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SUSY partners for spin-1/2 systems in nonrelativistic limits

Abstract: In this paper, we use some well-known techniques of Supersymmetric Quantum Mechanics (SUSYQM) namely the factorization method and shape invariance, to generate new analytically solvable potentials from some interacting fermionic models in nonrelativistic limits. These systems are described by the ordinary and the harmonically trapped Schrödinger-Pauli particle models and the Dirac-Coulomb Hamiltonian, this latter being set in its nonrelativistic limits. The spectrum for each of these models is obtained in a simple and transparent way. We then generate new solvable potentials that describe interactions between electromagnetic field and matter, paying due attention to the subtleties inherent in the application of SUSY to higher dimensional problems. SUSY breaking problems related to the partner singularities are dicussed along with the paper.

Keywords: supersymmetry (SUSY); Pauli-Schrödinger equation; Dirac-Coulomb system; SUSY partners; singularity

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1 Introduction

The solvable potentials play a fundamental role in many areas of physics and mathematics as they are simple and lead to much insight into the structure and property of more complicated phenomena. Solvable Hamiltonians provide toy expressions for primary and also sometimes for deep evaluations of physical systems and may also serve as laboratories for testing new quantization methods. Unfortunately there is not a lot of known solvable models in quantum mechanics and it remains interesting to take hold of a new analytically solvable potential since one expects to learn novel physics trough its spectrum.

SUSYQM is an important tool for the construction of solvable potentials among other things. It is a framework used to determine the spectrum of quantum mechanics Hamiltonians. This involves the development of various techniques to solve the Schrödinger equation with various interactions expressed by different form of potentials. One of these techniques is known as the factorization method. The factorization method to solve the Schrödinger equation is almost as old as quantum mechanics itself (for a review, see for instance Ref [1] in which an outstanding presentation of SUSYQM was published by Dutt, Khare and Sukhatme; see also Refs [2–5] for the basic formalism). This field of mathematical physics research, which still offers many open fascinating perspectives even for systems with a single degree of freedom [1], is being extended to higher dimensional systems as well as fermionic spin degrees of freedom [6–8].

Since the fundamental works by Witten [9] and Gendenshtein [8] the methods of supersymmetric nonrelativistic quantum mechanics have developed rapidly. It has been realized that there exist partner potentials with precisely the same energy spectra except for the ground state – whose wave function $\phi(x) = \psi_0(x)$ is used to define the superpotential $W(x)$ – and that if they are shape invariant, their spectra and wave functions can be exactly and analytically solved. The method may be sketched as follows. Through the factorization of the second order differential stationary Schrödinger equation into the composition of two first order differential operators which are adjoints of one another, a given Hamiltonian whose energy eigenspectrum of states would be known, is seen to belong to an infinite hierarchy of successive intertwined pairs of Hamiltonians ($H_\pm$) of which the energy eigenspectra may readily be identified starting from that of the first Hamiltonian. However, this requires care to be taken with the global SUSY breaking problems generated by singularities

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of partners. In each interval of the partition of intervals excluding the singularity points, all of these Hamiltonians share an identical spectrum of energy eigenvalues except for the lowest lying state which is removed as one moves from one member in the hierarchy to the next (the ground state energy E_0^- is missing in the partner Hamiltonian H_+, so that its ground state $E_0^+ = E_1^-$). In other words, given a quantum system characterized by a single degree of freedom, of which the energy eigenspectrum is known explicitly –namely both the values of its (discrete) energy eigenvalues and the corresponding quantum states, the latter say in terms of their (configuration space) wave functions–, there arises a semi-infinite hierarchy of integrable single degree of freedom quantum systems, all of whose energy eigenspectra are known likewise.

However some restrictions are required to enable the actual construction of such hierarchies leading to the concept of *Shape invariance* spanned by the following idea: if two successive intertwined members of the hierarchy are somehow related to one another, the induced recursion relations should allow for a solution which then identifies an integrable hierarchy of Hamiltonians. Specific classifications of shape invariant potentials have been achieved in the literature [8, 9]; see also Refs [7, 10, 11, 13–16] for a recent but not certainly exhaustive list. Hence SUSYQM for Schrödinger operator thus provide an important insight leading towards the construction of nontrivial integrable quantum potentials, starting from already known ones.

Since then, many researchers have dealt with quantization of integrable and even superintegrable Hamiltonians in the framework of SUSYQM and it has been shown that the nonrelativistic formalism of the SUSYQM may also be applied to the higher excited states of 1-D potentials, generating new partner potentials isospectral to the original potential, provided one is careful with singularities which may occur in the partners. The research has been further developed also in the direction of applying the WKB (Wentzel-Kramers-Brillouin) methods to such classes of Hamiltonians, including the search for improved simple quantization conditions which would be exact in case of SUSY shape invariant potentials[1, 8, 17–19], and also in direction of exploring the applicability of the path integral techniques[20]. It should be mentioned here that the ideas behind the SUSY property and shape invariance were formulated first by Infeld and Hull [10], where they were called the factorization method, and these authors refer further to the related ideas in the works of Schrödinger [11].

Our work is concerned with the application of the standard SUSY formalism for systems described by a single degree of freedom which must belong in principle to the full real line. We consider however three-dimensional problems (see Sections 2 and 4) and a two-dimensional problem (see Section 3) which we bring back to one-dimensional problems, being sure that the interaction described by the potential energy is such that the energy spectrum is discrete and bounded in each case. Our paper deals with the SUSY-n formalism for which there is only one superpotential W_n and the two partner Hamiltonians are separated by one level in the hierarchy i.e. a single gap. We present results of a straightforward further application of this formalism to a few, most important exactly solvable systems with potentials described by one degree of freedom, namely: the Schrödinger-Pauli equation which is a nonrelativistic limit of the Dirac equation in the presence of the magnetic field; the Schrödinger-Pauli equation with harmonic term, possibly in the presence of an electric background field, and the relativistic Dirac-Coulomb equation, taken here as far as its nonrelativistic limit. In all these models, singularities occurred, leading to the multiplication of isospectral families. Unavoidable difficulties, linked to the semi-infinite domain of the radial variable describing higher dimensional rotationally symmetric systems, are treated.

The paper is organized as follows. In section 2 we set and solve a Hamiltonian describing the three-dimensional Schrödinger-Pauli equation for a charged particle and generate one-dimensional partner potentials from a particular choice of magnetic field through the gauge potential freedom. We show that the alternative method based on the Darboux Transformations (DT) gives the same results. Section 3 is concerned with the Schrödinger-Pauli system for a charged particle in an harmonic trap set initially as the Penning trap to avoid dimensional ambiguities. Superpartners are constructed from the radial sector. Next an extension of this model to a background static electric field is considered. In section 4 the formalism of the Dirac-Coulomb problem has been used to obtain the radial solutions and the superparters of its potential are detailed in the nonrelativistic case. We close the paper in section 5 with a conclusion.

2 The Schrödinger-Pauli system for a charged particle

2.1 The model and its spectrum

Let us consider the simplest nontrivial situation of an electron of charge $q = -e$, interacting exclusively with an external homogeneous magnetics field $\mathbf{B}$ belonging *a priori* to

$\mathbb{R}^3$. The well-known Pauli equation has the following form with the convention $c = 1$,

$$\left[\frac{1}{2m}(\mathbf{p}+e\mathbf{A})^2+\mu\sigma\cdot\mathbf{B}\right]\psi = i\hbar\frac{\partial\psi}{\partial t}. \quad (1)$$

It describes, in three dimensional physical space, the nonrelativistic evolution of a charged particle of mass m, taken subsequently as the electron, undergoing the influence of a magnetic field $\mathbf{B}$ whose three-component vector gauge potential is denoted by $\mathbf{A}$. In this expression $\hbar$ is the reduced Planck constant, $\mu = \hbar e/(2m)$ is the Bohr magneton, σ is a three-component vector formed by the Pauli matrices represented as follows

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

In this notation the spinorial wave function ψ is represented as an element of $L^2(\mathbb{R}^3)\bigotimes\mathbb{C}^2$ i.e. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ and the operators appearing in (1) are assumed to be defined in this Hilbert space. We choose the Poincaré gauge in which the magnetic vector potential coincides with $\frac{1}{2}\mathbf{B}\wedge\mathbf{x}$. The formal expression of the stationary Hamiltonian may be written as follows

$$H_P = \frac{\mathbf{p}^2}{2m}+\frac{e^2}{2m}\left(\frac{1}{2}\mathbf{B}\wedge\mathbf{x}\right)^2+\frac{e}{2m}\mathbf{B}\cdot\mathbf{L}+\mu\sigma\cdot\mathbf{B}, \quad (3)$$

where the angular-orbital momentum reads $\mathbf{L} = \mathbf{x}\wedge\mathbf{p}$, and $\mathbf{p}^2 = p_1^2+p_2^2+p_3^2$, $p_i : -i\hbar\frac{\partial}{\partial x_i}$.

It is well known that the system described by the model (1) is integrable. For example, in Ref. [7], it has been proved that it possesses a $\mathcal{PT}$–symmetry and that the stationary partner commutes with the famous Runge-Leng vector given by $\mathbf{R} = \frac{1}{2m}(\mathbf{p}\wedge\mathbf{L}-\mathbf{L}\wedge\mathbf{p})+\mathbf{x}\phi$, ϕ being a scalar function describing an electric potential energy, $\phi = -\frac{e^2}{\|\mathbf{x}\|}$, with the commutation relations $[L_a, L_b] = i\hbar\varepsilon_{abc}L_c$, $[R_a, L_b] = i\hbar\varepsilon_{abc}R_c$, $[R_a, R_b] = -\frac{2i}{m}\hbar\varepsilon_{abc}L_cH$.

In the following development, $\mathbf{B} = B\mathbf{u}_z$ will stand for the uniform Abelian magnetic field which derives from the vector potential $\mathbf{A} = \frac{1}{2}\mathbf{B}\wedge\mathbf{x} = \frac{1}{2}B(x\mathbf{u}_y - y\mathbf{u}_x)$ exactly as in the ordinary Landau problem. Let us make the following gauge choice,

$$\mathbf{A} \to \mathbf{A}-\nabla\Lambda, \quad \Lambda(x, y) = \frac{1}{2}Bxy. \quad (4)$$

We obtain the vector potential $\mathbf{A} = -By\mathbf{u}_x$ which depends only on one variable, namely y. The stationary equation corresponding to (1) then becomes

$$E\psi(\mathbf{x}) = \frac{1}{2m}\left[(p_x-eBy)^2+p_y^2+p_z^2\right]\psi(\mathbf{x})+\mu B\sigma_z\psi(\mathbf{x}). \quad (5)$$

With respect to the Zeeman splitting, let us introduce the spinor describing the motion of the electron as follows

$$\psi(\mathbf{x}) = \begin{pmatrix} \psi_+(\mathbf{x}) \\ \psi_-(\mathbf{x}) \end{pmatrix}. \quad (6)$$

The stationary eigenvalue problem expresses as follows

$$\begin{aligned} H_\pm\psi_\pm(\mathbf{x}) &= E_{(\pm)}\psi_\pm(\mathbf{x}), \\ H_{(\pm)} &= \frac{1}{2m}\left[(p_x-eBy)^2+p_y^2+p_z^2\right]\pm\mu B, \end{aligned} \quad (7)$$

where the subscripts $\pm$ describe the Zeeman splitting generated by the term $\mu B\sigma_z$ in Equation (5).

From here our task is to solve the above eigenvalue equation. Let us recall that in the conventional quantization procedure, the p-spectrum in a one-dimensional problem is given by $\phi(q) = Ke^{(\frac{i}{\hbar})qp}$, K being a normalization constant to be specified under specific conditions. Since the operators p_x and p_z commute with $H_\pm$, they constitute a complete set of observables which commute. Consequently they share the same eigenfunctions which may be written as follows,

$$\psi_\pm(x, y, z) = \exp\left(\frac{i}{\hbar}(p_x x+p_z z)\right)\chi_\pm(y), \quad (8)$$

where p_x and p_z are the eigenvalues of the conjugate momenta.

It goes without saying that from now on, the quantities x, y, z, p_x, p_y and p_z stand for the eigenvalues of their corresponding operators previously represented by the same writing. Note that Equation (8) is the sort of consideration that brings us to a one-dimensional problem (the y-direction) favourable to the application of SUSYQM formalism. The functions $\chi_\pm$ satisfy the following differential equation

$$\begin{aligned} \frac{d^2\chi_\pm(y)}{dy^2} &+ \left[-\left(\frac{eB}{\hbar}\right)^2\left(y-\frac{p_x}{eB}\right)^2\right. \\ &+ \left.\frac{2m}{\hbar^2}\left(E^{(\pm)}-\frac{p_z^2}{2m}\mp\mu B\right)\right]\chi_\pm(y) = 0. \end{aligned} \quad (9)$$

The general solution of the above equation is given by [13]

$$\chi_{n\pm}(y) = C_\pm\exp\left[\frac{-(y-y_o)^2}{2}\right]H_n(y-y_o), \quad y_0 = \frac{p_x}{eB}, \quad (10)$$

where $n\in\mathbb{N}$ represents the radial quantum number, $C_\pm$ are integration constants and H_n are the Hermite polynomials. The corresponding eigenvalues are given by

$$E_{n\pm} = \frac{\hbar^2}{m}(n+\frac{1}{2})+\frac{p_z^2}{2m}\pm\frac{\hbar^2}{2m}. \quad (11)$$

Note finally that the effective potential, which corresponds to the response of the particle to the vibration modes of the magnetic field, in the y-direction, is expressed by

$$V(y) = \frac{\hbar^2}{2m}(y - y_0)^2. \tag{12}$$

This potential does not admit any point of singularity except for the singular points $\mp\infty$.

2.2 Construction of the superpartner potentials

The superpotential W_n may readily be obtained from the spectrum (10) using the general formula $W(y) = -\frac{\hbar}{\sqrt{2m}}\frac{\phi'(y)}{\phi(y)}$, $\phi(y) = \chi_{n\pm}(y)$. Let us denote by y_j the roots of the function ϕ. Since $\phi'(x) \neq 0$ at the nodes $x = y_j$, the superpotential $W(y)$ will have singularities at the nodes $y_j, j = 1, 2, \cdots, n$ of ϕ. However, this does not invalidate our derivation, but it merely means, as will become clear later on, that the partner potential generated by ϕ diverges to $+\infty$ as $x \longrightarrow y_j$, for any $j = 1, 2, \cdots, n$. This implies that the obtained SUSY partners are well defined between two consecutive singularities and that they do not communicate with solutions in the neighbouring wells. We have

$$\begin{aligned} W_n(y) &= -\frac{\hbar}{\sqrt{2m}}\left[(y - y_0) - \frac{H_{n+1}(y - y_0)}{H_n(y - y_0)}\right], \\ W_0(y) &= \frac{\hbar}{\sqrt{2m}}(y - y_0). \end{aligned} \tag{13}$$

The partner potentials are given by

$$\begin{aligned} V^-_{n\pm}(y) &= V(y) - E_{n\pm} = \frac{\hbar^2}{2m}(y - y_0)^2 \\ &- \frac{\hbar^2}{m}\left(n + \frac{1}{2}\right) - \frac{p_z^2}{2m} \mp \frac{\hbar^2}{2m}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} V^+_{n\pm}(y) &= V^-_{n\pm}(y) + \frac{2\hbar}{\sqrt{2m}}\frac{dW(y)}{dy}, \\ &= \frac{\hbar^2}{2m}(y - y_0)^2 - \frac{\hbar^2}{m}\left(n + \frac{1}{2}\right) - \frac{p_z^2}{2m} \mp \frac{\hbar^2}{2m} - \\ &\quad \frac{\hbar^2}{m}\left[1 + \frac{H_{n+2}(y - y_0)}{H_n(y - y_0)} - \frac{H^2_{n+1}(y - y_0)}{H^2_n(y - y_0)}\right]. \end{aligned} \tag{15}$$

These partner potentials report as the sum of the harmonic potential characterizing the magnetic field modes with an additive term. They are actually generalizations of the effective potential describing the original system. Each of these generalizations may be, for a fixed quantum level, possibly related to a more complex physical model than the initial one (see below the cases $n = 0, 1$).

Thus if $n = 0$ we have the common trivial case of usual SUSY potentials defined on $(-\infty, +\infty)$, showing just that the 1-D harmonic oscillator potential form studied is indeed SUSY-0 shape invariant. Things are different when one moves to excited states. For instance, if $n = 1$ we have two potential wells, each of them being defined on a semi-infinite domain. For $n = 2$ we have one infinite potential well on a finite domain between two nodes y_1 and y_2, and two potential wells on the two semi-infinite domains $(-\infty, y_1]$ and $[y_2, +\infty)$, and so on. A similar situation occurs in the next sections. The partner potentials constructed in this way are nontrivial and certainly interesting since they contribute to our list of solvable potentials which now becomes rich and large in its contents.

The shape invariance requirement for the ground states associated with the two partner potentials is fullfilled since we have,

$$\begin{aligned} V^+_0(y) &= V^-_0(y) - \frac{\hbar^2}{m}\left[1 + \frac{H_2(y - y_0)}{H_0(y - y_0)} - \frac{H^2_1(y - y_0)}{H^2_0(y - y_0)}\right] \\ &= V^-_0(y) - \frac{\hbar^2}{m}, \end{aligned} \tag{16}$$

where we have taken into account the following quantities $H_0 = 1$, $H_1 = 2(y - y_0)$, and $H_2 = 4(y - y_0)^2 - 2$. This invariance occurs only at the ground state level; for example at the two lowest excited states charaterized by $(n = 1, 2)$, we have

$$V^+_1(y) = V^-_1 + \frac{\hbar^2}{m}\left[1 + \frac{1}{(y - y_0)^2}\right], \tag{17}$$

$$\begin{aligned} V^+_2(y) &= V^-_2(y) - \frac{\hbar^2}{m}\Bigg[1 + \frac{16(y - y_0)^4 - 48(y - y_0)^2 + 12}{4(y - y_0)^2 - 2} \\ &- \frac{\left(8(y - y_0)^3 - 12(y - y_0)\right)^2}{\left(4(y - y_0)^2 - 2\right)^2}\Bigg]. \end{aligned} \tag{18}$$

The case $n = 1$ which has only one singularity at $y = y_0$ gives a new example, which nevertheless is well known as the one-dimensional problem of the harmonic potential, which is thus a 1D SUSY-1 partner of the potential of our initial system. We note interestingly that it is similar in its form to the effective potential for the harmonically trapped bidimensional charged Pauli particle (see Equation (40)) –though this potential describes the radial problem of a rotationally symmetric harmonic oscillator–, confirming that the excited level SUSY partners generalize the initial models.

Furthermore, for $n = 2$ we have another new nontrivial example of a specific potential which is the SUSY-2 partner of the initial potential. It has singularities at the two nodes $y_1 = y_0 - \frac{1}{\sqrt{2}}$ and $y_2 = y_0 + \frac{1}{\sqrt{2}}$ of the type $1/(y - y_j)$. Therefore it has three branches, namely $(-\infty, y_1[$, $]y_1, y_2[$, and $]y_2, +\infty)$.

For higher n we get new classes of potentials, all of them being isospectral to the initial harmonic oscillator potential in each of the $n+1$ branches induced by the n nodes y_j, $j = 1, 2, \cdots, m$.

The equidistant energy levels for the partner potentials $V^{\mp}_{n\pm}(y)$ are readily obtained as follows

$$E^{-}_{n\pm} = E_{n\pm} - E_0 = \frac{n\hbar^2}{m}, \; E^{+}_{n\pm} = E_{(n+1)\pm} - E_0 = \frac{(n+1)\hbar^2}{m}. \quad (19)$$

It is relevant to note that $E^{-}_{0} = 0$ which means that the symmetry is not broken, a fundamental requirement in this approach. Finally let us give the relations between the partners spectra using the ladder operators $(A, A^{\dagger})$ such as $A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dy} + W(y)$,

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dy} - \frac{\hbar}{\sqrt{2m}}\left[(y - y_0) - \frac{H_{n+1}(y - y_0)}{H_n(y - y_0)}\right]. \quad (20)$$

Consequently, the eigenfunctions and the eigenstate energies of the partners V^{+} and V^{-} are linked up by the following recurrence relations, with $\chi^{-}_{n} \equiv \chi_n$,

$$\begin{aligned} \chi^{+}_{n}(y) &= \frac{1}{\sqrt{E_{n+1}}}\left[\frac{\hbar}{\sqrt{2m}}\frac{d}{dy} - \frac{\hbar}{\sqrt{2m}}\left((y - y_0) \right.\right. \\ &\quad \left.\left. - \frac{H_{n+1}(y - y_0)}{H_n(y - y_0)}\right)\right]\chi^{-}_{n+1}(y), \quad (21) \\ E^{+}_{n} &= E^{-}_{n+1} = E_{n+1} - E_0. \quad (22) \end{aligned}$$

This completes our proof of isospectrality, generalized to the case that the generating function ϕ of the superpotential W_n, is a higher excited wave function, namely $\phi = \chi^{-}_{n}$, $n = 1, 2, \cdots$. The formalism of superpotential to generate SUSY partners works everywhere except at the singularities located at the nodal points y_i of ϕ, where the partner potential V^{+} diverges as $1/(y - y_i)^{2k}$, $k \geq 1$ thereby defining several branches of V^{+} well defined on their disjoint domains of definition. Similar properties apply to models described in the next sections.

Before closing this section let us point out that one can use the DT [21–23] of the Schrödinger equation to generate SUSY partner potentials as in SUSYQM, since it is well known that the DT of Schrödinger equation are related with SUSYQM. We are not going to deal with the issue of higher derivative SUSY with supercharges using DT, but we will introduce the transformation in its simplest approach to generate isospectral potentials.

In the one-dimensional stationary case the Schrödinger equation is given by

$$-\frac{d^2}{dx^2}\psi + u(x)\psi = \lambda\psi \quad (23)$$

with the notation $\frac{\hbar^2}{2m} = 1$, λ standing here for the energy spectrum related to the standard potential $u(x)$. With the useful DT we can generalize any specific standard potential and generate new interaction models with the same energy levels. The DT is linked to the Sturm-Liouville theory, and it is easy to see the implicit presence of DT in SUSYQM. Suppose that (23) accepts the particular solution ψ_1 for the eigenvalue λ_1,

$$-\psi_1^{''} + u(x)\psi_1 = \lambda_1\psi_1. \quad (24)$$

Then we employ ψ_1 as a *seed function* to construct the DT.

$$\phi(x) = \psi^{'} - \sigma_1(x)\psi, \quad \sigma_1 = \frac{d}{dx}\ln\psi_1. \quad (25)$$

Using (25), the Equation (23) takes the following form

$$-\frac{d^2}{dx^2}\phi + U(x)\phi = \lambda\phi, \quad (26)$$

with the generalized isospectral potential given by

$$U(x) = u(x) - 2\frac{d}{dx}\sigma_1. \quad (27)$$

Let us consider the model studied in section 2.1 described by the Equation (1) as an application of this approach. The resulting generalized potential (Darboux potential) which is the equivalent of V^{+}_{n} is written as follows,

$$U_n(y) = (y - y_0)^2 - 2\left[1 + \frac{H_{n+2}(y - y_0)}{H_n(y - y_0)} - \frac{H^2_{n+1}(y - y_0)}{H^2_n(y - y_0)}\right]. \quad (28)$$

We note that up to multiplicative and additive constants –which is anyway a matter of choice of physical constant normalization–, these potentials are similar to those given in equation(15) showing that the results are similar and that in the above form, the DT does not allow the avoidance of singularity problems.

3 SUSY partners for extended bidimensional Schrödinger - Pauli systems

3.1 Bidimensional Schrödinger-Pauli system in a harmonic trap

In this section, we go over to a more complex model –even though it will be set in the ordinary two-dimensional plane–consisting of the Schrödinger-Pauli model with Zeeman splitting term, under the influence of a two–dimensional harmonic trap expressed by $\frac{1}{2}m\omega_0^2\mathbf{x}^2 = \frac{1}{2}m\omega_0^2(x^2 + y^2)$. In this form however, there could occur some dimensional confusions in view of the other terms in the Hamiltonian (29) below. One way to avoid such confusions is to consider an axially symmetric Penning trap given by $\frac{1}{2}m\omega_0^2(x^2+y^2-2z^2)$. So doing, one could come down to the plane anyway by setting $z=0$ in the obtained solution. Note that this potential is a particular form of the harmonic potential in 3D [24, 25].

The stationary Hamiltonian describing the motion of an electron in this system is given by

$$H_P^1 = \frac{1}{2m}(\mathbf{p}-q\mathbf{A}(\mathbf{x}))^2-\frac{1}{2}g\mu\mathbf{B}\cdot\sigma+\frac{1}{2}m\omega_0^2(x^2+y^2-2z^2). \quad (29)$$

In this expression m is the effective electron mass and g is the gyromagnetic ratio such as $g\approx 2$. μ is the Bohr magneton and $q=-e(e>0)$ is the electron charge. $\mathbf{x}=(x,y,z)$ and $\mathbf{p}=(p_x,p_y,p_z)$ are the position and momentum operators respectively, while $\sigma=(\sigma_x,\sigma_y,\sigma_z)$ is the vector of the Pauli spin matrices. $\mathbf{A}(\mathbf{x})$ is the Abelian vector gauge potential linked up to the uniform external magnetic field $\mathbf{B}=(0,0,B)$ through the well-known Abelian relation $\mathbf{B}=\nabla\wedge\mathbf{A}$. This model may be related to the three-dimentional problem of a harmonically trapped two-level quantum particle submitted to a background magnetic field with Zeeman splitting term. The following choice of Poincaré gauge $\mathbf{A}=\frac{B}{2}(-y,x,0)$, allows us to write the Hamiltonian (29) as follows,

$$\begin{aligned} H_P^1 &= \frac{1}{2m}\left(p_x+\frac{1}{2}qBy\right)^2+\frac{1}{2m}\left(p_y-\frac{1}{2}qBx\right)^2 \\ &+ \frac{p_z^2}{2m}-\frac{1}{2}\mu gB\sigma_z+\frac{1}{2}m\omega_0^2(x^2+y^2-2z^2). \end{aligned} \quad (30)$$

This choice of the circular gauge makes manifest the invariance of the dynamics under $SO(2)$ rotation in the plane. Following the conventional canonical quantization procedure, the dynamics of the system is determined by the algebra spanned by the following commutation relations, $\mathbb{I}$ being the two-dimensional identity operator

$$[x,p_x]=i\hbar\mathbb{I},\quad [y,p_y]=i\hbar\mathbb{I},\quad [z,p_z]=i\hbar\mathbb{I},$$

$$[\sigma_x,\sigma_y]=2i\sigma_z,\quad [\sigma_y,\sigma_z]=2i\sigma_x,\quad [\sigma_z,\sigma_x]=2i\sigma_y. \quad (31)$$

One way to relate the Hamiltonian H with an appropiate Lie algebra is to construct its bosonic and fermionic representations. However, one can avoid this algebraic detour. Indeed for the sake of the main aim of this work, which is to construct solvable potentials expressed in terms of configuration space variables, it stands to reason to work in configuration space directly. Hence with respect to the circular $SO(2)$ symmetry, it is straightforward to work with the cylindical coordinates (r,θ,z) related with the cartesian coordinates as follows, $x=r\cos\theta$, $y=r\sin\theta$, $z=z$. We introduce the following normalization of the radial variable $u=r\sqrt{\frac{m\omega}{\hbar}}$, such as the appropiate measure for the integration of the configuration space, defining below the scalar product of quantum states in this space is given by

$$\begin{aligned} \iiint_{(\infty)} dxdydz &= \int_0^{+\infty} drr\int_0^{2\pi} d\theta\int_{-\infty}^{+\infty} dz \\ &= \frac{\hbar}{m\omega}\int_0^{\infty} duu\int_0^{2\pi} d\theta\int_{-\infty}^{+\infty} dz. \end{aligned} \quad (32)$$

By using the differential representation of the momentum operators, the Hamiltonian (30) reads

$$\begin{aligned} H_P^1 &= \frac{\hbar\omega}{2}\left[-\frac{\partial^2}{\partial u^2}-\frac{1}{u}\frac{\partial}{\partial u}+u^2\right] \\ &+ \frac{\hbar\omega}{2}\left[-\frac{1}{u^2}\frac{\partial^2}{\partial\theta^2}-\frac{i\omega_c}{\omega}\frac{\partial}{\partial\theta}\right] \\ &- \frac{1}{2}g\mu B\sigma_z-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}-m\omega_0^2z^2, \end{aligned} \quad (33)$$

where the cyclotron frequency of the electron ω_c and the effective frequency ω are given by

$$\omega=\sqrt{\omega_0^2+\frac{1}{4}\omega_c^2},\quad \omega_c=\frac{eB}{m}. \quad (34)$$

Taking always into account the cylindrical symmetry, let us introduce the following separation of variables, where ℓ is the azimuthal quantum number,

$$\psi_\pm(u,\theta)=e^{i\ell\theta}R_\pm(u)\zeta(z),\quad H\psi_\pm(u,\theta,z)=E\psi_\pm(u,\theta,z), \\ \ell=0,1,2\cdots. \quad (35)$$

We obtain the following two differential equations, γ being a constant,

$$\begin{aligned} &-\frac{\hbar^2}{2m}\frac{\partial^2R_\pm(r)}{\partial r^2}-\frac{\hbar^2}{2m}\frac{\partial R_\pm(r)}{\partial r} \\ &+\left[\frac{\hbar^2\ell^2}{2mr^2}+\frac{1}{2}m\omega^2r^2\mp\frac{1}{2}g\mu B-E-\gamma\right]R_\pm(r)=0, \quad (36) \\ &-\frac{\hbar^2}{2m}\frac{\partial^2\zeta(z)}{\partial z^2}-\left(m\omega_0^2z^2+\gamma\right)\zeta(z)=0. \quad (37) \end{aligned}$$

The general solution of (37) is readily given by

$$\zeta_n(z)=C\exp(-z^2/2)H_n(z), \quad (38)$$

n being the radial quantum number, C represents the integration constant and $H_n(z)$ the Hermite polynomials. Now we set $z=0$ in order to bring back our problem to the two-dimensional Euclidean plane. It immediately follows that n must be even for finite values of the special functions H_n. Henceforth the radial quantum number n takes the values $0,2,4,\cdots$.

Finally we put Equation (36) into a canonical form by setting $R_{\pm}(r) = r^{-\frac{1}{2}} f_{\pm}(r)$. This gives

$$-\frac{\hbar^2}{2m}\frac{\partial^2 f_{\pm}(r)}{\partial r^2} + \left[-\frac{\hbar^2(\frac{1}{4}-\ell^2)}{2mr^2} + \frac{1}{2}m\omega^2 r^2 + \frac{1}{2}\hbar\ell\omega_c \mp \frac{1}{2}g\mu B - E - \gamma\right] f_{\pm}(r) = 0, \quad (39)$$

which is in the form of a Schrödinger equation similar to that of the one-dimensional problems and can be subjected to a supersymmetric treatment even though some precautions will be required due to the domain of the radial variable, $r \in (0, \infty)$.

The effective potential is then given by

$$V_\ell(r) = \frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2(\ell^2 - \frac{1}{4})}{2mr^2}. \quad (40)$$

This potential is similar to that describing the isotropic harmonic oscillator potential and has a single point of singularity at $r = 0$. It is worth noting the similarity between this expression and the superpartner V_1^+ of the previous system (see Equation (17)). It consists of two terms; the first term corresponds to a recall force resulting from the magnetic field oscillating modes and the trapping force ($\omega = \sqrt{\omega_0^2 + \frac{1}{4}\omega_c^2}$); the second term corresponds to the action of a centrifugal force for $\ell > 1/2$ which boils down to $\ell = 1, 2, \cdots$, and that of a centripetal force for $\ell < 1/2$ i.e. for $\ell = 0$.

Using the notations $k_\pm^2 = -\frac{\ell\omega m}{\hbar} \pm \frac{g\mu B m}{\hbar^2} + \frac{2mE}{\hbar^2} + \frac{2m\gamma}{\hbar^2}$ and $\lambda = \frac{m\omega}{\hbar}$, we rewrite the radial equation as follows

$$\partial_r^2 f_\pm + \left[k_\pm^2 - \lambda^2 r^2 - (\ell - 1/4)r^{-2}\right] f_\pm = 0. \quad (41)$$

The solution of the radial problem and its energy eigenvalues are given by

$$\begin{aligned} f_{n\pm}(r) &= C_{n\pm} r^{\ell+\frac{1}{2}} e^{-\frac{1}{2}\lambda^2 r^2} F(-n, \ell{+}1; \lambda^2 r^2), \\ E_{n\pm} &= \hbar\omega(2n + \ell{+}\frac{3}{2}) + \frac{\ell\hbar\omega_c}{2} \mp \frac{g\mu B}{2m} - \gamma, \end{aligned} \quad (42)$$

n being the radial quantum number $n = 0, 2, 4 \cdots$. F is the confluent hypergeometric function [13].

The SUSY-n superpotential W_n could regularly be written as follows

$$\begin{aligned} W_n(r) &= -\frac{\hbar}{\sqrt{2m}}\left[\frac{\ell + \frac{1}{2}}{r} - \lambda^2 r \right. \\ &\quad \left. - \frac{2n\lambda^2 r}{\ell + 1}\frac{F(-n+1, \ell+2; \lambda^2 r^2)}{F(-n, \ell+1; \lambda^2 r^2)}\right], \end{aligned} \quad (43)$$

$$W_0(r) = \omega\sqrt{\frac{m}{2}}r - \frac{\hbar}{\sqrt{2m}}\frac{\ell + \frac{1}{2}}{r}. \quad (44)$$

However, since $r \in (0, +\infty)$ we have only a half-line problem and this may generate some difficulties [12]. To clarify the situation, let us introduce the principal quantum number N such that $N = n + \ell + 1$. One has to discuss the two cases whether N is fixed and ℓ varies, or N varies and ℓ is kept fixe. Let us mention that since the radial quantum number $n = 0, 2, 4, \cdots$, then the principal quantum number N runs as $N = \ell + 1, \ell + 3, \ell + 5, \cdots$.

Suppose N is fixed and ℓ varies. A straightforward computation gives the partners

$$\begin{aligned} V_{0\pm}^-(r) &= \frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2\left(\ell^2 - \frac{1}{4}\right)}{2mr^2} - \hbar\omega\left(\ell + \frac{3}{2}\right) \\ &\quad - \frac{\ell\hbar\omega_c}{2} \pm \frac{g\mu B}{2m}, \end{aligned} \quad (45)$$

$$\begin{aligned} V_{0\pm}^+(r) &= \frac{1}{2}m\omega^2 r^2 + \frac{\hbar^2\left(\ell + \frac{1}{2}\right)\left(\ell + \frac{3}{2}\right)}{2mr^2} - \hbar\omega\left(\ell + \frac{1}{2}\right) \\ &\quad - \frac{\ell\hbar\omega_c}{2} \pm \frac{g\mu B}{2m}, \end{aligned} \quad (46)$$

such as

$$V_0^+(r, \ell) - V_0^-(r, \ell + 1) = 2\hbar\omega + \frac{\hbar\omega_c}{2}, \quad (47)$$

showing suitably that SUSY furnishes a connection between levels ℓ and $\ell + 1$, which is not in accordance with $N = \ell + 1, \ell + 3, \ell + 5, \cdots$. In fact this confirms that SUSY can not be naively applied to higher dimensional systems and one needs to transform radial problems to full-line $(-\infty, +\infty)$ problems before proceeding further. So doing, one falls into the cases for which ℓ is fixed but N varies. Hence in the subsequent developments we shall restrict the rotationally symmetric problems to a given angular momentum sector before applying the SUSY formalism.

We consider the following transformation which switches $r \in (0, \infty)$ to $x \in (-\infty, \infty)$. Let us set $r = e^x$. As a result an equation of the type (41) gets transformed to

$$\frac{d^2\psi(x)}{dx^2} + \left[k^2 e^{2x} - \lambda^2 e^{4x} - \ell^2\right]\psi(x) = 0, \quad (48)$$

where we can identify the effective potential $V_{\text{eff}}(x) = k^2 e^{2x} - \lambda^2 e^{4x}$, which describes in a full-line range the Morse potential problem.

Finally, the solution of this problem and its energy eigenvalues are given by

$$\begin{aligned} \psi_n(x) &= C_n e^{\ell x} e^{-\frac{\lambda}{2}exp(2x)} F\left(-n, \ell + 1; \lambda e^{2x}\right), \\ E_{n\pm} &= \frac{\hbar\omega}{2}\left(n + \ell + \frac{1}{2}\right) + \frac{\ell\hbar\omega_c}{2} \mp \frac{g\mu B}{2} - \gamma, \end{aligned} \quad (49)$$

n being the radial quantum number $n = 0, 2, 4 \cdots$. $F(a, b; x)$ is the confluent hypergeometric function [13].

Following the same procedure as before we find the SUSY-n superpotential W_n written explicitly as follows

$$W_n(x) = -\frac{\hbar}{\sqrt{2m}}\left[N-n-1-\lambda e^{2x} - \frac{2n\lambda e^{2x}}{N-n}\frac{F\left(-n+1, N-n+1; \lambda e^{2x}\right)}{F\left(-n, N-n; \lambda e^{2x}\right)}\right], \quad (50)$$

whose special ground state value is

$$W_0(x) = -\frac{\hbar}{\sqrt{2m}}\left[N-1-\lambda e^{2x}\right]. \quad (51)$$

where N representes the principal quantum number related to the radial quantum number, n, and the azimuthal quantum number ℓ by: $N = n+\ell+1$.

The starting shifted potential, for a given radial quantum number n is written as follows

$$V^-_{n\pm}(x) = k_n^2 e^{2x} - \lambda e^{4x} - \frac{\hbar\omega}{2}\left(N-\frac{1}{2}\right) - \frac{(N-n-1)\hbar\omega_c}{2} \pm \frac{g\mu B}{2} + \gamma, \quad (52)$$

while the other partner potentials are given by

$$V^+_{n\pm}(x) = \left(k_n^2 - \frac{2\hbar^2\lambda}{m}\right)e^{2x} - \lambda e^{4x} - \frac{\hbar\omega}{2}\left(N-\frac{1}{2}\right) - \frac{(N-n-1)\hbar\omega_c}{2} \pm \frac{g\mu B}{2} + \gamma + I_n, \quad (53)$$

$$I_n = \frac{4nm\lambda\hbar\omega e^{2x}}{N-n}\frac{1}{F(0)}\left[\frac{(1-n)}{(N-n+1)}F(2) + \frac{n}{(N-n)}\frac{F^2(1)}{F(0)}\right], \quad (54)$$

where we have used the short-hand notation for the confluent hypergeometric function written as $F(i) = F(-n+i, N-n+i; \lambda e^{2x})$, $i = 0, 1, 2, \cdots$. It is readily seen that for $n = 0$ we get the shape invariant case of the corresponding two partner potentials such as

$$V^-_0(x) = k_0^2 e^{2x} - \lambda e^{4x} - \frac{\hbar\omega}{2}\left(N_0-\frac{1}{2}\right) - \frac{(N_0+1)\hbar\omega_c}{2} \pm \frac{g\mu B}{2} + \gamma, \quad (55)$$

$$V^+_0(x) = \left(k_0^2 - \frac{2\hbar^2\lambda}{m}\right)e^{2x} - \lambda e^{4x} - \frac{\hbar\omega}{2}\left(N_0-\frac{1}{2}\right) - \frac{(N_0+1)\hbar\omega_c}{2} \pm \frac{g\mu B}{2} + \gamma, \quad (56)$$

where $N_0 = \ell + 1$.

The isospectral partner potentials of $V^-_{2\pm}$ are calculated in a straightforward manner. We have

$$V^+_{2\pm}(x) = \left(k_2^2 - \frac{2\hbar^2\lambda^2}{m}\right)e^{2x} - \lambda e^{4x} - \frac{\hbar\omega}{2}\left(N_2-\frac{1}{2}\right) - \frac{(N_2-3)\hbar\omega_c}{2} \pm \frac{g\mu B}{2} + \gamma + I_2, \quad (57)$$

with

$$I_2 = 8\hbar\omega\beta_0\left(1-2\beta_0 z+\beta_0\beta_1 z^2\right)^{-1} \times \left[1-3\beta_1 z + 4\beta_0 z(1-\beta_1 z)^2 \times \left(1-2\beta_0 z+\beta_0\beta_1 z^2\right)^{-1}\right], \quad (58)$$

where $\beta_0 = (N_2-2)^{-1}$, $\beta_1 = (N_2-3)^{-1}$, $z = (\lambda e^x)^2$ and $N_2 = \ell + 3$.

We have with this result, the interesting generalization of a two-dimensional rotationally symmetric effective potential to a more complex but isospectral one in full-line. This new solvable potential presents two singularities, namely $x_1 = \frac{1}{2}\ln\left[\frac{\beta_0-\sqrt{\beta_0^2-\beta_0\beta_1}}{\lambda\beta_0\beta_1}\right]$ and $x_2 = \frac{1}{2}\ln\left[\frac{\beta_0+\sqrt{\beta_0^2-\beta_0\beta_1}}{\lambda\beta_0\beta_1}\right]$. As a fact we have a splitting into three different potentials defined within the three ranges $]-\infty, x_1[,]x_1, x_2[$ and $]x_2, +\infty[$. In other words in each of the previous range we have a new isospectral partner $V^+_{2\pm}$ of the shifted potential $V^-_{2\pm}$. The cases $V^+_{n\pm}$, $n = 4, 6, \cdots$ will describe more and more complex physical systems.

The partner potentials $V^{\mp}_{n\pm}(x)$ of equidistant energy levels are expressed as follows, leaving $E^-_0 = 0$,

$$E^-_{n\pm} = E_{n\pm} - E_0 = \frac{n\hbar\omega}{2}, \; E^+_{n\pm} = E_{(n+1)\pm} - E_0 = \frac{(n+1)\hbar\omega}{2}. \quad (59)$$

The relations between the normalized wave functions for the partner potentials $V^{\mp}_n$ may easily be obtained using the ladder operator given by $A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_n(x)$,

$$\psi^+_n = (E^-_{n+1})^{-1/2} A\psi^-_{n+1}, \; \psi^-_{n+1} = \psi_{n+1}, \; n = 0, 2, 4\cdots. \quad (60)$$

It is clear that they have the same energy levels, except for the $(n+1)$ lowest states of V^- for which there are no corresponding states of V^+, so that the ground state of the latter is $E^+_0 = E^-_1$.

3.2 Schrödinger-Pauli system in the presence of an electric background field with harmonic trap

Let us step forward towards an extended model by taking into account the presence of a background static electric field, in addition to the magnetic field and the harmonic potential. The stationary Hamiltonian of such a system is written as follows

$$H^2_P = \frac{1}{2m}(\mathbf{p}-q\mathbf{A}(\mathbf{x}))^2 - \frac{1}{2}g\mu\mathbf{B}\cdot\sigma + q\mathbf{E}\mathbf{x} + \frac{1}{2}m\omega_0^2(x^2+y^2-2z^2). \quad (61)$$

In the framework of the Poincaré gauge condition, this Hamiltonian becomes

$$
\begin{aligned}
H_P^2 &= \frac{1}{2m}\left(p_x + \frac{1}{2}qBy\right)^2 + \frac{1}{2m}\left(p_y - \frac{1}{2}qBx\right)^2 \\
&+ \frac{p_z^2}{2m} - \frac{1}{2}\mu gB\sigma_z + \\
&\quad q(E_x x + E_y y) + \frac{1}{2}m\omega_0^2(x^2 + y^2 - 2z^2). \qquad (62)
\end{aligned}
$$

Let us introduce the following changes of variables

$$
\begin{aligned}
X = x + \frac{qE_x}{m\omega_0^2}, &\qquad Y = y + \frac{qE_x}{m\omega_0^2}, \\
P_X = p_x - -\frac{q^2BE_y}{2m\omega_0^2}, &\quad P_Y = p_y + \frac{q^2BE_x}{2m\omega_0^2}. \qquad (63)
\end{aligned}
$$

We obtain the Hamiltonian which describes the background electric field influence in the trapped Schrödinger-Pauli system

$$
\begin{aligned}
H_P^2 &= \frac{1}{2m}\left(P_X^2 + P_Y^2\right) + \frac{1}{2}m\omega^2\left(X^2 + Y^2\right) - \frac{p_z^2}{2m} \\
&- \frac{1}{2}\omega_c L_z - m\omega_0^2 z^2 - \frac{q^2E^2}{2m\omega_0^2} - \frac{1}{2}g\mu B\sigma_z. \qquad (64)
\end{aligned}
$$

In this notation, $E^2 = E_x^2 + E_y^2$ stands for the electric field which must not been confused with the eigenstate energy that will be denoted as ε. $L_z = xp_y - yp_x$ is the z-component of the angular momentum.

By the same procedure as that followed in the previous section, we obtain the following differential equation in the new system of cylindical coordinates defined by (ρ, ϕ, z),

$$
\begin{aligned}
&-\frac{\hbar^2}{2m}\frac{\partial^2 f_\pm(\rho)}{\partial^2\rho} + \left[-\frac{\hbar^2(\frac{1}{4} - \ell^2)}{2m\rho^2} + \frac{1}{2}m\omega^2\rho^2 + \frac{1}{2}\hbar\ell\omega_c\right. \\
&\qquad\left.\mp\frac{1}{2}g\mu B - \frac{q^2E^2}{2m\omega_0^2} - \varepsilon\right] f_\pm(\rho) = 0, \qquad (65)
\end{aligned}
$$

the z-coordinate component of the wave function denoted by $\zeta(z)$ being identical to that obtained in the previous section (see Equation (38)). The coordinates ρ and ϕ are expressed in terms of the coordinates (ρ, θ) related to the former basis $\{\mathbf{u}_r, \mathbf{u}_\theta, \mathbf{u}_z\}$ as follows, with $K_1 = qE_x/(m\omega_0^2)$, $K_2 = qE_y/(m\omega_0^2)$,

$$
\begin{aligned}
\rho &= \left[r^2 + \frac{q^2E^2}{m^2\omega_0^4} + \frac{2qr}{m\omega_0^2}(E_x\cos\theta + E_y\sin\theta)\right]^{1/2}, \\
\tan\phi &= \frac{r\sin\theta + K_2}{r\cos\theta + K_1}. \qquad (66)
\end{aligned}
$$

The Equation (65) is similar to the one obtained in the absence of the electric background field (see relation (39)). It does mean that the presence of the external electric background field does not modify the nature of the symmetry but translates the movement of the particle. Consequently, we only give in what follows the radial solution and the SUSY-n potential with its partners, the remaining of the treatment being identical.

When we switch to the full-infinite domain as $\rho = e^u$, $u \in (-\infty, \infty)$, we obtain the u-dimensional solutions given in the limit $z \to 0$ by

$$
\begin{aligned}
f_n(u) &= C_n e^{\ell u} e^{-\frac{\lambda^2}{2}exp(2u)} F\left(-n, \ell + 1; \lambda e^{2u}\right), \qquad (67) \\
\varepsilon_{n\pm} &= \frac{\hbar\omega}{2}\left(n + \ell + \frac{1}{2}\right) + \frac{\ell\hbar\omega_c}{2} \mp \frac{g\mu B}{2} - \frac{q^2E^2}{2m\omega_0^2},
\end{aligned}
$$

with $n = 0, 2, 4, \cdots$. The super potential is written as follows, with $N = n + \ell + 1$,

$$
\begin{aligned}
W_n(u) &= -\frac{\hbar}{\sqrt{2m}}\left[N - n - 1 - \lambda e^{2u}\right. \\
&\left.- \frac{2n\lambda e^{2u}}{N - n}\frac{F\left(-n + 1, N - n + 1; \lambda e^{2u}\right)}{F\left(-n, N - n; \lambda e^{2u}\right)}\right], \qquad (68)
\end{aligned}
$$

and the isospectral SUSY-n partner potentials are given by

$$
\begin{aligned}
V_{n\pm}^-(u) &= k_n^2 e^{2u} - \lambda e^{4u} - \frac{\hbar\omega}{2}\left(N - \frac{1}{2}\right) \\
&- \frac{(N - n - 1)\hbar\omega_c}{2} \pm \frac{g\mu B}{2} + \frac{q^2E^2}{2m\omega_0^2}, \qquad (69)
\end{aligned}
$$

and

$$
\begin{aligned}
V_{n\pm}^+(u) &= \left(k_n^2 - \frac{2\hbar^2\lambda}{m}\right)e^{2u} - \lambda e^{4u} - \frac{\hbar\omega}{2}\left(N - \frac{1}{2}\right) \\
&- \frac{(N - n - 1)\hbar\omega_c}{2} \pm \frac{g\mu B}{2} - \frac{q^2E^2}{2m\omega_0^2} + I_n, \qquad (70)
\end{aligned}
$$

$$
I_n = \frac{4nm\lambda\hbar\omega e^{2u}}{(N - n)}\frac{1}{F(0)}\left[\frac{(1 - n)}{(N - n + 1)}F(2) + \frac{n}{(N - n)}\frac{F^2(1)}{F(0)}\right], \qquad (71)
$$

All the superparters satisfy properties of SUSY-broken and present many points of singularity as discussed in the previous section.

4 SUSYQM and the Dirac-Coulomb problem in nonrelativistic limits

In this section, we end the construction of SUSY partners for spin-1/2 systems in nonrelativistic limits by no longer taking into account an external magnetic field via the gauge potential $\mathbf{A}$ and the Zeeman splitting term $(1/2)\mu\sigma \cdot \mathbf{B}$, but by considering only the electric field via the Coulomb potential. This boils down to take only into account the timelike component of the electromagnetic quadrivector potential when one starts from a relativistic system. One of the models that has been widely studied in the literature, including in text books, but which still remains interesting for testing quantization methods among other things, is the Dirac-Coulomb Hamiltonian.

Let us mention that in the nonrelativistic limit, the Dirac-Coulomb problem is similar to the Schrödinger-Coulomb problem which has been known to be solvable for a long time. However with the intention of putting forward the spinorial character of our system and for the sake of the method, we choose the Dirac-Coulomb system set in its nonrelativistic limits to generate new solvable potentials. The standard way to solve these systems is to translate their eigenvalue problems in terms of hypergeometric equations (see for example the standard texbooks [14–16]). In this section, we will sketch the spectrum of our initial Hamiltonian following approaches known in the literature.

Let us consider the system in which the Dirac particle is charged and coupled to the four-component electromagnetic potential $A_\mu = (A_0, \mathbf{A})$. Gauge invariant coupling requires the "minimal" substitution $\partial_\mu \longrightarrow \partial_\mu + i\frac{e}{\hbar c}A_\mu$ in the free Dirac Hamiltonian. The formal Hamiltonian of such a system may be written as follows

$$H_{DC} = \begin{pmatrix} 1+\lambda^2 A_0 & -\lambda i\sigma\cdot\nabla + \lambda^2\sigma.\mathbf{A} \\ -\lambda i\sigma\cdot\nabla + \lambda^2\sigma.\mathbf{A} & -1+\lambda^2 A_0 \end{pmatrix}. \quad (72)$$

In this notation we have used atomic units $\hbar = 1 = \mu$, $e = 1$, $\lambda = 1/c$ (not to be confused with the one defined in Equation (41)) being the Compton wavelength standing for the relativistic parameter and μ is the mass of the particle of charge $-e$. σ is the vector of which the components are the usual Pauli matrices (σ_x,σ_y,σ_z) previously defined (see Equation (2)). Thus the eigenvalue wave equation reads $(H-\varepsilon)\chi = 0$, where ε is the relativistic energy, χ being the four-component spinor.

Now, we choose $\mathbf{A} = \mathbf{0}$ and impose spherical symmetry by taking $A_0 = V(r) = Q/r$ i.e. the Coulomb potential, Q being proportional to the particle charge depending on the normalization chosen. A separation of variables is then possible and so doing, the wave function reduces to a two-spinor,

$$\chi(\mathbf{r}) = \begin{pmatrix} i(\frac{g(r)}{r})\Omega^j_{\ell m} \\ (\frac{f(r)}{r})\sigma.\mathbf{r}\Omega^j_{\ell m} \end{pmatrix}, \quad (73)$$

$$\Omega^j_{\ell m}(\hat{r}) = \frac{1}{\sqrt{2\ell+1}} \begin{pmatrix} \sqrt{\ell \pm m + 1/2}\; Y^{m-1/2}_\ell(\hat{r}) \\ \pm\sqrt{\ell \mp m + 1/2}\; Y^{m+1/2}_\ell(\hat{r}) \end{pmatrix},$$

where f and g are real radial square-integrable function, $\hat{r}(\theta,\varphi)$ being the radial unit vector. For $j = \ell + 1/2$, $j = \ell - 1/2$, $Y^{m\pm1/2}_\ell$ is the spherical harmonic function –the angular wave function for the two-component spinor– and m stands for the integers in the range $-j, -j+1, \cdots, j$. We obtain the following 2×2 matrix equation for the radial two-spinor components

$$\begin{pmatrix} 1+\lambda^2\frac{Q}{r}-\varepsilon & \lambda\left(\frac{k}{r}-\frac{d}{dr}\right) \\ \lambda\left(\frac{k}{r}+\frac{d}{dr}\right) & -1+\lambda^2\frac{Q}{r}-\varepsilon \end{pmatrix}\begin{pmatrix} g(r) \\ f(r) \end{pmatrix} = 0, \quad (74)$$

where k is the spin-orbit quantum number defined as $k = \pm(j+1/2) = \pm1, \pm2, \cdots$ for $\ell = j \pm 1/2$.

To obtain a Schrödinger-like equation we proceed as follows. A global unitary transformation given by [26]

$$U(\eta) = \exp\left(\frac{i}{2}\lambda\eta\sigma_y\right) \quad (75)$$

is applied to the radial Equation (74), where η is a real constant parameter and σ_y is the 2×2 Pauli matrix. The Schrödinger-like requirement dictates that the parameter η satisfies the constraint $\sin(\lambda\eta) = \lambda Q/k$, where $-\frac{\pi}{2} \le \lambda\eta \le \frac{\pi}{2}$ depending on the signs of Q and k. Equation (74) is now transformed as follows

$$\begin{pmatrix} \frac{\gamma}{k}+2\lambda^2\frac{Q}{r}-\varepsilon & \lambda\left(-\frac{Q}{k}+\frac{\gamma}{r}-\frac{d}{dr}\right) \\ \lambda\left(-\frac{Q}{k}+\frac{\gamma}{r}+\frac{d}{dr}\right) & -\frac{\gamma}{r}-\varepsilon \end{pmatrix}\begin{pmatrix} \phi(r) \\ \theta(r) \end{pmatrix} = 0, \quad (76)$$

where $\gamma = k\sqrt{1-(\lambda Q/k)^2}$ and

$$\begin{pmatrix} \phi \\ \theta \end{pmatrix} = U\chi = \begin{pmatrix} \cos\frac{\lambda\eta}{2} & \sin\frac{\lambda\eta}{2} \\ -\sin\frac{\lambda\eta}{2} & \cos\frac{\lambda\eta}{2} \end{pmatrix}\begin{pmatrix} g \\ f \end{pmatrix}. \quad (77)$$

It is to be noted that the angular parameter of the unitary transformation $U(\eta)$ was intentionally split as $\lambda\eta$ and not collected into a single parameter, say φ. This is suggested by investigating the constraint $\sin(\varphi) = \lambda\eta/k$ in a non relativistic limit ($\lambda \longrightarrow 0$) where we should have $\sin(\varphi) = \lambda\eta \approx \varphi = \lambda\eta/k$. It also makes it obvious that in the nonrelativistic limit the transformation becomes the identity. Equation (77) gives the lower spinor component in terms of the upper as follows

$$\theta = \frac{\lambda}{\gamma/k+\varepsilon}\left(-\frac{Q}{k}+\frac{\gamma}{r}+\frac{d}{dr}\right)\phi \quad (78)$$

for $\varepsilon \neq -\gamma/k$, whereas the resulting Schrödinger-like wave equation for the upper component becomes

$$\left[-\frac{d^2}{dr^2}+\frac{\gamma(\gamma+1)}{r^2}+2\frac{Q\varepsilon}{r}-\frac{\varepsilon^2-1}{\lambda^2}\right]\phi(r) = 0. \quad (79)$$

Comparing this equation with that of the well-known nonrelativistic Coulomb problem given by

$$\left[-\frac{d^2}{dr^2}+\frac{\ell(\ell+1)}{r^2}+2\frac{Q}{r}-2E\right]\phi(r) = 0, \quad (80)$$

we obtain by correspondence, the following map between the parameters of the two problems:

$$Q \to Q\varepsilon,\quad E \to (\varepsilon^2-1)/2\lambda^2,\quad \ell \to (\gamma; -\gamma-1). \tag{81}$$

By using the parameter map (81) in the nonrelativistic energy spectrum, $E_n = -Q^2/2N^2$, we obtain the following relativistic spectrum for bound states

$$\varepsilon_n = \pm\left[1+\left(\frac{\lambda Q}{N}\right)^2\right]^{-1/2},\quad n=0,1,2,\cdots,N=n+\ell+1. \tag{82}$$

Following the prescription of transforming the half-line $(0,\infty)$ to $(-\infty,\infty)$ we employ $r=e^x$ to get

$$-\frac{1}{2}\frac{d^2\varphi(x)}{dx^2}+\left[\frac{1}{2}\left(\ell+\frac{1}{2}\right)^2+Qe^x-Ee^{2x}\right]\varphi(x)=0, \tag{83}$$

which describes a full-line problem for a Morse-like potential.

The upper one dimensional component of the spinor wave function is obtained using the same parameter map in the nonrelativistic wave function

$$\varphi_n(x) = C_n e^{(\ell+\frac{1}{2})x}e^{-\frac{\lambda_n}{2}e^x}F(-n,2\ell+2;\lambda_n e^x), \tag{84}$$

where $\lambda_n = -2Q/(n+\ell+1)$ and $F(-n,2\ell+2;\lambda_n e^x)$ are the hypergeometric polynomials.

The superpotential is readily obtained using the wave function (84), we have

$$\begin{aligned} W_n(x) &= -\frac{1}{\sqrt{2}}\left[\left(N-n-\frac{1}{2}\right)-\frac{\lambda_n}{2}e^x\right. \\ &\quad \left.-\frac{n\lambda_n}{2(N-n)}\frac{F(-n+1,2N-2n+1;\lambda_n e^x)}{F(-n,2N-2n;\lambda_n e^x)}\right], \end{aligned} \tag{85}$$

whose special SUSY-0 case is,

$$W_0(x) = -\frac{1}{\sqrt{2}}\left[N_0-\frac{1}{2}+\frac{Qe^x}{N_0}\right],\quad N_0=\ell+1. \tag{86}$$

We deduce the partner potentials $V_n^{\pm}$, starting from the shifted potential given by

$$V_n^-(x) = \frac{Q^2e^{2x}}{2N^2}+Qe^x+\frac{1}{2}\left(N-\frac{1}{2}\right)^2, \tag{87}$$

and the other partners are written as follows,

$$V_n^+(x) = \frac{Qe^{2x}}{2N^2}+Q\left(1-\frac{1}{N}\right)e^x+\frac{1}{2}\left(N-\frac{1}{2}\right)^2-I_n \tag{88}$$

$$\begin{aligned} I_n &= \frac{n\lambda_n^2}{(2N-2n)(2N-2n+1)} \\ &\times \frac{F(-n+2;2N-2n+2;\lambda_n e^x)}{F(-n;2N-2n;\lambda_n e^x)}+\frac{n^2\lambda_n^2}{(2N-2n)^2} \\ &\times \left\{\frac{F(-n+1;2N-2n+1;\lambda_n e^x)}{F(-n;2N-2n;\lambda_n e^x)}\right\}^2. \end{aligned} \tag{89}$$

The shifted potential is the sum of the effective potential and the nonrelativistic energy levels term. Note that this effective potential is nothing but the one related to the Morse potential.

For $n=0$ we get the shape invariance of the two partner potentials,

$$\begin{aligned} V_0^-(x) &= \frac{Qe^{2x}}{2N_0{}^2}+Qe^x+\frac{1}{2}\left(N_0-\frac{1}{2}\right)^2, \\ V_0^+(x) &= \frac{Qe^{2x}}{2N_0{}^2}+Q\left(1-\frac{1}{N_0}\right)e^x+\frac{1}{2}\left(N_0-\frac{1}{2}\right)^2. \end{aligned} \tag{90}$$

Let us give the explicit expressions for the two lowest excited states ($n=1,2$). We get

$$\begin{aligned} V_1^+(x) &= \frac{Q^2e^{2x}}{2N_1{}^2}+Qe^x+\frac{1}{2}\left(N_1-\frac{1}{2}\right)^2 \\ &- \frac{2Q^2}{(N_1-1)N_1}\left(1-\frac{Qe^x}{N_1-1}\right)^{-1} \end{aligned} \tag{91}$$

whith $N_1=\ell+2$ and

$$V_2^+(x) = \frac{Qe^{2x}}{2N_2{}^2}+Q\left(1-\frac{1}{N_2}\right)e^x+\frac{1}{2}\left(N_2-\frac{1}{2}\right)^2-I; \tag{92}$$

$$\begin{aligned} I &= \frac{4Q^2}{(N_2-2)(N_2+1)}\left[\frac{2\beta_2\beta_3}{(1-2\beta_2e^x+2\beta_2\beta_3e^{2x})}\right. \\ &+ \left.\frac{4\beta_2^2(1-\beta_3e^x)^2}{(1-2\beta_2e^x+2\beta_2\beta_3e^{2x})^2}\right] \end{aligned} \tag{93}$$

where $\beta_2=(2N_2-4)^{-1}$ and $\beta_3=(2N_2-3)^{-1}$, $N_2=\ell+3$ is the principal quantum number.

The case $n=1$ provides a new solvable modified Morse potential example which is thus the SUSY-1 nonrelativistic partner potential of the initial model. Contrary to the SUSY-0 case, it has singularities at the nodal point $x_0=\ln\left(\frac{N_1-2}{Q}\right)$, splitting in two branches its initial domain as follows: $(-\infty,x_0[$, and $]x_0,+\infty)$; the energy spectrum being identical in each of them.

The case $n=2$ is the nontrivial SUSY-2 which presents two singularities at the points $x_1=\ln\left[\frac{\beta_2-\sqrt{\beta_2^2-2\beta_2\beta_3}}{\beta_2\beta_3}\right]$ and $x_2=\ln\left[\frac{\beta_2+\sqrt{\beta_2^2-2\beta_2\beta_3}}{\beta_2\beta_3}\right]$.

Let us point out the following well-known important property of SUSY[12]. When we return to the r-dependent eigenvalue equation for the partner potential $V^+(x)$ (constructed for $n=0$), we obtain

$$\left[-\frac{1}{2}\frac{d^2}{dr^2}+\frac{Q^2}{2N^2}+Q\left(1-\frac{1}{N}\right)\frac{1}{r}+\frac{\ell(\ell+1)}{2r^2}\right]\chi(r)=0. \tag{94}$$

By comparing this equation whith (80), we note that the coefficient of $\frac{1}{r}$ in (94) has undergone a modification by a N-dependent factor. To interpret (94) we therefore need to redefine $\left(1-\frac{1}{N}\right) r$ as the new radial variable. This necessitates redefining the nuclear charge Q by bringing it out explicitly such as Q becomes $Q\left(1-\frac{1}{N}\right)$. We thus find a degeneracy to hold between states of the same ℓ but different N and Q. More specifically, while (80) is concerned with states possessing quantum numbers N, ℓ, and energies $-\frac{Q^2}{N^2}\frac{me^4}{\hbar^2}$, (94) accounts for states with quantum numbers $(N-1)$, ℓ and nuclear charge $Q\left(1-\frac{1}{N}\right)$ having same energies. With these results, new roles of SUSY are thus highlighted and the model may be used to establish supersymmetric interatomic connections between states of iso-electronic ions under the simultaneous change of the principal quantum number and nuclear charge.

Finally the relation between the wave functions can be calculated by means of the operator $A=\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}+W(x)$ and using the eigenfunction (84) and the superpotential (85); we have

$$\begin{aligned} \psi_n^+ &= \frac{1}{\sqrt{E_{n+1}^-}} A\psi_{n+1}^-, \\ E_{n+1}^- &= E_n^+ = \frac{Q^2}{2}\left[(\ell+1)^{-2}-(n+\ell+2)^{-2}\right]. \end{aligned} \tag{95}$$

5 Conclusion

In this paper, we mainly constructed new analytically solvable potentials which generalized isotropic harmonic potentials. We applied the SUSY formalism in quantum mechanics to higher excited states of different specific exactly solvable models, namely the Schrödinger-Pauli system for a charged particle, the bidimensional Schrödinger-Pauli system with harmonic trap, the bidimensional Schrödinger-Pauli system in the presence of an electric background field with harmonic trap and the Dirac-Coulomb problem in nonrelativistic limits. For the first model, we started from a three-dimensional problem with respect to the cartesian coordinates space which we reduced to a one-dimensional problem on the full-line $(-\infty,+\infty)$ –favourable for the construction of SUSY partners– by first taking advantage of the gauge freedom to maintain only the $\mathbf{u}_x$-component of the gauge field $\mathbf{A}$ which thus depends only of the variable y and then by factorizing the wave function. Concerning the other models, we took advantage of circular and spherical symmetries to construct the radial solutions for the two- and three-dimensional systems respectively. Throughout the work, the eigensolutions are obtained by a transparent means. In all cases we got valid new classes of isospectral partner potentials in restricting ourselves to a particular angular momentum sector, by switching the radial variable to a full-line variable. These partners thus also fall into the class of exactly solvable potentials although they can be sometimes quite complex; this is typically the case for the partner potentials V_n^+ for higher n. It suffices to look at their structure to imagine the complexity they may bring to the resolution of a Schrödinger equation containing these interactions. Nevertheless it is important to have a new solvable potential which may be a potential provider of new physics since it may be related to a concrete physical phenomenon. Moreover, the potential V_n^+ is isospectral to the shift partner potential V_n^- which looks generally more simple.

Another important specificity of the considered models is the singularities which occured in the partner potentials. From a global point of view, these are the sort of things to break the SUSY, since these singularities are new compared to the initial systems and the boundary conditions for the corresponding stationary equation will be different leading to a different spectrum of the Hamiltonian as well. However, within the range of a specific interval of the partition created by the nodal points, the initial Hamiltonian is isospectral with the SUSY-partner. Hence there will be as many families of partners as the number of intervals generated by singularities leading thus to what we can call degeneracy in the SUSY partners.

Let us close down this paper with some interesting perspectives. The SUSYQM method exposed in our work which aims at constructing superpotentials is of fundamental importance in the construction of the Hamiltonian coherent states. Indeed, the techniques exposed in this paper are widely involved in the construction of such a class of coherent states. There is no need to emphasize the importance and the usefulness of coherent states in quantum optics and more generally in quantum mechanics and its procedures, including new quantization techniques known as enhanced quantization methods (see for example Refs [27–32] for introductory developments). Our future works could involve this promising new research field. We shall also apply this SUSY formalism in future works with more generalizations, including for example solvable potentials in relativistic quantum mechanics.

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