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RESIDUAL-BASED A POSTERIORI ERROR ESTIMATES FOR A CONFORMING FINITE ELEMENT DISCRETIZATION OF THE NAVIER-STOKES/DARCY COUPLED PROBLEM

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Abstract

We consider in this paper, a new a posteriori residual type error estimator of a conforming mixed finite element method for the coupling of fluid flow with porous media flow on isotropic meshes. Flows are governed by the Navier-Stokes and Darcy equations, respectively, and the corresponding transmission conditions are given by mass conservation, balance of normal forces, and the Beavers-Joseph-Saffman law. The finite element subspaces consider Bernardi-Raugel and Raviart-

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Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise linear elements for a Lagrange multiplier defined on the interface. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. It is proven that the a posteriori error estimate provided in this paper is both reliable and efficient. In addition, our analysis can be extended to other finite element subspaces yielding a stable Galerkin scheme.

1. Introduction

There are many serious problems currently facing the world in which the coupling between groundwater and surface water is important. These include questions such as predicting how pollution discharges into streams, lakes, and rivers making its way into the water supply. This coupling is also important in technological applications involving filtration. In particular, for specific applications, we refer to flow in vuggy porous media appearing in petroleum extraction [1, 2], groundwater system in karst aquifers [19, 31], reservoir wellbore [3], industrial filtrations [25, 32], topology optimization [23], and blood motion in tumors and microvessels [35, 38]. We refer to the nice overview [18] and the references therein for its physical background, modelling, and standard numerical methods. One of the most popular models utilized to describe the aforementioned interaction is the Navier-Stokes/Darcy (or Stokes-Darcy) model, which consists in a set of differential equations where the Navier-Stokes (or Stokes) problem is coupled with the Darcy model through a set of coupling equations acting on a common interface given by mass conservation, balance of normal forces, and the so called Beavers-Joseph-Saffman condition. The Beavers-Joseph-Saffman condition was experimentally derived by Beavers and Joseph in [6], modified by Saffman in [37], and later mathematically justified in [28-30, 34].

A posteriori error estimators are computable quantities, expressed in terms of the discrete solution and of the data that measure the actual discrete errors without the knowledge of the exact solution. They are essential to design adaptive mesh refinement algorithms which equi-distribute the computational effort and optimize the approximation efficiency. Since the pioneering work of Babuška and Rheinboldt [5], adaptive finite element methods based on a posteriori error estimates have been extensively investigated.

A posteriori error estimations have been well-established for the coupled Stokes-Darcy problem on isotropic meshes, mainly for 2D domains [4, 13, 16, 21, 33] and recently on anisotropic meshes [26, 27]. However, only few works exist for the coupled Navier-Stokes/Darcy problem, see for instance [11, 24]. Up to the author's knowledge, the first work dealing with adaptive algorithms for the Navier-Stokes/Darcy coupling is [24], where an a posteriori error estimator for a discontinuous Galerkin approximation of this coupled problem with constant parameters is proposed. In [11], the authors have derived a reliable and efficient residual-based a posteriori error estimator for the three dimensional version of the augmented-mixed method introduced in [12]. The finite element subspaces that they have employed are piecewise constants, Raviart-Thomas elements of lowest order, continuous piecewise linear elements, and piecewise constants for the strain, Cauchy stress, velocity, and vorticity in the fluid, respectively, whereas Raviart-Thomas elements of lowest order for the velocity, piecewise constants for the pressure, and continuous piecewise linear elements for the traces, are considered in the porous medium. The authors in [17] consider the standard mixed formulation in the Navier-Stokes domain and the dual-mixed one in the Darcy region, which yields the introduction of the trace of the porous medium pressure as a suitable Lagrange multiplier. The finite element subspaces defining the discrete formulation employ Bernardi-Raugel and Raviart-Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise linear elements for the Lagrange multiplier. An a priori error analysis is performed with some numerical tests confirming the convergence rates.

In this work, we develop an a posteriori error analysis for the finite element method studied in [17]. The a posteriori error estimate is based on a suitable evaluation on the residual of the finite element solution. We further prove that our a posteriori error estimator is both reliable and efficient. These main results are summarized in Theorems 3.2 and 3.3. The difference between our paper and the reference [11] is that our analysis uses the standard mixed formulation in the Navier-Stokes domain and the dual-mixed one in the Darcy region, and another family of finite elements to approach the solution. In addition, it's independent of the finite elements employed to stabilize the scheme in [17]. Indeed, no interpolation operator for example linked to the finite elements used in this work is exploited in our a posteriori error analysis. Consequently, it can be extended to other finite element subspaces yielding a stable Galerkin scheme.

The rest of this work is organized as follows. Some preliminaries and notation are given in Section 2. In Section 3, the a posteriori error estimates are derived. The reliability analysis is carred out in Subsection 3.2, whereas in Subsection 3.3 we provide the efficiency analysis. Finally, we offer our conclusion and the further works in Section 4.

2. Preliminaries and Notation

2.1. Model problem. For simplicity of exposition we set the problem in $\mathbb{R}^2$. However, our study can be extended to the 3D case with few modifications [17, 33]. We consider the model of a flow in a bounded domain $\Omega \subset \mathbb{R}^2$, consisting of a porous medium domain Ω_D, where the flow is a Darcy flow, and an open region $\Omega_S = \Omega \setminus \overline{\Omega}_D$, where the flow is governed by the Navier-Stokes equations. The two regions are separated by an interface $\sum = \partial\Omega_D \bigcap \partial\Omega_S$. Let $\Gamma_* = \partial\Omega_* \setminus \sum$, $* \in \{S, D\}$. Each interface and boundary is assumed to be polygonal. We denote by $\mathbf{n}_S$ (resp., $\mathbf{n}_D$) the unit outward normal vector along $\partial\Omega_S$ (resp., $\partial\Omega_D$). Note that on the interface $\sum$, we have $\mathbf{n}_S = -\mathbf{n}_D$. The Figure 1 gives a schematic representation of the geometry.

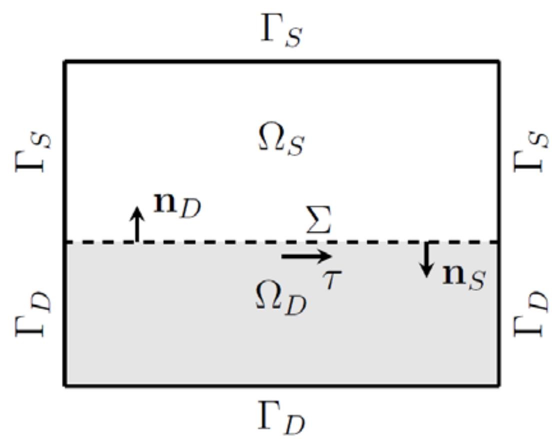

Figure 1. Domains for the 2D Navier-Stokes/Darcy model.

For any function v defined in Ω, since its restriction to Ω_S or to Ω_D could play a different mathematical roles (for instance their traces on $\sum$), we will set $v_S = v_{|\Omega_S}$ and $v_D = v_{|\Omega_D}$.

In Ω_*, $* \in \{S, D\}$ we denote by $\mathbf{u}_*$ the fluid velocity and by p_* the pressure. The motion of the fluid in Ω_S is described by the Navier-Stokes equations

$$\begin{cases} -2\mu \operatorname{div} \mathbf{e}(\mathbf{u}_S) + \nabla p_S + \rho(\mathbf{u}_S \cdot \nabla)\mathbf{u}_S = \mathbf{f}_S & \text{in} \quad \Omega_S, \\ \operatorname{div} \mathbf{u}_S = 0 & \text{in} \quad \Omega_S, \\ \mathbf{u}_S = \mathbf{0} & \text{on} \quad \Gamma_S, \end{cases} \tag{1}$$

while in the porous medium Ω_D, by Darcy's law

$$\begin{cases} \mathbf{K}^{-1}\mathbf{u}_D + \nabla p_D = \mathbf{f}_D & \text{in} \quad \Omega_D, \\ \operatorname{div} \mathbf{u}_D = 0 & \text{in} \quad \Omega_D, \\ \mathbf{u}_D \cdot \mathbf{n}_D = 0 & \text{on} \quad \Gamma_D. \end{cases} \tag{2}$$

Here, $\mu > 0$ is the dynamic viscosity of the fluid, ρ is its density, $\mathbf{f}_S$ is a given external force, $\mathbf{f}_D$ is a given external force that accounts for gravity, i.e., $\mathbf{f}_D = \rho\mathbf{g}$, where $\mathbf{g}$ is the gravity acceleration, div is the usual divergence operator and $\mathbf{e}$ is the strain rate tensor defined by:

$$\mathbf{e}(\psi)_{ij} := \frac{1}{2}\left(\frac{\partial \psi_i}{\partial x_j} + \frac{\partial \psi_j}{\partial x_i}\right), \quad 1 \leqslant i,\ j \leqslant 2,$$

and $\mathbf{K} \in [L^\infty(\Omega_D)]^{2\times 2}$ a symmetric and uniformly positive definite tensor in Ω_D representing the rock permeability κ of the porous medium divided by the dynamic viscosity μ of the fluid. Throughout the paper, we assume that there exits $C > 0$ such that

$$\xi \cdot \mathbf{K} \cdot \xi \geq C\|\xi\|^2_{\mathbb{R}^2},$$

for almost all $x \in \Omega_D$, and for all $\xi \in \mathbb{R}^2$.

Finally, we consider the following interface conditions on $\sum$:

$$\mathbf{u}_S \cdot \mathbf{n}_S + \mathbf{u}_D \cdot \mathbf{n}_D = 0, \tag{3}$$

$$p_S - 2\mu\mathbf{n}_S \cdot \mathbf{e}(\mathbf{u}_S) \cdot \mathbf{n}_S = p_D, \tag{4}$$

$$\frac{\sqrt{\tau \cdot \kappa \cdot \tau}}{\alpha_d \mu} \mathbf{n}_S \cdot \mathbf{e}(\mathbf{u}_S) \cdot \tau = -\mathbf{u}_S \cdot \tau, \tag{5}$$

where α_d is a dimensionless constant which depends only on the geometrical characteristics of the porous medium. Here, Equation (3) represents mass conservation, Equation (4) the balance of normal forces, and Equation (5) the Beavers-Joseph-Saffman conditions.

Equations (1) to (5) consist of the model of the coupled Navier-Stokes and Darcy flows problem that we will study below.

2.2. The variational formulation. In this subsection, we introduce the weak formulation derived in ([17], Subsection 2.2) for the coupled problem given by (1) to (5). To this end, let us first introduce further notations and definitions. In what follows, given $* \in \{S, D\}$, $u, v \in L^2(\Omega_*)$, $\mathbf{u}, \mathbf{v} \in [L^2(\Omega_*)]^2$, and $\mathbf{M}, \mathbf{N} \in [L^2(\Omega_*)]^{2\times 2}$, we set

$$(u, v)_* := \int_{\Omega_*} uv, \quad (\mathbf{u}, \mathbf{v})_* := \int_{\Omega_*} \mathbf{u}.\, \mathbf{v}, \quad \text{and} \quad (\mathbf{M}, \mathbf{N})_* := \int_{\Omega_*} \mathbf{M} : \mathbf{N},$$

where, given two arbitrary tensors **M** and **N**,

$$\mathbf{M} : \mathbf{N} := \mathrm{tr}(\mathbf{M}^t \mathbf{N}) = \sum_{i,j=1}^{2} M_{ij} N_{ij},$$

where the superscript t denotes transposition.

We use the standard terminology for Lebesgue and Sobolev spaces. In addition, if $\mathcal{O}$ is a domain, given and $r \in \mathbb{R}$ and $p \in [1, \infty[$, we define $\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2$ and $\mathbf{L}^p(\mathcal{O}) := [L^p(\mathcal{O})]^2$. For $r = 0$, we write $\mathbf{L}^2(\mathcal{O})$ and $L^2(\Gamma)$ instead of $\mathbf{H}^0(\mathcal{O})$ and $H^0(\Gamma)$, respectively, where Γ is a closed Lipschitz curve. The corresponding norms are denoted by $\|\cdot\|_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$ and $\mathbf{H}^r(\mathcal{O})$), $\|\cdot\|_{r,\Gamma}$ (for $H^r(\Gamma)$) and $\|\cdot\|_{L^p(\mathcal{O})}$ (if $p \neq 2$). Also, the Hilbert space

$$\mathbf{H}(\mathrm{div}; \mathcal{O}) := \{\mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \mathrm{div}\, \mathbf{w} \in L^2(\mathcal{O})\},$$

with norm $\|\cdot\|_{\mathrm{div},\mathcal{O}}$, is standard in the realm of mixed problems (see, e.g., [8]).

On the other hand, the symbol for the $L^2(\Gamma)$ inner product

$$\langle \xi, \lambda \rangle_\Gamma := \int_\Gamma \xi\lambda, \quad \forall \xi, \lambda \in L^2(\Gamma),$$

will also be employed for their respective extension as the duality product $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. In addition, given two Hilbert spaces H_1 and H_2, the product space $H_1 \times H_2$ will be endowed with the norm $\|\cdot\|_{H^1 \times H^1} = \|\cdot\|_{H_1} + \|\cdot\|_{H_2}$. Hereafter, given a nonnegative integer k and a subset S of $\mathbb{R}^2$, $\mathbb{P}^l(S)$ stands for the space of polynomials defined on S of degree $\leq l$. Finally, we employed $\mathbf{0}$ as a generic null vector.

The unknowns in the variational formulation of the Navier-Stokes/Darcy coupled problem and the corresponding spaces will be: $\mathbf{u}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, $p_S \in L^2(\Omega_S)$, $\mathbf{u}_D \in \mathbf{H}_{\Gamma_D}(\text{div}; \Omega_D)$, $p_D \in L^2(\Omega_D)$, where

$$\mathbf{H}^1_{\Gamma_S}(\Omega_S) := \{\mathbf{v} \in \mathbf{H}^1(\Omega_S): \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_S\},$$

$$\mathbf{H}_{\Gamma_D}(\text{div}; \Omega_D) := \{\mathbf{v} \in \mathbf{H}(\text{div}; \Omega_D): \quad \mathbf{v} \cdot \mathbf{n}_D = 0 \text{ on } \Gamma_D\}.$$

In addition, analogously to [22] we need to define a further unknown on the coupling boundary:

$$\lambda := p_D \in H^{1/2}(\textstyle\sum).$$

Note that, in principle, the space for p_D does not allow enough regularity for the trace λ to exist. However, the solution of Darcy equations has the pressure in $H^1(\Omega_D)$.

Next, for the derivation of the weak formulation of (1)-(5), we define the space

$$L^2_0(\Omega) := \left\{q \in L^2(\Omega): \int_\Omega q = 0\right\},$$

and we group the unkowns and spaces as follows:

$$\mathbf{u} := (\mathbf{u}_S, \mathbf{u}_D) \in \mathbf{H} := \mathbf{H}^1_{\Gamma_S}(\Omega_S) \times \mathbf{H}_{\Gamma_D}(\text{div}; \Omega_D);$$

$$(p, \lambda) \in \mathbf{Q} := L^2_0(\Omega) \times H^{1/2}(\textstyle\sum),$$

where $p := p_S \chi_{\Omega_S} + p_D \chi_{\Omega_D}$, with $\chi_{\Omega *}$ being the characteristic function for $* \in \{S, D\}$.

The weak formulation of the coupled problem (1)-(5) can be stated as follows [17]: Find $(\mathbf{u}, \psi) = ((\mathbf{u}_S, \mathbf{u}_D), (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$, such that

$$\begin{cases} \mathbf{a}(\mathbf{u}_S; \mathbf{u}, \mathbf{v}) + \mathbf{b}(\mathbf{v}, (p, \lambda)) = \mathbf{F}(\mathbf{v}), & \forall \mathbf{v} := (\mathbf{v}_S, \mathbf{v}_D) \in \mathbf{H}, \\ \mathbf{b}(\mathbf{u}, (q, \xi)) = 0, & \forall (q, \xi) \in \mathbf{Q}, \end{cases} \tag{6}$$

where $\mathbf{a} : \mathbf{H}^1_{\Gamma_S}(\Omega_S) \times (\mathbf{H} \times \mathbf{H}) \to \mathbb{R}$ and $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathbb{R}$ are the forms defined by

$$\mathbf{a}(\mathbf{w}_S;\, \mathbf{u},\, \mathbf{v}) := A_S(\mathbf{u}_S,\, \mathbf{v}_S) + O_S(\mathbf{w}_S;\, \mathbf{u}_S,\, \mathbf{v}_S) + A_D(\mathbf{u}_D,\, \mathbf{v}_D),$$

$$\mathbf{b}(\mathbf{v},\, (q,\, \xi)) := -(q,\, \operatorname{div} \mathbf{v}_S)_S - (q,\, \operatorname{div} \mathbf{v}_D)_D + \langle \mathbf{v}_S \cdot \mathbf{n}_S + \mathbf{v}_D \cdot \mathbf{n}_D,\, \xi \rangle_{\sum},$$

with

$$A_S(\mathbf{u}_S,\, \mathbf{v}_S) := 2\mu(\mathbf{e}(\mathbf{u}_S),\, \mathbf{e}(\mathbf{v}_S))_S + \left\langle \frac{\alpha_d \mu}{\sqrt{\tau \cdot \kappa \cdot \tau}} \mathbf{u}_S \cdot \tau,\, \mathbf{v}_S \cdot \tau \right\rangle_{\sum},$$

$$O_S(\mathbf{w}_S;\, \mathbf{u}_S,\, \mathbf{v}_S) := \rho((\mathbf{w}_S \cdot \nabla)\mathbf{u}_S,\, \mathbf{v}_S)_S,$$

$$A_D(\mathbf{u}_D,\, \mathbf{v}_D) := (\mathbf{K}^{-1}\mathbf{u}_D,\, \mathbf{v}_D)_D,$$

and $\mathbf{F}(\mathbf{v})$ is the linear functional $\mathbf{F} : \mathbf{H} \to \mathbb{R}$ defined as

$$\mathbf{F}(\mathbf{v}) = (\mathbf{f}_S,\, \mathbf{v}_S)_S + (\mathbf{f}_D,\, \mathbf{v}_D)_D, \quad \forall \mathbf{v} := (\mathbf{v}_S,\, \mathbf{v}_D) \in \mathbf{H}.$$

We define the bilinear form a_1 and the nonlinear form a_2 by :

$$[a_1(\mathbf{u}),\, \mathbf{v}] := A_S(\mathbf{u}_S,\, \mathbf{v}_S) + A_D(\mathbf{u}_D,\, \mathbf{v}_D), \tag{7}$$

$$[a_2(\mathbf{u}_S)(\mathbf{u}),\, \mathbf{v}] := O_S(\mathbf{u}_S;\, \mathbf{u}_S,\, \mathbf{v}_S), \tag{8}$$

and we set, for $\mathbf{u} = (\mathbf{u}_S,\, \mathbf{u}_D)$, $\mathbf{v} = (\mathbf{v}_S,\, \mathbf{v}_D)$ and $\phi = (q,\, \xi)$

$$[\mathbf{A}(\mathbf{u}_S)(\mathbf{u}_S,\, \mathbf{u}_D),\, (\mathbf{v}_S,\, \mathbf{v}_D)] := [a_1(\mathbf{u}),\, \mathbf{v}] + [a_2(\mathbf{u}_S)(\mathbf{u}),\, \mathbf{v}],$$

$$[\mathbf{B}(\mathbf{v}_S,\, \mathbf{v}_D),\, \phi] := \mathbf{b}(\mathbf{v},\, q) + \langle \mathbf{v}_S \cdot \mathbf{n}_S + \mathbf{v}_D \cdot \mathbf{n}_D,\, \xi \rangle_{\sum},$$

$$[\mathcal{F},\, \mathbf{v}] := \mathbf{F}(\mathbf{v}).$$

In all the foregoing terms, $[\cdot,\, \cdot]$ denotes the duality pairing induced by the corresponding operators. Then, the formulation (6) is equivalent to, find $(\mathbf{u},\, \psi) \in \mathbf{H} \times \mathbf{Q}$, with $\mathbf{u} = (\mathbf{u}_S,\, \mathbf{u}_D)$ and $\psi = (p,\, \lambda)$ such that:

$$\begin{cases} [\mathbf{A}(\mathbf{u}_S)(\mathbf{u}),\, \mathbf{v}] + [\mathbf{B}(\mathbf{v}),\, \psi] & = & [\mathcal{F},\, \mathbf{v}], & \forall \mathbf{v} \in \mathbf{H}, \\ [\mathbf{B}(\mathbf{u}),\, \phi] & = & 0, & \forall \phi \in \mathbf{Q}. \end{cases} \tag{9}$$

This problem has a unique solution as proved in ([17], Subsection 2.2).

Theorem 2.1 ([17], Subsection 2.2, Theorem 2)**.** *Assume that* $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ *and* $\mathbf{f}_D \in \mathbf{L}^2(\Omega_D)$ *satisfy the conditions* (36) *and* (43) *of the paper* [17]. *Then, there exists a unique solution* $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ *of* (9). *In addition, there exists a constant* $C > 0$, *independent of the solution, such that*

$$\|(\mathbf{u}, (p, \lambda))\|_{\mathbf{H}\times\mathbf{Q}} \le C(\|\mathbf{f}_S\|_{0,\Omega_S} + \|\mathbf{f}_D\|_{0,\Omega_D}).$$

2.3. Finite element discretization. Let $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ be respective triangulations of the domains Ω_S and Ω_D formed by shape-regular triangles of diameter h_T and denote by h_S and h_D their corresponding mesh sizes. Assume that they match on $\sum$ so that $\mathcal{T}_h := \mathcal{T}_h^S \cup \mathcal{T}_h^D$ is a triangulation of $\Omega := \Omega_S \cup \sum \cup \Omega_D$. Hereafter $h := \max\{h_S, h_D\}$.

For each $T \in \mathcal{T}_h^D$, we consider the local Raviart-Thomas space of the lowest order [36]:

$$RT_0(T) := \operatorname{span}\{(1, 0), (0, 1), (x_1, x_2)\},$$

where (x_1, x_2) is a generic vector in $\mathbb{R}^2$.

In addition, for each $T \in \mathcal{T}_h^S$, we denote by $BR(T)$ the local Bernardi-Raugel space (see [7]):

$$BR(T) := [\mathbb{P}^1(T)]^2 \oplus \operatorname{span}\{\eta_1\eta_2\mathbf{n}_3,\, \eta_1\eta_3\mathbf{n}_2,\, \eta_2\eta_3\mathbf{n}_1\},$$

where $\{\eta_1, \eta_2, \eta_3\}$ are the baricentric coordinates of T, and $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ are the unit outward normals to opposite sides of the corresponding vertices of T. Hence, we define the following finite element subspaces:

$$\mathbf{H}_h(\Omega_S) := \{\mathbf{v} \in \mathbf{H}^1(\Omega_S) : \mathbf{v}_{|T} \in BR(T),\ \forall T \in \mathcal{T}_h^S\},$$

$$\mathbf{H}_h(\Omega_D) := \{\mathbf{v} \in \mathbf{H}(\mathrm{div};\ \Omega_D) : \mathbf{v}_{|T} \in RT_o(T),\ \forall T \in \mathcal{T}_h^D\},$$

$$L_h(\Omega) := \{q \in L^2(\Omega) : q_{|T} \in \mathbb{P}_0(T),\ \forall T \in \mathcal{T}_h\}.$$

The finite element subspaces for the velocities and pressure are, respectively,

$$\mathbf{H}_{h,\Gamma_S}(\Omega_S) := \mathbf{H}_h(\Omega_S) \cap \mathbf{H}^1_{\Gamma_S}(\Omega_S),$$

$$\mathbf{H}_{h,\Gamma_D}(\Omega_D) := \mathbf{H}_h(\Omega_D) \cap \mathbf{H}_{\Gamma_D}(\mathrm{div};\ \Omega_D),$$

$$L_{h,0}(\Omega) := L_h(\Omega) \cap L^2_0(\Omega).$$

In turn, in order to define the discrete spaces for the unknowns on the interface $\sum$, we denote by $\sum_h$ the partition of $\sum$ inherited from $\mathcal{T}_h^S$ (or $\mathcal{T}_h^D$) and we assume, without loss of generality, that the number of edges of $\sum_h$ is even. Then, we let $\sum_{2h}$ be the partition of $\sum$ arising by joining pairs of adjacent edges of $\sum_h$. Note that since $\sum_h$ is inherited from the interior triangulations, it is automatically of bounded variation (i.e., the ratio of lengths of adjacent edges is bounded) and, therefore, so is $\sum_{2h}$. If the number of edges of $\sum_h$ is odd, we simply reduce it to the even case by joining any pair of two adjacent elements, and then construct $\sum_{2h}$ from this reduced partition. Then, we define the following finite element subspace for $\lambda \in H^{1/2}(\sum)$:

$$\Lambda_h(\textstyle\sum) := \{\xi_h \in C^0(\textstyle\sum) : \xi_{h|E} \in \mathbb{P}^1(E),\ \forall E \in \textstyle\sum_{2h}\}.$$

In this way, grouping the unknowns and spaces as follows:

$$\mathbf{u}_h := (\mathbf{u}_{h,S},\ \mathbf{u}_{h,D}) \in \mathbf{H}_{h,\Gamma_S}(\Omega_S) \times \mathbf{H}_{h,\Gamma_D}(\Omega_D);$$

$$(p_h, \lambda_h) \in \mathbf{Q}_h := L_{h,0}(\Omega) \times \Lambda_h(\sum),$$

where $p_h := p_{h,S}\chi_{\Omega_S} + p_{h,D}\chi_{\Omega_D}$, the Galerkin approximation of (6) reads: Find $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times Q_h$ such that,

$$\begin{cases} \mathbf{a}_h(\mathbf{u}_{h,S}; \mathbf{u}_h, \mathbf{v}) + \mathbf{b}(\mathbf{v}, (p_h, \lambda_h)) & = & \mathbf{F}(\mathbf{v}), \ \forall \mathbf{v} := (\mathbf{v}_S, \mathbf{v}_D) \in \mathbf{H}_h, \\ \mathbf{b}(\mathbf{u}_h, (q, \xi)) & = & 0, \quad \forall (q, \xi) \in \mathbf{Q}_h. \end{cases} \tag{10}$$

Here $\mathbf{a}_h : \mathbf{H}_{h,\Gamma_S}(\Omega_S) \times (\mathbf{H}_h \times \mathbf{H}_h) \to \mathbb{R}$ is the discrete version of $\mathbf{a}$ defined by

$$\mathbf{a}_h(\mathbf{w}_S; \mathbf{u}, \mathbf{v}) = \mathbf{a}(\mathbf{w}_S; \mathbf{u}, \mathbf{v}) + \mathbf{J}_S(\mathbf{w}_S; \mathbf{u}_S, \mathbf{v}_S), \tag{11}$$

for all $\mathbf{u}_S, \mathbf{v}_S, \mathbf{w}_S \in \mathbf{H}_h(\Omega_S)$, and where $\mathbf{J}_S(\mathbf{w}_S; \mathbf{u}_S, \mathbf{v}_S)$ is defined by:

$$\mathbf{J}_S(\mathbf{w}_S; \mathbf{u}_S, \mathbf{v}_S) := \frac{\rho}{2}(\mathbf{u}_S \operatorname{div} \mathbf{w}_S, \mathbf{v}_S)_S.$$

As before, we set, for $\mathbf{u}_h = (\mathbf{u}_{h,S}, \mathbf{u}_{h,D})$, $\mathbf{v}_h = (\mathbf{v}_{h,S}, \mathbf{v}_{h,D})$ and $\phi_h = (q_h, \xi_h)$

$$[\mathbf{A}_h(\mathbf{u}_{h,S})(\mathbf{u}_{h,S}, \mathbf{u}_{h,D}), (\mathbf{v}_{h,S}, \mathbf{v}_{h,D})] := [a_1(\mathbf{u}_h), \mathbf{v}_h] + [a_2^h(\mathbf{u}_{h,S})(\mathbf{u}_h), \mathbf{v}_h],$$

with,

$$[a_2^h(\mathbf{u}_{h,S})(\mathbf{u}_h), \mathbf{v}_h] := [a_2(\mathbf{u}_{h,S})(\mathbf{u}_h), \mathbf{v}_h] + \mathbf{J}_S(\mathbf{u}_{h,S}; \mathbf{u}_{h,S}, \mathbf{v}_{h,S}).$$

Thus, the formulation (10) is equivalent to, find $(\mathbf{u}_h, \psi_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, with $\mathbf{u}_h = (\mathbf{u}_{h,S}, \mathbf{u}_{h,D})$ and $\psi_h = (p_h, \lambda_h)$ such that:

$$\begin{cases} [\mathbf{A}_h(\mathbf{u}_{h,S})(\mathbf{u}_h), \mathbf{v}] + [\mathbf{B}(\mathbf{v}_h), \psi_h] & = [\mathcal{F}, \mathbf{v}_h], \ \forall \mathbf{v}_h \in \mathbf{H}_h, \\ [\mathbf{B}(\mathbf{u}_h), \phi_h] & = 0, \ \forall \phi_h \in \mathbf{Q}_h. \end{cases} \tag{12}$$

Theorem 2.2 (See [17], Subsection 3.2, Theorem 4 and Theorem 6)**.** *Assume that* $\boldsymbol{f}_S \in \boldsymbol{L}^2(\Omega_S)$ *and* $\mathbf{f}_D \in \mathbf{L}^2(\Omega_D)$ *satisfy the conditions* (71), (78), (82), *and* (86) *of the reference* [17]. *Then, there exists a unique* *solution* $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *to problem* (12) *and if the solution* $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ *of the continuous problem* (9) *is smooth enough, then we have*:

$$\|(\mathbf{u}, (p, \lambda)) - (\mathbf{u}_h, (p_h, \lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \lesssim h(\|\mathbf{u}_S\|_{2, \Omega_S} + \|\mathbf{u}_D\|_{1, \Omega_D} + \|\operatorname{div} \mathbf{u}_D\|_{1, \Omega_D}$$
$$+ \|p\|_{1, \Omega} + \|\lambda\|_{3/2, \sum}).$$

Here and below, in order to avoid excessive use of constants, the abbreviation $x \lesssim y$ stand for $x \leq cy$, with c a positive constant independent of x, y and $\mathcal{T}_h$.

For $\mathbf{v} = (\mathbf{v}_S, \mathbf{v}_D) \in \mathbf{H}_h$ and for $(q, \xi) \in \mathbf{Q}_h$, we can subtract (6) to (10) to obtain the Galerkin orthogonality relation:

$$A_s(\mathbf{e}_{\mathbf{u}_S}, \mathbf{v}_S) + A_D(\mathbf{e}_{\mathbf{u}_D}, \mathbf{v}_D) + O_S^h(\mathbf{u}_S; \mathbf{u}_S, \mathbf{v}_S) - O_S^h(\mathbf{u}_{h,S}; \mathbf{u}_{h,S}, \mathbf{v}_S)$$
$$+ \mathbf{b}(e_p, e_y) = 0,$$
$$\mathbf{b}((\mathbf{e}_{\mathbf{u}_S, \mathbf{e}_{\mathbf{u}_D}}), (q, \xi)) = 0,$$

where here and below, the errors in the velocity, in the pressure and in the Lagrange multiplier are respectively defined by

$$\mathbf{e}_{\mathbf{u}_*} := \mathbf{u}_* - \mathbf{u}_{h,*}; \quad e_p = p - p_h \text{ and } e_\lambda = \lambda - \lambda_h, * \in \{S, D\}. \tag{13}$$

We end this section with some notation again. For each $T \in \mathcal{T}_h$, we denoted by $\mathcal{E}(T)$ (resp., $\mathcal{N}(T)$) the set of its edges (resp., vertices) and set $\mathcal{E}_h = \bigcup_{T \in \mathcal{T}_h} \mathcal{E}(T)$, $\mathcal{N}_h = \bigcup_{T \in \mathcal{T}_h} \mathcal{N}(T)$. For $\mathcal{A} \subset \overline{\Omega}$, we define

$$\mathcal{E}_h(\mathcal{A}) := \{E \in \mathcal{E}_h : E \subset \mathcal{A}\} \text{ and } \mathcal{N}_h(\mathcal{A}) := \{x \in \mathcal{N}_h, x \in \mathcal{A}\}.$$

With every edge $E \in \mathcal{E}_h$, we introduce the outer normal vector by $\mathbf{n} = (n_x, n_y)^\top$. Furthermore, for each face E, we fix one of the two normal vectors and denote it by $\mathbf{n}_E$. In addition, we introduce the tangent vector

$\tau = \mathbf{n}^{\top} := (-n_y, n_x)^{\top}$ such that it is oriented positively (with respect to T). Similarly, set $\tau_E := \mathbf{n}_E^{\top}$.

For any $E \in \mathcal{E}_h$ and any piecewise continuous function φ, we denote by $[\varphi]_E$ its jump across E in the direction of $\mathbf{n}_E$:

$$[\varphi]_E(x) := \begin{cases} \lim\limits_{t\to 0+} \varphi(x + t\mathbf{n}_E) - \lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for an interior edge/face } E, \\ -\lim\limits_{t\to 0+} \varphi(x - t\mathbf{n}_E) & \text{for a boundary edge/face } E. \end{cases}$$

Furthermore one requires local subdomains (also known as patches). As usual, let w_T be the union of all elements having a common face with T. Similarly, let w_E be the union of both elements having E as face (with appropriate modifications for a boundary face). By $w_{\mathbf{x}}$ we denote the union of all elements having $\mathbf{x}$ as node.

In the sequel, we will also make use of the following differential operator:

$$\operatorname{curl} \mathbf{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \text{ for } \mathbf{v} = (v_1, v_2).$$

3. Error Estimators

In order to solve the Navier-Stokes/Darcy coupled problem by efficient adaptive finite element methods, reliable and efficient a posteriori error analysis is important to provide appropriated indicators. In this section, we first define the local and global indicators and then the upper and lower error bounds are derived.

3.1. Residual error estimator. The general philosophy of residual error estimators is to estimate an appropriate norm of the correct residual by terms that can be evaluated easier, and that involve the data at hand. To this end denote the exact element residuals by

$$\mathbf{R}_{S,T} = \mathbf{f}_S + 2\mu \operatorname{div}(\mathbf{e}(\mathbf{u}_{h,S})) - \nabla p_{h,S} - \rho(\mathbf{u}_{h,S} \cdot \nabla)\mathbf{u}_{h,S}$$

$$- \frac{\rho}{2} \mathbf{u}_{h,S} \mathrm{div} \mathbf{u}_{h,S} \text{ in } T \in \mathcal{T}_h^S,$$

$$\mathbf{R}_{D,T} = \mathbf{f}_D - \mathbf{K}^{-1} \mathbf{u}_{h,D} - \nabla p_{h,D} \text{ in } T \in \mathcal{T}_h^D.$$

As it is common, these exact residuals are replaced by some finite-dimensional approximation called approximate element residual $\mathbf{r}_{*,T}$, $* \in \{S, D\}$

$$\mathbf{r}_{*,T} \in [\mathbb{P}^k(T)]^2 \text{ on } T \in \mathcal{T}_h^*.$$

This approximation is here achieved by projecting $\mathbf{f}_S$ on the space of piecewise constant functions in Ω_S and piecewise $\mathbb{P}^1$ functions in Ω_D, more precisely for all $T \in \mathcal{T}_h^S$, we take

$$\mathbf{f}_{T,S} = \frac{1}{|T|} \int_T \mathbf{f}(x) dx,$$

while for all $T \in \mathcal{T}_h^D$, we take $\mathbf{f}_{T,D}$ as the unique element of $[\mathbb{P}^1(T)]^2$ such that

$$\int_T \mathbf{f}_{T,D}(x) \cdot \mathbf{q}(x) dx = \int_T \mathbf{f}(x) \cdot \mathbf{q}(x) dx, \quad \forall \mathbf{q} \in [\mathbb{P}^1(T)]^2.$$

Finally, the global function $\mathbf{f}_h$ is defined by

$$\mathbf{f}_{h,*} = \mathbf{f}_{T,*} \text{ in } T, \quad \forall T \in \mathcal{T}_h^*, * \in \{S, D\}.$$

Hence

$$\mathbf{r}_{S,T} = \mathbf{f}_{T,S} + 2\mu \,\mathrm{div}\, \mathbf{e}(\mathbf{u}_{h,S}) - \nabla p_{h,S} - \rho(\mathbf{u}_{h,S} \cdot \nabla)\mathbf{u}_{h,S}$$

$$- \frac{\rho}{2} [\mathrm{div}\, (\mathbf{u}_{h,S}) \mathbf{u}_{h,S}] \text{ in } T \in \mathcal{T}_h^S,$$

$$\mathbf{r}_{D,T} = \mathbf{f}_{T,D} - \mathbf{K}^{-1} \mathbf{u}_{h,D} - \nabla p_{h,D} \text{ in } T \in \mathcal{T}_h^D.$$

Next introduce the gradient jump in normal direction by

$$\mathbf{J}_{E,\mathbf{n}_E} := \begin{cases} [(2\mu\,\mathbf{e}(\mathbf{u}_{h,S}) - p_{h,S}\mathbf{I})\cdot\mathbf{n}_E]_E & \text{if} \quad E \in \mathcal{E}_h(\Omega_S), \\ \mathbf{0} & \text{if} \quad E \in \mathcal{E}_h(\partial\Omega_S), \end{cases}$$

where $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$.

Definition 3.1 (Residual error estimator)**.** The residual error estimator is globally defined by:

$$\Theta := \left\{ \sum_{T\in\mathcal{T}_h^S} \Theta_{S,T}^2 + \sum_{T\in\mathcal{T}_h^D} \Theta_{D,T}^2 \right\}^{1/2}, \tag{14}$$

where the local error indicators $\Theta_{S,T}^2$ (with $T \in \mathcal{T}_h^S$) and $\Theta_{D,T}^2$ (with $T \in \mathcal{T}_h^D$) are given by

$$\begin{aligned} \Theta_{S,T}^2 := {} & \|\mathbf{r}_{S,T}\|_{0,T}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h(\overline{\Omega}_S)} \|\mathbf{J}_{E,\mathbf{n}_E}\|_{0,E}^2 \\ & + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h(\overline{\Sigma})} \|-p_{h,S} + p_{h,D} - 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\mathbf{n}_S\|_{0,E}^2 \\ & + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h(\overline{\Sigma})} \left\| \frac{\alpha_d\mu}{\sqrt{\tau\cdot\kappa\cdot\tau}}\,\mathbf{u}_{h,S}\cdot\tau + 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\tau \right\|_{0,E}^2 \\ & + \sum_{E\in\mathcal{E}_h(\overline{\Sigma})} \|\mathbf{u}_{h,S}\cdot\mathbf{n}_S + \mathbf{u}_{h,D}\cdot\mathbf{n}_D\|_{0,E}^2 \\ & + \|\operatorname{div}\mathbf{u}_{h,S}\|_{0,T}^2, \end{aligned} \tag{15}$$

and

$$\begin{aligned} \Theta_{D,T}^2 := {} & h_T^2(\|\boldsymbol{f}_{h,D} - \nabla p_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D}\|_{0,T}^2 + \|\operatorname{curl}(\boldsymbol{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D}\|_{0,T}^2) \\ & + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h(\Omega_D)} h_E\|[(\boldsymbol{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D})\cdot\tau_E]_E\|_{0,E}^2 \end{aligned}$$

$$+ \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h(\partial \Omega_D)} h_E \|(\boldsymbol{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D}) \cdot \tau_E\|_{0,E}^2$$

$$+ \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h(\overline{\Sigma})} h_E \|p_{h,D} - \lambda_h\|_{0,E}^2$$

$$+ \|\operatorname{div} \mathbf{u}_{h,D}\|_{0,T}^2. \tag{16}$$

Furthermore denote the local and global approximation terms by

$$\zeta_T := \begin{cases} \|\mathbf{f}_S - \mathbf{f}_{h,S}\|_{0,T,} & \forall T \in \mathcal{T}_h^S, \\ h_T(\|\mathbf{f}_D - \mathbf{f}_{h,D}\|_{0,T} + \|\operatorname{curl}(\mathbf{f}_D - \mathbf{f}_{h,D})\|_{0,T,} & \forall T \in \mathcal{T}_h^D, \end{cases}$$

$$\zeta := (\zeta_S^2 + \zeta_D^2)^{1/2}, \tag{17}$$

where

$$\zeta_S := \left(\sum_{T \in \mathcal{T}_h^S} \zeta_T^2 \right)^{1/2} \text{ and } \zeta_D := \left(\sum_{T \in \mathcal{T}_h^D} \zeta_T^2 \right)^{1/2}. \tag{18}$$

Remark 3.1. The residual character of each term on the right-hand sides of (15) and (16) is quite clear since if $(\mathbf{u}_h, (p_h, \lambda_h))$ would be the exact solution of (9), then they would vanish.

3.2. Reliability of the a posteriori error estimator. Recall the notation for the velocity error $\mathbf{e_u} = \mathbf{u} - \mathbf{u}_h$, the pressure error $e_p = p - p_h$ and the Lagrange multiplier $e_\lambda = \lambda - \lambda_h$. The a posteriori error estimator Θ is consider reliable if it satisfies

$$\|(\mathbf{e_u}, (e_p, e_\lambda))\|_{\mathbf{H} \times \mathbf{Q}} \lesssim \Theta + \zeta. \tag{19}$$

In this subsection, we shall prove this estimate. But before, we remains some analytical tools. We introduce the Clément interpolation operator $I_h^* : H^1(\Omega_*) \to H_h^1(\Omega_*)$, with

$$H_h^1(\Omega_*) := \{v \in C^0(\overline{\Omega}_*) : v_{|T} \in \mathbb{P}^1(T), \forall T \in \mathcal{T}_h^*\},$$

approximates optimally non-smooth functions by continuous piecewise linear functions. In addition, we will make use of a vector valued version of I_h^*, that is, $\mathbf{I}_h^* : H^1(\Omega_*) \to H_h^1(\Omega_*)$, which is defined componentwise by I_h^*. The following lemma establishes the local approximation properties of I_h^* (and hence of $\mathbf{I}_h^*$), for proof, see ([14], Section 3).

Lemma 3.1 (Clément operator)**.** *For each* $* \in \{S, D\}$, *there exist constants* $c_1, c_2 > 0$, *independent of h, such that for all* $v \in H^1(\Omega_*)$ *there holds*

$$\|v - I_h^* v\|_{0,T} \leq c_1 h_T \|v\|_{1,\Delta_*(T)}, \quad \forall T \in \mathcal{T}_h^*, \tag{20}$$

and

$$\|v - I_h^* v\|_{0,E} \leq c_2 h_E^{1/2} \|v\|_{1,\Delta_*(E)}, \quad \forall T \in \mathcal{T}_h^*, \forall E \in \mathcal{E}_h, \tag{21}$$

where

$$\Delta_*(T) := \cup\{T' \in \mathcal{T}_h^* : T' \cap T \neq \emptyset\} \text{ and } \Delta_*(E) := \cup\{T' \in \mathcal{T}_h^* : T' \cap T \neq \emptyset\}.$$

Proceeding analogously to ([12], Subsection 2.5) (see also [11]), we first let $\mathbf{P} : \mathbf{X} := \mathbf{H} \times \mathbf{Q} \to \mathbf{X}' := \mathbf{H}' \times \mathbf{Q}'$ and $\mathbf{P}_h : \mathbf{X}_h := \mathbf{H}_h \times \mathbf{Q}_h \to \mathbf{X}'_h := \mathbf{H}'_h \times \mathbf{M}'_h$ be the nonlinear operator suggested by the left hand sides of (6) and (10) with the velocity solutions $\mathbf{u}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ and $\mathbf{u}_{S,h} \in \mathbf{H}_{h,\Gamma_S}(\Omega_S)$, that is,

$$[\mathbf{P}(\mathbf{U}), \mathbf{V}] := [(a_1 + a_2(\mathbf{u}_S))(\mathbf{u}), \mathbf{v}] + [\mathbf{B}(\mathbf{u}_S, \mathbf{u}_D), \phi] + [\mathbf{B}(\mathbf{v}_S, \mathbf{v}_D), \psi], \tag{22}$$

for all $\mathbf{U} = ((\mathbf{u}_S, \mathbf{u}_D, \psi), \mathbf{V} = ((\mathbf{v}_S, \mathbf{v}_{D)}), \phi)$, where $\psi = (p, \lambda)$, $\phi = (q, \xi)$; and

$$[\mathbf{P}_h(\mathbf{U}_h),\ \mathbf{V}_h] := [a_1 + a_2^h(\mathbf{u}_{h,S})(\mathbf{u}_{h,S},\ \mathbf{u}_{h,D})] + [\mathbf{B}(\mathbf{u}_{h,S},\ \mathbf{u}_{h,D}),\ \phi_h]$$

$$+ [\mathbf{B}(\mathbf{v}_{h,S},\ \mathbf{v}_{h,D}),\ \psi_h], \tag{23}$$

for all $\mathbf{U}_h = ((\mathbf{u}_{h,S},\ \mathbf{u}_{h,D}),\ \psi_h),\ \mathbf{V}_h = ((\mathbf{v}_{h,S},\ \mathbf{v}_{h,D}),\ \phi_h)$. Then, setting,

$\mathcal{F} := (\mathbf{F},\ \mathbf{O}) \in \mathbf{H}' \times \mathbf{Q}'$ with $\mathbf{O} \equiv 0$ on $\mathbf{H} \times \mathbf{Q}$, it is clear from (9) and (12) that $\mathbf{P}$ and $\mathbf{P}_h$ satisfy

$$[\mathbf{P}(\mathbf{U}),\ \mathbf{V}] = [\mathcal{F},\ \mathbf{V}], \quad \forall \mathbf{V} \in \mathbf{H} \times \mathbf{Q}, \tag{24}$$

and

$$[\mathbf{P}_h(\mathbf{U}_h),\ \mathbf{V}_h] = [\mathcal{F},\ \mathbf{V}_h], \quad \forall \mathbf{V}_h \in \mathbf{H}_h \times \mathbf{Q}_h, \tag{25}$$

respectively. In addition, we find, as explained in ([12], Subsection 5.2), that a_1 has hemi-continuous first order Gâteaux derivative $\mathcal{D}_{a_1} : \mathbf{X} \to \mathcal{L}(\mathbf{X},\ \mathbf{X}')$. Is this way, the Gâteaux derivative of $\mathbf{P}$ at $\mathbf{W} \in \mathbf{X}$ is obtained by replacing $[a_1(\cdot),\ \cdot\ ]$ in (22) by $\mathcal{D}_{a_1}(\mathbf{W})(.,\ .)$ (see [12], Subsection 5.2 for details), that is,

$$\mathcal{D}\mathbf{P}(\mathbf{W})(\mathbf{U},\ \mathbf{V}) := \mathcal{D}_{a_1}(\mathbf{w})(\mathbf{u},\ \mathbf{v}) + [a_2(\mathbf{u}_S)(\mathbf{u}),\ \mathbf{v}] + [\mathbf{B}(\mathbf{v}_S,\ \mathbf{v}_D),\ \psi]$$

$$+ [\mathbf{B}(\mathbf{u}_S,\ \mathbf{u}_D),\ \phi],$$

for all $\mathbf{U} = ((\mathbf{u}_S,\ \mathbf{u}_D),\ \psi),\ \mathbf{V} = ((\mathbf{v}_S,\ \mathbf{v}_D),\ \phi) \in \mathbf{H} \times \mathbf{Q}$. We deduce (see [12], Subsection 5.2) the existence of a positive constant $C_{\mathbf{P}}$, independent of $\mathbf{W}$ and the continuous and discrete solutions, such that the following global inf-sup condition holds:

$$C_{\mathbf{P}} \|\mathbf{U}\|_{\mathbf{H}\times\mathbf{Q}} \leq \sup_{\mathbf{V} \in (\mathbf{H}\times\mathbf{Q})^*} \frac{\mathcal{D}\mathbf{P}(\mathbf{W})(\mathbf{U},\ \mathbf{V})}{\|\mathbf{V}\|_{\mathbf{H}\times\mathbf{Q}}}, \quad \forall \mathbf{W} \in \mathbf{H} \times \mathbf{Q}. \tag{26}$$

We are now in position of establishing the following preliminary a posteriori error estimate.

Theorem 3.1. *The following estimation holds*:

$$\|\mathbf{U}-\mathbf{U}_h\|_{\mathbf{H}\times\mathbf{Q}} \lesssim \|R\|_{(\mathbf{H}\times\mathbf{Q})'}, \tag{27}$$

where $R : \mathbf{H}\times\mathbf{Q} \to \mathbb{R}$ *is the residual functional given by* $R(\mathbf{V}) := [\mathcal{F} - \mathbf{P}_h(\mathbf{U}_h), \mathbf{V}]$, *for all* $\mathbf{V} \in \mathbf{H}\times\mathbf{Q}$, *which satisfies*

$$R(\mathbf{V}_h) = 0, \quad \forall \mathbf{V}_h \in \mathbf{H}_h \times \mathbf{Q}_h. \tag{28}$$

Proof. The proof is similar to ([11], page 955, proof of Theorem 3.5). Further details are omitted. □

According to the upper bound (27) provided by the previous theorem, it only remains now to estimate $\|R\|_{(\mathbf{H}\times\mathbf{Q})'}$. To this end, we first observe that the functional R can be decomposed as follows:

$$R(\mathbf{V}) := R_1(\mathbf{v}_S) + R_2(\mathbf{v}_D) + R_3(q_S) + R_4(q_D) + R_5(\xi) + R_6(\mathbf{v}), \tag{29}$$

for all $\mathbf{V} = (\mathbf{v}, (q, \xi)) \in \mathbf{H}\times\mathbf{Q}$, with $\mathbf{v} = (\mathbf{v}_S, \mathbf{v}_D)$, $q = (q_S, q_D)$; and where,

$$R_1(\mathbf{v}_S) :=$$

$$\left(\mathbf{f}_S + 2\mu\,\mathrm{div}(\mathbf{e}(\mathbf{u}_{h,S})) - \nabla p_{h,S} - \rho(\mathbf{u}_{h,S}\cdot\nabla)\mathbf{u}_{h,S} - \frac{\rho}{2}\mathbf{u}_{h,S}\,\mathrm{div}\,\mathbf{u}_{h,S}, \mathbf{v}_S\right)_S$$

$$+ ([p_{h,S}\mathbf{I} - 2\mu\mathbf{e}(\mathbf{u}_{h,S}))\cdot\mathbf{n}_S], \mathbf{v}_S)_{\partial\Omega_S}$$

$$+ \langle -p_{h,S} + p_{h,D} - 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\mathbf{n}_S, \mathbf{v}_S\cdot\mathbf{n}_S\rangle_\Sigma$$

$$- \left\langle \frac{\alpha_d\mu}{\sqrt{\tau\cdot\kappa\cdot\tau}}\mathbf{u}_{h,S}\cdot\tau + 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\tau, \mathbf{v}_S\cdot\tau\right\rangle_\Sigma,$$

$$R_2(\mathbf{v}_D) := (\mathbf{f}_D - \nabla p_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D}, \mathbf{v}_D)_D,$$

$$R_3(q_S) := (q_S, \nabla\cdot\mathbf{u}_{h,S})_S,$$

$$R_4(q_D) := (q_D, \nabla\cdot\mathbf{u}_{h,D})_D,$$

$$R_5(\xi) := -\langle \mathbf{u}_{h,S}\cdot\mathbf{n}_S + \mathbf{u}_{h,D}\cdot\mathbf{n}_D, \xi\rangle_\Sigma,$$

$$R_6(\mathbf{v}_D) := \langle p_{h,D} - \lambda_h, \mathbf{v}_S \cdot \mathbf{n}_S + \mathbf{v}_D \cdot \mathbf{n}_D \rangle_\Sigma.$$

In this way, it follows that

$$\|R\|_{(\mathbf{H}\times\mathbf{Q})'} \leq \Big\{\|R_1\|_{\mathbf{H}_{\Gamma_S}(\Omega_S)'} + \|R_2\|_{\mathbf{H}(\mathrm{div},\Omega_D)'} + \|R_3\|_{L^2(\Omega_S)'} + \|R_4\|_{L^2(\Omega_D)'} + \|R_5\|_{H^{-1/2}(\Sigma)} + \|R_6\|H^{-1/2}(\Sigma)\Big\} \tag{30}$$

Hence, our next purpose is to derive suitable upper bounds for each one of the terms on the right hand side of (30). We start with the following lemma, which is a direct consequence of the Cauchy-Schwarz inequality and the trace inequality.

Lemma 3.2. *The following estimation holds*:

$$\|R_1\|_{\mathbf{H}_{\Gamma_S}(\Omega_S)'} \lesssim \Bigg\{\sum_{T\in\mathcal{T}_h^S}\Bigg(\|\mathbf{r}_{S,T}\|_{0,T}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h(\overline{\Omega}_S)} \|\mathbf{J}_{E,\mathbf{n}_E}\|_{0,E}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h(\overline{\Sigma})} \|-p_{h,S} + p_{h,D} - 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\mathbf{n}_S\|_{0,E}^2 + \sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h(\overline{\Sigma})} \left\|\frac{\alpha_d\mu}{\sqrt{\tau\cdot\kappa\cdot\tau}}\mathbf{u}_{h,S}\cdot\tau + 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\tau\right\|_{0,E}^2\Bigg)\Bigg\}^{1/2} + \zeta_S.$$

In addition, *there holds*

$$\|R_3\|_{L^2(\Omega_S)'} \lesssim \left\{\sum_{T\in\mathcal{T}_h^S}\|\mathrm{div}\,\mathbf{u}_{h,S}\|_{0,T}^2\right\}^{1/2},$$

$$\|R_4\|_{L^2(\Omega_D)'} \lesssim \left\{\sum_{T\in\mathcal{T}_h^D}\|\mathrm{div}\,\mathbf{u}_{h,D}\|_{0,T}^2\right\}^{1/2},$$

$$\|R_5\|_{H^{-1/2}(\Sigma)} \lesssim \left\{ \sum_{E \in \mathcal{E}_h(\overline{\Sigma})} \|\mathbf{u}_{h,S} \cdot \mathbf{n}_S + \mathbf{u}_{h,D} \cdot \mathbf{n}_D\|_{0,E}^2 \right\}^{1/2}.$$

Next, we derive the upper error bound for R_2 and R_6. We have the following lemma:

Lemma 3.3. *There holds*:

$$\|R_2\|_{\mathbf{H}(\mathrm{div};\Omega_D)'} \lesssim \left\{ \sum_{T \in \mathcal{T}_h^D} h_T^2 \left(\|\boldsymbol{f}_{h,D} - \nabla p_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D}\|_{0,T}^2 + \|\mathrm{curl}(\boldsymbol{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D})\|_{0,T}^2 \right) \right.$$

$$+ \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h(\Omega_D)} h_E \| [(\boldsymbol{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D}) \cdot \tau_E]_E \|_{0,E}^2$$

$$\left. + \sum_{E \in \mathcal{E}(T) \cap \mathcal{E}_h(\partial\Omega_D)} h_E \|(\boldsymbol{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D}) \cdot \tau_E\|_{0,E}^2 \right\}^{1/2}$$

$$+ \zeta_D, \tag{31}$$

and

$$\|R_6\|_{H^{-1/2}(\Sigma)} \lesssim \left\{ \sum_{E \in \mathcal{E}_h(\overline{\Sigma})} h_E \|p_{h,D} - \lambda_h\|_{0,E}^2 \right\}^{1/2}. \tag{32}$$

Proof. The estimate (32) follows directly from ([11], page 953, Lemma 3.4). Our next goal is to derive the upper bound for R_2, for which, given $\mathbf{v}_D \in \mathbf{H}(\mathrm{div}\ \Omega_D)$, we consider its Helmholtz decomposition provided in ([21], page 1882, Lemma 3.3). More precisely, there is $C_D > 0$ such that each $\mathbf{v}_D \in \mathbf{H}(\mathrm{div}\ \Omega_D)$ can be decomposed as $\mathbf{v}_D = \mathbf{w}_D + \text{curl}\,\beta_D$, where $\mathbf{w}_D \in \mathbf{H}^1(\Omega_D)$ and $\beta_D \in H^1(\Omega_D)$ with $\int_{\Omega_D} \beta_D = 0$, and $\|\mathbf{w}\|_{\mathbf{H}^1(\Omega_D)} + \|\beta\|_{H^1(\Omega_D)} \leq C_D \|\mathbf{v}_D\|_{\mathbf{H}(\text{div};\, \Omega_D)}$.

Then, defining $\mathbf{v}_{h,D} := I_h^D(\mathbf{w}_D) + \text{curl}(I_h^D \beta_D) \in \mathbf{H}_h(\text{div}\ \Omega_D)$, which can be seen as a discrete Helmholtz decomposition of $\mathbf{v}_{h,D}$, and applying from (28) that $R_2(\mathbf{v}_{h,D}) = 0$, we can write

$$R_2(\mathbf{v}_D) = R_2(\mathbf{v}_D - \mathbf{v}_{h,D}) = R_2(\mathbf{w}_D - \mathbf{w}_{h,D}) + R_2(\text{curl}(\beta_D - \beta_{h,D})), \tag{33}$$

with $\mathbf{w}_{h,D} = I_h^D(\mathbf{w}_D)$ and $\beta_{h,D} = \text{curl}(I_h^D \beta_D)$. Note that $\text{curl}(\nabla p_{h,D}) = 0$. Thus, we have now by standard Green's formula in two spatial dimensions on each T, the inequality:

$$\begin{aligned} R_2(\mathbf{v}_D) \lesssim \sum_{T \in \mathcal{T}_h^D} \{ & (\mathbf{f}_D - \nabla p_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D},\, \mathbf{w}_D - \mathbf{w}_{h,D})_T \\ & - (\text{curl}(\mathbf{f}_D - \mathbf{K}^{-1}\mathbf{u}_{h,D}),\, \beta_D - \beta_{h,D})_T \\ & + ((\mathbf{f}_D - \nabla p_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D}) \cdot \tau_D,\, \beta_D - \beta_{h,D})_{\partial T} \}. \end{aligned}$$

We introduce the approximation $\mathbf{f}_{h,D}$ of $\mathbf{f}_D$, and then the estimate (31) follows by applying Cauchy-Schwarz inequality and the Clément operator estimations of Lemma 3.1. □

We have now the main result of this subsection:

Theorem 3.2. *Assume that* $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ *and* $\mathbf{f}_D \in \mathbf{L}^2(\Omega_D)$ *satisfy the conditions of Theorems* 2.1 *and* 2.2. *Let* $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ *be the exact solution and* $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the finite element solution of coupled problem Navier-Stokes/Darcy. Then, the a posteriori error estimator* Θ *satisfies* (19).

Proof. Follows directly from Lemmas 3.2 and 3.3. □

3.3. Efficiency of the a posteriori error estimator. In order to derive the local lower bounds, we proceed similarly as in [9] and [10] (see also [20]), by applying inverse inequalities, and the localization technique based on simplex-bubble and face-bubble functions. To this end, we recall some notation and introduce further preliminary results. Given $T \in \mathcal{T}_h$, and $E \in \mathcal{E}(T)$, we let b_T and b_E be the usual simplex-bubble and face-bubble functions, respectively (see (1.5) and (1.6) in [40]). In particular, b_T satisfies $b_T \in \mathbb{P}^3(T)$, $\operatorname{supp}(b_T) \subseteq T$, $b_T = 0$ sur ∂T, and $0 \le b_T \le 1$ on T. Similarly, $b_E \in \mathbb{P}^2(T)$, $\operatorname{supp}(b_E) \subseteq \omega_E := \{T' \in \mathcal{T}_h : E \in \mathcal{E}(T')\}$, $b_E = 0$ on $\partial T \setminus E$ and $0 \le b_E \le 1$ in ω_E. We also recall from [39] that, given $k \in \mathbb{N}$, there exists an extension operator $L : C(E) \to C(T)$ that satisfies $L(p) \in \mathbb{P}^k(T)$ and $L(p)_{|E} = p, \forall p \in \mathbb{P}^k(E)$. A corresponding vectorial version of L, that is, the componentwise application of L, is denoted by $\mathbf{L}$. Additional properties of b_T, b_E, and L are collected in the following lemma (see [39]):

Lemma 3.4. *Given $k \in \mathbb{N}^*$, there exist positive constants depending only on k and shape-regularity of the triangulations (minimum angle condition), such that for each simplex T and $E \in \mathcal{E}(T)$ there hold*

$$\|q\|_{0,T} \lesssim \|q b_T^{1/2}\|_{0,T} \lesssim \|q\|_{0,T}, \quad \forall q \in \mathbb{P}^k(T), \tag{34}$$

$$\|\nabla(q b_T)\|_{0,T} \lesssim h_T^{-1}\|q\|_{0,T}, \quad \forall q \in \mathbb{P}^k(T), \tag{35}$$

$$\|p\|_{0,E} \lesssim \|b_E^{1/2} p\|_{0,E} \lesssim \|p\|_{0,E}, \quad \forall p \in \mathbb{P}^k(E), \tag{36}$$

$$\|L(p)\|_{0,T} + h_E\|\nabla(L(p))\|_{0,T} \lesssim h_E^{1/2}\|p\|_{0,E}, \quad \forall p \in \mathbb{P}^k(E). \tag{37}$$

To prove local efficiency for $\omega \subset \Omega := \Omega_S \cup \sum \cup \Omega_D$, let us denote by

$$\|(\mathbf{v}, (q, \xi))\|_{h,w}^2 := \sum_{E \in \mathcal{E}_h(\overline{\omega} \cap \overline{\Omega}_S)} h_E^{-1}\left(\|\mathbf{v}\|_{1,\omega_E}^2 + \|q_S\|_{\omega_E}^2\right) + \|\mathbf{v}_D\|_{\mathbf{H}(\mathrm{div};\,\omega \cap \Omega_D)}^2$$

$$+ \|q_D\|_{L^2(\omega \cap \Omega_D)}^2 + \|\xi\|_{1/2, \Sigma \cap \overline{\omega}}^2,$$

where

$$\omega_E := \bigcup\{T' \in \mathcal{T}_h^S : E \in \mathcal{E}(T)\}. \tag{38}$$

Recall further the notation for the velocity error $\mathbf{e_u} = \mathbf{u} - \mathbf{u}_h$, the pressure error $e_p = p - p_h$ and the Lagrange multiplier $e_\lambda = \lambda - \lambda_h$. The main result of this subsection can be stated as follows:

Theorem 3.3. *Assume that* $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ *and* $\mathbf{f}_D \in \mathbf{L}^2(\Omega_D)$ *satisfy the conditions of Theorem* 2.1 *and* 2.2. *Let* $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$ *be the exact solution and* $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the finite element solution of coupled problem Navier-Stokes/Darcy. Then, the local error estimator* Θ_T *satisfies*:

$$\Theta_T \lesssim \|(\mathbf{e_u}, (e_p, e_\lambda))\|_{h, \widetilde{\omega}_T} + \sum_{T' \subset \widetilde{\omega}_T} \zeta_{T'}, \quad \forall T \in \mathcal{T}_h, \tag{39}$$

where $\widetilde{\omega}_T$ *is a finite union of neighbouring elements of T.*

Proof. To establish the lower error bound (39), we will make extensive use of the original system of equations given by (1) to (5), which is recovered from the mixed formulation (6) by choosing suitable test functions and integrating by parts backwardly the corresponding equations. Thereby, we bound each term of the residual separately.

(1) **Element residual in** Ω_S**.** Set $\mathbf{w}_T := \mathbf{r}_{S,T} b_T \in [H_0^1(T)]^2$ and consider

$(\mathbf{r}_{S,T}, \mathbf{w}_T)_T =$

$$\int_T \left(\mathbf{f}_S + 2\mu \operatorname{div}(\mathbf{e}(\mathbf{u}_{h,S})) - \nabla p_{h,S} - \rho(\mathbf{u}_{h,S} \cdot \nabla)\mathbf{u}_{h,S} - \frac{\rho}{2}\mathbf{u}_{h,S} \operatorname{div} \mathbf{u}_{h,S}\right) \cdot \mathbf{w}_T.$$

Introduce $\mathbf{f}_S$ and use the formulation (6) to get

$$\begin{aligned}\int_T \mathbf{r}_{S,T} \cdot \mathbf{w}_T = & \int_T (\mathbf{f}_S - \mathbf{f}_{S,h}) \cdot \mathbf{w}_T \\ & + \int_T (2\mu\mathbf{e}(\mathbf{u}_S) : \nabla\mathbf{w}_T) + \rho\int_T [(\mathbf{u}_S \cdot \nabla)\mathbf{u}_S] \cdot \mathbf{w}_T - p_S \operatorname{div} \mathbf{w}_T \\ & + \int_T [2\mu \operatorname{div}(\mathbf{e}(\mathbf{u}_{h,S})) - \nabla p_{h,S} - \rho(\mathbf{u}_{h,S} \cdot \nabla)\mathbf{u}_{h,S}] \cdot \mathbf{w}_T \\ & - \int_T \left[\frac{\rho}{2}\mathbf{u}_{h,S} \operatorname{div} \mathbf{u}_{h,S}\right] \cdot \mathbf{w}_T.\end{aligned}$$

Integrating by parts we get

$$\begin{aligned}\int_T \mathbf{r}_{S,T} \cdot \mathbf{w}_T = & \int_T (\mathbf{f}_S - \mathbf{f}_{S,h}) \cdot \mathbf{w}_T + 2\mu\int_T \mathbf{e}(\mathbf{e}_{\mathbf{u}_S}) : \nabla(\mathbf{w}_T) - \int_T e_p \operatorname{div} \mathbf{w}_T \\ & + \int_T \left[\rho(\mathbf{u}_S \cdot \nabla)\mathbf{u}_S - \frac{\rho}{2}\mathbf{u}_{h,S} \operatorname{div} \mathbf{u}_{h,S} - \rho(\mathbf{u}_{h,S} \cdot \nabla)\mathbf{u}_{h,S}\right] \cdot \mathbf{w}_T.\end{aligned}$$

Cauchy-Schwarz inequality implies that

$$\begin{aligned}\int_T \mathbf{r}_{S,T} \cdot \mathbf{w}_T \lesssim & \|\mathbf{f}_S - \mathbf{f}_{S,h}\|_{0,T}\|\mathbf{w}_T\|_{0,T} + (2\mu\|\mathbf{e}_{\mathbf{u}_S}\|_{1,T} + \|e_{p_S}\|_{0,T})\|\nabla\mathbf{w}_T\|_{0,T} \\ & + \left|\int_T \left[\rho(\mathbf{u}_S \cdot \nabla)\mathbf{u}_S - \frac{\rho}{2}\mathbf{u}_{h,S} \operatorname{div} \mathbf{u}_{h,S} - \rho(\mathbf{u}_{h,S} \cdot \nabla)\mathbf{u}_{h,S}\right] \cdot \mathbf{w}_T\right|.\end{aligned}$$

The inverse inequalities (34) and (35), the obvious relation $\|\mathbf{w}_T\|_{0,T} \leqslant \|\mathbf{r}_{s,T}\|_{0,T}$ and the fact that $\|\mathbf{u}_S\|_{1,\Omega_S}$ (see [17], Lemma 5 and Lemma 6) and $\|\mathbf{u}_{S,h}\|_{1,\Omega_S}$ (see [17], Lemma 12) are both bounded lead to

$$\|\mathbf{r}_{S,T}\|_{0,T} \lesssim \|\mathbf{f}_S - \mathbf{f}_{S,h}\|_{0,T} + h_T^{-1}\|\nabla\mathbf{e}_{\mathbf{u}_S}\|_{0,T} + h_T^{-1}\|e_{p_S}\|_{0,T}.$$

As $h_T^{-1} \leq h_E^{-1}$, $\forall E \in \mathcal{E}(T)$, then we deduce

$$\|\mathbf{r}_{S,T}\|_{0,T} \lesssim \|(\mathbf{e_u}, (e_p, e_\lambda))\|_{h,w_T} + \zeta_S. \tag{40}$$

(2) **Element residual in Ω_D.** Set $\mathbf{w}_T := \mathbf{r}_{D,T} b_T \in [H_0^1(T)]^2$, we use (6) and integrate by parts to obtain:

$$\begin{aligned}
\int_T \mathbf{r}_{D,T} \cdot \mathbf{w}_T &= \int_T (\mathbf{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D}) \cdot \mathbf{w}_T \\
&= \int_T (\mathbf{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D}) \cdot \mathbf{w}_T \\
&\quad + \int_T (\mathbf{K}^{-1}\mathbf{u}_D - \mathbf{f}_D) \cdot \mathbf{w}_T - p_D \operatorname{div} \mathbf{w}_T \\
&= \int_T -(\mathbf{f}_D - \mathbf{f}_{h,D}) \cdot \mathbf{w}_T + \int_T (\mathbf{K}^{-1}\mathbf{e}_{\mathbf{u}_D} \cdot \mathbf{w}_T + e_{p_D} \operatorname{div} \mathbf{w}_T).
\end{aligned}$$

As before Cauchy-Schwarz inequality and the inverse inequalities (34)-(35) lead to,

$$h_T\|\mathbf{r}_{D,T}\|_{0,T} \lesssim h_T\|\mathbf{f}_D - \mathbf{f}_{h,D}\|_{0,T} + \|\mathbf{K}^{-1}\mathbf{e}_{\mathbf{u}_D}\|_{0,T} + \|e_{p_D}\|_{0,T}.$$

Thereby,

$$h_T\|\mathbf{r}_{D,T}\|_{0,T} \lesssim \|(\mathbf{e_u}, (e_p, e_\lambda))\|_{h,w_T} + \zeta_D. \tag{41}$$

(3) **Curl element residual in Ω_D.** For $T \in \mathcal{T}_h^D$, we set $C_T = \operatorname{curl}(\mathbf{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D})$ and $w_T = C_T b_T$. Hence we notice that $\operatorname{curl}(w_T)$ belongs to $\mathbf{H}$ and is divergence free, therefore by (6) we have

$$\mathbf{a}(\mathbf{u}_D, \operatorname{curl}(w_T)) = (\mathbf{f}_D, \operatorname{curl}(w_T))_D,$$

or equivalently,

$$\int_T (\mathbf{K}^{-1}\mathbf{u}_D - \mathbf{f}_D) \cdot \operatorname{curl}(w_T) = 0. \tag{42}$$

But by Green's formula, we may write

$$\int_T C_T w_T = \int_T \operatorname{curl}(\mathbf{f}_{h,D} - \mathbf{f}_D) w_T + \int_T (\mathbf{f}_D - \mathbf{K}^{-1}\mathbf{u}_{h,D}) \cdot \operatorname{curl}(w_T),$$

and by using (42), we deduce that

$$\int_T C_T w_T = \int_T \operatorname{curl}(\mathbf{f}_{h,D} - \mathbf{f}_D) w_T + \int_T [\mathbf{K}^{-1}(\mathbf{u}_D - \mathbf{u}_{h,D})] \cdot \operatorname{curl}(w_T).$$

By Cauchy-Schwarz inequality, we obtain

$$\int_T C_T w_T \leq \|\operatorname{curl}(\mathbf{f}_{h,D} - \mathbf{f}_D)\|_{0,T} \|w_T\|_{0,T} + \|\mathbf{K}^{-1}\mathbf{e}_{\mathbf{u}_D}\|_{0,T} \|\operatorname{curl} w_T\|_{0,T}.$$

Again the inverse inequalities (34)-(35) allows to get

$$h_T \|\operatorname{curl}(\mathbf{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D})\|_{0,T} \lesssim \|\mathbf{K}^{-1}\mathbf{e}_{\mathbf{u}_D}\|_{0,T} + h_T \|\operatorname{curl}(\mathbf{f}_{h,D} - \mathbf{f}_D)\|_{0,T},$$

let

$$h_T \|\operatorname{curl}(\mathbf{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D})\|_{0,T} \lesssim \|(\mathbf{e}_{\mathbf{u}}, (e_p, e_\lambda))\|_{h,w_T} + \zeta_D. \tag{43}$$

(4) **Divergence element in** Ω_*, $* \in \{S, D\}$**.** We directly see that

$$\operatorname{div}(\mathbf{u}_* - \mathbf{u}_{h,*}) = -\operatorname{div} \mathbf{u}_{h,*}, \quad \forall * \in \{S, D\},$$

hence by Cauchy-Schwarz inequality, we conclude

$$\|\operatorname{div} \mathbf{u}_{h,*}\|_{0,T} \lesssim \|\operatorname{div} \mathbf{e}_{\mathbf{u}_*}\|_{0,T}, \quad * \in \{S, D\}. \tag{44}$$

(5) **Normal jump in** Ω_S**.** For each edge $E \in \mathcal{E}_h(\Omega_S)$, we consider $w_E = T_1 \cup T_2$. As $\mathbf{J}_{E,\mathbf{n}_E} \in [\mathbb{P}^0(E)]^2$, we set

$$\mathbf{w}_E := -\mathbf{J}_{E,\mathbf{n}_E} b_E \in [H_0^1(w_E)]^2.$$

First the weak formulation (6) yields

$$\mathbf{a}(\mathbf{u}, \mathbf{w}_E) + \mathbf{b}(\mathbf{w}_E, p) = (\mathbf{f}, \mathbf{w}_E)_{w_E},$$

that is equivalent to

$$\int_{w_E} \mathbf{f}_S \cdot \mathbf{w}_E = \int_{w_E} [2\mu \mathbf{e}(\mathbf{u}_S) - p_S \mathbf{I}] : \mathbf{e}(\mathbf{w}_E)$$

$$+ \int_{w_E} [\rho(\mathbf{u}_S \cdot \nabla)\mathbf{u}_S] \cdot \mathbf{w}_E$$

$$+ \int_{\partial\omega_E} [p_S \mathbf{I} - 2\mu \mathbf{e}(\mathbf{u}_S)]\mathbf{n}_E \cdot \mathbf{w}_E. \tag{45}$$

By elementwise partial integration, we further have

$$-\int_E \mathbf{J}_{E, \mathbf{n}_E} \cdot \mathbf{w}_E = \int_{\omega_E} (2\mu \mathbf{e}(\mathbf{u}_{h,S}) - p_{h,S}\mathbf{I}) : \mathbf{e}(\mathbf{w}_E)$$

$$- \sum_{i=1}^{2} \int_{T_i} (-2\mu \operatorname{div} \mathbf{e}(\mathbf{u}_{h,S}) + \nabla p_{h,S}) \cdot \mathbf{w}_E.$$

Hence, by previous identity (45), we get

$$-\int_E \mathbf{J}_{E, \mathbf{n}_E} \cdot \mathbf{w}_E = \sum_{i=1}^{2} \int_{T_i} [\mathbf{f}_S - (-2\mu \operatorname{div} \mathbf{e}(\mathbf{u}_{h,S}) + \nabla p_{h,S})] \cdot \mathbf{w}_E$$

$$- \int_{w_E} [2\mu \mathbf{e}(\mathbf{e}_{\mathbf{u}_{h,S}}) - e_{p_S}\mathbf{I}] : \mathbf{e}(\mathbf{w}_E)$$

$$+ \int_{w_E} [\rho(\mathbf{u}_S \cdot \nabla)\mathbf{u}_S] \cdot \mathbf{w}_E.$$

We introduce the approximation $\mathbf{f}_{S,h}$ of $\mathbf{f}_S$, use the Cauchy-Schwarz inequality, the inverse inequalities (36)-(37) and the fact that $\|\mathbf{u}_S\|_{1,\Omega_S}$ (see [17], Lemma 5 and Lemma 6) is bounded, to get

$$\|\mathbf{J}_{E,\mathbf{n}_E}\|_{0,E} \lesssim h_E^{1/2} \left(\sum_{i=1}^{2} (\|\mathbf{f}_S - \mathbf{f}_{h,S}\|_{0,T_i} + \|\mathbf{r}_{S,T_i}\|_{0,T_i}) \right)$$

$$+ h_E^{-1/2} (\|\nabla(\mathbf{e}_{\mathbf{u}_S})\|_{0,w_E} + \|e_{p_S}\|_{0,w_E}).$$

As $h_E \leq 1$, then by (40), we obtain

$$\|\mathbf{J}_{E,\mathbf{n}_E}\|_{0,E} \lesssim \|(\mathbf{e}_\mathbf{u}, (e_p, e_\lambda))\|_{h,w_E} + \zeta_S. \tag{46}$$

(6) **Interface elements on $\sum$.** To estimate the interface elements, we fix an edge E in $\sum$ and for a constant r_E fixed later on and a unit vector $\mathbf{N}$, we consider $\mathbf{w}_E = (\mathbf{w}_{E,S}, \mathbf{w}_{E,D})$ such that:

$$\mathbf{w}_E = r_E b_E \mathbf{N}. \tag{47}$$

The vector $\mathbf{w}_E$ clearly, belongs to $\mathbf{H}$. Hence the weak formulation (6) yields

$$\mathbf{a}(\mathbf{u}, \mathbf{w}_E) + \mathbf{b}(\mathbf{w}_E, p) = (\mathbf{f}, \mathbf{w}_E)_{w_E},$$

that is equivalent to

$$\int_{T_S} (2\mu \mathbf{e}(\mathbf{u}_S) : \mathbf{e}(\mathbf{w}_E) - p_S \operatorname{div} \mathbf{w}_E) + \int_{T_S} (\mathbf{K}^{-1}\mathbf{u}_D \cdot \mathbf{w}_E - p_D \operatorname{div} \mathbf{w}_E)$$

$$+ \frac{\mu\alpha_d}{\sqrt{\tau \cdot \kappa \cdot \tau}} (\mathbf{u}_S \cdot \tau, \mathbf{w}_{E,S} \cdot \tau)_E + \int_{w_E} [\rho(\mathbf{u}_S \cdot \nabla)\mathbf{u}_S] \cdot \mathbf{w}_E = (\mathbf{f}, \mathbf{w}_E)_{\omega_E}, \tag{48}$$

where T_S (resp., T_D) is the unique triangle included in $\overline{\Omega}_S$ (resp., $\overline{\Omega}_D$) having E as an edge.

On the other hand, integrating by parts in T_S and T_D yields

$$\int_{T_S} (2\mu \mathbf{e}(\mathbf{u}_{h,S}) : \mathbf{e}(\mathbf{w}_{E,S}) - p_{h,S} \operatorname{div} \mathbf{w}_{E,S})$$

$$+ \int_{T_D} (\mathbf{K}^{-1}\mathbf{u}_{h,D} \cdot \mathbf{w}_{E,D} - p_{h,D} \operatorname{div} \mathbf{w}_{E,D})$$

$$+ \frac{\mu\alpha_d}{\sqrt{\tau \cdot \kappa \cdot \tau}} (\mathbf{u}_{h,S} \cdot \tau, \mathbf{w}_{E,S} \cdot \tau)_E$$

$$= -\int_{T_S} (2\mu \operatorname{div} \mathbf{e}(\mathbf{u}_{h,S}) - \nabla p_{h,S}) \cdot \mathbf{w}_{E,S}$$

$$+ \int_{T_D} (\mathbf{K}^{-1}\mathbf{u}_{h,D} \cdot \mathbf{w}_{E,D} + \nabla p_{h,D}) \cdot \mathbf{w}_{E,D}$$

$$+\frac{\mu\alpha_d}{\sqrt{\tau\cdot\kappa\cdot\tau}}(\mathbf{u}_{h,S}\cdot\tau,\ \mathbf{w}_{E,S}\cdot\tau)_E$$

$$-\int_E([p_{h,S}]_E\mathbf{w}_{E,S}\cdot\mathbf{n}_E-2\mu[\mathbf{e}(\mathbf{u}_{h,S})\mathbf{n}_E]\cdot\mathbf{w}_{E,S}).$$

Subtracting this identity to (48) we find

$$\int_E([p_h]_E\mathbf{w}_E\cdot\mathbf{n}_E-2\mu(\mathbf{e}(\mathbf{u}_{h,S})\mathbf{n}_E\cdot\mathbf{w}_{E,S})-\frac{\mu\alpha_d}{\sqrt{\tau\cdot\kappa\cdot\tau}}(\mathbf{u}_{h,S}\cdot\tau,\ \mathbf{w}_{E,S}\cdot\tau)_E$$

$$=\int_{T_S}(2\mu\mathbf{e}(\mathbf{e}_{\mathbf{u}_S}):\mathbf{e}(\mathbf{w}_{E,S})-e_{p_S}\operatorname{div}\mathbf{w}_{E,S})$$

$$+\int_{T_D}(\mathbf{K}^{-1}\mathbf{e}_{\mathbf{u}_D}\cdot\mathbf{w}_{E,D}-e_{p_D}\operatorname{div}\mathbf{w}_{E,D})$$

$$+\frac{\mu\alpha_d}{\sqrt{\tau\cdot\kappa\cdot\tau}}(\mathbf{e}_{\mathbf{u}_S}\cdot\tau,\ \mathbf{w}_{E,s}\cdot\tau)_E-\int_{T_S}[\rho(\mathbf{u}_S\cdot\nabla)\mathbf{u}_S]\cdot\mathbf{w}_E$$

$$-\int_{T_S}(\mathbf{f}_S+2\mu\operatorname{div}\mathbf{e}(\mathbf{u}_{h,S})-\nabla p_{h,S})\cdot\mathbf{w}_{E,S}$$

$$-\int_{T_D}(\mathbf{f}_D-\mathbf{K}^{-1}\mathbf{u}_{h,D}-\nabla p_{h,D})\cdot\mathbf{w}_{E,D}$$

In that last terms introducing the element residual $\mathbf{r}_{*,T}$, $*\in\{S, D\}$, we arrive at

$$\int_E([p_{h,S}]_E\mathbf{w}_E\cdot\mathbf{n}_E-2\mu(\mathbf{e}(\mathbf{u}_{h,S})\mathbf{n}_E\cdot\mathbf{w}_{E,S})-\frac{\mu\alpha_d}{\sqrt{\tau\cdot\kappa\cdot\tau}}(\mathbf{u}_{h,S}\cdot\tau,\ \mathbf{w}_{E,S}\cdot\tau)_E$$

$$=\int_{T_S}(2\mu\mathbf{e}(\mathbf{e}_{\mathbf{u}_S}):\mathbf{e}(\mathbf{w}_{E,S})-e_{p_S}\operatorname{div}\mathbf{w}_{E,S})$$

$$+\int_{T_D}(\mathbf{K}^{-1}\mathbf{e}_{\mathbf{u}_D}\cdot\mathbf{w}_{E,D}-e_{p_D}\operatorname{div}\mathbf{w}_{E,D})$$

$$+\frac{\mu\alpha_d}{\sqrt{\tau\cdot\kappa\cdot\tau}}(\mathbf{e}_{\mathbf{u}_S}\cdot\tau,\ \mathbf{w}_{E,S}\cdot\tau)_E-\int_{T_S}[\rho(\mathbf{u}_S\cdot\nabla)\mathbf{u}_S]\cdot\mathbf{w}_E$$

$$-\int_{T_S}(\mathbf{f}_S-\mathbf{f}_{h,S}+\mathbf{r}_{S,T}).\ \mathbf{w}_{E,S}-\int_{T_D}(\mathbf{f}_D-\mathbf{f}_{h,D}+\mathbf{r}_{D,T}).\ \mathbf{w}_{E,D}. \quad (49)$$

(a) To estimate the term

$$\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h(\overline{\Sigma})} \left\| \frac{\alpha_d \mu}{\sqrt{\tau\cdot\kappa\cdot\tau}} \mathbf{u}_{h,S}\cdot\tau + 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\tau \right\|_{0,E}^2,$$

we take

$$r_E = \frac{\alpha_d \mu}{\sqrt{\tau\cdot\kappa\cdot\tau}} \mathbf{u}_{h,S}\cdot\tau + 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\tau \text{ and } \mathbf{N} = \tau.$$

With this choice, $\mathbf{w}_E\cdot\mathbf{n}_S + \mathbf{w}_E\cdot\mathbf{n}_D = 0$ on $\sum$. And thus, the identity (49) and the inverse inequality (36) yield,

$$\begin{aligned}
\|r_E\|_E^2 \lesssim{} & \int_{T_S} (2\mu\mathbf{e}(\mathbf{e}_{\mathbf{u}_S}) : \mathbf{e}(\mathbf{w}_{E,S}) - e_{p_S}\operatorname{div}\mathbf{w}_{E,S}) \\
& + \int_{T_D} (\mathbf{K}^{-1}\mathbf{e}_{\mathbf{u}_D}\cdot\mathbf{w}_{E,D} - e_{p_D}\operatorname{div}\mathbf{w}_{E,D}) \\
& + \frac{\mu\alpha_d}{\sqrt{\tau\cdot\kappa\cdot\tau}}(\mathbf{e}_{\mathbf{u}_S}\cdot\tau, \mathbf{w}_{E,S}\cdot\tau)_E - \int_{T_S} [\rho(\mathbf{u}_S\cdot\nabla)\mathbf{u}_S]\cdot\mathbf{w}_{E,S} \\
& - \int_{T_S} (\mathbf{f}_S - \mathbf{f}_{h,S} + \mathbf{r}_{S,T})\cdot\mathbf{w}_{E,S} - \int_{T_D} (\mathbf{f}_D - \mathbf{f}_{h,D} + \mathbf{r}_{D,T})\cdot\mathbf{w}_{E,D}.
\end{aligned}$$

Hence Cauchy-Schwarz inequality, the inverse inequalities (37), the upper error bound of $\|\mathbf{r}_{*,T}\|_{0,T}$ [i.e., estimates (40) and (41)], and the fact that $\|\mathbf{u}_S\|_{1,\Omega_S}$ (see [17], Lemma 5 and Lemma 6) is bounded lead to

$$\left\| \frac{\alpha_d \mu}{\sqrt{\tau\cdot\kappa\cdot\tau}} \mathbf{u}_{h,S}\cdot\tau + 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\tau \right\|_{0,E} \lesssim \|(\mathbf{e}_\mathbf{u}, (e_p, e_\lambda))\|_{h,\omega_E} + \sum_{T'\subset\omega_E} \zeta_{T'}, \tag{50}$$

with $\omega_E = T_S \cup T_D$.

(b) To estimate the term

$$\sum_{E\in\mathcal{E}(T)\cap\mathcal{E}_h(\overline{\Sigma})} \| - p_{h,S} + p_{h,D} - 2\mu\mathbf{n}_S\cdot\mathbf{e}(\mathbf{u}_{h,S})\cdot\mathbf{n}_S\|_{0,E}^2,$$

we take

$$r_E = p_{h,D} - p_{h,S} + 2\mu \mathbf{n}_s \cdot \mathbf{e}(\mathbf{u}_{h,S}) \cdot \mathbf{n}_S \text{ and } \mathbf{N} = \mathbf{n}_S.$$

As before the identity (49), the inverse inequalities (36) and (37), the upper bounds of $\|\mathbf{r}_{*,T}\|_{0,T}$, $* \in \{S, D\}$ and of (46), and the fact that $\|\mathbf{u}_S\|_{1,\Omega_S}$ (see [17], Lemma 5 and Lemma 6) is bounded lead to

$$\|p_{h,D} - p_{h,S} + 2\mu \mathbf{n}_S \cdot \mathbf{e}(\mathbf{u}_{h,S}) \cdot \mathbf{n}_S\|_{0,E} \lesssim \|(\mathbf{e}_{\mathbf{u}}, (e_p, e_\lambda))\|_{h,\omega_E} + \sum_{T' \subset \omega_E} \zeta_{T'}. \tag{51}$$

(c) For $E \in \mathcal{E}_h(\overline{\Sigma})$, the term $\sum_{E \in \mathcal{E}_h(\overline{\Sigma})} h_E \|p_{h,D} - \lambda_h\|_{0,E}^2$ is bounded as follows:

$$\sum_{E \in \mathcal{E}_h(\overline{\Sigma})} h_E \|p_{h,D} - \lambda_h\|_{0,E}^2 \lesssim \sum_{E \in \mathcal{E}_h(\overline{\Sigma})} h_E (\|\lambda - \lambda_h\|_{0,E}^2 + \|\lambda - p_{h,D}\|_{0,E}^2)$$

$$\lesssim h\|\lambda - \lambda_h\|_{1/2,\Sigma}^2 \lesssim \|(\mathbf{e}_{\mathbf{u}}, (e_p, e_\lambda))\|_{h,\omega_E}^2. \tag{52}$$

(d) Analogously to ([4], Lemma 4.7), the term

$\sum_{E \in \mathcal{E}(\overline{\Sigma})} \|\mathbf{u}_{h,S} \cdot \mathbf{n}_S + \mathbf{u}_{h,D} \cdot \mathbf{n}_D\|_{0,E}$ can be bounded by

$$\|\mathbf{u}_{h,S} \cdot \mathbf{n}_S + \mathbf{u}_{h,D} \cdot \mathbf{n}_D\|_{0,E}^2 \lesssim h_{T_S}^{-1} \|\mathbf{u}_S - \mathbf{u}_{h,S}\|_{0,T_S}^2 + h_{T_S}^2 \|\nabla(\mathbf{u}_S - \mathbf{u}_{h,S})\|_{0,T}^2$$

$$+ \|\mathbf{u}_D - \mathbf{u}_{h,D}\|_{0,T_D}^2 + h_{T_D}^2 \|\mathrm{div}(\mathbf{u}_D - \mathbf{u}_{h,D})\|_{0,T}^2,$$

where $E = \partial T_S \cap \partial T_D$. Now, as for each $* \in \{S, D\}$, $h_{T_*} \leq 1$ and $h_{T_S}^{-1} \leq h_E^{-1}$, then we have the estimate

$$\|\mathbf{u}_{h,S} \cdot \mathbf{n}_S + \mathbf{u}_{h,D} \cdot \mathbf{n}_D\|_{0,E} \lesssim \|(\mathbf{e}_{\mathbf{u}}, (e_p, e_\lambda))\|_{h,\omega_E}, \text{ with } \omega_E = T_S \cup T_D. \tag{53}$$

(7) **Tangential jump in $\overline{\Omega}_D$.** Finally, for $E \in \mathcal{E}_h(\overline{\Omega}_D)$, the terms

$$\sum_{E \in \mathcal{E}_h(\Omega_D)} h_E \| [(\mathbf{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D}) \cdot \tau_E]_E \|_{0,E}^2,$$

and $\sum_{E \in \mathcal{E}_h(\partial\Omega_D)} h_E \| (\mathbf{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D}) \cdot \tau_E \|_{0,E}^2$, respectively, are bounded analogously as in ([11], Lemma 3.16) by:

$$h_E \| [(\mathbf{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D}) \cdot \tau_E]_E \|_{0,E}^2 \lesssim \| \mathbf{u}_D - \mathbf{u}_{h,D} \|_{0,w_E}^2 \tag{54}$$

$$\lesssim \| (\mathbf{e_u}, (e_p, e_\lambda)) \|_{h,\omega_E}^2,$$

for all $E \in \mathcal{E}_h(\Omega_D)$, where the set w_E is given by

$$w_E := \bigcup \{T' \in \mathcal{T}_h^D : E \in \mathcal{E}(T')\};$$

and

$$h_E \| (\mathbf{f}_{h,D} - \mathbf{K}^{-1}\mathbf{u}_{h,D} - \nabla p_{h,D}) \cdot \tau_E \|_{0,E}^2 \lesssim \| \mathbf{u}_D - \mathbf{u}_{h,D} \|_{0,T_E}^2 \tag{55}$$

$$\lesssim \| (\mathbf{e_u}, (e_p, e_\lambda)) \|_{h,\omega_E}^2,$$

for all $E \in \mathcal{E}_h(\partial\Omega_D)$, with T_E, the triangle of $\mathcal{T}_h^D$ having E as an edge.

The estimates (40), (41), (43), (44), (46), (50), (51), (52), (53), (54), and (55) provide the desired local lower error bound of Theorem 3.3. □

4. Summary

In this paper, we have discussed a posteriori error estimates for a finite element approximation of the Navier-Stokes/Darcy system. A residual type a posteriori error estimator is provided, that is both reliable and efficient. Many issues remain to be addressed in this area, let us mention other types of a posteriori error estimators or implementation and convergence analysis of adaptive finite element methods. Further, it is well known that an internal layer appears at the interface $\sum$ as the permeability tensor degenerates, in that case anisotropic meshes have to be used in this layer (see for instance [15, 27]). Hence we intend to extend our results to such anisotropic meshes.

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