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CHARACTERIZATION OF ELECTRE I CHOICE PROCEDURES

Mustapha Balewa Sanni[1,*], Carlos Ogouyandjou[2] and Freedath Djibril Moussa[3]

Abstract. This paper discusses choice procedures that select the set of best alternatives taking into account reflexive binary relations (called pseudo-tournaments in the paper), such as those that can be obtained when constructing an outranking relation à la Electre. The paper contains interesting results which link together the second "exploitation" step in the Electre I outranking method with two choice procedures (Gocha and Getcha choice procedures also known in the literature as Schwartz set and Smith set respectively). A set of results that characterize some properties of the two outranking methods (ElectI and ElectIP choice procedures) is also presented.

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1. Introduction

One of the basic structure used in decision theory for preference modelling is composed of the pair (A, R) where R is a binary relation (preference relation) defined on a set A of alternatives (projects, candidates, sports teams, etc...). The binary relation is then used to make recommandations (selection of a subset of alternatives, ranking of all alternatives or sorting of alternatives into different categories). In some areas (social choice, economics, operational research, ...) , R is generaly complete (*i.e.* alternatives are comparable two by two) and very often antisymmetric (we say that R is a tournament). Sometimes the binary relation R is simply complete (R is then reffered to as weak tournament). More recently situations involving incomplete binary relations (some alternatives are not comparable) have been studied see [4]. In the area of Multiple Criteria Decision Making (MCDM), Roy [11] proposed an outranking method which leads to incomplete and reflexive binary relations called outranking relations (useful references regarding these methods include [13, 18]). These methods have often been criticized for their lack of theoretical foundations. To overcome this issue, Bouysssou and Pirlot [3] adopt an axiomatization approach within a general framework for conjoint measurement in order to obtain characterizations of reflexive concordance-discordance relations.

Keywords. Choice procedure, outranking method, Electre I, top cycle, Gocha, Getcha.

[1] Institut National Supérieur de Technologie Industriel (INSTI) de Lokossa, Université Nationale des Sciences, Technologies, Ingénierie et Mathématiques (UNSTIM) d'Abomey, Abomey, Benin.

[2] Institut de Mathématiques et de Sciences Physiques, Université d'Abomey-Calavi, Cotonou, Benin.

[3] Faculté des Sciences et Techniques, Université d'Abomey-Calavi, Cotonou, Benin.

*Corresponding author: musanni@yahoo.fr

The most popular and most ancient outranking approach is $ElectreI$ method where the outranking relation R is used for selecting a subset containing good alternatives after circuit reduction and, determining the kernel of a new binary relation (see [5,9] for overviews). Bouyssou [2] shows that any reflexive binary relation (pseudo tournament) on a finite set of alternatives may be obtained as the result of the construction technique of $ElectreI$.

$ElectreI$ method proceeds in two steps: the construction step in which an outranking relation (pseudo tournament) is built and the exploitation phase in which the outranking relation is used to produce a choice set. This paper is concerned with the second step and studies the mechanism by which the choice is made with the $ElectreI$ method. This problem is deeply related to the classic problems in the field of social choice theory where literature abounds with studies of appropriate choice procedures (*i.e.* a functions which map each binary relation to a non empty set of alternatives called the choice set) concerning tournaments and weak tournaments (see [8,10]). Choice procedures for incomplete binary relations have been less studied in the litterature [6,14] and a common approach used is to treat incomparability as indifference relations and to apply a particular choice procedure for weak tournaments.

The aim of this paper is two folds: first we give an axiomatization of outranking methods or more precisely of $ElectreI$ method (see here as a choice procedure) and then we propose another way of characerizing choice procedures for incomplete binary relations that differs from the common approach mentioned earlier.

The rest of the paper is organized as follows. Section 2 introduces our notation and definitions. Section 3 introduces the concept of kernels and Electre I choice procedures. Our main results are presented in Section 4. A final section concludes.

2. Notation and definitions

A binary relation R on a set A of alternatives (projects, candidates, sports teams, etc...) is a subset of $A \times A$. We mostly write aRb instead of $(a,b) \in R$. For a non empty set B of A, $R|B$ represents the restriction of R on B, *i.e.* $R|B = \{(x,y) \in B \times B / xRy\}$. We supose for pratical purposes that A is a finite set. A binary relation R on A is said to be reflexive if aRa, for all $a \in A$. It is symmetric if $aRb \implies bRa$, for all $a,b \in A$. Relation R is asymmetric if $aRb \implies \text{not}(bRa)$, for all $a,b \in A$ with $a \neq b$. It is antisymmetric if $(aRb \text{ and } bRa) \implies a = b$, for all $a,b \in A$. It is transitive if $(aRb \text{ and } bRc) \implies aRc$, for all $a,b,c \in A$. It is complete if aRb or bRa, for all $a,b \in A$. A tournament is a complete and antisymmetric relation. A weak tournament is a complete relation. A pseudo tournament is any reflexive binary relation (the relation may be complete or not). These definitions imply that tournaments and weak (pseudo) tournaments are reflexive[4].

Three other binary relations ($\alpha(R)$: strict preference relation, $\beta(R)$: indifference relation and $\gamma(R)$: incomparability relation) are defined from a pseudo tournament R as follow:

For all $a,b \in A$, $a\alpha(R)b \iff aRb$ and $\text{not}(bRa)$, $a\beta(R)b \iff aRb$ and bRa and $a\gamma(R)b \iff \text{not}(aRb)$ and $\text{not}(bRa)$. It can be noticed that $\gamma(R)$ is reflexive and symmetric, $\alpha(R)$ is asymmetric ($\alpha(R)$ is also called the asymmetric part of R) and $\gamma(R)$ is symmetric. $a\alpha(R)b$ (resp. $a\gamma(R)b$) can be interpreted as a beats or is better than (resp. a is indifferent to) b.

Example 2.1. Let R be the pseudo tournament defined on $A = \{a,b,c,d,e,f,g,h\}$ by $b\alpha(R)a$, $a\alpha(R)c$, $c\beta(R)b$, $c\alpha(R)d$, $d\alpha(R)e$, $e\alpha(R)f$, $f\alpha(R)d$, $f\beta(R)g$ and $g\alpha(R)h$. We also have xRx, $\forall x \in A$. The graph of R[5] is represented by:

$$\begin{tikzcd}
a \arrow[dr] & & e \arrow[dr] & & h \\
b \arrow[u] \arrow[r] & c \arrow[r] & d \arrow[u] & f \arrow[l] \arrow[r, dash] & g \arrow[u]
\end{tikzcd}$$

An alternative $a \in A$ is a Condorcet winner if $a\alpha(R)b$, $\forall b \in A$ (with $a \neq b$). If a Condorcet winner exists then it is unique.

[4]The classical definition supposes that a tournament is an irreflexive relation.

[5]Loops (reflexive arcs) are omitted from the graph.

A subset $\{x_1, x_2, \ldots, x_k\}$ (with $k \geq 2$) of A is a R-circuit if there exists a permutation s of $\{1, 2, \ldots, k\}$ such that $x_{s(1)} R x_{s(2)} R \ldots R x_{s(k)}$ and $x_{s(k)} R x_{s(1)}$. Also a subset $\{x_1, x_2, \ldots, x_k\}$ of A is a αR-circuit if there exists a permutation s of $\{1, 2, \ldots, k\}$ such that $x_{s(1)} \alpha(R) x_{s(2)} \alpha(R) \ldots \alpha(R) x_{s(k)}$ and $x_{s(k)} \alpha(R) x_{s(1)}$. R is acyclic (resp. αR-acyclic) if it contains no R-circuit (resp. no αR-circuit).

A subset $\{x_1, x_2, \ldots, x_k\}$ of A is a maximal R-circuit (or a weak coalition) if $k = 1$ or if $\{x_1, x_2, \ldots, x_k\}$ is a R-circuit contained in any other R-circuit. Similarly a subset $\{x_1, x_2, \ldots, x_k\}$ of A is a maximal αR-circuit (or a strong coalition) if $k = 1$ or if $\{x_1, x_2, \ldots, x_k\}$ is a αR-circuit contained in any other αR-circuit.

It is clear that two weak (or strong) coalitions have an empty intersection. We can then determine the reduction (*i.e.* the partition) of A into maximal R-circuits or maximal αR-circuits. We denote by $A_R = \{a_{1R}, a_{2R}, \ldots, a_{lR}\}$ (resp. $A_{\alpha R} = \{a_{1P}, a_{2P}, \ldots, a_{tP}\}$) $(l, t \in \mathbb{N})$ the set of maximal R-circuits (resp. maximal αR-circuits) of A. On the set $A_R = \{a_{1R}, a_{2R}, \ldots, a_{lR}\}$ (resp. $A_{\alpha R} = \{a_{1P}, a_{2P}, \ldots, a_{tP}\}$), we define the pseudo tournament R_σ (resp.$R_{(\alpha\sigma)}$) by $\forall a_{iR}, a_{jR} \in A_R$ $(a_{iR} \neq a_{jR})$, $a_{iR} R_\sigma a_{jR} \Leftrightarrow \exists x \in a_{iR}, \exists y \in a_{jR}$ such that $x \alpha(R) y$ (resp. $\forall a_{iP}, a_{jP} \in A_{\alpha R}$ $(a_{iP} \neq a_{jP})$, $a_{iP} R_{(\alpha\sigma)} a_{jP} \Leftrightarrow \exists x \in a_{iP}, \exists y \in a_{jP}$ such that $x\alpha(R)y$). Both R_σ and $R_{(\alpha\sigma)}$ are asymmetric.

$a_{iR} R_\sigma a_{jR}$ (resp. $a_{iR} R_{(\alpha\sigma)} a_{jR}$) can be interpreted as weak coalition a_{iR} beats weak coalition a_{jR} (resp. strong coalition a_{iP} beats strong coalition a_{jP}). So a weak (resp. strong) coalition beats another weak (resp. strong) coalition if and only if there exists an alternative in the first one that beats an alternative in the second[6].

We can also notice that if R is a weak tournament then $a_{iR} R_\sigma a_{jR} \Leftrightarrow \forall x \in a_{iR}, \forall y \in a_{jR}, x\alpha(R)y$.

Example 2.1 (Continued)**.** For the pseudo tournament of the example given above, we have $A_R = \{\{a, b, c\}, \{d, e, f, g\}, \{h\}\}$ and $\{a, b, c\} R_\sigma \{d, e, f, g\}, \{d, e, f, g\} R_\sigma \{h\}$. We also have $A_{\alpha R} = \{\{a\}, \{b\}, \{c\}, \{d, e, f\}, \{g\}, \{h\}\}$ and $\{b\} R_{(\alpha\sigma)} \{a\}, \{a\} R_{(\alpha\sigma)} \{c\}, \{c\} R_{(\alpha\sigma)} \{d, e, f\}, \{g\} R_{(\alpha\sigma)} \{h\}$.

The transitive closure R^* of R is the binary relation defined on A as follows: $\forall x, y \in A, xR^*y$ if and only if $\exists k \in \mathbb{N}$ with $k \geq 1, \exists x_1, x_2, \ldots, x_k \in A$, such that $\forall i \in \{1, 2, \ldots, k-1\}, x_i R x_{i+1}, x_1 = x$ et $x_k = y$. In other words xR^*y if and only if there exists at least a path of length k from x to y (we also say that y is reachable from x or x is indirectely at least as good as y). The transitive closure $\alpha(R)^*$ of $\alpha(R)$ can also be defined in the same way (we then say that y is $\alpha(R)$-reachable from x) or x indirectely beats y.

An alternative x is maximal w.r.t. R if we have $\mathrm{not}(y\alpha(R)x), \forall y \in A$. The set of maximal alternatives is denoted by $M(R)$: $M(R) = \{x \in A / \mathrm{not}(y\alpha(R)x),\ \forall y \in A\}$. $M(R)$ may be empty. If R is without αR-circuit (*i.e.* the asymmetric part of R is without circuit) then $M(R)$ is non empty.

A choice procedure is a function C that maps each pseudo tournament R to a nonempty subset $C(R)$ of A called the choice set.

If R is a tournament the choice procedure is called a tournament solution. We say that a choice procedure C is contained in a choice procedure C' if $C(R) \subseteq C'(R)$ for every pseudo tournament R defined on A (we write $C \subseteq C'$). For the pseudo tournament R_σ (resp. $R_{(\alpha\sigma)}$) defined above any weak (resp. strong) coalition of $\mathrm{C}(R_\sigma)$ (resp. $\mathrm{C}(R_{(\alpha\sigma)})$)will be qualified as winning weak (resp. strong) coalition w.r.t. to the choice procedure C.

One of the most studied tournament solutions studied in the literature is the Top cycle (TC) choice procedure [15] defined, for every tournament R on A, by $\mathrm{TC}(R) = M(R^*)$. Schwartz [15,16] defined two extensions of Top cycle choice procedure: *Gocha* and *Getcha*[7] choice procedures for weak tournaments. For every weak tournament R on A, we have, $Gocha(R) = M(\alpha(R)^*)$ and $Getcha(R) = M(R^*)$. In this paper, we adopt the same definition of *Gocha* and *Getcha* choice procedures for pseudo tournaments[8].

Definition 2.2. For R a pseudo tournament defined on A Gocha and Getcha choice procedures are defined as: $Gocha(R) = M(\alpha(R)^*)$ and $Getcha(R) = M(R^*)$.

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[6]Maximal weak (resp. strong) coalition corresponds to weak (resp. strong) connected component in a directed graph.

[7]These sets are also known as Schwartz set and Smith set respectively.

[8]For pseudo tournaments, Sanni [14] gives four extensions of the Top cycle choice procedure including the two that are considered in this paper.

Example 2.1 (Continued)**.** For the pseudo tournament R of the example given above, we have $Gocha(R) = \{b, g\}$, $Getcha(R) = \{a, b, c\}$. We also have $Gocha(R_\sigma) = Getcha(R_\sigma) = \{\{a, b, c\}\}$, $Gocha(R_{(\alpha\sigma)}) = Getcha(R_{(\alpha\sigma)}) = \{\{b\}, \{g\}\}$.

Lemma 2.3. *For every pseudo tournament R defined on A, the following results hold.*

(1) $Getcha(R) = \bigcup a_{iR}$, *with* $a_{iR} \in M(R_\sigma)$.
(2) $Gocha(R) = \bigcup a_{iP}$, *with* $a_{iP} \in M(R_{(\alpha\sigma)})$.

Proof. (1) Let show that $Getcha(R) = \bigcup a_{iR}$ where R is a pseudo tournament defined on A.
($\subseteq$): Let $x \in Getcha(R)$ and $a_{iR} \in A_R$ such that $x \in a_{iR}$. If $a_{iR} \notin M(R_\sigma)$, then $\exists a_{jR} \in A_R$ such that $a_{jR} R_\sigma a_{iR}$ with $x \notin a_{jR}$. So $\exists y \in a_{jR}$ such that yR^*x and not (xR^*y). Which contradicts $x \in Getcha(R)$.
($\supseteq$): Let $x \in a_{iR}$ with $a_{iR} \in M(R_\sigma)$and suppose $x \notin Getcha(R)$. Then $\exists x \in A$ such that yR^*x and not (xR^*y). So $\exists a_{jR} \in A_R$ such that $a_{jR} R_\sigma a_{iR}$. Since relation R_σ is asymmetric, we have not $(a_{iR} R_\sigma a_{jR})$. Which is a contradiction because $a_{iR} \in M(R_\sigma)$.

(2) The proof is similar to the previous one.

□

3. Kernel and Electre I choice procedures

We now introduce two definitions of *ElectreI* choice procedures based on the notion of kernel.

Definition 3.1. Let R be a pseudo tournament defined on A. A subset $N(R)$ of A is a R-kernel (or simply a kernel) of R if the two following conditions hold.

(1) $\forall y \notin N(R)$, $\exists x \in N(R)$ such that xRy.
(2) $\forall x, y \in N(R)$ $(x \neq y)$, we have $x\gamma(R)y$.

von Neumann and Morgestern [17] define $N(R)$ as a stable set.

A drawback of a kernel is that it may not exist for some pseudo tournaments. Sometimes the kernel is not unique. Berge [1] and Roy [12,13] show that any acyclic relation R contains a unique kernel. So when a binary relation R is acyclic, the kernel of R is a good candidate for choice sets. When the relation R contains circuits Roy and Bouyssou [13] suggest to proceed as follows:

– Identification of maximal circuits of R.
– Definition of the pseudo tournament R_σ on A_R.
– Determination of $N(R_\sigma)$, the R_σ-kernel of R_σ.

Let us notice that the reduction of R_σ into circuits is like treating alternatives in the circuit as equivalent. Doing so may lead to lost of informations from the pseudo tournament. In practice, alternatives of the kernel are analysed (robustesse analysis, weight criteria, etc…) in order to make a difference between them.

From the R_σ-kernel of A_R, we define *ElectreI* (*ElectI*) choice procedure as follow:

Definition 3.2. The *ElectI* choice procedure applied to a pseudo tournament R defined on A yields the choice set $ElectI(R) = \{a \in A : a \in a_{iR}$ with $a_{iR} \in N(R_\sigma)\}$, where $N(R_\sigma)$ represents the R_σ-kernel of A_R.

It is obvious that $ElectI(R_\sigma) = N(R_\sigma)$; so $ElectI(R) = \bigcup a_{iR}$, with $a_{iR} \in ElectI(R_\sigma)$.

Alternatives of the kernel of R (or of the R_σ-kernel of R_σ) are not comparable. We envisage the case where these alternatives are indifferent or incomparable. This leads to the concepts of P-kernel (or $R_{(\alpha\sigma)}$-kernel) and *ElectreIP* choice procedure.

Definition 3.3. Let R be a pseudo tournament defined on A. A P-kernel of R is the subset $N'(R) = N(\alpha(R))$.

So for a P-kernel $N'(R))$ the following conditions hold:

(1) $\forall y \notin N'(R)$, $\exists x \in N'(R)$ such that $x\alpha(R)y$.
(2) $\forall x, y \in N'(R)$, we have not $(x\alpha(R)y)$.

Condition 2 is equivalent to $\forall x, y \in N'(R)$ $(x \neq y)$, $x\beta(R)y$ or $x\gamma(R)y$. This condition tells that alternatives of the P-kernel are either incomparable or indifferent.

A P-kernel of R may not exist and is not always unique. Lahiri [7] studies conditions of existence and unicity of P-kernel, in the case of weak tournaments. For pseudo tournaments, Berge [1] shows that any αR-acyclic relation contains a unique P-kernel. When a binary relation contains αR-circuits, we proceed the way we did for the kernel by finding the $R_{(\alpha\sigma)}$-kernel of $R_{(\alpha\sigma)}$. $ElectreIP$ ($ElectIP$) choice procedure is then defined by:

Definition 3.4. The $ElectIP$ choice procedure applied to a pseudo tournament R defined on A yields the choice set $ElectIP(R) = \{a \in A : a \in a_{iP}\}$ with $a_{iP} \in N'(R_{(\alpha\sigma)})$, where $N'(R_{(\alpha\sigma)})$ represents the $R_{(\alpha\sigma)}$-kernel of $R_{(\alpha\sigma)}$.

It is obvious that $ElectIP(R_{(\alpha\sigma)}) = N'(R_{(\alpha\sigma)})$; so $ElectIP(R) = \bigcup a_{iP}$, with $a_{iP} \in ElectIP(R_{(\alpha\sigma)})$.

When R is an asymmetric or an antisymmetric relation, we also have $N'(R) = N(R)$, which implies that $ElectI(R) = ElectIP(R)$.

Example 2.1 (continued)**.** For the pseudo tournament given above, we have:

$A_R = \{\{a,b,c\},\{d,e,f,g\},\{h\}\}$, $\{a,b,c\}R_\sigma\{d,e,f,g\}$ and $\{d,e,f\}R_\sigma\{h\}$. So $ElectI(R) = \{\{a,b,c\}\cup\{h\}\} = \{a,b,c,h\}$, $ElectI(R_\sigma) = ElectIP(R_\sigma) = \{\{a,b,c\},\{h\}\}$.

We also have $A_{\alpha R} = \{\{a\},\{b\},\{c\},\{d,e,f\},\{g\},\{h\}\}$ and $\{b\}R_{(\alpha\sigma)}\{a\}$, $\{a\}R_{(\alpha\sigma)}\{c\}$, $\{c\}R_{(\alpha\sigma)}\{d,e,f\}$, $\{g\}R_{(\alpha\sigma)}\{h\}$. So $ElectIP(R) = \{\{b\}\cup\{c\}\cup\{g\}\} = \{b,c,g\}$, $ElectI(R_{(\alpha\sigma)}) = ElectIP(R_{(\alpha\sigma)}) = \{\{b\},\{c\},\{g\}\}$.

Lemma 3.5. *For every pseudo tournament R defined on A, the following results are true.*

(1) *$M(R_\sigma) \neq \emptyset$ and $M(R_\sigma) \subseteq N(R_\sigma)$. Therefore $\forall a_{iR} \in M(R_\sigma)$, we have $a_{iR} \subseteq ElectI(R)$.*
(2) *$M(R_{(\alpha\sigma)}) \neq \emptyset$ and $M(R_{(\alpha\sigma)}) \subseteq N'(R_{(\alpha\sigma)})$. Therefore $\forall a_{iP} \in M(R_{(\alpha\sigma)})$, we have $a_{iP} \subseteq ElectIP(R)$.*
(3) *$ElectIP(R_\sigma) = ElectI(R_\sigma)$.*
(4) *$ElectIP(R_{(\alpha\sigma)}) = ElectI(R_{(\alpha\sigma)})$.*

Proof. (1) Suppose that $M(R_\sigma) = \emptyset$. Then for $a_{1R} \in A_R$, $\exists a_{2R} \in A_R$ such that $a_{2R}R_\sigma a_{1R}$. Since R_σ is asymmetric, we have $a_{1R} \neq a_{2R}$. $a_{2R} \in A_R$, then $\exists a_{3R} \in A_R \setminus \{a_{1R}, a_{2R}\}$ such that $a_{3R}R_\sigma a_{2R}$. By continuing this reasoning and since A_R is a finite set, there exists $k \in \mathbb{N}$ such that for $t \in \{1,\ldots,k-1\}$, $a_{(t+1)R}R_\sigma a_{tR}$, $a_{(t+1)R} \in A_R \setminus \{a_{1R},\ldots,a_{tR}\}$ and for all $a_{jR} \in A_R$, we have not$(a_{jR}R_\sigma a_{kR})$: a contradiction. Now let's show that $M(R_\sigma) \subseteq N(R_\sigma)$. For $a_{iR} \notin N(R_\sigma)$, there exists $a_{jR} \in N(R_\sigma)$ such that $a_{jR}R_\sigma a_{iR}$. Since R_σ is asymmetric we also have not$(a_{iR}R_\sigma a_{jR})$. So $a_{iR} \notin M(R)$.
(2) The proof is similar to the previous one.
(3) Obvious because R_σ is asymmetric.
(4) Obvious because $R_{(\alpha\sigma)}$ is asymmetric.

□

Lemma 3.6. *If R is a pseudo tournament defined on A then every maximal R-circuit contains a maximal αR-circuit.*

Proof. Consider σ_R a maximal R-circuit of A. If $\{a_{iP}, i = 1, \ldots\}$ is the family of maximal αR-circuit of σ_R then every a_{iP} is also a maximal αR-circuit of A. Otherwise $\exists k, l \in \mathbb{N}$, a_{kP} a maximal αR-circuit of σ_R, a_{lP} a maximal αR-circuit of A such that $a_{lP} \subset a_{lP}$ (strict inclusion): this is not possible. □

4. Results

In this section, we present our results in two directions: we first consider comparaison results in which we analyze the inclusion relation between the choice procedures described above. We then analyze the axiomatic characterization of each of the choice procedures.

4.1. Comparison results

In this subsection, we study the set-theoretic relationships between the choice sets mentioned in the paper. Proposition 4.1 below indicates that both *ElectI* and *ElectIP* choice procedures coincide when the binary relation is a tournament.

Proposition 4.1. *If R is a tournament defined on A then, $ElectI(R) = ElectIP(R) = \mathrm{TC}(R)$.*

Proof. Let R be a tournament defined on A.

– $ElectI(R) = ElectIP(R)$ because for every tournament, weak and strong coalitions are identical.
– $ElectI(R) = \mathrm{TC}(R)$. Notice that for a tournament R, $\mathrm{TC}(R) = M(R^*)$ is a weak (strong) coalition. $\mathrm{TC}(R)$ is also a R_σ-Condorcet winner, so $\mathrm{TC}(R) = ElectI(R_\sigma)$, which implies $ElectI(R) = \mathrm{TC}(R)$.

□

The next proposition shows that for weak tournaments, *ElectreI* and *Getcha* choice procedures coincide. We also have $Gocha \subseteq ElectIP \subseteq ElectI$.

Proposition 4.2. *If R is a weak tournament defined on A, then we have:*

(1) $ElectI(R) = Getcha(R)$.
(2) $Gocha(R) \subseteq ElectIP(R) \subseteq ElectI(R)$.

Proof. (1) The binary relation R_σ is asymmetric and complete so the R_σ-kernel of A_R is the R_σ-Condorcet winner, which is also the set of maximal elements of R^*.

(2) – $Gocha(R) \subseteq ElectIP(R)$
Let $x \in Gocha(R)$ and suppose that $x \notin ElectIP(R)$. Then $\exists a_{iP}, a_{jP} \in A_{\alpha R}$, such that $x \in a_{jP}$ with $a_{iP} \in N(R_{(\alpha\sigma)})$, $a_{jP} \notin N(R_{(\alpha\sigma)})$ and $a_{iP} R_{(\alpha\sigma)} a_{jP}$. So $\exists z \in a_{iP}$, $\exists y \in a_{jP}$ such that zPy. Since $x, y \in a_{jP}$, we then have yP^*x. Which in turn impies zP^*x. $x \in Gocha(R)$, so, by definition of *Gocha*, we have xP^*z. We can then deduce that $a_{iP} \cup a_{jP}$ is a maximal αR-circuit, contradicting the maximality of a_{iP} and/or a_{jP}. This inclusion may be strict as shown by the pseudo tournament of Example 2.1 for which $ElectIP(R) = \{b, c, g\}$ and $Gocha(R) = \{b, g\}$.
– $ElectIP(R) \subseteq ElectI(R)$
Let $x \in ElectIP(R)$. Suppose $x \notin ElectI(R)$. Then there exists a maximal R-circuit $a'_{jR} \in A_R$ ($a'_{jR} \neq N(R_\sigma)$) such that $x \in a'_{jR}$. We also have $N(R_\sigma) R_\sigma a'_{jR}$. Now consider $\{a_{1P}, a_{2P}, \ldots, a_{tP}\}$) the decomposition of $N(R_\sigma)$ into maximal αR-circuits. This decomposition admits a maximal element, say a_{kP}, for $R_{(\alpha\sigma)}|N(R_\sigma)$. a_{kP} is also a maximal elements for $R_{(\alpha\sigma)}$. So $a_{kP} \in N(R_{(\alpha\sigma)})$. Let $a'_{jP} \in N(R_{(\alpha\sigma)})$ such $x \in a'_{jP}$ ($a'_{jP} \subseteq a_{jR}$). We have $a_{kP} R_{(\alpha\sigma)} a'_{jP}$: a contradiction since $a_{kP} \in N(R_{(\alpha\sigma)})$ and $a'_{jP} \in N(R_{(\alpha\sigma)})$.

□

Proposition 4.3 below deals with the general case of pseudo tournaments. It shows that the *Gocha* and *Getcha* choice sets are respectively contained in *ElectIP* and *ElectI* choice sets. The four other pairs of choice sets are not disjoint.

Proposition 4.3. *For a pseudo tournament R defined on A, we have the following results.*

(1) $ElectI(R) \cap ElectIP(R) \neq \emptyset$.
(2) $ElectI(R) \cap Gocha(R) \neq \emptyset$.
(3) $Getcha(R) \subseteq ElectI(R)$.
(4) $Gocha(R) \subseteq ElectIP(R)$.
(5) $Getcha(R) \cap ElectIP(R) \neq \emptyset$.
(6) $Getcha(R) \cap Gocha(R) \neq \emptyset$.

Proof. (1) Lemma 3.6 shows that $ElectIP$ and $ElectI$ choice procedures always intersect.

(2) Let a_{iR} be a maximal element of R_σ and A_{iP} the decomposition of a_{iR} into maximal αR-circuits of A. a_{iR} is contained in $ElectI(R)$ and every maximal element of $R_{(\alpha\sigma)}|A_{iP}$ (which is also a maximal element of $R_{(\alpha\sigma)}$) is contained in $Gocha(R)$.

(3) Let $x \in Getcha(R)$ and suppose that $x \notin ElectI(R)$. Then $\exists a_{iR}, a_{jR} \in A_R$, such that $x \in a_{jR}$ with $a_{iR} \in N(R_\sigma)$, $a_{jR} \notin N(R_\sigma)$ and $a_{iR} R_\sigma a_{jR}$. So $\exists z \in a_{iR}$, $\exists y \in a_{jR}$ such that zRy. Since $x, y \in a_{jR}$, we then have yR^*x, which in turn implies zR^*x. $x \in Getcha(R)$, so, by definition of $Getcha$, we have xR^*z. We can then deduce that $a_{iR} \cup a_{jR}$ is a maximal R-circuit, contradicting the maximality of a_{iR} and/or a_{jR}.

(4) Let $x \in Gocha(R)$ and suppose that $x \notin ElectIP(R)$. Then $\exists a_{iP}, a_{jP} \in A_{\alpha R}$, such that $x \in a_{jP}$ with $a_{iP} \in N(R_{(\alpha\sigma)})$, $a_{jP} \notin N(R_{(\alpha\sigma)})$ and $a_{iP} R_{(\alpha\sigma)} a_{jP}$. So $\exists z \in a_{iP}$, $\exists y \in a_{jP}$ such that $z\alpha(R)y$. Since $x, y \in a_{jP}$, we then have $y\alpha(R)^*x$. Which in turn impies $z\alpha(R)^*x$. $x \in Gocha(R)$, so, by definition of $Gocha$, we have $x\alpha(R)^*z$. We can then deduce that $a_{iP} \cup a_{jP}$ is a maximal αR-circuit, contradicting the maximality of a_{iP} and/or a_{jP}.

(5) Let a_{iR} be a maximal R-circuit contained in $Getcha(R)$. By Lemma 2.3, we have $a_{iR} \in M(R_\sigma)$. If A_{iP} represents the decomposition of a_{iR} into maximal R-circuits of A then every maximal element of $R_{(\alpha\sigma)}|A_{iP}$ (which is also a maximal element of $R_{(\alpha\sigma)}$) is contained in $ElectIP(R)$.

(6) We know, from Lemma 2.3, that $Getcha(R) = \bigcup a_{iR}$ and $Gocha(R) = \bigcup a_{iP}$, with $a_{iR} \in M(R_\sigma)$ and $a_{iP} \in M(R_{(\alpha\sigma)})$. We have also seen from Lemma 3.6 that every maximal R-circuit contains a maximal αR-circuit. So for every pseudo tournament R defined on A, we have $Getcha(R) \cap Gocha(R) \neq \emptyset$.

□

4.2. Characterization results

We now give a characterization of the various choice procedures studied in this article. For this purpose, we consider six properties (properties τ_1, τ_2, τ_3, τ_{1P}, τ_{2P} and τ_{3P}). We also introduce IC (Independence of Completeness) property in order to check wether incomparability and indifference are treated in the same way or not by each of the choice procedures.

Definition 4.4. A choice procedure C defined on A endowed with a pseudo tournament R

– Satisfies property τ_1 if for every $x \in A$, $[x \in C(R) \Leftrightarrow \exists a_{iR} \in C(R_\sigma)$ such that $x \in a_{iR}]$.
– Satisfies property τ_2 if $\forall a_{jR} \in A_R|C(R_\sigma)$, $\exists a_{iR} \in C(R_\sigma)$ such that $a_{iR} R_\sigma a_{jR}$.
– Satisfies property τ_3 if for $a_{iR}, a_{jR} \in A_R$ such that $a_{iR} R_\sigma a_{jR}$, we have $[a_{iR} \in C(R_\sigma) \Rightarrow a_{jR} \notin C(R_\sigma)]$.
– Satisfies property τ_{1P} if for every $x \in A$, $[x \in C(R) \Leftrightarrow \exists a_{iP} \in C(R_{(\alpha\sigma)})$ such that $x \in a_{iP}]$.
– Satisfies property τ_{2P} if $\forall a_{jP} \in A_{\sigma_P}|C(R_{(\alpha\sigma)})$, $\exists a_{iP} \in C(R_{(\alpha\sigma)})$ such that $a_{iP} R_{(\alpha\sigma)} a_{jP}$.
– Satisfies property τ_{3P} if for $a_{iP}, a_{jP} \in A_{\alpha R}$ such that $a_{iP} R_{(\alpha\sigma)} a_{jP}$, we have $[a_{iP} \in C(R_{(\alpha\sigma)}) \Rightarrow a_{jP} \notin C(R_{(\alpha\sigma)})]$.
– Satisfies Independence of completeness (IC) property if $C(R) = C(R')$, where R' is the pseudo tournament defined on A such that for all $a, b \in A$,we have $a\beta(R)b \Rightarrow a\beta(R')b$, $a\alpha(R)b \iff a\alpha(R')b$ and $a\gamma(R)b \Rightarrow a\beta(R')b$.

Property τ_1 indicates that an alternative is chosen if and only if it belongs to a weak winning coalition. Property τ_2 tells that a non winning weak coalition is always beaten by at least one winning weak coalition. According to τ_3 if a weak coalition beats another weak coalition then if one is a winning weak coalition the other is not. Property τ_{1P} indicates that an alternative is chosen if and only if it belongs to a strong winning coalition. Property τ_{2P} says that a non winning strong coalition is always beaten by at least one winning strong coalition. According to τ_{3P} if a strong coalition beats another strong coalition then if one is a winning strong coalition the other is not. IC prescribes that a chosen alternative remains chosen when all the incomparabilities are replaced by indifferences.

Theorem 4.5. *The Table 1 gives properties (in columns) satisfy or not by each of the choice procedures (in rows).*

TABLE 1. Axiomatisation of the four choice procedures.

	τ_1	τ_2	τ_3	τ_{1P}	τ_{2P}	τ_{3P}	IC
ElectI	$\surd^{1.}$	$\surd^{5.}$	$\surd^{9.}$	$X^{13.}$	$\surd^{17.}$	$\surd^{21.}$	$X^{25.}$
ElectIP	$X^{2.}$	$\surd^{6.}$	$\surd^{10.}$	$\surd^{14.}$	$\surd^{18.}$	$\surd^{22.}$	$\surd^{26.}$
Gocha	$X^{3.}$	$X^{7.}$	$\surd^{11.}$	$\surd^{15.}$	$X^{19.}$	$\surd^{23.}$	$\surd^{27.}$
Getcha	$\surd^{4.}$	$X^{8.}$	$\surd^{12.}$	$X^{16.}$	$X^{20.}$	$\surd^{24.}$	$X^{28.}$

Legend: The symbol $\surd$ *(resp. X) means that the choice procedure satisfies (resp. does not satisfy) the property in the column.*

We notice from the table that all the choice procedures studied satisfy properties τ_3 and τ_{3P}. In addition, *ElectI* and *Getcha* choice procedures do not satisfy IC property, showing that they do not treat incomparability and indifference in the same way.

Proof. (1) Let $x \in A$. $x \in ElectI(R) \Leftrightarrow \exists a_{1R} \in R_\sigma$-kernel of A_R such that $x \in a_{1R} \Leftrightarrow \exists a_{1R} \in ElectI(R_\sigma)$ such that $x \in a_{1R}$ [because $ElectI(R) = \bigcup a_{iR}$, with $a_{iR} \in ElectI(R_\sigma)$].

(2) Consider the pseudo tournament R of the Example 2.1. We have $ElectIP(R) = \{b, c, g\}$, $ElectIP(R_\sigma) = \{\{a, b, c\}, \{h\}\}$. So $g \in ElectIP(R)$ but $g \notin \{a, b, c\}$ and $g \notin \{h\}$.

(3) Consider the pseudo tournament R of Example 2.1 given above. We have $Gocha(R) = \{b, g\}$ and $Gocha(R_\sigma) = \{\{a, b, c\}\}$. So $g \in Gocha(R)$ but $g \notin \{a, b, c\}$.

(4) Let $x \in A$. According to Lemma 2.3, $Getcha(R) = \bigcup a_{iR}$ with $a_{iR} \in M(R_\sigma)$. So $x \in Getcha(R) \Leftrightarrow \exists i_0 \in \mathbb{N}$, $a_{i_0R} \in M(R_\sigma)$ such that $x \in a_{i_0R} \Leftrightarrow \exists i_0 \in \mathbb{N}$, $a_{i_0R} \in M(R^*_\sigma)$ such that $x \in a_{i_0R}$. [Since A_R is without $R_\sigma - circuit$, we have $M(R_\sigma) = M(R^*_\sigma)$]. So $x \in Getcha(R) \Leftrightarrow \exists i_0 \in \mathbb{N}$, $a_{i_0R} \in Getcha(R_\sigma)$ such that $x \in a_{i_0R}$.

(5) By construction, ElectI choice procedure satisfies propery τ_2.

(6) Let $a_{jR} \in A_R|ElectIP(R_\sigma)$. From Lemma 3.5, $ElectIP(R_\sigma) = ElectI(R_\sigma)$, so $a_{jR} \in A_R|ElectI(R_\sigma)$. Since *ElectI* choice procedure satisfies property τ_2 then $\exists a_{iR} \in ElectI(R_\sigma)$ such that $a_{iR}R_\sigma a_{jR}$. So $\exists a_{iR} \in ElectIP(R_\sigma)$ such that $a_{iR}R_\sigma a_{jR}$.

(7) In the example given above, $Gocha(R_\sigma) = \{\{a, b, c\}\}$. So $\{h\} \in A_R|Gocha(R_\sigma)$ but we have not $(\{a, b, c\}R_\sigma\{h\})$.

(8) Same as 7, since in Example 2.1 $Getcha(R_\sigma) = Gocha(R_\sigma)$.

(9) Let $a_{iR}, a_{jR} \in A_R$ such that $a_{iR}R_\sigma a_{jR}$. By definition of the kernel of R_σ, a_{iR} and a_{jR} cannot both belong to $ElectI(R_\sigma) = N(R_\sigma)$. If $a_{iR} \in ElectI(R_\sigma)$, then $a_{jR} \notin ElectI(R_\sigma)$ (otherwise a_{iR} and a_{jR} would not be comparable by R_σ).

(10) Since $ElectIP(R_\sigma) = ElectI(R_\sigma)$, by Lemma 3.5, the proof of 10. results from that of 9.

(11) Let $a_{iR}, a_{jR} \in A_R$ such that $a_{iR}R_\sigma a_{jR}$. If $a_{iR} \in Gocha(R_\sigma)$, then $a_{iR} \in M(R_{(\alpha\sigma)})$. Since $a_{iR}R_\sigma a_{jR}$, there exists a_{kP} a maximal αR-circuit of A ($a_{kP} \subseteq a_{jR}$) such that $a_{iR}R_{(\alpha\sigma)}a_{kP}$. Since $R_{(\alpha\sigma)}$ is acyclic, we then have $a_{kP} \notin Gocha(R_\sigma)$: which also implies that $a_{jR} \notin Gocha(R_\sigma)$

(12) Let $a_{iR}, a_{jR} \in A_R$ such that $a_{iR}R_\sigma a_{jR}$. If $a_{iR} \in Getcha(R_\sigma)$, then $a_{iR} \in M(R_\sigma)$. Since $a_{iR}R_\sigma a_{jR}$ and R_σ is acyclic, we then have $a_{jR} \notin Getcha(R_\sigma)$.

(13) For the pseudo tournament R of the example given above, we have $ElectI(R) = \{a, b, c, h\}$, $ElectI(R_{(\alpha\sigma)}) = \{\{b\}, \{c\}\}, \{g\}\}$. So $h \in ElectI(R)$ but $h \notin \{b\}$, $h \notin \{c\}$ and $h \notin \{g\}$.

(14) Let $x \in A$ such that $x \in ElectIP(R)$. $\exists a_{1P} \in R_{(\alpha\sigma)}$-kernel of $A_{\alpha R}$ such that $x \in a_{1P}$. Since $ElectIP(R) = \bigcup a_{iP}$, with $a_{iP} \in ElectI(R_{(\alpha\sigma)})$, we then have $a_{1P} \in ElectI(R_{(\alpha\sigma)})$.

(15) Let $x \in A$. According to Lemma 2.3 $Gocha(R) = \bigcup a_{iP}$, with $a_{iP} \in M(R_{(\alpha\sigma)})$. So $x \in Gocha(R) \Leftrightarrow \exists i_0 \in \mathbb{N}$, $a_{i_0P} \in M(R_{(\alpha\sigma)})$ such that $x \in a_{i_0P} \Leftrightarrow \exists i_0 \in \mathbb{N}$, $a_{i_0P} \in M(R^*_{(\alpha\sigma)})$ such that $x \in a_{i_0P}$. [Since $R_{(\alpha\sigma)}$ is

acyclic, we have $M(R_{(\alpha\sigma)}) = M(R^*_{(\alpha\sigma)})$]. So $x \in Gocha(R) \Leftrightarrow \exists i_0 \in \mathbb{N}$, $a_{i_0P} \in Gocha(R_{(\alpha\sigma)})$ such that $x \in a_{i_0P}$.

(16) Consider the pseudo tournament R of the example given above. We have $Getcha(R) = \{a, b, c\}$ and $Getcha(R_{(\alpha\sigma)}) = \{\{b\}, \{g\}\}$. So $a \in Getcha(R)$ but $a \notin \{b\}$ and $a \notin \{g\}$.

(17) Let $a_{jP} \in A_{\alpha R}|ElectI(R_{(\alpha\sigma)})$. From Lemma 3.5, $ElectIP(R_{(\alpha\sigma)}) = ElectI(R_{(\alpha\sigma)})$, so $a_{jP} \in A_{\alpha R}|ElectIP(R_{(\alpha\sigma)})$. Since $ElectIP$ choice procedure satisfies property τ_{2P} (see proof below) then $\exists a_{iP} \in ElectIP(R_{(\alpha\sigma)})$ such that $a_{iP}R_{(\alpha\sigma)}a_{jP}$. So $\exists a_{iP} \in ElectI(R_{(\alpha\sigma)})$ such that $a_{iP}R_{(\alpha\sigma)}a_{jP}$.

(18) By construction $ElectIP$ choice procedure satisfies property τ_{2P}.

(19) In the example given above, $Gocha(R_{(\alpha\sigma)}) = \{\{b\}, \{g\}\}$. So $\{c\} \in A_{\alpha R}|Gocha(R_{(\alpha\sigma)})$ but we have not $(\{b\}R_{(\alpha\sigma)}\{c\})$ and not $(\{g\}R_{(\alpha\sigma)}\{c\})$.

(20) Same as 19. since in Example 2.1, $Getcha(R_{(\alpha\sigma)}) = Gocha(R_{(\alpha\sigma)})$.

(21) Let $a_{iP}, a_{jP} \in A_{\alpha R}$ such that $a_{iP}R_{(\alpha\sigma)}a_{jP}$. If $a_{iP} \in ElectI(R_{(\alpha\sigma)})$, then $a_{jP} \notin ElectI(R_{(\alpha\sigma)})$ (otherwise [a_{iP} and a_{jP} would not be comparable by $R_{(\alpha\sigma)}$).

(22) Let $a_{iP}, a_{jP} \in A_{\alpha R}$ such that $a_{iP}R_{(\alpha\sigma)}a_{jP}$. a_{iP} and a_{jP} cannot both belong to $ElectIP(R_{(\alpha\sigma)})$ by definition of the kernel of $R_{(\alpha\sigma)}$.
If $a_{iP} \in ElectIP(R_{(\alpha\sigma)})$, then $a_{iP} \in ElectI(R_{(\alpha\sigma)})$ (because according to Lem. 3.5, $ElectIP(R_{(\alpha\sigma)}) = ElectI(R_{(\alpha\sigma)})$). Since $ElectI$ choice procedure satisfies property τ_{3P} then $a_{jP} \notin ElectI(R_{(\alpha\sigma)})$. Finally, again from Lemma 3.5, we have $a_{jP} \notin ElectIP(R_{(\alpha\sigma)})$.

(23) Let $a_{iP}, a_{jP} \in A_{\alpha R}$ such that $a_{iP}R_{(\alpha\sigma)}a_{jP}$. If $a_{iP} \in Gocha(R_{(\alpha\sigma)})$, then $a_{iP} \in M(R_{(\alpha\sigma)})$. Since $a_{iP}R_{(\alpha\sigma)}a_{jP}$, and $R_{(\alpha\sigma)}$ is acyclic, we then have $a_{jP} \notin Gocha(R_{(\alpha\sigma)})$

(24) Let $a_{iP}, a_{jP} \in A_{\alpha R}$ such that $a_{iP}R_{(\alpha\sigma)}a_{jP}$. If $a_{iP} \in Getcha(R_{(\alpha\sigma)})$, then $a_{iP} \in M(R_{(\alpha\sigma)})$. Since $a_{iP}R_{(\alpha\sigma)}a_{jP}$ and $R_{(\alpha\sigma)}$ is acyclic, we then have $a_{jP} \notin Getcha(R_{(\alpha\sigma)})$.

(25) Let R be the pseudo tournament defined in Example 2.1. The pseudo tournament R' defined as the relation containing R and the following pairs $b\alpha(R')a$, $a\alpha(R')c$, $c\beta(R')b$, $c\alpha(R')d$, $d\alpha(R')e$, $e\alpha(R')f$, $f\alpha(R')d$, $f\alpha(R')d$, $f\beta(R')h$ $g\alpha(R')h$, $a\beta(R')d$, $a\beta(R')e$, $a\beta(R')f$, $a\beta(R')g$, $a\beta(R')h$, $b\beta(R')d$, $b\beta(R')e$, $b\beta(R')f$, $b\beta(R')g$, $b\beta(R')h$, $c\beta(R')e$, $c\beta(R')f$, $c\beta(R')g$, $c\beta(R')h$, $d\beta(R')g$, $d\beta(R')h$, $e\beta(R')g$ and $e\beta(R')h$. So $ElectI(R) = \{a, b, c, h\}$ and $ElectI(R') = \{a, b, c, d, e, f, g, h\}$.

(26) Let R, R' be two pseudo tournaments defined on A such that for all $a, b \in A$, $a\beta(R)b \Rightarrow a\beta(R')b$, $a\alpha(R)b \Longleftrightarrow a\alpha(R')b$ and $a\gamma(R)b \Rightarrow a\beta(R')b$. We then have $R_{(\alpha\sigma)} = R'_{(\alpha\sigma)}$. So $ElectIP(R) = ElectIP(R')$.

(27) Let R, R' be two pseudo tournaments defined on A such that for all $a, b \in A$, $a\beta(R)b \Rightarrow a\beta(R')b$, $a\alpha(R)b \Longleftrightarrow a\alpha(R')b$ and $a\gamma(R)b \Rightarrow a\beta(R')b$. $x \in Gocha(R) \Leftrightarrow [\forall y \in A, y\alpha(R)^*x \Rightarrow x\alpha(R)^*y] \Leftrightarrow [\forall y \in A, y\alpha(R')^*x \Rightarrow x\alpha(R')^*y] \Leftrightarrow x \in Gocha(R')$.

(28) Let R be the pseudo tournament defined in Example 2.1. The pseudo tournament R' defined as the relation containing R and the following pairs $b\alpha(R')a$, $a\alpha(R')c$, $c\beta(R')b$, $c\alpha(R')d$, $d\alpha(R')e$, $e\alpha(R')f$, $f\alpha(R')d$, $f\alpha(R')d$, $f\beta(R')h$ $g\alpha(R')h$, $a\beta(R')d$, $a\beta(R')e$, $a\beta(R')f$, $a\beta(R')g$, $a\beta(R')h$, $b\beta(R')d$, $b\beta(R')e$, $b\beta(R')f$, $b\beta(R')g$, $b\beta(R')h$, $c\beta(R')e$, $c\beta(R')f$, $c\beta(R')g$, $c\beta(R')h$, $d\beta(R')g$, $d\beta(R')h$, $e\beta(R')g$ and $e\beta(R')h$. So $Getcha(R) = \{a, b, c\}$ and $Getcha(R') = \{a, b, c, d, e, f, g, h\}$.

□

Proposition 4.6. (1) *$ElectI$ choice procedure is the only choice procedure satisfying properties τ_1, τ_2 and τ_3.*
(2) *$ElectIP$ choice procedure is the only choice procedure satisfying properties τ_{1P}, τ_{2P} and τ_{3P}.*

Proof. We give the proof only for the first point. The proof for the second point is similar. We know, from Proposition 4.6, that $ElectI$ choice procedure satisfies properties τ_1, τ_2 and τ_3. Let C be a choice procedure satisfying properties τ_1, τ_2 and τ_3 and R a pseudo tournament defined on A.

– Let $x \in C(R)$ and suppose $x \notin ElectI(R)$. $\exists a'_{1R} \notin ElectI(R_\sigma)$ such that $x \in a'_{1R}$ (because $ElectI(R) = \bigcup a_{iR}$, with $a_{iR} \in ElectI(R_\sigma)$). $ElectI$ choice procedure satisfies property τ_2, so $\exists a_{1R} \in ElectI(R_\sigma)$ such that $a_{1R}R_\sigma a'_{1R}$. Since C satisfies property τ_1, $x \in C(R) \Rightarrow a'_{1R} \in C(R_\sigma)$. C satisfies property τ_3, so $[a_{1R}R_\sigma a'_{1R}$ and $a'_{1R} \in C(R_\sigma)] \Rightarrow a_{1R} \notin C(R_\sigma)$. C satisfies property τ_2, so $\exists a'_{2R} \in C(R_\sigma)$ such that $a'_{2R}R_v a_{1R}$. $ElectI$

satisfies property τ_3, so $[a'_{2R}R_\sigma a_{1R}$ and $a_{1R} \in ElectI(R_\sigma)] \Rightarrow a'_{2R} \notin ElectI(R_\sigma)$. $ElectI$ choice procedure satisfies property τ_2, so $\exists a_{2R} \in ElectI(R_\sigma)$ such that $a_{2R}R_\sigma a'_{2R}$. By continuing with the same reasoning, $\exists a_{1R}, a_{2R}, \ldots, a_{tR} \ldots \in ElectI(R_\sigma)$, $\exists a'_{1R}, a'_{2R} \ldots a'_{kR} \in C(R_\sigma)$ such that $a_{tR}R_\sigma a'_{tR}$ and $a'_{(t+1)R}R_\sigma a_{tR}$. Since A_R is a finite set, then $\exists i, r, n \in \mathbb{N}$ such that $a_{iR}R_\sigma a'_{iR}R_\sigma a_{(i-1)R}R_\sigma a'_{(i-1)R} \ldots R_\sigma a'_{(i-n)R}$ and $a'_{(i-n)R}R_\sigma a_{iR}$: which is not posible because R_σ is acyclic.

– Let $x \in ElectI(R)$ and suppose $x \notin C(R)$. Since C satisfies property τ_1, $\exists a'_{1R} \notin C(R_\sigma)$ such that $x \in a'_{1R}$. The choice procedure C satisfies property τ_2, so $\exists a_{1R} \in C(R_\sigma)$ such that $a_{1R}R_\sigma a'_{1R}$. $ElectI$ choice procedure satisfies property τ_1, so $x \in ElectI(R) \Rightarrow a'_{1R} \in ElectI(R_\sigma)$. Also $ElectI$ choice procedure satisfies property τ_3, so $[a_{1R}R_\sigma a'_{1R}$ and $a'_{1R} \in ElectI(R_\sigma)] \Rightarrow a_{1R} \notin ElectI(R_\sigma)$. $ElectI$ choice procedure satisfies property τ_2, so $\exists a'_{2R} \in ElectI(R_\sigma)$ such that $a'_{2R}R_\sigma a_{1R}$. The choice procedure C satisfies property τ_3, so $[a'_{2R}R_\sigma a_{1R}$ and $a_{1R} \in C(R_\sigma)] \Rightarrow a'_{2R} \notin C(R_\sigma)$. The choice procedure C satisfies property τ_2, so $\exists a_{2R} \in C(R_\sigma)$ such that $a_{2R}R_\sigma a'_{2R}$. By continuing with the same reasoning, $\exists a_{1R}, a_{2R}, \ldots, a_{tR} \ldots \in C(R_\sigma)$, $\exists a'_{1R}, a'_{2R} \ldots a'_{kR} \in ElectI(R_\sigma)$ such that $a_{tR}R_\sigma a'_{tR}$ and $a'_{(t+1)R}R_\sigma a_{tR}$. Since A_R is a finite set, then $\exists i, r, n \in \mathbb{N}$ such that $a_{iR}R_\sigma a'_{iR}R_\sigma a_{(i-1)R}R_\sigma a'_{(i-1)R} \ldots R_\sigma a'_{(i-n)R}$ and $a'_{(i-n)R}R_\sigma a_{iR}$: which is not posible because R_σ is acyclic.

$\square$

5. Conclusion

The main contribution of this paper is to offer an axiomatic characterization of the exploitation phase of Electre I method after the outranking relation has been obtained. This characterization is different from those given by Bouyssou and Pirlot [3] made within a general framework for conjoint measurement. Our approach is based on the exploitation step of *ElectreI* and consists in finding properties that characterize *ElectreI* choice procedures. For this reason we considered two versions of *ElectreI* method (*ElectreI* and *ElectreIP*) viewed as choice procedures. To better characterize each of them, we also defined a family of six τ properties (τ_1, τ_2, τ_3, τ_{1P}, τ_{2P} and τ_{3P}). These properties have been interpreted in terms of weak or strong coalitions which correspond to weak or strong connected component in the resulting graph. For example, property τ_1 indicates that an alternative is chosen if and only if it belongs to a weak winning coalition. Property τ_2 shows that a non winning weak coalition is always beaten by at least one winning weak coalition. According to τ_3 if a weak coalition beats another weak coalition then if one is a winning weak coalition the other is not. We then showed in Proposition 4.6 that *ElectreI* (resp *ElectreIP*) choice pocedure is the only choice procedure that satisfies properties τ_1, τ_2 and τ_3 (resp. properties τ_{1P}, τ_{2P} and τ_{3P}). *ElectreI* and *ElectreIP* choice pocedures are based on outranking methods which deal with non necessarily complete binary relations. So, we introduced property IC to check if the choice procedures studied are sensitive to incomparability or indifference. We prove in Theorem 4.5 that *ElectI* choice procedure does not satisfy IC property showing that this choice procedure does not deal with incomparability and indifference in the same way.

By analyzing the set-theoretic relationships between the two choice procedures, we noticed that they both coincide with the Top cycle choice procedure when the binary relation is a tournament. The comparison of the two proposed choice procedures (*ElectI* and *ElectIP*) with two extensions of the Top cycle (*Gocha* and *Getcha*), defined in the litterature, has shown that, for weak tournaments, $Gocha \subseteq ElectIP \subseteq ElectI$ and, that, *ElectI* choice procedure is equivalent to *Getcha* choice procedure. For the general case concerning pseudo tournaments, we prove in Proposition 4.3 that *ElectI* and *ElectIP* choice procedures always intersect. We also see in the same proposition that $Gocha \subseteq ElectIP$ and $Getcha \subseteq ElectI$. *ElectI* and *ElectIP* choice procedures have been well characterized through Theorem 4.5 and Proposition 4.6, and, as *ElectI* does not satisfies IC property, this choice procedure can be considered as a good extension of the Top cycle choice procedures in the general case.

Note that we did not study the computational complexity of each of the procedures. This will be done in a future work.

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