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Research Article

Hom-Lie Triple System and Hom-Bol Algebra Structures on Hom-Maltsev and Right Hom-Alternative Algebras

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Every multiplicative Hom-Maltsev algebra has a natural multiplicative Hom-Lie triple system structure. Moreover, there is a natural Hom-Bol algebra structure on every multiplicative Hom-Maltsev algebra and on every multiplicative right (or left) Hom-alternative algebra.

1. Introduction

The study of Lie triple systems (Lts) on their own as algebraic objects started from Jacobson's work [1] and developed further by, for example, Lister [2], Yamaguti [3], and other mathematicians. Lts constitute examples of ternary algebras. If $(g, [,])$ is a Lie algebra, then $(g, [,,])$ is a Lts, where $[x, y, z] := [[x, y], z]$ (see [1, 4, 5]). Another construction of Lts from binary algebras is the one from Maltsev algebras found by Loos [6].

Maltsev algebras were introduced by Maltsev [7] in a study of commutator algebras of alternative algebras and also as a study of tangent algebras to local smooth Moufang loops. Maltsev used the name "Moufang-Lie algebras" for these nonassociative algebras while Sagle [8] introduced the term "Malcev algebras." Equivalent defining identities of Maltsev algebras are pointed out in [8].

Alternative algebras, Maltsev algebras, and Lts (among other algebras) received a twisted generalization in the development of the theory of Hom-algebras during these latest years. The forerunner of the theory of Hom-algebras is the Hom-Lie algebra introduced by Hartwig et al. in [9] in order to describe the structure of some deformation of the Witt algebra and the Virasoro algebra. It is well-known that Lie algebras are related to associative algebras via the commutator bracket construction. In the search of a similar construction for Hom-Lie algebras, the notion of a Hom-associative algebra is introduced by Makhlouf and Silvestrov in [10], where it is proved that a Hom-associative algebra gives rise to a Hom-Lie algebra via the commutator bracket construction. Since then, various Hom-type structures are considered (see, e.g., [11–23]). Roughly speaking, Hom-algebraic structures are corresponding ordinary algebraic structures whose defining identities are twisted by a linear self-map. A general method for constructing a Hom-type algebra from the ordinary type of algebra with a linear self-map is given by Yau in [24].

In [11, 21], n-ary Hom-algebra structures generalizing n-ary algebras of Lie type or associative type were considered. In particular, generalizations of n-ary Nambu or Nambu-Lie algebras, called n-ary Hom-Nambu and Hom-Nambu-Lie algebras, respectively, were introduced in [11] while Hom-Jordan algebras were defined in [18] and Hom-Lie triple systems (Hom-Lts) were introduced in [21] (another definition of a Hom-Jordan algebra is given in [20]). It is shown [21] that Hom-Lts are ternary Hom-Nambu algebras with additional properties and that Hom-Lts arise also from Hom-Jordan triple systems or from other Hom-type algebras.

Motivated by the relationships between some classes of binary algebras and some classes of binary-ternary algebras, a study of Hom-type generalization of binary-ternary algebras is initiated in [16] with the definition of Hom-Akivis algebras.

Further, Hom-Lie-Yamaguti algebras are considered in [14] and Hom-Bol algebras [12] are defined as a twisted generalization of Bol algebras which are introduced and studied in [25–27] as infinitesimal structures tangent to smooth Bol loops (some aspects of the theory of Bol algebras are discussed in [28–30]).

In this paper, we will be concerned with right (or left) Hom-alternative algebras, Hom-Maltsev algebras, Hom-Lts, and Hom-Bol algebras. We extend Loos' construction of Lts from Maltsev algebras ([6], Satz 1) to the Hom-algebra setting (Section 3). Specifically, we prove (Theorem 14) that every multiplicative Hom-Maltsev algebra is naturally a multiplicative Hom-Lts by a suitable definition of the ternary operation. As a tool in the proof of this fact, we point out a kind of compatibility relation between the original binary operation of a given Hom-Maltsev algebra and the ternary operation mentioned above (Lemma 13). Moreover, we obtain that every multiplicative Hom-Maltsev algebra has a natural Hom-Bol algebra structure (Theorem 17). In [31] Mikheev proved that every right alternative algebra has a natural (left) Bol algebra structure. In [29] Hentzel and Peresi proved that not only a right alternative algebra but also a left alternative algebra has left Bol algebra structure. In Section 4 we prove that the Hom-analogue of these results holds. Specifically, every multiplicative right (or left) Hom-alternative algebra is a Hom-Bol algebra (Theorem 23). It could be observed that the methods used in the proof of results in [6, 29, 31] cannot be reported in the Hom-algebra setting at the present stage of the theory of Hom-algebras. In Section 5 we specify Theorem 23 to recover the construction of left Bol algebras from right alternative algebras (Theorem 26; one observes that, in our proof, we use essentially some fundamental properties of right alternative algebras). In Section 2 we recall some basic definitions and facts about Hom-algebras. We define the Hom-Jordan associator of a given Hom-algebra and point out that every Hom-algebra is a Hom-triple system with respect to the Hom-Jordan associator. This observation is used in the proof of Theorem 23.

All vector spaces and algebras are meant over an algebraically closed ground field $\mathbb{K}$ of characteristic 0.

2. Some Basics on Hom-Algebras

We first recall some relevant definitions about binary and ternary Hom-algebras. In particular, we recall the notion of a Hom-Maltsev algebra as well as some of its equivalent defining identities. Although various types of n-ary Hom-algebras are introduced and discussed in [11, 21], for our purpose, we will consider ternary Hom-algebras (ternary Hom-Nambu algebras and Hom-Lts) and Hom-Bol algebras. For fundamentals on Hom-algebras, one may refer, for example, to [9–11, 13, 17, 24, 32]. Some aspects of the theory of binary Hom-algebras are considered in [33], while some classes of binary-ternary Hom-algebras are defined and discussed in [12, 14, 16].

Definition 1. (i) A *Hom-algebra* is a triple $(A, *, \alpha)$ in which A is a $\mathbb{K}$-vector space, $* : A \times A \to A$ a bilinear map (the binary operation), and $\alpha : A \to A$ a linear map (the twisting map). The Hom-algebra A is said to be *multiplicative* if $\alpha(x * y) = \alpha(x) * \alpha(y)$ for all $x, y \in A$.

(ii) The *Hom-Jacobian* in $(A, *, \alpha)$ is the trilinear map $J_\alpha : A \times A \times A \to A$ defined as $J_\alpha(x, y, z) := \circlearrowleft_{x,y,z}(x * y) * \alpha(z)$, where $\circlearrowleft_{x,y,z}$ denotes the sum over cyclic permutation of x, y, z.

(iii) The *Hom-associator* of a Hom-algebra $(A, *, \alpha)$ is the trilinear map as : $A^{\otimes 3} \to A$ defined as $\mathrm{as}(x, y, z) = (x * y) * \alpha(z) - \alpha(x) * (y * z)$. If $\mathrm{as}(x, y, z) = 0$ for all $x, y, z \in A$, then $(A, *, \alpha)$ is said to be *Hom-associative*.

Remark 2. If $\alpha = \mathrm{id}$ (the identity map), then a Hom-algebra $(A, *, \alpha)$ reduces to an ordinary algebra $(A, *)$, the Hom-Jacobian J_α is the ordinary Jacobian J, and the Hom-associator is the usual associator for the algebra $(A, *)$. One observes that, in general, the map α is not always injective nor surjective (see [13, 15] for discussions on the subject). So, for example, a given algebra can be twisted into zero algebra and some properties of Hom-algebras may not be valid for corresponding ordinary algebras.

As for ordinary algebras, to each Hom-algebra $\mathscr{A} := (A, *, \alpha)$ are attached two Hom-algebras: the *commutator Hom-algebra* $\mathscr{A}^- := (A, [,], \alpha)$, where $[x, y] := x * y - y * x$ (the commutator of x and y), and the *plus Hom-algebra* $\mathscr{A}^+ := (A, \circ, \alpha)$, where $x \circ y := x * y + y * x$ (the *Jordan product*) for all $x, y \in A$.

For our purpose, we provide the following.

Definition 3. The *Hom-Jordan associator* of a Hom-algebra $\mathscr{A} := (A, *, \alpha)$ is the trilinear map $\mathrm{as}^J : A^{\otimes 3} \to A$ defined as $\mathrm{as}^J(x, y, z) = (x \circ y) \circ \alpha(z) - \alpha(x) \circ (y \circ z)$, where "$\circ$" is the Jordan product on A.

If $\alpha = \mathrm{id}$, the Hom-Jordan associator reduces to the usual Jordan associator.

Definition 4. (i) A *Hom-Lie algebra* is a Hom-algebra $(A, *, \alpha)$ such that the binary operation "$*$" is anticommutative and the *Hom-Jacobi identity*

$$J_\alpha(x, y, z) = 0 \tag{1}$$

holds for all x, y, and z in A ([9]).

(ii) A *Hom-Maltsev algebra* is a Hom-algebra $(A, *, \alpha)$ such that the binary operation "$*$" is anticommutative and that the *Hom-Maltsev identity*

$$J_\alpha(\alpha(x), \alpha(y), x * z) = J_\alpha(x, y, z) * \alpha^2(x) \tag{2}$$

holds for all x, y, z in A ([20]).

(iii) A *Hom-Jordan algebra* is a Hom-algebra $(A, *, \alpha)$ such that $(A, *)$ is a commutative algebra and the *Hom-Jordan identity*

$$\mathrm{as}(x * x, \alpha(y), \alpha(x)) = 0 \tag{3}$$

is satisfied for all x, y in A ([20]).

(iv) A Hom-algebra $(A, *, \alpha)$ is called a *right Hom-alternative algebra* if

$$\mathrm{as}(x, y, y) = 0 \tag{4}$$

for all x, y in A. A Hom-algebra $(A, *, \alpha)$ is called a *left Hom-alternative algebra* if

$$\text{as}(x, x, y) = 0 \tag{5}$$

for all x, y in A. A Hom-algebra $(A, *, \alpha)$ is called a *Hom-alternative algebra* if it is both right and left Hom-alternative [18].

Remark 5. When $\alpha = \text{id}$, the Hom-Jacobi identity (1) is the usual *Jacobi identity* $J(x, y, z) = 0$. Likewise, for $\alpha = \text{id}$, the Hom-Maltsev identity (2) reduces to the *Maltsev identity* $J(x, y, x * z) = J(x, y, z) * x$. Therefore a Lie (resp., Maltsev) algebra $(A, *)$ may be seen as a Hom-Lie (resp., Hom-Maltsev) algebra with the identity map as the twisting map. Also Hom-Maltsev algebras generalize Hom-Lie algebras in the same way that Maltsev algebras generalize Lie algebras. For $\alpha = \text{id}$ in the Hom-Jordan identity, we recover the usual Jordan identity. Observe that the definition of the Hom-Jordan identity in [20] is slightly different from the one formerly given in [18].

Hom-Maltsev algebras are introduced in [20] in connection with a study of Hom-alternative algebras introduced in [18]. In fact it is proved ([20], Theorem 3.8) that every Hom-alternative algebra is *Hom-Maltsev admissible;* that is, the commutator Hom-algebra of any Hom-alternative algebra is a Hom-Maltsev algebra (this is the Hom-analogue of Maltsev's construction of Maltsev algebras as commutator algebras of alternative algebras [7]). This result is also mentioned in [16], Section 4, using an approach via Hom-Akivis algebras (this approach is close to the one of Maltsev in [7]). Also, every Hom-alternative algebra is *Hom-Jordan admissible;* that is, its plus Hom-algebra is a Hom-Jordan algebra ([20]). Examples of Hom-alternative algebras and Hom-Jordan algebras could be found in [18, 20]. An example of a right Hom-alternative algebra that is not left Hom-alternative is given in [23].

Equivalent to (2) defining identities of Hom-Maltsev algebras are found in [20] where, in particular, it is shown that the identity

$$\begin{aligned} &J_\alpha(\alpha(x), \alpha(y), w * z) + J_\alpha(\alpha(w), \alpha(y), x * z) \\ &\quad = J_\alpha(x, y, z) * \alpha^2(w) + J_\alpha(w, y, z) * \alpha^2(x) \end{aligned} \tag{6}$$

is equivalent to (2) in any anticommutative Hom-algebra $(A, *, \alpha)$ ([20], Proposition 2.7). In [34], it is proved that, in any anticommutative Hom-algebra $(A, *, \alpha)$, the Hom-Maltsev identity (2) is equivalent to

$$\begin{aligned} &J_\alpha(\alpha(x), \alpha(y), u * v) \\ &\quad = \alpha^2(u) * J_\alpha(x, y, v) + J_\alpha(x, y, u) * \alpha^2(v) \\ &\qquad - 2J_\alpha(\alpha(u), \alpha(v), x * y). \end{aligned} \tag{7}$$

Therefore, apart from (2), identities (6) and (7) may be taken as defining identities of a Hom-Maltsev algebra.

The Hom-algebras mentioned above are *binary* Hom-algebras. The first generalization of binary algebras was the ternary algebras introduced in [1]. Ternary algebraic structures also appeared in various domains of theoretical and mathematical physics (see, e.g., [35]). Likewise, binary Hom-algebras are generalized to n-ary Hom-algebra structures in [11] (see also [21]).

Definition 6 (see [11]). A *ternary Hom-Nambu algebra* is a triple $(A, [,,], \alpha)$ in which A is a $\mathbb{K}$-vector space, $[,,] : A \times A \times A \to A$ is a trilinear map, and $\alpha = (\alpha_1, \alpha_2)$ is a pair of linear maps (the twisting maps) such that the identity

$$\begin{aligned} &[\alpha_1(x), \alpha_2(y), [u, v, w]] \\ &\quad = [[x, y, u], \alpha_1(v), \alpha_2(w)] \\ &\qquad + [\alpha_1(u), [x, y, v], \alpha_2(w)] \\ &\qquad + [\alpha_1(u), \alpha_2(v), [x, y, w]] \end{aligned} \tag{8}$$

holds for all u, v, w, x, and y in A. Identity (8) is called the *ternary Hom-Nambu identity*.

Remark 7. When $\alpha_1 = \text{id} = \alpha_2$ one recovers the usual ternary Nambu algebra. One may refer to [35] for the origins of Nambu algebras. In [11], examples of n-ary Hom-Nambu algebras that are not Nambu algebras are provided.

Definition 8 (see [21]). A *Hom-Lie triple system* (Hom-Lts) is a ternary Hom-algebra $(A, [,,], \alpha = (\alpha_1, \alpha_2))$ such that

$$[x, y, z] = -[y, x, z], \tag{9}$$

$$\circlearrowleft_{x,y,z} [x, y, z] = 0, \tag{10}$$

and the ternary Hom-Nambu identity (8) holds in $(A, [,,], \alpha = (\alpha_1, \alpha_2))$.

One notes that when the twisting maps α_1, α_2 are both equal to the identity map id, then we recover the usual notion of a Lie triple system [2, 3]. Examples of Hom-Lts could be found in [21].

A particular situation, interesting for our setting, occurs when the twisting maps α_i are all equal, $\alpha_1 = \alpha_2 = \alpha$ and $\alpha([x, y, z]) = [\alpha(x), \alpha(y), \alpha(z)]$ for all x, y, and z in A. The Hom-Lie triple system $(A, [,,], \alpha)$ is then said to be *multiplicative* [21]. In case of multiplicativity, the ternary Hom-Nambu identity (8) then reads

$$\begin{aligned} &[\alpha(x), \alpha(y), [u, v, w]] \\ &\quad = [[x, y, u], \alpha(v), \alpha(w)] \\ &\qquad + [\alpha(u), [x, y, v], \alpha(w)] \\ &\qquad + [\alpha(u), \alpha(v), [x, y, w]]. \end{aligned} \tag{11}$$

In [14] a (multiplicative) *Hom-triple system* is defined as a (multiplicative) ternary Hom-algebra $(A, [,,], \alpha)$ such that (9) and (10) are satisfied (thus a multiplicative Hom-Lts is seen as a Hom-triple system in which identity (11) holds; observe that this definition of a Hom-triple system is different from the one formerly given in [21], where a

Hom-triple system is just the Hom-algebra $(A, [,,], \alpha)$). With this vision of a Hom-triple system, it is shown [14] that every multiplicative non-Hom-associative algebra (i.e., not necessarily Hom-associative algebra) has a natural Hom-triple system structure if defining $[x, y, z] \coloneqq [[x, y], \alpha(z)] - \mathrm{as}(x, y, z) + \mathrm{as}(y, x, z)$. We note here that we get the same result if defining another ternary operation on a given Hom-algebra. Specifically, we have the following result.

Proposition 9. *Let $\mathscr{A} = (A, *, \alpha)$ be a multiplicative Hom-algebra. Define on $\mathscr{A}$ the ternary operation*

$$(x, y, z) \coloneqq as^{J}(y, z, x) \tag{12}$$

for all x, y, and $z \in A$. Then $(A, (,,), \alpha)$ is a multiplicative Hom-triple system.

Proof. A proof follows from the straightforward checking of identities (9) and (10) for "$(,,)$" using the commutativity of the Jordan product "$\circ$." □

Since our results here depend on multiplicativity, in the rest of this paper we assume that all Hom-algebras (binary or ternary) are multiplicative and while dealing with the binary operation "$*$" and where there is no danger of confusion, we will use juxtaposition in order to reduce the number of braces; that is, for example, $xy * \alpha(z)$ means $(x * y) * \alpha(z)$.

Various results and constructions related to Hom-Lts are given in [21]. In particular, it is shown that every Lts L can be twisted along any self-morphism of L into a multiplicative Hom-Lts. For our purpose we just mention the following result.

Proposition 10 (see [21]). *Let $(A, *)$ be a Maltsev algebra and $\alpha : A \to A$ an algebra morphism. Then $A_\alpha \coloneqq (A, [,,]_\alpha, \alpha)$ is a multiplicative Hom-Lts, where $[x, y, z]_\alpha = \alpha(2xy * z - yz * x - zx * y)$, for all x, y, and z in A.*

One observes that the product $[x, y, z] = 2xy * z - yz * x - zx * y$ is the one defined in [6] providing a Maltsev algebra $(A, *)$ with a Lts structure. A construction describing another view of Proposition 10 above will be given in Section 3 (see Proposition 16) via Hom-Maltsev algebras. For the time being, we point out the following slight generalization of the result above, producing a sequence of multiplicative Hom-Lts from a given Maltsev algebra.

Proposition 11. *Let $(A, *)$ be a Maltsev algebra and $\alpha : A \to A$ an algebra morphism. Let $\alpha^0 = id$ and, for any integer $n \geq 1$, $\alpha^n = \alpha \circ \alpha^{n-1}$. If one defines on A a trilinear operation $[,,]_{\alpha^n}$ by*

$$[x, y, z]_{\alpha^n} = \alpha^n (2xy * z - yz * x - zx * y) \tag{13}$$

for all x, y, and z in A, then $(A, [,,]_{\alpha^n}, \alpha^n)$ is a multiplicative Hom-Lts.

Proof. Let $[x, y, z] = 2xy * z - yz * x - zx * y$ and then $[x, y, z]_{\alpha^n} = \alpha^n([x, y, z])$. We shall use the fact that $(A, [,,])$ is a Lts [6]. Identities (9) and (10) for $[x, y, z]_{\alpha^n}$ are quite obvious. Next,

$$\begin{aligned}
&[\alpha^n(x), \alpha^n(y), [u, v, w]_{\alpha^n}]_{\alpha^n} \\
&\quad = [\alpha^n(x), \alpha^n(y), \alpha^n([u, v, w])]_{\alpha^n} \\
&\quad = \alpha^{2n}([x, y, [u, v, w]]) \\
&\quad = \alpha^{2n}([[x, y, u], v, w]) + \alpha^{2n}([u, [x, y, v], w]) \\
&\qquad + \alpha^{2n}([u, v, [x, y, w]]) \\
&\quad = [\alpha^n([x, y, u]), \alpha^n(v), \alpha^n(w)]_{\alpha^n} \\
&\qquad + [\alpha^n(u), \alpha^n([x, y, v]), \alpha^n(w)]_{\alpha^n} \\
&\qquad + [\alpha^n(u), \alpha^n(v), \alpha^n([x, y, w])]_{\alpha^n} \\
&\quad = [[x, y, u]_{\alpha^n}, \alpha^n(v), \alpha^n(w)]_{\alpha^n} \\
&\qquad + [\alpha^n(u), [x, y, v]_{\alpha^n}, \alpha^n(w)]_{\alpha^n} \\
&\qquad + [\alpha^n(u), \alpha^n(v), [x, y, w]_{\alpha^n}]_{\alpha^n}
\end{aligned} \tag{14}$$

and so (11) holds for $[,,]_{\alpha^n}$. Thus $(A, [,,]_{\alpha^n}, \alpha^n)$ is a multiplicative Hom-Lts. □

In [12] we defined a Hom-Bol algebra as a twisted generalization of a (left) Bol algebra. For the introduction and original studies of Bol algebras, we refer to [25–27] (the defining identities of left Bol algebras are recalled in Section 5 of the present paper). Bol algebras are further considered in, for example, [29, 30].

Definition 12 (see [12]). A *Hom-Bol algebra* is a quadruple $(A, [,], (,,), \alpha)$ in which A is a vector space, "$[,]$" a binary operation, "$(,,)$" a ternary operation on A, and $\alpha : A \to A$ a linear map such that

(HB1) $\alpha([x, y]) = [\alpha(x), \alpha(y)]$.

(HB2) $\alpha((x, y, z)) = (\alpha(x), \alpha(y), \alpha(z))$.

(HB3) $[x, y] = -[y, x]$.

(HB4) $(x, y, z) = -(y, x, z)$.

(HB5) $\circlearrowleft_{x,y,z}(x, y, z) = 0$.

(HB6) $(\alpha(x), \alpha(y), [u, v]) = [(x, y, u), \alpha^2(v)] + [\alpha^2(u), (x, y, v)] + (\alpha(u), \alpha(v), [x, y]) - [[\alpha(u), \alpha(v)], [\alpha(x), \alpha(y)]]$.

(HB7) $(\alpha^2(x), \alpha^2(y), (u, v, w)) = ((x, y, u), \alpha^2(v), \alpha^2(w)) + (\alpha^2(u), (x, y, v), \alpha^2(w)) + (\alpha^2(u), \alpha^2(v), (x, y, w))$

for all u, v, w, x, y, and $z \in A$.

Identities (HB1) and (HB2) mean the multiplicativity of $(A, [,], (,,), \alpha)$. It is built into our definition for convenience.

One observes that for $\alpha = id$ identities (HB3)–(HB7) reduce to the defining identities of a (left) Bol algebra [25] (see also [29, 30]). If $[x, y] = 0$ for all $x, y \in A$, then $(A, [,], (,,), \alpha)$ becomes a (multiplicative) Hom-Lts $(A, (,,), \alpha^2)$.

Construction results and some examples of Hom-Bol algebras are given in [12]. In particular, Hom-Bol algebras can be constructed from Maltsev algebras. The Hom-analogues of the construction of Bol algebras from Maltsev algebras [25] or from right alternative algebras [31] (see also [29]) are considered in this paper.

3. Hom-Lts and Hom-Bol Algebras from Hom-Maltsev Algebras

In this section, we prove that every multiplicative Hom-Maltsev algebra has a natural multiplicative Hom-Lts structure (Theorem 14) and, moreover, a natural Hom-Bol algebra structure (Theorem 17). Theorem 14 could be seen as the Hom-analogue of Loos' result ([6], Satz 1) although his proof cannot be reproduced here. Besides identities (6) and (7), Lemma 13 below is a tool in the proof of this result. Theorem 17 could be seen as the Hom-analogue of a construction by Mikheev [25] of Bol algebras from Maltsev algebras. Proposition 16 is another view of a result in [21] (see Proposition 10 above).

In his work [6], Loos considered in a Maltsev algebra $(A, *)$ the following ternary operation:

$$\{x, y, z\} = 2xy * z - yz * x - zx * y. \tag{15}$$

Then $(A, \{,,\})$ turns out to be a Lts. This result, in the Hom-algebra setting, looks as in Theorem 14 below. Similarly as in the Loos construction, our investigations are based on the following ternary operation in a Hom-Maltsev algebra $(A, *, \alpha)$:

$$\{x, y, z\}_\alpha = 2xy * \alpha(z) - yz * \alpha(x) - zx * \alpha(y). \tag{16}$$

From (16) it clearly follows that $\{,,\}_\alpha$ can also be written as

$$\{x, y, z\}_\alpha = -J_\alpha(x, y, z) + 3xy * \alpha(z). \tag{17}$$

One observes that when $\alpha = \mathrm{id}$, we recover product (15). First, we prove the following.

Lemma 13. *Let $(A, *, \alpha)$ be a Hom-Maltsev algebra. If one defines on $(A, *, \alpha)$ a ternary operation "$\{,,\}_\alpha$" by (16), then*

$$\begin{aligned}\{\alpha(x), \alpha(y), u * v\}_\alpha &= \alpha^2(u) * \{x, y, v\}_\alpha \\ &\quad + \{x, y, u\}_\alpha * \alpha^2(v) \\ &\quad - J_\alpha(\alpha(u), \alpha(v), x * y)\end{aligned} \tag{18}$$

for all u, v, x, and y in A.

Proof. Let us write (7) as

$$\begin{aligned}&-J_\alpha(\alpha(x), \alpha(y), u * v) \\ &\quad = -J_\alpha(x, y, u) * \alpha^2(v) + \alpha^2(u) * (-J_\alpha(x, y, v)) \\ &\qquad + 3J_\alpha(\alpha(u), \alpha(v), x * y) \\ &\qquad - J_\alpha(\alpha(u), \alpha(v), x * y).\end{aligned} \tag{19}$$

That is,

$$\begin{aligned}&-J_\alpha(\alpha(x), \alpha(y), u * v) \\ &\quad = -J_\alpha(x, y, u) * \alpha^2(v) + \alpha^2(u) * (-J_\alpha(x, y, v)) \\ &\qquad + 3\alpha(u)\alpha(v) * \alpha(x * y) + 3(\alpha(v) * xy) \\ &\qquad * \alpha^2(u) + 3(xy * \alpha(u)) * \alpha^2(v) \\ &\qquad - J_\alpha(\alpha(u), \alpha(v), x * y).\end{aligned} \tag{20}$$

Therefore, by multiplicativity, we have

$$\begin{aligned}&-J_\alpha(\alpha(x), \alpha(y), u * v) + 3\alpha(x)\alpha(y) * \alpha(u * v) \\ &\quad = (-J_\alpha(x, y, u) + 3xy * \alpha(u)) * \alpha^2(v) + \alpha^2(u) \\ &\qquad * (-J_\alpha(x, y, v) + 3xy * \alpha(v)) \\ &\qquad - J_\alpha(\alpha(u), \alpha(v), x * y)\end{aligned} \tag{21}$$

and so, we get (18) by (17). □

We now prove the following.

Theorem 14. *Let $(A, *, \alpha)$ be a multiplicative Hom-Maltsev algebra. If one defines on $(A, *, \alpha)$ a ternary operation "$\{,,\}_\alpha$" by (16), then $(A, \{,,\}_\alpha, \alpha^2)$ is a multiplicative Hom-Lts.*

Proof. We must prove the validity of (9), (10), and (11) for operation (16) in the Hom-Maltsev algebra $(A, *, \alpha)$.

First observe that the multiplicativity of $(A, *, \alpha)$ implies that $\alpha^2(\{x, y, z\}_\alpha) = \{\alpha^2(x), \alpha^2(y), \alpha^2(z)\}_\alpha$, with x, y, and z in A.

From the skew-symmetry of "$*$" and $J_\alpha(x, y, z)$, it clearly follows from (17) that $\{x, y, z\}_\alpha = -\{y, x, z\}_\alpha$ which is (9) for "$\{,,\}_\alpha$."

Next, using (17) and the skew-symmetry of $J_\alpha(x, y, z)$ where applicable, we compute

$$\begin{aligned}&\{x, y, z\}_\alpha + \{y, z, x\}_\alpha + \{z, x, y\}_\alpha \\ &\quad = -J_\alpha(x, y, z) + 3xy * \alpha(z) - J_\alpha(y, z, x) + 3yz \\ &\qquad * \alpha(x) - J_\alpha(z, x, y) + 3zx * \alpha(y) \\ &\quad = -3J_\alpha(x, y, z) + 3J_\alpha(x, y, z) = 0\end{aligned} \tag{22}$$

and thus $\circlearrowleft_{x,y,z}\{x, y, z\}_\alpha = 0$, so we get (10) for "$\{,,\}_\alpha$."

Consider now $\{\alpha^2(x), \alpha^2(y), \{u, v, w\}_\alpha\}_\alpha$ in $(A, *, \alpha)$. Then

$$\begin{aligned}&\{\alpha^2(x), \alpha^2(y), \{u, v, w\}_\alpha\}_\alpha = \{\alpha^2(x), \alpha^2(y), 2uv \\ &\quad * \alpha(w) - vw * \alpha(u) - wu * \alpha(v)\}_\alpha \quad \text{(by (16))} \\ &= \{\alpha^2(x), \alpha^2(y), 2uv * \alpha(w)\}_\alpha - \{\alpha^2(x), \alpha^2(y), \\ &\quad vw * \alpha(u)\}_\alpha - \{\alpha^2(x), \alpha^2(y), wu * \alpha(v)\}_\alpha \\ &\quad = \{\alpha(x), \alpha(y), 2u * v\}_\alpha * \alpha^3(w) + \alpha^2(2u * v)\end{aligned}$$

$$
\begin{aligned}
&* \{\alpha(x), \alpha(y), \alpha(w)\}_\alpha - J_\alpha\left(\alpha(2u * v), \alpha^2(w),\right.\\
&\left.\alpha(x * y)\right) - \{\alpha(x), \alpha(y), v * w\}_\alpha * \alpha^3(u)\\
&- \alpha^2(v * w) * \{\alpha(x), \alpha(y), \alpha(u)\}_\alpha + J_\alpha\left(\alpha(v\right.\\
&\left.* w), \alpha^2(u), \alpha(x * y)\right) - \{\alpha(x), \alpha(y), w * u\}_\alpha\\
&* \alpha^3(v) - \alpha^2(w * u) * \{\alpha(x), \alpha(y), \alpha(v)\}_\alpha\\
&+ J_\alpha\left(\alpha(w * u), \alpha^2(v), \alpha(x * y)\right) \quad (\text{by } (18))\\
={}& \left(2\{x, y, u\}_\alpha * \alpha^2(v) + 2\alpha^2(u) * \{x, y, v\}_\alpha\right.\\
&\left.- 2J_\alpha(\alpha(u), \alpha(v), x * y)\right) * \alpha^3(w) + 2\alpha^2(u * v)\\
&* \{\alpha(x), \alpha(y), \alpha(w)\}_\alpha - J_\alpha\left(\alpha(2u * v), \alpha^2(w),\right.\\
&\left.\alpha(x * y)\right) - \left(\{x, y, v\}_\alpha * \alpha^2(w) + \alpha^2(v)\right.\\
&\left.* \{x, y, w\}_\alpha - J_\alpha(\alpha(v), \alpha(w), x * y)\right) * \alpha^3(u)\\
&- \alpha^2(v * w) * \{\alpha(x), \alpha(y), \alpha(u)\}_\alpha + J_\alpha\left(\alpha(v\right.\\
&\left.* w), \alpha^2(u), \alpha(x * y)\right) - \left(\{x, y, w\}_\alpha * \alpha^2(u)\right.\\
&\left.+ \alpha^2(w) * \{x, y, u\}_\alpha - J_\alpha(\alpha(w), \alpha(u), x * y)\right)\\
&* \alpha^3(v) - \alpha^2(w * u) * \{\alpha(x), \alpha(y), \alpha(v)\}_\alpha\\
&+ J_\alpha\left(\alpha(w * u), \alpha^2(v), \alpha(x * y)\right)\\
&\qquad\qquad (\text{again by } (18))\\
={}& 2\{x, y, u\}_\alpha \alpha^2(v) * \alpha^3(w) + 2\alpha^2(u)\{x, y, v\}_\alpha\\
&* \alpha^3(w) - 2J_\alpha(\alpha(u), \alpha(v), x * y) * \alpha^3(w)\\
&+ 2\alpha^2(u * v) * \{\alpha(x), \alpha(y), \alpha(w)\}_\alpha - J_\alpha\left(\alpha(2u\right.\\
&\left.* v), \alpha^2(w), \alpha(x * y)\right) - \{x, y, v\}_\alpha \alpha^2(w) * \alpha^3(u)\\
&- \alpha^2(v)\{x, y, w\}_\alpha * \alpha^3(u) + J_\alpha(\alpha(v), \alpha(w), x\\
&* y) * \alpha^3(u) - \alpha^2(v * w) * \{\alpha(x), \alpha(y), \alpha(u)\}_\alpha\\
&+ J_\alpha\left(\alpha(v * w), \alpha^2(u), \alpha(x * y)\right) - \{x, y, w\}_\alpha\\
&\cdot \alpha^2(u) * \alpha^3(v) - \alpha^2(w)\{x, y, u\}_\alpha * \alpha^3(v)\\
&+ J_\alpha(\alpha(w), \alpha(u), x * y) * \alpha^3(v) - \alpha^2(w * u)\\
&* \{\alpha(x), \alpha(y), \alpha(v)\}_\alpha + J_\alpha\left(\alpha(w * u), \alpha^2(v), \alpha(x\right.\\
&\left.* y)\right) = 2\{x, y, u\}_\alpha \alpha^2(v) * \alpha^3(w) - \alpha^2(v * w)\\
&* \alpha(\{x, y, u\}_\alpha) - \alpha^2(w)\{x, y, u\}_\alpha * \alpha^3(v)\\
&+ 2\alpha^2(u)\{x, y, v\}_\alpha * \alpha^3(w) - \{x, y, v\}_\alpha \alpha^2(w)\\
&* \alpha^3(u) - \alpha^2(w * u) * \alpha(\{x, y, v\}_\alpha) + 2\alpha^2(u\\
&* v) * \alpha(\{x, y, w\}_\alpha) - \alpha^2(v)\{x, y, w\}_\alpha * \alpha^3(u)\\
&- \{x, y, w\}_\alpha \alpha^2(u) * \alpha^3(v) - 2J_\alpha(\alpha(u), \alpha(v), x\\
&* y) * \alpha^3(w) - J_\alpha\left(\alpha(2u * v), \alpha^2(w), \alpha(x * y)\right)\\
&+ J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u) + J_\alpha\left(\alpha(v * w),\right.\\
&\left.\alpha^2(u), \alpha(x * y)\right) + J_\alpha(\alpha(w), \alpha(u), x * y)\\
&* \alpha^3(v) + J_\alpha\left(\alpha(w * u), \alpha^2(v), \alpha(x * y)\right)\\
&\qquad\qquad (\text{rearranging terms})\\
={}& \left\{\{x, y, u\}_\alpha, \alpha^2(v), \alpha^2(w)\right\}_\alpha + \left\{\alpha^2(u), \{x, y, v\}_\alpha,\right.\\
&\left.\alpha^2(w)\right\}_\alpha + \left\{\alpha^2(u), \alpha^2(v), \{x, y, w\}_\alpha\right\}_\alpha\\
&+ \left[-2\left(J_\alpha(\alpha(u), \alpha(v), x * y) * \alpha^3(w)\right.\right.\\
&\left.+ J_\alpha\left(\alpha(u * v), \alpha^2(w), \alpha(x * y)\right)\right)\\
&+ J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u)\\
&+ J_\alpha\left(\alpha(v * w), \alpha^2(u), \alpha(x * y)\right)\\
&+ J_\alpha(\alpha(w), \alpha(u), x * y) * \alpha^3(v)\\
&\left.+ J_\alpha\left(\alpha(w * u), \alpha^2(v), \alpha(x * y)\right)\right].
\end{aligned}
\tag{23}
$$

In this latest expression, denote by $N(u, v, w, x, y)$ the expression in "$[\cdots]$"; to conclude, we proceed to show that $N(u, v, w, x, y) = 0$.

Observe first that, by (6), we have

$$
\begin{aligned}
&J_\alpha(\alpha(u), x * y, \alpha(w)) * \alpha^2(\alpha(v))\\
&\quad + J_\alpha(\alpha(v), x * y, \alpha(w)) * \alpha^2(\alpha(u))\\
&= J_\alpha\left(\alpha^2(u), \alpha(x * y), \alpha(v) * \alpha(w)\right)\\
&\quad + J_\alpha\left(\alpha^2(v), \alpha(x * y), \alpha(u) * \alpha(w)\right).
\end{aligned}
\tag{24}
$$

That is,

$$
\begin{aligned}
&J_\alpha(\alpha(w), \alpha(u), x * y) * \alpha^3(v)\\
&\quad + J_\alpha\left(\alpha(w * u), \alpha^2(v), \alpha(x * y)\right)\\
&= J_\alpha\left(\alpha(v * w), \alpha^2(u), \alpha(x * y)\right)\\
&\quad + J_\alpha(\alpha(v), \alpha(w), x * y) * \alpha^3(u).
\end{aligned}
\tag{25}
$$

With this observation, the expression $N(u, v, w, x, y)$ is transformed as follows:

$$
\begin{aligned}
N(u, v, w, x, y) &= 2\left[-J_\alpha(\alpha(u), \alpha(v), x * y)\right.\\
&\left.* \alpha^3(w) - J_\alpha\left(\alpha(u * v), \alpha^2(w), \alpha(x * y)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+2\left[J_\alpha\left(\alpha(v*w),\alpha^2(u),\alpha(x*y)\right)\right.\\
&\left.+J_\alpha(\alpha(v),\alpha(w),x*y)*\alpha^3(u)\right]\\
&=2\left[-J_\alpha\left(\alpha^2(w),\alpha(x*y),\alpha(u)*\alpha(v)\right)\right.\\
&-J_\alpha\left(\alpha^2(u),\alpha(x*y),\alpha(w)*\alpha(v)\right)\\
&-J_\alpha(\alpha(u),\alpha(v),x*y)*\alpha^3(w)\\
&\left.+J_\alpha(\alpha(v),\alpha(w),x*y)*\alpha^3(u)\right]\\
&=2\left[-J_\alpha(\alpha(w),x*y,\alpha(v))*\alpha^3(u)\right.\\
&-J_\alpha(\alpha(u),x*y,\alpha(v))*\alpha^3(w)\\
&-J_\alpha(\alpha(u),\alpha(v),x*y)*\alpha^3(w)\\
&\left.+J_\alpha(\alpha(v),\alpha(w),x*y)*\alpha^3(u)\right]\\
&\Big(\text{applying (6) to}\\
&-J_\alpha\left(\alpha^2(w),\alpha(x*y),\alpha(u)*\alpha(v)\right)\\
&-J_\alpha\left(\alpha^2(u),\alpha(x*y),\alpha(w)*\alpha(v)\right)\Big)\\
&=0\,(\text{by the skew-symmetry of } J_\alpha(x,y,z)).
\end{aligned}
\tag{26}
$$

Therefore, we obtain that (11) holds for "$\{,,\}_\alpha$" and we conclude that $(A,\{,,\}_\alpha,\alpha^2)$ is a Hom-Lts. □

Remark 15. In the proof of his result, Loos ([6], Satz 1) used essentially the fact that the left translations $L(x)$ in a Maltsev algebra $(A,*)$ are derivations with respect to the ternary operation "$\{,,\}$" defined by (15). Unfortunately, for Hom-Maltsev algebras such a tool is still not available at hand.

From [20] (Theorem 2.12) we know that any Maltsev algebra A can be twisted into a Hom-Maltsev algebra along any linear self-map of A. Consistent with this result, we recall the following method for constructing Hom-Lts which, in fact, is a result in [21] (see also Propositions 10 and 11 above) but using a Hom-Maltsev algebra construction in our proof (as a consequence of Theorem 14).

Proposition 16. *Let $(A,*)$ be a Maltsev algebra and α any self-morphism of $(A,*)$. If one defines on $(A,*)$ a ternary operation "$\{,,\}_\alpha$" by*

$$\{x,y,z\}_\alpha=\alpha^2(2xy*z-yz*x-zx*y),\tag{27}$$

then $(A,\{,,\}_\alpha,\alpha^2)$ is a multiplicative Hom-Lts.

Proof. One knows ([20], Theorem 2.12) that, from $(A,*)$ and any self-morphism α of $(A,*)$, we get a (multiplicative) Hom-Maltsev algebra $(A,\tilde{*},\alpha)$, where $x\tilde{*}y=\alpha(x*y)$ for all x,y in A. Next, if one defines on $(A,\tilde{*},\alpha)$ a ternary operation

$$
\begin{aligned}
\{x,y,z\}_\alpha &:= 2(x\tilde{*}y)\tilde{*}\alpha(z)-(y\tilde{*}z)\tilde{*}\alpha(x)\\
&-(z\tilde{*}x)\tilde{*}\alpha(y),
\end{aligned}
\tag{28}
$$

then, by Theorem 14, $(A,\{,,\}_\alpha,\alpha^2)$ is a Hom-Lts and "$\{,,\}_\alpha$" is expressed through "$*$" as

$$
\begin{aligned}
\{x,y,z\}_\alpha &= 2\alpha(\alpha(x*y)*\alpha(z))\\
&-\alpha(\alpha(y*z)*\alpha(x))\\
&-\alpha(\alpha(z*x)*\alpha(y))\\
&=2\alpha^2(xy*z)-\alpha^2(yz*x)-\alpha^2(zx*y)\\
&=\alpha^2(2xy*z-yz*x-zx*y).
\end{aligned}
\tag{29}
$$

□

Observe that though constructed in quite a different way, the operation "$\{,,\}_\alpha$" in Proposition 16 above coincides with "$[,,]_{\alpha^n}$" in Proposition 11 for $n=2$.

Combining Lemma 13 and Theorem 14, we get the following result.

Theorem 17. *Let $(A,*,\alpha)$ be a multiplicative Hom-Maltsev algebra. If one defines on $(A,*,\alpha)$ a ternary operation $(,,)_\alpha$ by*

$$(x,y,z)_\alpha:=\frac{1}{3}\{x,y,z\}_\alpha,\tag{30}$$

where "$\{,,\}_\alpha$" is defined by (17), then $(A,*,(,,)_\alpha,\alpha)$ is a Hom-Bol algebra.

Proof. Definition (30) and Theorem 14 imply that $(A,(,,)_\alpha,\alpha^2)$ is a multiplicative Hom-Lts; that is, (HB4), (HB5), and (HB7) hold for $(A,*,(,,)_\alpha,\alpha)$. Now, (HB1), (HB2), and (HB3) are, respectively, the multiplicativity and skew-symmetry of "$*$"; next, we are done if we prove (HB6) for $(A,*,(,,)_\alpha,\alpha)$.

From (17) and multiplicativity we have

$$
\begin{aligned}
&-J_\alpha(\alpha(u),\alpha(v),x*y)\\
&=\{\alpha(u),\alpha(v),x*y\}_\alpha-3(\alpha(u)\alpha(v))\\
&*(\alpha(x)\alpha(y))
\end{aligned}
\tag{31}
$$

and then (18) takes the form

$$
\begin{aligned}
&\{\alpha(x),\alpha(y),u*v\}_\alpha\\
&=\{x,y,u\}_\alpha*\alpha^2(v)+\alpha^2(u)*\{x,y,v\}_\alpha\\
&+\{\alpha(u),\alpha(v),x*y\}_\alpha-3(\alpha(u)\alpha(v))\\
&*(\alpha(x)\alpha(y)).
\end{aligned}
\tag{32}
$$

Multiplying by $1/3$ each member of this latter equality and using (30), we get

$$
\begin{aligned}
&(\alpha(x),\alpha(y),u*v)_\alpha\\
&=(x,y,u)_\alpha*\alpha^2(v)+\alpha^2(u)*(x,y,v)_\alpha\\
&+(\alpha(u),\alpha(v),x*y)_\alpha-(\alpha(u)\alpha(v))\\
&*(\alpha(x)\alpha(y))
\end{aligned}
\tag{33}
$$

which is (HB6) for $(A, *, (,,)_\alpha, \alpha)$. So $(A, *, (,,)_\alpha, \alpha)$ is a Hom-Bol algebra. □

Example 18. Let A be a vector space with basis $\{e_1, e_2, e_3, e_4\}$. From [20] (Example 2.13) we know that if one considers the linear map $\alpha : A \to A$ given by

$$\alpha(e_1) = e_1 + e_3;$$
$$\alpha(e_2) = 2e_2 + 2e_4;$$
$$\alpha(e_3) = -e_3; \tag{34}$$
$$\alpha(e_4) = -2e_4$$

and the multiplication table given by

$$e_1 * e_2 = -2e_2 - 2e_4 \ \ (= -e_2 * e_1);$$
$$e_1 * e_3 = e_3 \ \ (= -e_3 * e_1);$$
$$e_1 * e_4 = -2e_4 \ \ (= -e_4 * e_1); \tag{35}$$
$$e_2 * e_3 = -4e_4 \ \ (= -e_3 * e_2)$$

(only nonzero products are specified), then $(A, *, \alpha)$ is a multiplicative Hom-Maltsev algebra. It is observed that $(A, *, \alpha)$ is not a Hom-Lie algebra nor a Maltsev algebra.

Now, by (17) and (30), one checks that the only nonzero ternary products $(x, y, z)_\alpha$ on A with respect to the basis elements are

$$(e_1, e_2, e_1)_\alpha = -4e_2 \left(= -(e_2, e_1, e_1)_\alpha\right);$$
$$(e_1, e_3, e_1)_\alpha = -e_3 \left(= -(e_3, e_1, e_1)_\alpha\right); \tag{36}$$
$$(e_1, e_4, e_1)_\alpha = -4e_4 \left(= -(e_4, e_1, e_1)_\alpha\right).$$

By Theorem 17 we get that $(A, *, (,,)_\alpha, \alpha)$ is a Hom-Bol algebra.

Since any Hom-alternative algebra is Hom-Maltsev admissible ([20], Theorem 3.8), from Theorem 17 we have the following.

Corollary 19. *Let $(A, *, \alpha)$ be a multiplicative Hom-alternative algebra. Then $(A, [,], (,,)_\alpha, \alpha)$ is a Hom-Bol algebra, where $(x, y, z)_\alpha := -(1/3)(2[[x, y], \alpha(z)] - [[y, z], \alpha(x)] - [[z, x], \alpha(y)])$, for all x, y, and $z \in A$.*

The aim of Section 4 is a generalization of Corollary 19 to multiplicative right (or left) Hom-alternative algebras.

Various constructions of Hom-Lts are offered in [21] starting from either Hom-associative algebras, Hom-Lie algebras, Hom-Jordan triple systems, ternary totally Hom-associative algebras, Maltsev algebras, or alternative algebras. In practice, it is easier to construct Hom-Lts or Hom-Bol algebras from well-known (binary) algebras such as alternative algebras or Maltsev algebras. From this point of view, our construction results (Theorem 14, Proposition 16, and Theorem 17) have rather a theoretical feature (the extension to Hom-algebra setting of Loos' result [6] and a result by Mikheev [25]) than a practical method for constructing Hom-Lts or Hom-Bol algebras. However, it could be of some interest to get a Hom-Lts or a Hom-Bol algebra from a given Hom-Maltsev algebra without resorting to the corresponding Maltsev algebra.

4. Hom-Lts and Hom-Bol Algebras from Right (or Left) Hom-Alternative Algebras

In this section we prove that every multiplicative right (or left) Hom-alternative algebra has a natural Hom-Bol algebra structure (and, subsequently, a natural Hom-Lts structure). This is the Hom-analogue of a result by Mikheev [31] and by Hentzel and Peresi [29] although with a different scheme of proof.

First we recall some few basic properties of right Hom-alternative algebras that could be found in [18, 23].

The linearized form of the right Hom-alternative identity $\mathrm{as}(x, y, y) = 0$ is given by the following result.

Lemma 20 (see [18]). *If $(A, *, \alpha)$ is a Hom-algebra, then the following statements are equivalent.*

(i) $(A, *, \alpha)$ is right Hom-alternative.

(ii) $(A, *, \alpha)$ satisfies

$$as(x, y, z) = -as(x, z, y) \tag{37}$$

for all x, y, and $z \in A$.

(iii) $(A, *, \alpha)$ satisfies

$$\alpha(x) * (yz + zy) = xy * \alpha(z) + xz * \alpha(y) \tag{38}$$

for all x, y, and $z \in A$.

Observe that if $(A, *, \alpha)$ is a right Hom-alternative algebra, then $(A, *^{op}, \alpha)$ is a left Hom-alternative algebra, where $x *^{op} y := y * x$. So the mirrors of (37) and (38) hold for $(A, *^{op}, \alpha)$:

$$\mathrm{as}(x, y, z) = -\mathrm{as}(y, x, z), \tag{39}$$

$$\begin{aligned}((x *^{op} y) + (y *^{op} x)) *^{op} \alpha(z) \\ = \alpha(x) *^{op} (y *^{op} z) + \alpha(y) *^{op} (x *^{op} z).\end{aligned} \tag{40}$$

Now we have the following.

Lemma 21. *In any multiplicative right Hom-alternative algebra $(A, *, \alpha)$, the identity*

$$\begin{aligned} &as([u, v], \alpha(x), \alpha(y)) \\ &= [as(u, x, y), \alpha^2(v)] + [\alpha^2(u), as(v, x, y)] \\ &+ as(\alpha(v), \alpha(u), [x, y]) \\ &- as(\alpha(u), \alpha(v), [x, y]) \end{aligned} \tag{41}$$

holds for all x, y, and $z \in A$.

Proof. The identity

$$\begin{aligned} \mathrm{as}(uv, \alpha(x), \alpha(y)) = {} & \mathrm{as}(u, x, y)\,\alpha^2(v) \\ & + \alpha^2(u)\,\mathrm{as}(v, x, y) \\ & - \mathrm{as}(\alpha(u), \alpha(v), [x, y]) \end{aligned} \tag{42}$$

holds in any right Hom-alternative algebra (see [23], Theorem 7.1 (7.1.1c)). Next, in this identity, switching u and v, we have

$$\begin{aligned} \mathrm{as}(vu, \alpha(x), \alpha(y)) = {} & \mathrm{as}(v, x, y)\,\alpha^2(u) \\ & + \alpha^2(v)\,\mathrm{as}(u, x, y) \\ & - \mathrm{as}(\alpha(v), \alpha(u), [x, y]). \end{aligned} \tag{43}$$

Then, subtracting memberwise this latter equality from the one above and using the linearity of as, we get (41). □

Note that in the case when $(A, *, \alpha)$ is a left Hom-alternative algebra, identity (41) reads as

$$\begin{aligned} & \mathrm{as}(\alpha(x), \alpha(y), [u, v]) \\ & \quad = [\mathrm{as}(x, y, u), \alpha^2(v)] + [\alpha^2(u), \mathrm{as}(x, y, v)] \\ & \qquad + \mathrm{as}([x, y], \alpha(v), \alpha(u)) \\ & \qquad - \mathrm{as}([x, y], \alpha(u), \alpha(v)). \end{aligned} \tag{44}$$

In any multiplicative right (or left) Hom-alternative algebra $(A, *, \alpha)$ we consider the ternary operation defined by (12); that is,

$$(x, y, z) \coloneqq \mathrm{as}^J(y, z, x), \tag{45}$$

where as^J is the Hom-Jordan associator defined in Section 2. Observe that for $\alpha = \mathrm{id}$ the ternary operation "$(,,)$" is precisely the one defined in [29] (see also [31], Remark 2) and that makes any right (or left) alternative algebra into a left Bol algebra. In [29], Hentzel and Peresi used the approach of Mikheev [31] who formerly proved that the commutator algebra of any right alternative algebra has a left Bol algebra structure.

Proposition 22. *(i) If $(A, *, \alpha)$ is a multiplicative right Hom-alternative algebra, then*

$$(x, y, z) = [[x, y], \alpha(z)] - 2as(z, x, y) \tag{46}$$

for all x, y, and $z \in A$.

(ii) If $(A, *, \alpha)$ is a multiplicative left Hom-alternative algebra, then

$$(x, y, z) = [[x, y], \alpha(z)] - 2as(x, y, z) \tag{47}$$

for all x, y, and $z \in A$.

Proof. (i) From (12) we have

$$\begin{aligned} (x, y, z) &= (y \circ z) \circ \alpha(x) - \alpha(y) \circ (z \circ x) \\ &= ((y * z) + (z * y)) * \alpha(x) \\ &\quad + [\alpha(x) * ((y * z) + (z * y))] \\ &\quad - [\alpha(y) * ((z * x) + (x * z))] \\ &\quad - ((z * x) + (x * z)) * \alpha(y) \\ &= ((y * z) + (z * y)) * \alpha(x) \\ &\quad + [(x * y) * \alpha(z) + (x * z) * \alpha(y)] \\ &\quad - [(y * z) * \alpha(x) + (y * x) * \alpha(z)] \\ &\quad - ((z * x) + (x * z)) * \alpha(y) \quad (\text{by } (38)) \\ &= (z * y) * \alpha(x) + (x * y) * \alpha(z) \\ &\quad - (y * x) * \alpha(z) - (z * x) * \alpha(y) \\ &= (z * y) * \alpha(x) - (z * x) * \alpha(y) + [x, y] \\ &\quad * \alpha(z) \\ &= (z * y) * \alpha(x) - (z * x) * \alpha(y) \\ &\quad + [[x, y], \alpha(z)] + \alpha(z) * [x, y] \\ &= [[x, y], \alpha(z)] + (z * y) * \alpha(x) - \alpha(z) \\ &\quad * (y * x) - (z * x) * \alpha(y) + \alpha(z) \\ &\quad * (x * y) \\ &= [[x, y], \alpha(z)] + \mathrm{as}(z, y, x) - \mathrm{as}(z, x, y) \\ &= [[x, y], \alpha(z)] - 2\mathrm{as}(z, x, y) \quad (\text{by } (37)) \end{aligned} \tag{48}$$

and so we get (46).

(ii) Proceeding as above, but using (40) and then (39), one gets (47). □

We are now in a position to prove the main result of this section.

Theorem 23. *Let $(A, *, \alpha)$ be a multiplicative right (resp., left) Hom-alternative algebra. If one defines on A a ternary operation "$(,,)$" by (46) (resp., (47)), then $(A, (,,), \alpha^2)$ is a Hom-Lts and $(A, [,], (,,), \alpha)$ is a Hom-Bol algebra.*

Proof. We prove the theorem for a multiplicative right Hom-alternative algebra $(A, *, \alpha)$ (the proof of the left case is the mirror of the right one).

Identities (HB1) and (HB2) follow from the multiplicativity of $(A, *, \alpha)$. Identities (HB3) and (HB4) are obvious from the definition of "$[,]$" and "$(,,)$"; identity (HB5) follows from Proposition 9.

In [22] Yau showed that if, on a multiplicative Hom-Jordan algebra $(A, \circ, \alpha)$, define a ternary operation by

$$[x, y, z] \coloneqq 2(\alpha(x) \circ (y \circ z) - \alpha(y) \circ (x \circ z)), \tag{49}$$

then $(A, [,,], \alpha^2)$ is a multiplicative Hom-Lts (see [22], Corollary 4.1). Now, observe that $[x, y, z] = 2\mathrm{as}^J(y, z, x)$; that is, $[x, y, z] = 2(x, y, z)$. Therefore, since every multiplicative right Hom-alternative algebra is Hom-Jordan admissible (see [23], Theorem 4.3), we conclude that $(A, (,,), \alpha^2)$ is a multiplicative Hom-Lts and so identity (HB7) holds for $(A, [,], (,,), \alpha)$.

Next, $(A, [,], (,,), \alpha)$ is a Hom-Bol algebra if we prove that (HB6) additionally holds.

Write (46) as

$$-2\mathrm{as}(z, x, y) = (x, y, z) - [[x, y], \alpha(z)]. \qquad (50)$$

Multiplying each member of (41) by −2 and next using (50), we get

$$\begin{aligned}(\alpha(x), \alpha(y), [u, v]) &- [[\alpha(x), \alpha(y)], \alpha([u, v])] \\ &= [(x, y, u) - [[x, y], \alpha(u)], \alpha^2(v)] \\ &+ [\alpha^2(u), (x, y, v) - [[x, y], \alpha(v)]] \\ &+ (\alpha(u), [x, y], \alpha(v)) \\ &- [[\alpha(u), [x, y]], \alpha^2(v)] \\ &- (\alpha(v), [x, y], \alpha(u)) \\ &+ [[\alpha(v), [x, y]], \alpha^2(u)].\end{aligned} \qquad (51)$$

That is,

$$\begin{aligned}(\alpha(x), \alpha(y), [u, v]) = {} & [(x, y, u), \alpha^2(v)] \\ &+ [\alpha^2(u), (x, y, v)] \\ &- ([x, y], \alpha(u), \alpha(v)) \\ &+ ([x, y], \alpha(v), \alpha(u)) \\ &+ \alpha([[x, y], [u, v]]).\end{aligned} \qquad (52)$$

Observe that

$$\begin{aligned}&-([x, y], \alpha(u), \alpha(v)) + ([x, y], \alpha(v), \alpha(u)) \\ &\quad = (\alpha(u), [x, y], \alpha(v)) + ([x, y], \alpha(v), \alpha(u)) \\ &\quad = -(\alpha(v), \alpha(u), [x, y]) \\ &(\text{since } \circlearrowleft_{a,b,c}(a, b, c) = 0 \text{ by (HB5)}) = (\alpha(u), \alpha(v), [x, y]).\end{aligned} \qquad (53)$$

Therefore, (52) now reads

$$\begin{aligned}(\alpha(x), \alpha(y), [u, v]) = {} & [(x, y, u), \alpha^2(v)] \\ &+ [\alpha^2(u), (x, y, v)] \\ &+ (\alpha(u), \alpha(v), [x, y]) \\ &- \alpha([[u, v], [x, y]])\end{aligned} \qquad (54)$$

and so (HB6) holds for $(A, [,], (,,), \alpha)$. Thus we conclude that $(A, [,], (,,), \alpha)$ is a Hom-Bol algebra. One gets the same result in the case when $(A, *, \alpha)$ is a multiplicative left Hom-alternative algebra and essentially using (47) and (44). This finishes the proof. □

Example 24. Let A be a five-dimensional vector space with basis $\{e, u, v, w, z\}$ and let $\alpha : A \to A$ be a linear map given by

$$\begin{aligned}\alpha(e) &= e + u + v; \\ \alpha(u) &= -u; \\ \alpha(v) &= -v; \\ \alpha(w) &= -w; \\ \alpha(z) &= -z.\end{aligned} \qquad (55)$$

Define on A a binary operation "$*$" by

$$\begin{aligned}e * e &= e + u + v; \\ e * u &= -v; \\ e * w &= -w + z; \\ e * z &= -z; \\ u * e &= -u; \\ z * e &= -z\end{aligned} \qquad (56)$$

(again, only nonzero products are specified). Then $(A, *, \alpha)$ is a multiplicative right Hom-alternative algebra (see [23], Example 2.9). Then, using $[x, y] = x * y - y * x$ and (46), one could find (although the computation is somewhat lengthy) all the nonzero products "[,]" and "(, ,)" with respect to the basis elements e, u, v, w, and z of A. We just point out that they are nonzero products; for example, $[e, u] = u - v$, $[e, w] = -w + z$, $(e, u, e) = -u - v$, and $(e, w, e) = -w - z$. Therefore, Theorem 23 implies that $(A, [,], (,,), \alpha)$ is a Hom-Bol algebra.

5. The Construction of Bol Algebras from Right Alternative Algebras Revisited

As already mentioned in Section 2, for $\alpha = \mathrm{id}$ in Definition 12 we get the definition of a left Bol algebra.

Definition 25 (see [25, 27]). A *left Bol algebra* is a triple $(A, [,], (,,))$ in which A is a vector space, "[,]" a binary operation, and "(, ,)" a ternary operation on A such that

(B1) $[x, y] = -[y, x]$,

(B2) $(x, y, z) = -(y, x, z)$,

(B3) $\circlearrowleft_{x,y,z}(x, y, z) = 0$,

(B4) $(x, y, [u, v]) = [(x, y, u), v] + [u, (x, y, v)] + (u, v, [x, y]) - [[u, v], [x, y]]$,

(B5) $(x, y, (u, v, w)) = ((x, y, u), v, w) + (u, (x, y, v), w) + (u, v, (x, y, w))$,

for all $u, v, w, x, y, z \in A$.

In this section we show how the construction of Hom-Bol algebras from right or left Hom-alternative algebras described in Section 4 can be specified to the ordinary untwisted case of construction of (left) Bol algebras from right or left alternative algebras ([29, 31]). In fact, for $\alpha = \mathrm{id}$ in Theorem 23 and specifying the right alternative case, we get the following.

Theorem 26. *Let $(A, *)$ be a right alternative algebra. If one defines on A a ternary operation "$(,,)$" by*

$$(x, y, z) = [[x, y], z] - 2as(z, x, y), \tag{57}$$

where $as(u, v, w) = uv * w - u * vw$, then $(A, (,,))$ is a Lts and $(A, [,], (,,))$ is a left Bol algebra.

Proof. Identities (B1) and (B2) are obvious. For $\alpha = \mathrm{id}$, the Hom-Jordan associator (see Definition 3) reduces to the usual Jordan associator $as^J(u, v, w) := (u \circ v) \circ w - u \circ (v \circ w)$ in $(A, *)$. The fact that (B3) and (B5) hold in $(A, *)$ follows from the equality $(x, y, z) = as^J(y, z, x)$ that holds in right alternative algebras (the untwisted form of (46)) and from that right alternative algebras are Jordan admissible [36]. Therefore $(A, (,,))$ is a Lts since any Jordan algebra is a Lts with respect to the operation $(x, y, z) = as^J(y, z, x)$ (see [1]). So we get the untwisted version of (HB5) and (HB7).

In order to show that (B4) holds in $(A, *)$, we proceed as follows. First, recall that the identity

$$\begin{aligned} as(uv, y, x) &= u as(v, y, x) + as(u, v, y)x \\ &\quad + as(u, vy, x) - as(u, v, yx) \end{aligned} \tag{58}$$

holds in any algebra. Also, in a right alternative algebra $(A, *)$ (over a ground field of characteristic different from 2), the following identity holds [37]:

$$as(u, v, v * y) = as(u, v, y) * v; \tag{59}$$

that is, by linearization and right alternativity,

$$\begin{aligned} as(u, v * y, x) &= as(u, v, x * y) - as(u, v, y) * x \\ &\quad - as(u, x, y) * v. \end{aligned} \tag{60}$$

Putting (60) in (58), we get

$$\begin{aligned} as(u * v, y, x) &= u * as(v, y, x) - as(u, x, y) * v \\ &\quad + as(u, v, [x, y]); \end{aligned} \tag{61}$$

that is, by right alternativity,

$$\begin{aligned} as(u * v, x, y) &= as(u, x, y) * v + u * as(v, x, y) \\ &\quad - as(u, v, [x, y]). \end{aligned} \tag{62}$$

Now, in (62) switching u and v and then subtracting the obtained equality from (62), one gets

$$\begin{aligned} as([u, v], x, y) &= [as(u, x, y), v] + [u, as(v, x, y)] \\ &\quad + as(v, u, [x, y]) - as(u, v, [x, y]) \end{aligned} \tag{63}$$

(observe that (63) is the untwisted form of (41)). Next, write (57) as

$$-2as(z, x, y) = (x, y, z) - [[x, y], z]. \tag{64}$$

Then multiplying (63) by -2 and using the equality above, and next proceeding as in the proof of Theorem 23, one proves the validity of (B4) for $(A, [,], (,,))$. Thus we get that $(A, [,], (,,))$ is a left Bol algebra. □

Remark 27. (i) The process of constructing left Bol algebras from right alternative algebras described in Theorem 26 above is different from the ones given in [29, 31]. In our approach here, we rely essentially on fundamental properties of right alternative algebras (see, e.g., [36, 37]) without subsidiary constructions.

(ii) If $(A, *)$ is a left alternative algebra, it is also possible to get a natural left Bol algebra structure on $(A, *)$. Indeed, one needs to consider the counterparts of (x, y, z) and (63) that looks, respectively, as

$$(x, y, z) = [[x, y], z] - as(x, y, z) \tag{65}$$

(the untwisted version of (47)) and

$$\begin{aligned} as(x, y, [u, v]) &= [as(x, y, u), v] + [u, as(x, y, v)] \\ &\quad + as([x, y], v, u) \\ &\quad - as([x, y], u, v). \end{aligned} \tag{66}$$

Next one proceeds as in Theorem 26 observing that a left alternative algebra is also Jordan-admissible (see [36], Theorem 2, for right alternative algebras).

Conflicts of Interest

The authors declare that they have no conflicts of interests.

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