



**GENERALIZATION AND RESOLUTION OF
THE HOMOGENEOUS REACTION-DIFFUSION
EQUATIONS BY THE METHOD OF FACTORIZATION
OF ORDINARY DIFFERENTIAL OPERATORS**

**S. Moumouni^{1,*}, V. M. S. Soumanou², S. Massou², A. L. Essoun² and
M. Tchoffo³**

¹Higher Teachers' Training School of Natitingou
University of Parakou
06 BP 1716 Cotonou (République du Bénin)
Benin
e-mail: sounma.moumouni@gmail.com

²Department of Physics
Faculty of Science and Technology
University of Abomey-Calavi
Benin

³Department of Physics
University of Dschang
Cameroon

Received: May 29, 2016; Accepted: August 14, 2016

2010 Mathematics Subject Classification: 34A05, 34C20, 35K57, 92D25.

Keywords and phrases: homogeneous reaction-diffusion equations, method of factorization of ordinary differential operators, population dynamics.

*Corresponding author

Communicated by Yansheng Liu

Abstract

The objective of this article is the generalization of the homogeneous reaction-diffusion equations of the monostable or bistable types. From the non-linear second order differential equation of the front profile (solution of the homogeneous reaction-diffusion equations), a method of generalization of the homogeneous reaction-diffusion equations, based on the theory of factorization of ordinary differential operators [9], was proposed. This method enables us to build two generalized homogeneous reaction-diffusion equations: the first with one species and the second with two species.

1. Introduction

Equations of models of diffusion intervene in varied fields like combustion [1], chemistry [2], biology or ecology [3]. In population dynamics [4], these equations model the evolution of species which interact between themselves and move in an environment. When we consider an isolated species of density $u(t, x)$ at time t , with the position x and which moves in the direction (Ox) , its equation of diffusion can be written in the following general form:

$$\frac{\partial u(x, t)}{\partial t} = D_x(u(x, t)) + f(x, u(x, t)), \quad t > 0, x \in \mathbb{R}. \quad (1.1)$$

In this expression, $f(x, u)$ represents the reaction term which defines interactions between the species and its environment. This term is a function of the density $u(x, t)$ of species and their position x . The movement of the species is described by the dispersion operator D_x . According to the mode of displacement of the species, this operator is local or non-local. Thus, we distinguish two types of non-linear equations of diffusion: reaction-diffusion equations and integro-differential equations.

In the case of the reaction-diffusion equations, the dispersion operator D_x is local because it is supposed that the species at the time t and at position x diffuse only towards its immediate neighbour. Thus, this operator

is an elliptic differential operator of the second order. The first models of reaction-diffusion introduced by [5, 6] in population genetics are written in the form:

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + f(u(x, t)), \quad t > 0, x \in \mathbb{R}. \tag{1.2}$$

These reaction-diffusion equations are non-linear parabolic partial derivative equations. In these equations, the non-linearity shown in the reaction term f does not depend on the position x in space. In the case where the environment of the species is homogeneous, we consider the regular reaction term f in the interval $[0, 1]$ and null at the ends ($f(0) = f(1) = 0$).

This approach of reaction-diffusion becomes inaccurate when we study species whose individuals can carry out long distance displacements. These types of dispersions at long distances intervene in phenomena like the pollination of certain plants or the propagation of epidemics. These dispersions are often due to external vectors (human, animal, ...) which foster the transport of species far from their place of origin. A manner of taking into account these types of dispersions at long distances is to use integro-differential equations [7, 8] in the form:

$$\frac{\partial u(x, t)}{\partial t} = \int_{-\infty}^{+\infty} J(|x - y|)u(y, t)dy - u(x, t) + f(u(x, t)),$$

$$t > 0 \text{ and } x \in \mathbb{R}. \tag{1.3}$$

In these equations, J is a density of probability also called core of dispersion. Thus, the dispersion operator D_x is described by an integral operator of the form $J \star u - u$. The study of these types of equations is beyond the scope of this work.

The objective of this article is to generalize and solve the homogeneous equations of reaction-diffusion (1.2) by using the method of factorization of differential operators suggested in [9]. In Section 2, we describe the various types of homogeneous reaction-diffusion equations. In Section 3, we present

the methodology of resolution of the reaction-diffusion equations by the factorization method of operators. Sections 4 and 5 are dedicated to the results of generalizations of the homogeneous reaction-diffusion equations, with one species and two species, respectively. Finally, in Section 6, we report the conclusion of work.

2. Various Types of Homogeneous Reaction-diffusion Equations

Generally, the reaction term f in equation (1.2), used in population dynamics is monostable or bistable.

2.1. Monostable reaction term

The reaction term f is of the monostable type if:

$$f'(0) > 0, f'(1) < 0 \text{ and } f > 0 \text{ on }]0, 1[. \quad (2.1)$$

The current traditional example of monostable term is:

$$f(u) = (1 - u)(au + 1)u \text{ with } a \geq 0. \quad (2.2)$$

If $a \leq 1$, then the line of equation $y = f'(0)u$ is above the first bisectrix which corresponds to a reaction term of type KPP (for Kolmogorov, Petrovsky and Piskunov [6]). If on the other hand, $1 > a$, then the line of equation $y = f'(0)u$ is below the first bisectrix, which in ecology, corresponds to a weak Allee effect [10].

2.2. Bistable reaction term

The reaction term f is of the bistable type if:

$$\int_0^1 f(s)ds > 0 \text{ with } f'(0) < 0, \text{ and } f'(1) < 0. \quad (2.3)$$

The traditional example of bistable term the most common is the cubic function:

$$f(u) = (1 - u)(u - \rho)u \text{ with } \rho \in]0, 1/2[. \quad (2.4)$$

Thus, $f < 0$ on $]0, \rho[$ and $f > 0$ on $]\rho, 1[$.

3. Resolution of the Homogeneous Reaction-diffusion Equations by the Method of Factorization of Ordinary Differential Operators

Several authors [6, 11-13] showed that the reaction-diffusion models of the monostable or bistable type have solutions of the fronts type, in uniform translation connecting the state of balance in $-\infty$ to the state of balance in $+\infty$, which makes it possible for these models to describe the invasion at speed and constant profile of a virgin space by a population. These solutions are $u(x, t) = U(x - ct)$, where c is the speed of the front and U is the front profile connecting the two states of balance of the equation. The front profile U of equation (1.2) satisfies the following non-linear elliptic equation:

$$\frac{d^2U(z)}{dz^2} + c \frac{dU(z)}{dz} + f(U(z)) = 0, \text{ where } (z = x - ct) \in \mathbb{R}$$

and $0 < U < 1$. (3.1)

According to [9], equation (3.1) can be factorized in form:

$$\left(\frac{d}{dz} - \Psi_1(U(z))\right)\left(\frac{d}{dz} - \Psi_2(U(z))\right)U(z) = 0. \tag{3.2}$$

The expansion of (3.2) leads to the following equation:

$$\frac{d^2U(z)}{dz^2} - \left(\Psi_1 + \Psi_2 + \frac{d\Psi_2}{dU}U\right)\frac{dU(z)}{dz} + \Psi_1\Psi_2U(z) = 0. \tag{3.3}$$

Thus, the particular solution of equation (3.1) is one of the solutions of the equations below:

$$\left(\frac{d}{dz} - \Psi_1(U(z))\right)U(z) = 0, \tag{3.4a}$$

$$\left(\frac{d}{dz} - \Psi_2(U(z))\right)U(z) = 0 \tag{3.4b}$$

provided that Ψ_1 and Ψ_2 , solutions of system (3.5) below exist:

$$\begin{cases} \Psi_1 \Psi_2 U = f(U), \\ \Psi_1 + \Psi_2 + \frac{d\Psi_2}{dU} U = -c. \end{cases} \quad (3.5)$$

In the continuation of this work, we will use this approach to generalize and solve the homogeneous reaction-diffusion of equations for one species and two species, respectively.

4. Generalization of the Homogeneous Reaction-diffusion Equations to One Species

4.1. First stage of generalization

In the expression of the reaction term f of the traditional reaction-diffusion models (monostable or bistable), the growth rate $\frac{f(u)}{u}$ comprises two terms: the first term is in favor of the growth of the species; and the second term is against its growth. This second term still called function of braking is often generalized by several authors including [14] in the form: $(1 - u^n)$, $n \in \mathbb{R}_+$.

In this work, we adopt this generalized form of the function of braking, which leads us to write the reaction term in equation (1.2) in the form:

$$f(u) = (1 - u^n)g(u)u. \quad (4.1)$$

With this stage, our objective is to determine the expression of the function so that the reaction-diffusion equation (1.2) with the reaction term (4.1) can admit a particular solution with the step described in Section 3. Thus, by taking into account the system (3.5), we can choose the functions Ψ_1 and Ψ_2 such as:

$$\Psi_1(U) = k(1 - U^n) \quad \text{and} \quad \Psi_2 = \frac{1}{k} g(U), \quad k \in \mathbb{R}^*. \quad (4.2)$$

And g must satisfy the following differential equation:

$$Ug'(U) + g(U) + k^2(1 - U^n) + kc = 0. \tag{4.3}$$

By solving this equation and taking into account that $f(0) = f(1) = 0$, we obtain:

$$g(U) = \frac{k^2}{n+1} U^n - k(k+c). \tag{4.4}$$

Thus, we obtain a generalized expression of the reaction term in the following form:

$$f(u) = (1 - u^n)(au^n - \rho)u, \text{ where } a \in \mathbb{R}_+^* \text{ and } \rho \in \mathbb{R}^*. \tag{4.5}$$

By considering the expression (4.5) for the reaction term, the functions Ψ_1 and Ψ_2 are as follows:

$$\Psi_1(U) = k(1 - U^n) \quad \text{and} \quad \Psi_2 = \frac{1}{k}(aU^n - \rho), \quad k \in \mathbb{R}^*. \tag{4.6}$$

Thus, we use these expressions to solve equations (3.4a) and (3.4b). Nevertheless, only the solution obtained with equation (3.4b) is not valid and is as follows:

$$U^n(z) = \frac{\rho A \exp\left(-\frac{n\rho}{k}z\right)}{a\left(A \exp\left(-\frac{n\rho}{k}z\right) - 1\right)}, \text{ where } k^2 = a(n+1) \text{ and } \rho = k(c+k). \tag{4.7}$$

In summary, the homogeneous reaction-diffusion equation that we generalized is structured as follows:

$$\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + (1 - u^n(x, t))(au^n(x, t) - \rho)u(x, t)$$

$$t > 0, x \in \mathbb{R}, \text{ where } a \in \mathbb{R}_+^* \text{ and } \rho \in \mathbb{R}^*. \tag{4.8}$$

If moreover, at initial time, $u(x, 0) = \mu(x)$, the particular solution of equation (4.8) obtained by the method of factorization of ordinary

differential operators [9] is as follows:

$$u^n(x, t) = \frac{\rho \mu^n(x) \exp(\omega_n t)}{\rho + a \mu^n(x) (\exp(\omega_n t) - 1)}, \text{ where } \omega_n = \frac{n\rho(\rho - a(n+1))}{a(n+1)}. \quad (4.9)$$

4.2. Second stage of generalization

In the first stage, we carried out the generalization of the reaction term which represents the only term of non-linearity in the traditional reaction-diffusion equation. In the second stage, we add a second term of non-linearity and rewrite equation (4.8) in this new form:

$$h(u(x, t)) \frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + (1 - u^n(x, t))(a u^n(x, t) - \rho) u(x, t),$$

$$t > 0, x \in \mathbb{R}, \text{ where } a \in \mathbb{R}_+^* \text{ and } \rho \in \mathbb{R}^*. \quad (4.10)$$

By applying the step described in Section 3, the objective at this stage is to determine the expression of the function h . The front profile U of equation (4.10) satisfies the following non-linear elliptic equation:

$$\frac{d^2 U(z)}{dz^2} + h(U(z))c \frac{dU(z)}{dz} + (1 - U^n(z))(a U^n(z) - \rho)U(z) = 0,$$

where $(z = x - ct)$. (4.11)

By taking into account the system (3.5), we obtain the following system:

$$\begin{cases} \Psi_1(U) = k(1 - U^n), \\ \Psi_2 = \frac{1}{k}(a U^n - \rho), \\ \Psi_1 + \Psi_2 + \frac{d\Psi_2}{dU} U = -h(U)c, \end{cases} \quad k \in \mathbb{R}^*. \quad (4.12)$$

The resolution of the last equation of this system gives:

$$h(U) = \frac{\rho - k^2}{kc} + \frac{k^2 - a(n+1)}{kc} U^n, \quad (4.13)$$

we can write the function in the form:

$$h(u) = \alpha + \beta u^n, \text{ where } \alpha \in \mathbb{R}_+^* \text{ and } \beta \in \mathbb{R}. \tag{4.14}$$

Thus, the second reaction-diffusion equation that we have just generalized is as follows:

$$(\alpha + \beta u^n(x, t)) \frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + (1 - u^n(x, t))(a u^n(x, t) - \rho) u(x, t),$$

$$t > 0, x \in \mathbb{R}, \text{ where } (a, \alpha) \in \mathbb{R}_+^{*2}, \rho \in \mathbb{R}^* \text{ and } \beta \in \mathbb{R}. \tag{4.15}$$

By supposing that $u(x, 0) = \mu(x)$, the particular solution of equation (4.15) obtained by the method of factorization of ordinary differential operators [9] is in the form:

$$u^n(x, t) = \frac{\rho \mu^n(x) \exp(\omega_n t)}{\rho + a \mu^n(x) (\exp(\omega_n t) - 1)}, \text{ where } \omega_n = \frac{n\rho(\rho - a(n + 1))}{\beta\rho + \alpha a(n + 1)}. \tag{4.16}$$

4.3. Analysis of particular solutions of the generalized homogeneous reaction-diffusion equations

All our analysis will relate to equation (4.8) which corresponds to equation (4.15) when $\alpha = 1$ and $\beta = 0$. With equation (4.8), we obtain successively from the reaction term:

$$f'(u) = (n + 1)(a + \rho)u^n - a(2n + 1)u^{2n} - \rho, \tag{4.17}$$

$$f'(0) = -\rho \text{ and } f'(1) = n(\rho - a), \tag{4.18}$$

$$\int_0^1 f(s) ds = \frac{n(a - (n + 1)\rho)}{2(n + 1)(n + 2)}. \tag{4.19}$$

4.3.1. Monostable reaction-diffusion model

The reaction-diffusion model is of the monostable type if the reaction term f checks: $f'(0) > 0$, $f'(1) < 0$ and $f > 0$ on $]0, 1[$.

Thus, for the model (4.8), it is important that $\rho < 0$ and $a > 0$ (Figure 1). Under these conditions, $\omega_n > 0, \forall n > 0$ then, this gives rise to a process of growth of the species as illustrated in Figure 2.

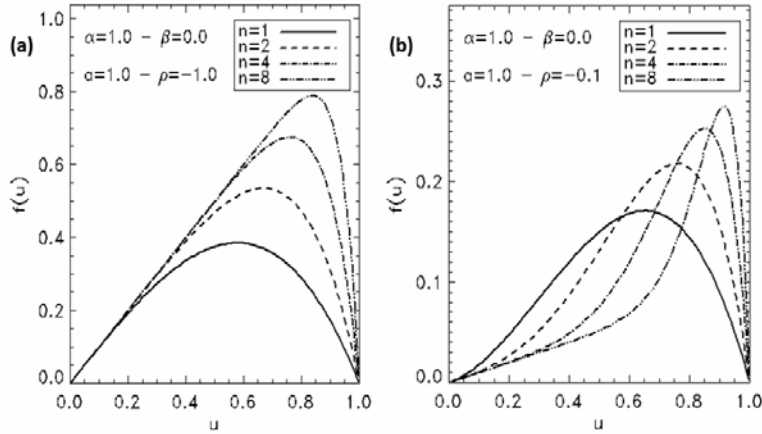


Figure 1. Shape of the reaction term (monostable) of the reaction-diffusion model (4.8), for various values of the parameters. Figure (a) corresponds to the KPP type [6] and Figure (b) corresponds to a weak Allee effect [10].

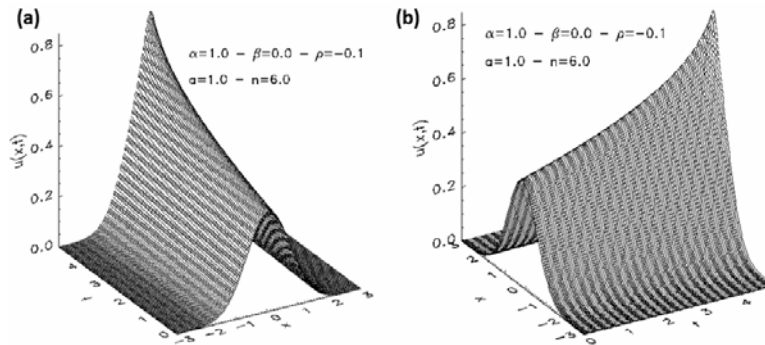


Figure 2. Shape of the solution (4.9): case of the monostable reaction term. It is about a process of growth of the species because $\omega_n > 0, \forall n > 0$. Here $u(x, 0) = \mu(x) = \exp(-x^2)/\sqrt{2\pi}$. (a) and (b) are drawn for the same values of parameters, but we swapped the directions of the (Ox) and (Ot) axes with one another.

4.3.2. Bistable reaction-diffusion model

The reaction-diffusion model is of the bistable type if the reaction term f checks: $\int_0^1 f(s)ds > 0$, $f'(0) < 0$, $f'(1) < 0$. Thus, in equation (4.8), it is necessary to have: $\rho > 0$ and $a > (n+1)\rho$ (Figure 3).

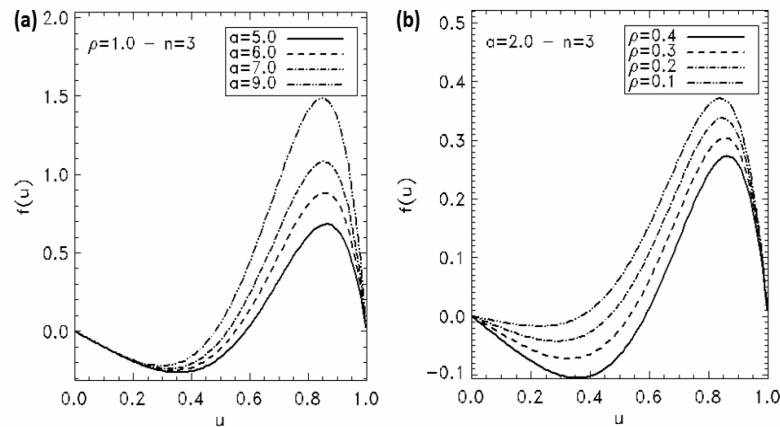


Figure 3. Shape of the reaction term (bistable) of the reaction-diffusion model (4.8), for various values of the parameters. Figure (a) corresponds to the case of the variation of the parameter a and Figure (b) corresponds to the case of the variation of the parameter ρ .

Under these conditions, we attain two types of processes:

- Growth of the species when $\omega_n > 0$ if $\rho > (n+1)a$ as illustrated in Figure 4.
- Decrease of the species when $\omega_n < 0$ if $\rho < (n+1)a$ as illustrated in Figure 5.

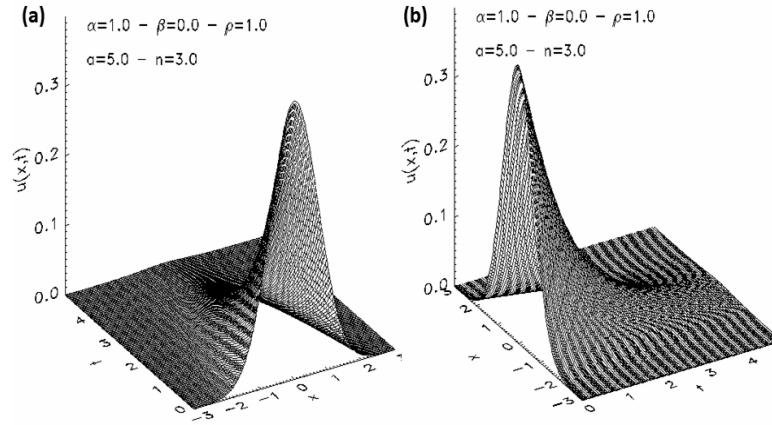


Figure 4. Shape of the solution (4.9): case of the bistable reaction term. The process is a growth when $\rho > (n+1)a$. Here $u(x, 0) = \mu(x) = \exp(-x^2)/\sqrt{2\pi}$. (a) and (b) are drawn for the same values of parameters, but we swapped the directions of the (Ox) and (Ot) axes with one another.

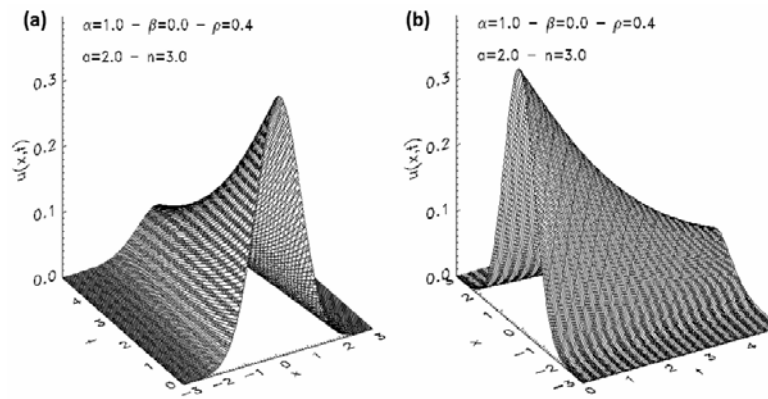


Figure 5. Shape of the solution (4.9): case of the bistable reaction term. The process is a decline when $\rho < (n+1)a$. Here $u(x, 0) = \mu(x) = \exp(-x^2)/\sqrt{2\pi}$. (a) and (b) are drawn for the same values of parameters, but we swapped the directions of the (Ox) and (Ot) axes with one another.

For example, when the process of diffusion is decreasing, one is interested in the half-life time noted T^* such as: $u(x, T^*) = u(x, 0)/2$. For

equation (4.8) whose solution is (4.9), we obtain:

$$T^*(x) = \frac{1}{\omega_n} \ln \left(\frac{\rho - a\mu^n(x)}{2^n \rho - a\mu^n(x)} \right). \tag{4.20}$$

Figure 6(a) describes the variation of half-life time at a particular point with the parameter. On the other hand, Figure 6(b) describes the variation of half-life time with x .

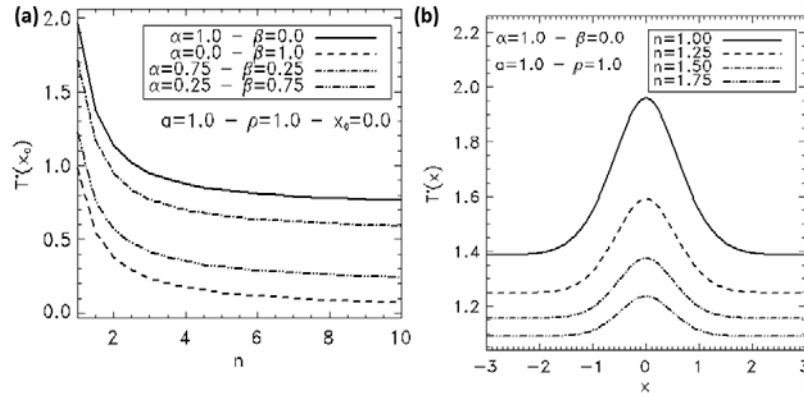


Figure 6. Shape of the half-life time (4.20). Figure (a) describes its variation at one particular point $x_0 = 0$ with the parameter n . On the other hand, Figure (b) describes its variation with x . Here $u(x, 0) = \mu(x) = \exp(-x^2)/\sqrt{2\pi}$.

5. Generalization of the Homogeneous Reaction-diffusion Equations to Two Species

The objective of this section is to propose a generalization of the homogeneous reaction-diffusion equation to two species and to solve this equation by using the method of factorization of ordinary differential operators [9]. In population dynamics, there are various types of interactions between two species. Three most significant and documented interactions are: the predation, the competition and the mutualism. In case of our study, we choose to write a general system, without specifying the type of

interaction:

$$\begin{cases} g_1(u(x, t), v(x, t)) \frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} + f_1(u(x, t), v(x, t))u(x, t), \\ g_2(u(x, t), v(x, t)) \frac{\partial v(x, t)}{\partial t} = \frac{\partial^2 v(x, t)}{\partial x^2} + f_2(u(x, t), v(x, t))v(x, t), \\ t > 0, \quad x \in \mathbb{R}, \end{cases} \quad (5.1)$$

where $u(x, t)$ and $v(x, t)$ are the densities of each species at time t and position x . As in the case with a species, we are interested in the front profiles of the same front speed c such as: $u(x, t) = U(x - ct)$ and $v(x, t) = V(x - ct)$. We also place ourselves under the conditions of Cauchy such as: $u(x, 0) = \mu(x)$ and $v(x, 0) = \vartheta(x)$. The differential connection of the front profiles to the system (5.1) is:

$$\begin{cases} \frac{d^2 U(z)}{dz^2} + cg_1(U, V) \frac{dU(z)}{dz} + f_1(U, V)U(z) = 0, \\ \frac{d^2 V(z)}{dz^2} + cg_2(U, V) \frac{dV(z)}{dz} + f_2(U, V)V(z) = 0, \end{cases} \quad (5.2)$$

where $(z = x - ct)$.

By applying the approach of factorization of ordinary differential operators [5] to the system (5.2), we have:

$$\begin{cases} \left[\frac{d}{dz} - \Psi_{11}(U, V) \right] \left[\frac{d}{dz} - \Psi_{12}(U, V) \right] U(z) = 0, \\ \left[\frac{d}{dz} - \Psi_{21}(U, V) \right] \left[\frac{d}{dz} - \Psi_{22}(U, V) \right] U(z) = 0, \end{cases} \quad (5.3)$$

we obtain, after expansion of (5.3), the following system:

$$\begin{cases} \frac{d^2U(z)}{dz^2} - \left(\Psi_{11} + \Psi_{12} + \frac{\partial \Psi_{12}}{\partial U} U \right) \frac{dU(z)}{dz} + \Psi_{11}\Psi_{12}U(z) = 0, \\ \frac{d^2V(z)}{dz^2} - \left(\Psi_{21} + \Psi_{22} + \frac{\partial \Psi_{22}}{\partial V} V \right) \frac{dV(z)}{dz} + \Psi_{21}\Psi_{22}V(z) = 0. \end{cases} \quad (5.4)$$

By comparing the systems (5.2) and (5.4), we successively obtain:

$$\begin{cases} \Psi_{11}(U, V)\Psi_{12}(U, V) = f_1(U, V), \\ \Psi_{11} + \Psi_{12} + \frac{\partial \Psi_{12}}{\partial U} U = -cg_1(U, V), \end{cases} \quad (5.5a)$$

$$\begin{cases} \Psi_{21}(U, V)\Psi_{22}(U, V) = f_2(U, V), \\ \Psi_{21} + \Psi_{22} + \frac{\partial \Psi_{22}}{\partial V} V = -cg_2(U, V). \end{cases} \quad (5.5b)$$

The generalization that we propose in this case is similar to that of the process with one species. Thus, we define the functions f_1 and f_2 as follows:

$$\begin{aligned} f_1 &: [0, 1] \times [0, 1] \rightarrow \mathbb{R} \\ (U, V) &\mapsto (a_1U^n - \rho_1)(b_1V^m - \gamma_1), \end{aligned} \quad (5.6a)$$

$$\begin{aligned} f_2 &: [0, 1] \times [0, 1] \rightarrow \mathbb{R} \\ (U, V) &\mapsto (a_2U^n - \rho_2)(b_2V^m - \gamma_2), \end{aligned} \quad (5.6b)$$

$(n, m) \in \mathbb{R}_+^{*2}$, $(a_1, a_2, b_1, b_2) \in \mathbb{R}^{*4}$ and $(\rho_1, \rho_2, \gamma_1, \gamma_2) \in \mathbb{R}^{*4}$ are the parameters of the model. At this stage, the objective of the work is to determine the functions g_1 and g_2 , and the conditions on the parameters so that the system (5.2) admits at least a particular solution by the method of factorization of ordinary differential operators [9]. By replacing the expressions (5.6a) and (5.6b), respectively, in the systems (5.5a) and (5.5b) and after resolution, the following expressions for the functions are obtained:

$$f_1 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$$

$$(U, V) \mapsto (aU^n - \rho)(aV^n - \gamma), \quad (5.7a)$$

$$f_2 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$$

$$(U, V) \mapsto (bU^n - \gamma)(bV^n - \rho), \quad (5.7b)$$

$$g_1 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$$

$$(U, V) \mapsto \delta - \varepsilon a(U^n + V^n), \quad (5.7c)$$

$$g_2 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$$

$$(U, V) \mapsto \delta - \varepsilon b(U^n + V^n). \quad (5.7d)$$

With $n \in \mathbb{R}_+^*$, $(a, b) \in \mathbb{R}_+^{*2}$, $(\rho, \gamma, \delta) \in \mathbb{R}^{*3}$ and $\varepsilon \in \mathbb{R}$ which are the new parameters of the model. In addition:

$$\begin{cases} \Psi_{11}(U, V) = \frac{1}{k_1}(aV^n - \gamma) \text{ and } \Psi_{12}(U, V) = k_1(aU^n - \rho), \\ \Psi_{21}(U, V) = \frac{1}{k_2}(bU^n - \gamma) \text{ and } \Psi_{22}(U, V) = k_2(bV^n - \rho), \end{cases}$$

$$\text{where } (k_1, k_2) \in \mathbb{R}^{*2}. \quad (5.8)$$

By introducing the expressions (5.8) into equations (5.4), we obtain the solution of system (5.2). Thus, the system of reaction-diffusion equations to two species that we have just built is as follows:

$$\begin{cases} [\delta - \varepsilon a(u^n + v^n)] \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (au^n - \rho)(av^n - \gamma)u, \\ [\delta - \varepsilon b(u^n + v^n)] \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (bu^n - \gamma)(bv^n - \rho)v, \\ t > 0, \quad x \in \mathbb{R}. \end{cases} \quad (5.9)$$

And its particular solution is:

$$\begin{cases} u^n(x, t) = \frac{\rho \mu^n(x) \exp(\theta_n t)}{\rho + a \mu^n(x) (\exp(\theta_n t) - 1)}, \\ v^n(x, t) = \frac{\rho \vartheta^n(x) \exp(\theta_n t)}{\rho + b \vartheta^n(x) (\exp(\theta_n t) - 1)}, \end{cases} \quad (5.10)$$

where $\theta_n = \frac{n\rho(\rho + \gamma(n+1))}{\delta(n+1)}$.

6. Conclusion

This work has led to the analysis of the homogeneous reaction-diffusion equations whose terms of reactions are of the monostable or bistable type, and where the density of the species is denoted by $u(x, t)$. Several former works had shown that these reaction-diffusion equations have solutions of front type, in uniform translation connecting the state of balance at $-\infty$ to the state of balance at $+\infty$. These solutions are $u(x, t) = U(x - ct)$, where c is the speed of the front and U is the front profile. The front profile U checks a non-linear differential equation of the second order. In addition, the method of factorization of ordinary differential operators, recently, published by [9], opened a new step for the resolution of the differential equations. By applying this step to the differential equation of the front profile U , we proposed a method of generalization of the traditional homogeneous reaction-diffusion equations (monostable and bistable). This method has enabled us to build two generalized homogeneous reaction-diffusion equations: the first with one species and the second with two species. It also made it possible to determine the particular solution of each equation. These generalizations made it possible to increase the number of parameters of the traditional reaction-diffusion equations, thus, offering more flexibility for the adjustment of the solutions $u(x, t)$ on measured data.

References

- [1] H. Berestycki and B. Larrouturou, Quelques aspects mathématiques de la propagation des flammes prémélangées, *Pitman Res.* 220 (1991), 65-129.
- [2] J. Billingham and D. J. Needham, The development of traveling waves in quadratic and cubicautocatalysis with unequal diffusion rates. I. Permanent form of traveling waves, *Philos. Trans. R. Soc. Lond. Ser. A* 334 (1991), 1-24.
- [3] N. Kawasaki and K. Shigesada, *Biological Invasions: Theory and Practice*, Oxford University Press, 1997.
- [4] Jimmy Garnier, *Analyse mathématique de modèles de dynamique des populations: équations aux dérivées partielles paraboliques et équations intégral-différentielles*, Analysis of PDEs, Aix-Marseille Université, 2012.
- [5] R. A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* 7 (1937), 335-369.
- [6] N. S. Kolmogorov, N. Petrovsky and I. G. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bull. Univ. Moscou, Sér. A* 1 (1937), 1-26.
- [7] J. S. Clark, C. Fastie, G. Hurtt, S. T. Jackson, C. Johnson, G. A. King, M. Lewis, J. Lynch, S. Pacala, C. Prentice, E. W. Schupp, T. Webb III and P. Wyckoff, Reid's paradox of rapid plant migration, *BioScience* 48 (1998), 13-24.
- [8] D. Mollison, Spatial contact models for ecological and epidemic spread, *J. R. Stat. Ser. B Stat. Methodol.* 39 (1977), 283-326.
- [9] L. M. Berkovich, Method of factorization of ordinary differential operators and some of its applications, *AADM* 1 (2007), 122-149.
- [10] T. Turchin, *Quantitative Analysis of Movement: Measuring and Modeling Population Redistribution in Animals and Plants*, Sinauer Associates, 1998.
- [11] H. Berestycki, B. Nicolaenko and B. Scheurer, Traveling wave solutions to combustion models and their singular limits, *SIAM J. Math. Anal.* 16 (1985), 1207-1242.
- [12] H. G. Aronson and D. G. Weinberger, Multidimensional non-linear diffusion arising in population-genetics, *Adv. Math.* 30(1) (1978), 33-76.

- [13] P. C. Fife and J. B. McLeod, The approach of solutions of nonlinear diffusion equations to traveling front solutions, *Arch. Ration. Mech. Anal.* 65 (1977), 335-361.
- [14] S. Moussiliou, S. Massou, S. Moumouni and M. Tchoffo, Stochastic chaos in a class of Fokker-Planck equations derived from population dynamics, *Advances in Differential Equations and Control Processes* 10(2) (2012), 89-111.