3.5. Also, in ambient Lorentzian case, we prove that there is a strong interplay between Einstein–Weyl screen structures on the conformal lightlike hypersurface and the (induced) one on the (Riemannian) screen foliation.

2. Preliminaries on lightlike hypersurfaces

It is well known that the normal bundle $TM^\perp$ of the lightlike hypersurface M^{n+1} of a semi-Riemannian manifold $\overline{M}^{n+2}$ is a rank 1 vector subbundle of the tangent bundle TM. A complementary bundle of $TM^\perp$ in TM is a rank n nondegenerate distribution over M, called a *screen distribution* of M, denoted by $\mathrm{S}(TM)$, such that

$$TM = \mathrm{S}(TM) \oplus_{\mathrm{Orth}} TM^\perp, \tag{1}$$

where $\oplus_{\mathrm{Orth}}$ denotes the orthogonal direct sum. Existence of $\mathrm{S}(TM)$ is secured provided M is paracompact. A lightlike hypersurface with a specific screen distribution is denoted by $(M, g, \mathrm{S}(TM))$. We know [4] that for such a triplet, there exists a unique rank 1 vector subbundle $\mathrm{tr}(TM)$ of $T\overline{M}$ over M, such that for any nonzero section ξ of $TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $\mathrm{tr}(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(N,\xi) = 1, \qquad \overline{g}(N,N) = \overline{g}(N,W) = 0, \qquad W \in \Gamma(ST(M)\restriction_{\mathcal{U}}). \tag{2}$$

Then $T\overline{M}$ is decomposed as follows:

$$T\overline{M}\restriction_M \;=\; TM \oplus \mathrm{tr}(TM) \;=\; \left\{TM^\perp \oplus \mathrm{tr}(TM)\right\} \oplus_{\mathrm{Orth}} \mathrm{S}(TM). \tag{3}$$

We call $\mathrm{tr}(TM)$ a (*null*) *transversal vector bundle* along M. In fact, from (2) and (3) one can show that, conversely, a choice of a transversal bundle $\mathrm{tr}(TM)$ determines uniquely the screen distribution $\mathrm{S}(TM)$. A vector field N as in (2) is called a *null transversal vector field* of M. It is noteworthy that the choice of a null transversal vector field N along M determines both the null transversal vector bundle, the screen distribution and a unique radical vector field, say ξ, satisfying (2). Whence, from now on, by a *normalized lightlike hypersurface* we mean a triplet (M, g, N) where g is the induced metric on M along with a null transversal vector field N. In fact, in case the ambient manifold $\overline{M}$ has Lorentzian signature, at an arbitrary point x in M, a real lightlike cone C_x is invariantly defined in the (ambient) tangent space $T_x\overline{M}$ and is tangent to M along a generator emanating from x. This generator is exactly the radical fiber $\Delta_x = T_xM^\perp$. Each null vector field N, $x \mapsto N_x \in C_x \setminus \Delta_x$, determines a normalization of M. Let (M, g, N) be a normalized lightlike hypersurface. A null vector field $\widetilde{N}$ is a normalizing field for (M, g) if and only if $\widetilde{N} = \phi N + \zeta$, for some nowhere vanishing $\phi \in C^\infty(M)$ and $\zeta \in \Gamma(TM)$.

Now, on a normalized lightlike hypersurface (M, g, N), the local Gauss and Weingarten equations are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \qquad \overline{\nabla}_X N = -A_N X + \tau(X)N, \qquad \nabla_X \xi = -\overset{\star}{A}_\xi X - \tau(X)\xi,$$
$$\nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{4}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}, \nabla$ and $\overset{\star}{\nabla}$ denote the Levi-Civita connection on $(\overline{M}, \overline{g})$, the induced connection on M and the connection on the screen distribution $S(TM)$ respectively, P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1). The $(0, 2)$-tensors B and C are the local second fundamental forms on TM and $S(TM)$ respectively, $\overset{\star}{A}_\xi$ the local shape operator on $S(TM)$ and τ a 1-form on TM defined by $\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi)$. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/TM^\perp$ [7]. As per [4, p. 83], the second fundamental form B of M is independent of the choice of a screen distribution and satisfies for all $X, Y \in \Gamma(TM)$,

$$B(X, \xi) = 0, \qquad \text{and} \qquad B(X, Y) = g(\overset{\star}{A}_\xi X, Y).$$

The linear connection $\overset{\star}{\nabla}$ from (4) is a metric connection on $S(TM)$. But for ∇ we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y),$$

for all tangent vectors fields X, Y and Z in $\Gamma(TM)$, with

$$\eta(\cdot) = \overline{g}(N, \cdot). \tag{5}$$

It follows that the induced connection ∇ is torsion-free, but not necessarily g-metric. It is the case if and only if M is totally geodesic. Equivalently, the local second fundamental form B vanishes identically. In fact, this is also equivalent to saying that $M \ni x \mapsto T_x M^\perp$ is a Killing distribution on M.

Finally, we recall from [2] the following results. Consider on M a normalizing pair $\{\xi, N\}$ satisfying (2) and the 1-form η as in (5). For $X \in \Gamma(TM)$, we have $X = PX + \eta(X)\xi$ and $\eta(X) = 0$ if and only if $X \in \Gamma(S(TM))$. Now, we define $\flat$ by

$$\flat : \Gamma(TM) \to \Gamma(T^*M), \qquad X \mapsto X^\flat = g(X, \cdot) + \eta(X)\eta(\cdot). \tag{6}$$

Clearly, such $\flat$ is an isomorphism of $\Gamma(TM)$ onto $\Gamma(T^*M)$, and can be used to generalize the usual nondegenerate definition. In the latter case, $\Gamma(S(TM))$ coincides with $\Gamma(TM)$, and as a consequence the 1-form η vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$. We let $\#$ denote the inverse of the isomorphism $\flat$ given by (6). For $X \in \Gamma(TM)$ (resp. $\omega \in T^*M$), $X^\flat$ (resp. $\omega^\#$) is called the dual 1-form of X (resp. the dual vector field of ω) with respect to the degenerate metric g. It follows from (6) that if ω is a 1-form on M, we have for $X \in \Gamma(TM)$,

$$\omega(X) = g(\omega^\#, X) + \omega(\xi)\eta(X). \tag{7}$$

Define a $(0, 2)$-tensor $\widetilde{g}$ by

$$\widetilde{g}(X, Y) = X^\flat(Y), \qquad X, Y \in \Gamma(TM).$$

Clearly, $\widetilde{g}$ defines a nondegenerate metric on M which plays an important role in defining the usual differential operators gradient, divergence, Laplacian with respect to degenerate metric g on lightlike hypersurfaces, see [2] for details. Also, observe that $\widetilde{g}$ coincides with g if the latter is nondegenerate. The $(0, 2)$-tensor $g^{[\cdot,\cdot]}$, inverse of $\widetilde{g}$ is called *the pseudo-inverse of* g. With respect to the quasi orthonormal local frame field $\{X_0 = \xi, X_1, \ldots, X_n, X_{n+1} = N\}$ adapted to the decompositions (1) and (3) we have

$$\widetilde{g}(\xi, \xi) = 1, \qquad \widetilde{g}(\xi, X) = \eta(X), \qquad \widetilde{g}(X, Y) = g(X, Y), \qquad X, Y \in \Gamma(S(TM)),$$

and the following is proved [2].

Proposition 2.1.
For any smooth function $f\colon \mathcal{U}\subset M\to\mathbb{R}$ *we have*

$$\operatorname{grad}^g f = g^{[\alpha\beta]} f_\alpha X_\beta, \qquad \text{where} \quad f_\alpha = X_\alpha\cdot f, \quad \alpha,\beta = 0,\dots,n.$$

For any vector field X *on* $\mathcal{U}\subset M$*,*

$$\operatorname{div}^g X = \sum_{\alpha=0}^{n} \varepsilon_\alpha \widetilde{g}(\nabla_{X_\alpha} X, X_\alpha), \qquad \varepsilon_0 = 1.$$

For a smooth function f *defined on* $\mathcal{U}\subset M$*,*

$$\Delta^g f = \sum_{\alpha=0}^{n} \varepsilon_\alpha \widetilde{g}(\nabla_{X_\alpha}\operatorname{grad}^g f, X_\alpha).$$

In particular, ρ being an endomorphism, resp. a symmetric bilinear form, on $(M, g, S(TM))$, we have

$$\operatorname{tr}\rho = \operatorname{trace}_g \rho = \sum_{\alpha,\beta=0}^{n} g^{[\alpha\beta]}\widetilde{g}(\rho(X_\alpha), X_\beta), \qquad \text{resp.} \quad \operatorname{trace}_g\rho = \sum_{\alpha,\beta=0}^{n} g^{[\alpha\beta]}\rho_{\alpha\beta}.$$

All manifolds will be assumed connected, paracompact and smooth.

In index free notation, the relation $\widetilde{g}(\nabla^g f, X) = df(X)$ defines the gradient of the scalar function f with respect to the degenerate metric g. With nondegenerate g, one has $\widetilde{g} = g$ so that Proposition 2.1 generalizes the usual known formulae to the degenerate case.

From now on, unless otherwise stated, the ambient manifold $(\overline{M}, \overline{g})$ has a Lorentzian signature so that all ligthlike hypersurfaces considered are of signature $(0, n)$. In particular, it follows that any screen distribution is Riemannian.

Let us mention here some remarkable facts that highly motivated and influenced some of our choices below. In most cases, semi-Riemannien manifolds admitting nontrivial (Einstein–)Weyl structure do admit domains or horizons which are (nontrivial) totally geodesic lightlike hypersurfaces. As it is well known [8], there is a natural one-to-one correspondence between smooth, space-time oriented conformally compact, globally hyperbolic, Lorentzian Einstein–Weyl 3-manifolds $(M, [g], \nabla)$ and orientation reversing diffeomorphisms $\psi\colon \mathbb{C}P_1\to\mathbb{C}P_1$. Example of such an Einstein–Weyl manifold is the three dimensional de Sitter space $SL(2,\mathbb{C})/SL(2,\mathbb{R})$ (the mass hyperboloid). But it is well known that a slice of the latter with null hyperplanes give rise to a family of nontrivial totally geodesic hypersurfaces. Also, consider (M, g) to be a black hole event horizon in a C^∞ Lorentzian manifold $(\overline{M}, \overline{g})$ satisfying natural hypothesis, using the well-known regularity and area theorem by Chruściel et al. [3]. Let Σ_a, $a = 1, 2$, be two achronal C^2 embedded spacelike hypersurfaces, $S_a = \Sigma_a\cap M$ and M_{12} the part of M between S_1 and S_2. If S_1 belongs to the past of S_2 with area S_1 = area S_2, then M_{12} is a totally geodesic lightlike hypersurface. The list is not exhaustive and there are many other interesting examples we can cite. Also, as stated above, only totally geodesic lightlike hypersurfaces do have their induced connection metric and torsion-free. Motivated by the above observations, it is tempting for both physical and (technical) geometric reasons to study Einstein–Weyl structures on totally geodesic lightlike hypersurfaces, for a first step. Therefore, in the remainder of the text, only such lightlike hypersurfaces will be in consideration.

Although being lightlike for (M, g_0) is invariant under conformal change of the metric, for a totally geodesic (M, g_0), not all metrics in the conformal class of g_0 guarantee this geometric condition on M. In this respect, we consider below appropriate conformal structure on a given totally geodesic (M, g_0).

3. Weyl screen structures

Let (M, g_0) be a totally geodesic hypersurface in a $(n+2)$-dimensional pseudo-Riemannian manifold $(\overline{M}, \overline{g})$. Consider on M conformal metrics of the form $g = e^{-2f} g_0$ with $X(f) = 0$ for $X \in TM^\perp = \operatorname{span}\{\xi\}$, i.e f is constant on ξ-orbits. These metrics endow M with a special conformal structure we denote it by $c = [g_0]_0$. For each metric $g\in c$, (M, g) is also totally geodesic, and there exists a g-compatible torsion-free connection ∇^g. Throughout the text, M endowed with this conformal structure is denoted as (M, c).

Definition 3.1.

A Weyl structure relative to (M, c) is a symmetric linear connection D on M that preserves the structure. More precisely:

(i) D is torsion-free.

(ii) For g in the conformal class c, there exists a unique 1-form θ on M such that

$$Dg = -2\theta \otimes g. \tag{8}$$

Remark 3.2.

Conditions (i) and (ii) in Definition 3.1 determine a Weyl structure modulo $\mathrm{S}^2(T^*M) \otimes TM^{\perp}$.

Lemma 3.3.

The kernel $TM^{\perp}$ $(= \mathrm{Rad}(TM) = \mathrm{Ker}\, g)$ of g is parallel with respect to any Weyl structure D on (M, c). Furthermore, up to a renormalization, one can choose $\xi \in TM^{\perp}$ such that $D_\xi \xi = 0$ and for any $g \in c$, there exists a torsion-free g-compatible linear connection D^g with $D^g_\xi \xi = 0$.

Proof. Let $X, Y, Z \in \Gamma(TM)$ and $g \in c$. From (8) we have

$$X \cdot g(Y, Z) - g(D_X Y, Z) - g(Y, D_X Z) = -2\theta(X) g(Y, Z).$$

Then for $Z \in \mathrm{Rad}(TM)$, one has $g(Y, D_X Z) = 0$ for all $Y \in \Gamma(TM)$. Thus $D_X Z \in \mathrm{Rad}(TM)$ for all $Z \in \mathrm{Rad}(TM)$. Now let $\xi \in \mathrm{Rad}(TM)$, we have $D_\xi \xi = \psi(\xi)\xi$. If $\psi(\xi) = 0$ then there is nothing more to prove. Otherwise, choose on the null integral curve $\mathcal{C}$ of ξ a new parameter $t^*(t)$ such that

$$\frac{d^2 t^*}{dt^2} - \psi\left(\frac{d}{dt}\right)\frac{dt^*}{dt} = 0$$

with $d/dt = \xi$. Such a parameter always exists on $\mathcal{C}$ and one has $D_{d/dt^*}\, d/dt^* = 0$. Now, let $g \in c$ and D^g_1 be a torsion-free g-compatible connection. Then, $0 = D_\xi \xi = D^g_{1\xi}\xi + S(\xi, \xi)\xi$ where $S \in \mathrm{S}^2(T^*M)$. If $D^g_{1\xi}\xi = 0$ then there is nothing more to prove. Otherwise, change D^g_1 in $D^g_2 = D^g_1 + S \otimes \xi$. Such a D^g_2 is a torsion-free linear g-compatible connection on M and $D^g_{2\xi}\xi = 0$ and the proof is complete. □

Remark 3.4.

From Lemma 3.3 it follows that the element $S \in \mathrm{S}^2(T^*M)$ modulo which the Weyl structure is determined satisfies $S(\xi, \xi) = 0$ for a suitable choice of the torsion-free g-compatible linear connection D^g of g. The element $S \in \mathrm{S}^2(T^*M)$ is entirely determined by the following.

Lemma 3.3 is true locally, but this may not hold globally. In fact, we just need this to hold on the domain $\mathcal{U}$ of the characteristic section ξ; what we assume from now on.

Definition 3.5.

Let $(M, c, \mathrm{S}(TM))$ be a totally geodesic lightlike hypersurface (M, g_0) endowed with the conformal structure $c = [g_0]_0$. A *Weyl screen structure* D relative to $(M, c, \mathrm{S}(TM))$ is a Weyl structure for which $\mathrm{S}(TM)$ is parallel, that is for all tangent vector fields X and Y in TM, $D_X PY \in \Gamma(\mathrm{S}(TM))$.

Note.

Throughout the text, we sometimes consider the quadruplet $(M, c, D, \mathrm{S}(TM))$ (as in Definition 3.5) as the Weyl screen structure. Also, as it is parallel, the screen distribution involved in this definition is integrable. Vector fields tangent to its leaves are said to be *horizontal*.

Lemma 3.6.

Let D be a Weyl screen structure on $(M, c, \mathrm{S}(TM))$. Let $\Omega^1_{\mathrm{hor}}(M)$ denote the space of horizontal 1-form on M, that is $\omega \in \Omega^1_{\mathrm{hor}}(M)$ if and only if $\omega(X) = 0$ for all $X \in \mathrm{Rad}(TM)$.

(i) *For any $g \in c$, $\theta_g \in \Omega^1_{\mathrm{hor}}(M)$.*

(ii) *For $g \in c$ there exists a unique $\theta_g \in \Omega^1_{\mathrm{hor}}(M)$ and a unique $S \in S^2(T^*M)$ such that for $X, Y \in \Gamma(TM)$,*

$$D_X Y = D^g_X Y + \theta_g(X)Y + \theta_g(Y)X - g(X,Y)\theta_g^{\#_g} - S(X,Y)\xi, \tag{9}$$

where $\theta_g^{\#_g}$ is the dual of θ_g with respect to the degenerate metric g and the screen distribution $\mathrm{S}(TM)$. Furthermore,

$$S(X,Y) = \begin{cases} 0 & \text{if} \quad X, Y \in \mathrm{Rad}(TM) \\ C(X,Y) + \eta(X)\theta_g(Y) & \text{if} \quad (X,Y) \in \Gamma(TM) \times \Gamma(\mathrm{S}(TM)), \end{cases} \tag{10}$$

where C denotes the second fundamental form of $\mathrm{S}(TM)$ in (M, g).

Proof. Let $X \in \mathrm{Rad}(TM)$, $Y, Z \in \Gamma(TM)$. From (8) and Lemma 3.3 we have $L_X g_0(Y,Z) = -2\theta_{g_0}(X) g_0(Y,Z)$. But (M, g_0) is totally geodesic and $L_X g_0 = 0$. Thus, $\theta_{g_0}(X) = 0$, $X \in \mathrm{Rad}(TM)$. For $g = e^{-2f} g_0 \in c$, we have $\theta_g = \theta_{g_0} + df$ with $df(X) = 0$, $X \in \mathrm{Rad}(TM)$. Thus, $\theta_g(X) = \theta_{g_0}(X) + df(X) = 0$, $X \in \mathrm{Rad}(TM)$ and (i) is proved.

Now, let us write for a choice of $g \in c$ and for all $X, Y \in \Gamma(TM)$,

$$D_X Y = D^g_X Y + \widetilde{\theta}_X Y, \tag{11}$$

where $D^g_X Y$ is the torsion-free g-compatible linear connection pointed out in Lemma 3.3. As D and D^g are torsion-free, one has

$$\widetilde{\theta}_X Y = \widetilde{\theta}_Y X. \tag{12}$$

Taking into account (11), (12) and the g-compatibility of D^g one has

$$g(\widetilde{\theta}_X Y, Z) + g(Y, \widetilde{\theta}_X Z) = 2\theta_g(X) g(Y,Z). \tag{13}$$

By circular permutation in (13) and taking into account (12) one has

$$g(\widetilde{\theta}_X Y, Z) = \theta_g(X) g(Y,Z) + \theta_g(Y) g(X,Z) - \theta_g(Z) g(X,Y).$$

As θ_g is horizontal (from (i)) its g-dual $\theta_g^{\#_g}$ is a horizontal vector field and from (7) one can write $\theta_g(Z) = g(Z, \theta_g^{\#_g})$. It follows that

$$\widetilde{\theta}_X Y = \theta_g(X)Y + \theta_g(Y)X - g(X,Y)\theta_g^{\#_g} - S(X,Y)\xi$$

for some $S \in S^2(T^*M)$. Also, from (4) we have

$$D^g_X PY = \overset{\star}{\nabla}{}^g_X PY + C^g(X, PY)\xi,$$

where $\overset{\star}{\nabla}{}^g$ is the induced Levi-Civita connection by D^g on the screen distribution and C^g the second fundamental form of the screen distribution in (M, g). Thus

$$D_X PY = \overset{\star}{\nabla}{}^g_X PY + \theta_g(X)PY + \theta_g(Y)PX - g(X,Y)\theta_g^{\#_g} + \left[C^g(X,PY) + \eta(X)\theta_g(Y) - S(X,PY)\right]\xi.$$

Observe that, since θ_g is a horizontal 1-form, one has $\theta_g^{\#_g} \in \Gamma(\mathrm{S}(TM))$. From condition $D_X PY \in \Gamma(S(TM))$ in Definition 3.5, $\mathrm{S}(TM)$ is D-parallel if and only if the term in bracket vanishes identically on M. It follows that for $X, Y \in \Gamma(TM)$,

$$S(X, PY) = C(X, PY) + \eta(X)\theta_g(Y). \tag{14}$$

In particular, for all $Y \in \Gamma(TM)$,

$$S(\xi, PY) = S(PY, \xi) = C^g(\xi, PY) + \theta_g(Y). \tag{15}$$

Finally, $S(\xi, \xi) = 0$ follows from Remark 3.4 and the proof is complete. □

Remark 3.7.

From $S(\xi,\xi)=0$ and (15) one can write

$$S(\xi,PY)=S(PY,\xi)=C^g(\xi,PY)+\theta_g(Y),\qquad Y\in\Gamma(TM). \tag{16}$$

Clearly, for a given $g\in c$, among all g-compatible torsion-free linear connections, there is only one which satisfies (9). Thus, if we take our data for a Weyl screen structure on $(M,\mathrm{S}(TM))$ to be $g\in c$ and the 1-form θ_g, $D=D^g+\widetilde{\theta}$ is uniquely determined.

The curvature tensor of the Weyl screen structure D is defined by

$$R^D(X,Y)=D_{[X,Y]}-[D_X,D_Y] \tag{17}$$

and we let Ric^D denote the Ricci curvature of D. It is defined to be the trace of the map $Z\mapsto R^D(X,Z)Y$. For a representative $g\in c$ and a g-quasiorthonormal frame field $(X_\alpha)_\alpha$ on M,

$$\mathrm{Ric}^D(X,Y)=g^{[\alpha\beta]}\widetilde{g}\big(R^D(X,X_\alpha)Y,X_\beta\big) \tag{18}$$

and clearly, the right hand side of (18) does not change under conformal rescaling in c. The scalar curvature Scal^D of D is defined by

$$\mathrm{Scal}^D=\mathrm{tr}_c\,\mathrm{Ric}^D.$$

Observe that Scal^D is not a function on M, but for a choice of a metric $g\in c$, it is defined by $\mathrm{Scal}^D_g=\mathrm{tr}_g\,\mathrm{Ric}^D$.

Proposition 3.8.

Suppose $D=D^g+\widetilde{\theta}$ where $g\in c$ and θ_g is the 1-form associated to the pair $\{D,g\}$. Then

$$\begin{aligned} R^D(X,Y) &= R^g(X,Y)-2d\theta_g(X,Y)\,\mathrm{id}+\Big(D^g_Y\theta_g^{\#_g}-\theta(Y)\theta_g^{\#_g}+\frac{1}{2}\big|\theta_g^{\#_g}\big|_g^2Y\Big)\wedge X\\ &\quad-\Big(D^g_X\theta_g^{\#_g}-\theta(X)\theta_g^{\#_g}+\frac{1}{2}\big|\theta_g^{\#_g}\big|_g^2X\Big)\wedge Y-\big(\mathcal{K}^g(X,Y)-\mathcal{K}^g(Y,X)\big)\xi \end{aligned} \tag{19}$$

with $\mathcal{K}^g(X,Y)=i_Y(D^g_XS)+S\big(Y,\theta_g^{\#_g}\big)i_Xg+S(Y,\xi)i_XS+\varphi_g(X)i_YS$, where ($(M,g)$ being totally geodesic) the 1-form φ_g is defined by $D^g_X\xi=\varphi_g(X)\xi$, and $X\wedge Y=g(X,\cdot)Y-g(Y,\cdot)X$.

This is a standard computation using (9) and the curvature formula (17). The following lemma gives expression of $\mathcal{K}^g(X,Y)-\mathcal{K}^g(Y,X)$ for horizontal X and Y in terms of the second fundamental form C of the screen distribution $\mathrm{S}(TM)$.

Lemma 3.9.

For $X,Y\in\Gamma(\mathrm{S}(TM))$, we have

$$\begin{aligned} \mathcal{K}^g(X,Y)-\mathcal{K}^g(Y,X) &= \eta\big(\overline{R}(X,Y)Z\big)+\big[g(X,Z)c\big(Y,\theta_g^{\#_g}\big)-g(Y,Z)c\big(X,\theta_g^{\#_g}\big)\big]\\ &\quad+\big[C(X,Z)C(\xi,Y)-C(Y,Z)C(\xi,X)\big]+\big[\theta_g(Y)C(X,Z)-\theta_g(X)C(Y,Z)\big], \end{aligned}$$

where $\overline{R}$ is the ambient Riemannian curvature of $\big(\overline{M},e^{-2\overline{f}}\overline{g}\big)$, with $\overline{f}\restriction_M=f$ and C the second fundamental form of the screen distribution $\mathrm{S}(TM)$.

This is a result of direct use of (14), (16) and the Gauss–Codazzi equation for the screen distribution,

$$\overline{g}\big(\overline{R}(X,Y)Z,N\big) = \big(D^g_X C\big)(Y,Z) - \big(D^g_Y C\big)(X,Z)\varphi_g(X)C(Y,Z) - \varphi_g(Y)C(X,Z).$$

Taking into account (19) and (18), we get

Proposition 3.10.
The Ricci curvature of D is given by

$$\begin{aligned}\mathrm{Ric}^D(X,Y) = {}& \mathrm{Ric}^g(X,Y) - 2d\theta_g(X,Y) + (1-n)\big(D^g_X\theta_g\big)(Y)\\ &+ (n-1)\theta_g(X)\theta_g(Y) + (1-n)g(X,Y)\big|\theta_g^{\#_g}\big|_g^2 - g(X,Y)\delta^g\theta_g\\ &+ \Big(\big[\big(D^g_X S\big)(\xi,Y) - \big(D^g_\xi S\big)(X,Y)\big] + g(X,Y)S\big(\xi,\theta_g^{\#_g}\big) - S(\xi,X)S(\xi,Y) + \varphi_g(X)S(\xi,Y)\Big).\end{aligned} \tag{20}$$

Proposition 3.11.
Let D be a Weyl structure on $(M,c,S(TM))$, then, for $g\in c$,

$$\begin{aligned}\mathrm{Scal}^D_g = {}& \mathrm{scal}^g - (n-1)^2\big|\theta_g^{\#_g}\big|_g^2 + (1-2n)\delta^g\theta_g + (n-1)\varphi_g\big(\theta_g^{\#_g}\big) + \mathrm{div}^g i_\xi S - \mathrm{tr}\, g(D^g_\xi S)\\ &+ nS\big(\xi,\theta_g^{\#_g}\big) - \big|(i_\xi S)^{\#_g}\big|_g^2 + g\big(\varphi_g^{\#_g},(i_\xi S)^{\#_g}\big).\end{aligned} \tag{21}$$

Proof. We have $\mathrm{Scal}^D_g = g^{[\alpha\beta]}\mathrm{Ric}^D(X_\alpha,X_\beta)$, where $(X_\alpha)_\alpha$ is a quasiorthonormal frame field on M adapted to the decomposition (2). Then using the above Ricci formula leads to

$$\begin{aligned}\mathrm{Ric}^D(X_\alpha,X_\beta) = {}& \mathrm{Ric}^g(X_\alpha,X_\beta) - 2d\theta_g(X_\alpha,X_\beta) + (1-n)\big(D^g_{X_\alpha}\theta_g\big)(X_\beta)\\ &+ (n-1)\theta_g(X_\alpha)\theta_g(X_\beta) + (1-n)g_{\alpha\beta}\big|\theta_g^{\#_g}\big|_g^2 - g_{\alpha\beta}\delta^g\theta_g\\ &+ \Big(\big[\big(D^g_{X_\alpha}S\big)(\xi,X_\beta) - \big(D^g_\xi S\big)(X_\alpha,X_\beta)\big] + g_{\alpha\beta}S\big(\xi,\theta_g^{\#_g}\big) - S(\xi,X_\alpha)S(\xi,X_\beta) + \varphi_g(X_\alpha)S(\xi,X_\beta)\Big)\end{aligned}$$

with $\delta^g\theta_g = \mathrm{div}^g\theta_g^{\#_g}$. Contracting with $g^{[\alpha\beta]}$ and a straightforward computation give relation (21). □

4. Einstein–Weyl screen structures

Note that as D is not a metric connection on M, its Ricci curvature is not necessarily symmetric. The quadruplet $(M,c,S(TM),D)$ defines an Einstein–Weyl screen structure if D is a Weyl screen structure on $(M,c,S(TM))$ and the symmetrised Ricci tensor of D is proportional to g pointwise. Equivalently, there exists a function $\wedge\in C^\infty(M)$ such that

$$\mathrm{Ric}^D(X,Y) + \mathrm{Ric}^D(Y,X) = \wedge g(X,Y), \tag{22}$$

for all tangent vectors $X,Y\in TM$. The function $\wedge$ (depends on $g\in c$) is called the Einstein–Weyl function of the structure with respect to g. By (20) one has

$$\mathrm{Ric}^D(X,Y) + \mathrm{Ric}^D(Y,X) = \mathrm{Ric}^g(X,Y) + \mathrm{Ric}^g(Y,X) + \mathcal{D}(\theta_g)(X,Y) + 2g(X,Y)\Big\{(1-n)\big|\theta_g^{\#_g}\big|_g^2 - \delta^g\theta_g + S\big(\xi,\theta_g^{\#_g}\big)\Big\},$$

where

$$\begin{aligned}\mathcal{D}(\theta_g)(X,Y) = {}& (1-n)\big[\big(D^g_X\theta_g\big)(Y) + \big(D^g_Y\theta_g\big)(X) - 2\theta_g(X)\theta_g(Y)\big] + \big[\big(D^g_X S\big)(\xi,Y) + \big(D^g_Y S\big)(\xi,X)\big]\\ &+ \big[\varphi_g(X)S(\xi,Y) + \varphi_g(Y)S(\xi,X)\big] - \big[\big(D^g_\xi S\big)(X,Y) + S(\xi,X)S(\xi,Y)\big].\end{aligned} \tag{23}$$

Also, on the symmetry of Ric^g note that

$$\mathrm{Ric}^g(X,Y) - \mathrm{Ric}^g(Y,X) = 2d\varphi_g(X,Y) \tag{24}$$

for all tangent vectors X, Y in TM. Then, it follows from (23) and (24)

Proposition 4.1.
The quadruplet $(M, c, S(TM), D)$ defines a Einstein–Weyl screen structure if and only if D is defined by (9) for all $g \in c$ and the Ricci curvature of g satisfies

$$\mathrm{Ric}^g = d\varphi_g - \frac{1}{2}\mathcal{D}(\theta_g) + \overline{\wedge} g, \tag{25}$$

where $\overline{\wedge}$ is related to $\wedge$ in (22) by $\overline{\wedge} = \frac{1}{2}\wedge - \Big[(1-n)\big|\theta_g^{\#_g}\big|_g^2 - \delta^g\theta_g + S\big(\xi, \theta_g^{\#_g}\big)\Big]$ with $\mathcal{D}(\theta_g)$ given by (23).

5. A generic example

Let (N, g_N) and (F, g_F) be a lightlike and a Riemannian manifold of dimension n and m respectively. Let $\pi\colon N\times F \to N$ and $\varrho\colon N\times F \to F$ denote the projection maps given by $\pi(x,y) = x$ and $\varrho(x,y) = y$ for $(x,y) \in N\times F$, respectively, where the projection π on N is done with respect to a nondegenerate screen distribution $S(TN)$. The product manifold $M = N\times F$, endowed with the degenerate metric defined by

$$g(X,Y) = g_N(\pi_\star X, \pi_\star Y) + f(\pi(x,y))\, g_F(\varrho_\star X, \varrho_\star Y),$$

for all X, Y tangent to M, where $\star$ is the symbol of the tangent linear map and $f\colon N \to \mathbb{R}_+^\star$ is some positive smooth function on N, is called a lightlike warped product and denoted as $M = (N\times_f F, g)$. In case $f = 1$ such a product is called a *lightlike product* and denoted as $M = (N\times F, g)$.

Let $(M = \mathbb{L}\times_f N, g)$ be a totally geodesic lightlike warped product hypersurface (in a Lorentzian Einstein manifold $(\overline{M}, \langle\cdot,\cdot\rangle)$, with f a smooth positive function on $\mathbb{L}$, a (one dimensional) null integral curve of a global null section ξ on $\overline{M}$, $(N, \overset{\star}{c}, \mathcal{D})$ a Riemannian manifold (N, g_N) equipped with a Einstein–Weyl structure $\mathcal{D}$, $\overset{\star}{c}$ being the whole conformal class of the Riemannian metric g_N. The (induced) degenerate metric g_0 on M can be written as

$$g_0|_x(X,Y) = \big((f\circ\pi_1)(x)\big)^2 g_N\big(\pi_{2\star}X, \pi_{2\star}Y\big),$$

where π_1 and π_2 denote the projections on the factors $\mathbb{L}$ and N of M respectively. On $\big(M = \mathbb{L}\times_f N, g_0\big)$ consider the conformal class $c_M = \{e^{-2\sigma}g_0 : \xi\cdot\sigma = 0\}$, and let D be defined on M by

$$D_X Y = \mathcal{D}_{X_2} Y_2 + \frac{1}{2}\big[\big(X_1\cdot\phi\big)Y_2 + \big(Y_1\cdot\phi\big)X_2\big],$$

with $X = (X_1, 0) + (0, X_2) = (X_1, X_2)$, $Y = (Y_1, 0) + (0, Y_2) = (Y_1, Y_2)$ on $\mathbb{L}\times N$, $\phi = \ln f$. Consider on M the distribution $\mathcal{S}$ given by $\mathcal{S}_x = T_{\pi_2(x)}N$, $x \in M$. Clearly, $\mathcal{S}$ defines a screen distribution on M which is D-parallel and $(M, c_M, D, \mathcal{S})$ is a Weyl screen structure on M. Moreover, for this Weyl screen structure, we find out that the tensor S is taken to be identically zero according to Lemma 3.6 and from (10) of the same lemma, we have $C(X,Y) = 0$ for horizontal vector fields (that is tangent to the screen distribution $\mathcal{S}$) and $\theta_g(Z) = -C(\xi, PZ)$ for all $Z \in \Gamma(TM)$. Note that as for a given $g \in c_M$, (M, g) is totally geodesic, the associate Ricci tensor Ric^g is symmetric and it follows from (24) that $d\varphi_g = 0$. Moreover, as the ambient manifold $\overline{M}$ is assumed to be Einstein and for each g in the conformal class c_M, (M, g) is totally geodesic, Ric^g is pointwise proportional to the metric g on M. It follows from Proposition 4.1 that our Weyl-screen structure is a Einstein–Weyl screen structure, and in particular $(\mathcal{D}\theta_g)$ given by (23) vanishes identically.

The screen distribution $\mathcal{S}$ is integrable and its leaves are totally geodesic in M (due to $C \equiv 0$ on $\mathcal{S}\times\mathcal{S}$). In fact, in ambient (Lorentzian) four dimension, under a reduced holonomy assumption, the (Riemannian) Einstein–Weyl structure factor N should be flat.

6. Totally umbilical screen foliation

The screen distribution $S(TM)$ is said to be totally umbilical if there exists a function $\lambda \in C^\infty(M)$ such that

$$C(X, PY) = \lambda g(X, Y), \tag{26}$$

for all tangent vectors X, Y in TM. Then, (10) becomes

$$S(X, Y) = \lambda g(X, Y) + \eta(X)\theta_g(Y), \tag{27}$$

for $(X, Y) \in \Gamma(TM) \times \Gamma(S(TM))$. In particular, $S(\xi, X) = S(X, \xi) = \theta_g(X)$ for all X in $\Gamma(TM)$.

Lemma 6.1.
For $g \in c$ and for all tangent vectors X, Y and Z in $\Gamma(TM)$,

$$(D^g_Z S)(X, Y) = (Z \cdot \lambda) g(X, Y) + \left[\theta_g(X)\left(D^g_Z \eta\right)(Y) + \theta_g\left(Y\right)(D^g_Z \eta)(X)\right] + \left[\eta(X)\left(D^g_Z \theta_g\right)(Y) + \eta(Y)\left(D^g_Z \theta_g\right)(X)\right]. \tag{28}$$

Proof. Let $(X, Y) \in \Gamma(TM) \times \Gamma(S(TM))$, it is immediate using (27) that, for $Z \in \Gamma(TM)$,

$$\left(D^g_Z S\right)(X, Y) = (Z \cdot \lambda) g(X, Y) + \left(D^g_Z \eta\right)(X)\theta_g(Y) + \left(D^g_Z \theta_g\right)(Y)\eta(X). \tag{29}$$

Now for $(X, Y) \in \Gamma(TM) \times \Gamma(TM)$ observe that

$$\left(D^g_Z S\right)(X, Y) = \left(D^g_Z S\right)(X, PY) + \eta(Y)\left(D^g_Z S\right)(\xi, PX)$$

and then, using (29) and the fact that θ_g and η are horizontal and vertical respectively, lead to relation (28). □

In particular, for all tangent vectors X, Y in TM,

$$\left(D^g_X S\right)(\xi, Y) = \left(D^g_X \theta_g\right)(Y) - \varphi_g(X)\theta_g(Y), \tag{30}$$

$$\left(D^g_\xi S\right)(X, Y) = (\xi \cdot \lambda) g(X, Y) + \left(D^g_\xi \theta_g\right)(X)\eta(Y) + \left(D^g_\xi \theta_g\right)(Y)\eta(X), \tag{31}$$

which arises from (28) and the fact that η is parallel along the ξ-orbits. We also have the following fact.

Proposition 6.2.
Assume that $(M, c, S(TM), D)$ is an Einstein–Weyl screen structure with totally umbilical $S(TM)$, then

$$\left(D^g_\xi \theta_g\right)(X) = 0, \quad X \in \Gamma(TM), \qquad \text{and} \tag{32a}$$

$$\left(D^g_\xi S\right)(X, Y) = (\xi \cdot \lambda) g(X, Y) \tag{32b}$$

for all tangent vectors X, Y in $\Gamma(TM)$. Moreover,

$$\begin{aligned} \mathrm{Ric}^D(X, Y) = {} & \mathrm{Ric}^g(X, Y) - 2d\theta_g(X, Y) + (2-n)\left(D^g_X \theta_g\right)(Y) + (n-2)\theta_g(X)\theta_g(Y) \\ & + (2-n)\left|\theta_g^{\#_g}\right|^2_g g(X, Y) - g(X, Y)\delta^g\theta_g - (\xi \cdot \lambda) g(X, Y), \end{aligned} \tag{33}$$

$$\mathrm{Scal}^D_g = \mathrm{scal}^g + (2-n)(n-1)\left|\theta_g^{\#_g}\right|^2_g + 2(1-n)\delta^g\theta_g + n\varphi_g\left(\theta_g^{\#_g}\right) - n(\xi \cdot \lambda). \tag{34}$$

where λ is given by (26).

Proof. Note that $\mathrm{Ric}^g(\xi, Y) = \mathrm{Ric}^g(Y, \xi) = 0$ and $2d\varphi_g(\xi, Y) = \mathrm{Ric}^g(\xi, Y) - \mathrm{Ric}^g(Y, \xi) = 0$. Then, as the structure is Einstein–Weyl, by (25), we have $\mathcal{D}(\theta_g)(\xi, X) = 0$ for all tangent vectors X in $\Gamma(TM)$. Thus, (32a) follows from (31) by setting $Y = \xi$, and substitution in (23). Thereafter, (31) reduces to (32b) and (32) is proved. Now, (33) and (34) are just rewriting of (20) and (21) respectively, taking into account (32) and (27) and the proof is complete. □

Note.

The metric $g \in c = [g_0]_0$ will be called the *trivial extension* of its restriction $g' \in c' = [g'_0]$ on the horizontal.

Lemma 6.3.

If (M, g) is totally geodesic in flat $(\overline{M}, \overline{g})$ then for all horizontal vector fields X and Y, one has

$$\mathrm{Ric}^g(X, Y) = \mathrm{Ric}^{g'}(X, Y),$$

where g' is the restriction of g on the horizontal.

Proof. For horizontal vector fields X and Y, one has

$$\mathrm{Ric}^g(X, Y) = g^{[\alpha\beta]}\widetilde{g}\big(R^g(X, X_\alpha)Y, X_\beta\big) = g^{ij}g\big(R^g(X, X_i)Y, X_j\big) + \widetilde{g}\big(R^g(X, \xi)Y, \xi\big).$$

On the other hand, for horizontal X, Y and Z one has

$$R^g(X, Y)Z = \overset{\star}{R}(X, Y)Z + \Big\{\big[(\overset{\star}{\nabla}{}^{g'}_Y C)(X, Z) - (\overset{\star}{\nabla}{}^{g'}_X C)(Y, Z)\big] + \big[C(X, Z)\varphi_g(Y) - C(Y, Z)\varphi_g(X)\big]\Big\}\xi,$$

where $\overset{\star}{R}$ denotes the curvature tensor of the induced Levi-Civita connection $\overset{\star}{\nabla}{}^{g'}$ on the horizontal. Hence,

$$\begin{aligned}\mathrm{Ric}^g(X, Y) &= g^{ij}g\big(\overset{\star}{R}(X, X_i)Y, X_j\big) + \widetilde{g}\big(R^g(X, \xi)Y, \xi\big)\\ &= g'^{ij}g'\big(\overset{\star}{R}(X, X_i)Y, X_j\big) + \widetilde{g}\big(R^g(X, \xi)Y, \xi\big) = \mathrm{Ric}^{g'}(X, Y) + \widetilde{g}\big(R^g(X, \xi)Y, \xi\big).\end{aligned}$$

Finally, as (M, g) is totally geodesic in $\overline{M}$ which is flat, we have [4, p. 97], $R^g(X, \xi)Y = \overline{R}(X, \xi)Y = 0$. Thus,

$$\mathrm{Ric}^g\restriction_{\mathrm{hor}} = \mathrm{Ric}^{g'}. \tag{35}$$

□

Remark 6.4.

Under hypothesis of Lemma 6.3, since $\mathrm{Ric}^g(\xi, X) = \mathrm{Ric}^g(X, \xi)$, it follows from (35) that on leaves of the integrable screen distribution, $\mathrm{scal}^g\restriction_{M'} = \mathrm{scal}^{g'}$, where M' is any leaf of $S(TM)$.

Note.

For a Weyl screen structure D relative to $(M, c, S(TM))$, as for any $g \in c$, the associate 1-form θ_g is horizontal,we will indistinctly note by θ_g its restriction on the horizontal. Thus, for horizontal vectors X, Y, we have

$$\big(D^g_X\theta_g\big)(Y) = X \cdot \theta_g(Y) - \theta_g\big(\overset{\star}{\nabla}{}^{g'}_X Y + C(X, Y)\xi\big) = (\overset{\star}{\nabla}{}^{g'}_X\theta_g)(Y). \tag{36}$$

Now, we state the following.

Theorem 6.5.

Let $(M, c, S(TM), D)$ be an Einstein–Weyl screen structure quadruplet in the Lorentzian space $\mathbb{R}_1^{n+2}$ and D' the (Riemannian) induced Weyl structure by D on the conformal structure (M', c') where M' is a leaf of the totally umbilical integrable screen distribution $S(TM)$ and $c' = c\restriction_{M'}$. Then,

(a) *D' is a (Riemann) Einstein–Weyl structure relative to (M', c'). Furthermore, the Einstein–Weyl functions $\wedge$ and $\wedge'$ relative to $g \in c$ and $g' = g\restriction_{M'} \in c'$ respectively, are related along M' by*

$$\frac{1}{2}(\wedge - \wedge') = \varphi_g\left(\theta_g^{\#_g}\right) + 2(\xi \cdot \lambda). \tag{37}$$

(b) *If the screen foliation is compact and the Cotton–York tensor [6] of D' vanishes identically, then the Weyl screen structure D relative to $(M, c, S(TM))$ is closed.*

(c) *Along compact leaves of $S(TM)$, the trivial extension g to (M, c) of the Gauduchon metric [5] associated to (M', c', D') satisfies*

 (i) $\mathrm{Scal}^g - (n+2)\left|\theta_g^{\#_g}\right|_g^2 = G,$

 (ii) $\mathrm{Scal}_g^D + n(n-4)\left|\theta_g^{\#_g}\right|_g^2 - (3-2n)\varphi_g(\theta_g^{\#_g}) + n(\xi\cdot\lambda) = G$, *where G is the Gauduchon constant [5].*

Proof. Let X, Y be horizontal vector fields. By the use of (25), Lemma 6.3, (30) and (32b), we get

$$\mathrm{Ric}^{g'}(X,Y) = \mathrm{Ric}^g(X,Y) = d\varphi_g(X,Y) - \frac{1}{2}\mathcal{D}(\theta_g)(X,Y) + \left[\frac{1}{2}\wedge - \left[(2-n)\left|\theta_g^{\#_g}\right|_g^2 - \delta^g\theta_g - 2(\xi\cdot\lambda)\right]\right]g(X,Y).$$

where $\wedge$ is the Einstein–Weyl function with respect to $g \in c$. Hence, from (36),

$$\mathrm{Ric}^{g'}(X,Y) = d\varphi_g(X,Y) - \frac{1}{2}\mathcal{D}'(\theta_g)(X,Y) + \left[\frac{1}{2}\wedge - \left[(2-n)\left|\theta_g^{\#_g}\right|_g^2 - \delta^g\theta_g - 2(\xi\cdot\lambda)\right]\right]g'(X,Y)$$

with

$$\mathcal{D}'(\theta_g)(X,Y) = (2-n)\left[\left(\overset{\star}{\nabla}{}^{g'}_X\theta_g\right)(Y) + \left(\overset{\star}{\nabla}{}^{g'}_Y\theta_g\right)(X) - 2\theta(X)\theta(Y)\right].$$

The symmetry of the $(0,2)$-tensors $\mathrm{Ric}^{g'}$, $\mathcal{D}'(\theta_g)$ and g' leads to $d\varphi_g(X,Y) = 0$ and

$$\mathrm{Ric}^{g'}(X,Y) = -\frac{1}{2}\mathcal{D}'(\theta_g)(X,Y) + + \left[\frac{1}{2}\wedge' - \left[(2-n)\left|\theta_g^{\#_g}\right|_{g'}^2 - \delta^{g'}\theta_{g'}\right]\right]g'(X,Y) \tag{38}$$

with $\wedge' = \wedge - 2\left[\varphi_g\left(\theta_g^{\#_g}\right) + 2(\xi\cdot\lambda)\right]$. It follows from (38) that (M', c', D') is an Einstein–Weyl structure on the Riemannian leaf M' [9] with Einstein–Weyl function $\wedge'$ relative to g' as given in (37).

Now, let $g \in c$ denote the trivial extension of the standard metric of (M', c', D') and θ_g the associated 1-form. We show that $d\theta_g = 0$. Suppose M' is a compact leaf of $S(TM)$ and that the Cotton–York tensor of D' vanishes identically. Then, we know by the Ivanov result in [6] that $\overset{\star}{\nabla}{}^{g'}\theta_g = 0$ where $\overset{\star}{\nabla}{}^{g'}$ is the Levi-Civita connection of $g' = g\restriction_{M'}$ the standard metric of (M', c', D'). Then using (36), we have for horizontal vector fields X and Y,

$$\left(D_X^g\theta_g\right)(Y) = \left(\overset{\star}{\nabla}{}^{g'}_X\theta_g\right)(Y) = 0.$$

Finally, using (32a) and the fact that θ_g is horizontal, we deduce that θ_g is parallel with respect to D^g, that is $D^g = 0$. Hence, $d\theta_g = 0$ and θ_g is closed and (b) is proved.

Note that we have [5] on (M', c', D')

$$\mathrm{Scal}_{g'}^{D'} = \mathrm{Scal}^{g'} + 2(n-1)\delta^{g'}\theta_{g'} - (n-1)(n-2)\left|\theta_{g'}^{\#_{g'}}\right|_{g'}^2. \tag{39}$$

Also, the following relation defines the Gauduchon constant G:

$$\mathrm{Scal}_{g'}^{D'} + n(n-4)\left|\theta_{g'}^{\#_{g'}}\right|_{g'}^{2} = G. \tag{40}$$

Then, (i) in (c) is a simple consequence of Remark 6.4. On the other hand, using (34), Remark 6.4 and (39), one has along M',

$$\mathrm{Scal}_g^D\restriction_{M'} = \mathrm{Scal}_{g'}^{D'} - 4(n-1)\delta^{g'}\theta_g + (3-2n)\varphi_g\left(\theta_g^{\#_g}\right) - n(\xi\cdot\lambda).$$

So,

$$\mathrm{Scal}_g^D\restriction_{M'} + n(n-4)\left|\theta_g^{\#_g}\right|_g^2 + 4(n-1)\delta^{g'}\theta_g - (3-2n)\varphi_g\left(\theta_g^{\#_g}\right) + n(\xi\cdot\lambda) = \mathrm{Scal}_{g'}^{D'} + n(n-4)\left|\theta_{g'}^{\#_{g'}}\right|_{g'}^2. \tag{41}$$

Then, (ii) follows from (41) and (40) and the proof is complete. □

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